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The universal representation group of Huybrechts's dimensional dual hyperoval

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Abstract

A d-dimensional dual hyperoval can be regarded as the image $\mathcal{S} = \rho(\Sigma)$ of a full d-dimensional projective embedding ρ of a dual circular space Σ. The affine expansion $\mathrm{Exp}(\rho)$ of ρ is a semibiplane and its universal cover is the expansion of the abstract hull $\widetilde{\rho}$ of ρ.

In this paper we consider Huybrechts's dual hyperoval, namely $\rho(\Sigma)$ where Σ is the dual of the affine space $\mathsf{AG}(n, 2) \subset \mathsf{PG}(n, 2)$ and ρ is induced by the embedding of the line grassmannian of $\mathsf{PG}(n, 2)$ in $\mathsf{PG}\left(\binom{n+1}{2} - 1, 2\right)$.

It is known that the universal cover of $\mathrm{Exp}(\rho)$ is a truncation of a Coxeter complex of type D_{2^n} and that, if $\widetilde{U}$ is the codomain of the abstract hull $\widetilde{\rho}$ of ρ, then $\widetilde{U}$ is a subgroup of the Coxeter group D of type D_{2^n}, $|\widetilde{U}| = 2^{2^n - 1}$ but $\widetilde{U}$ is non-commutative. This information does not explain what the structure of $\widetilde{U}$ is and how $\widetilde{U}$ is placed inside D. These questions will be answered in this paper.

Keywords: dimensional dual hyperovals, semibiplanes, embeddings, Coxeter groups, exterior algebras

MSC 2000: primary: 20F55; secondary: 51E20, 51E45, 15A75

1 Introduction

Throughout this paper we adopt the notation of [5] for groups, thus using the colon ":" for split extensions, the symbol 2^n to denote an elementary abelian 2-group of order 2^n, and so on. We refer to [12] for basics on diagram geometry.

1.1 Dimensional dual hyperovals

Dimensional dual hyperovals have been introduced by Yoshiara [18] and Huybrechts and Pasini [10], mainly in connection with the investigation of $\mathsf{AG}.c^*$-geometries (in particular $c.c^*$-geometries with (IP), namely semibiplanes; see Pasini and Yoshiara [14]). In particular, a 2-dimensional dual hyperoval of size 22, living inside the hermitian variety $\mathcal{H}(5,4)$ and admitting $3^{\cdot}\mathrm{Aut}(\mathsf{M}_{22})$ as its automorphism group, naturally gives rise to an $\mathsf{AG}.c^*$-geometry for $2^{12}{:}3^{\cdot}\mathrm{Aut}(\mathsf{M}_{22})$, thoroughly investigated in [10]. However, dimensional dual hyperovals have soon appeared to be relevant in other respects than strict diagram geometry or the investigation of semibiplanes. For instance, they play a role in the characterization of Veronesean varieties by Thas and Van Maldeghem [16], [17]. They are quite naturally linked with distance regular graphs, semi-partial geometries and codes (Cooperstein and Thas [6], Pasini and Yoshiara [15]). In particular, new distance regular graphs have been discovered with the aid of dimensional dual hyperovals (Pasini and Yoshiara [15]). Interesting connection also exist between certain dimensional dual hyperovals and Steiner systems (Huybrechts [9], Buratti and Del Fra [4], Del Fra and Yoshiara [8]).

In view of the above, it is not surprising that dimensional dual hyperovals have soon been regarded as objects interesting in themselves. The earliest thoroughful investigation of their geometrical properties is due to Del Fra [7], but many progresses have been done since then. We refer the reader to Yoshiara [20] for a survey of this topic, updated at 2005. We shall only recall a few essentials here.

Let $V := V(N,q)$, the N-dimensional vector space over $\mathsf{GF}(q)$. According to the usual definition, for a positive integer $d < N$, a *d-dimensional dual hyperoval* of $\mathsf{PG}(V)$ is a family $\mathcal{S}$ of d-dimensional subspaces of $\mathsf{PG}(V)$ such that:

(DH1) $|X \cap Y| = 1$ for any two distinct members $X, Y \in \mathcal{S}$;

(DH2) every point of $\mathsf{PG}(V)$ belongs to either none or just two members of $\mathcal{S}$;

(DH3) $\langle \bigcup_{X\in\mathcal{S}} X \rangle = \mathsf{PG}(V)$.

Clearly, (DH3) implies $N \geq 2d+1$. An upper bound for N has been discovered by Yoshiara [19] (see also Del Fra [7] for the case of $d = 2$).

Proposition 1.1. *If either $q > 2$ or $d \leq 2$ then $N \leq \binom{d+2}{2}$. If $q = 2$ and $d > 2$ then $N \leq \binom{d+2}{2} + 2$.*

Actually, $N \leq \binom{d+2}{2}$ in all known examples. So, one might conjecture that $N \leq \binom{d+2}{2}$ even when $q = 2$.

1.2 Dimensional projective embeddings

It readily follows from (DH1) and (DH2) that $\mathcal{S}$, regarded as an abstract combinatorial structure, is nothing but the dual Σ of the circular space on $\theta := q^d + q^{d-1} + \cdots + q + 2$ points. (We recall that Σ is the point-block geometry with $\theta(\theta-1)/2$ points and θ blocks, where every point belongs to precisely two blocks, every block has $\theta - 1$ points and any two blocks have just one point in common.) Accordingly, we may regard $\mathcal{S}$ as the image of a (full) d-dimensional projective embedding $\rho : \Sigma \to \mathsf{PG}(V)$ of Σ in $\mathsf{PG}(V)$. Explicitly, ρ is a mapping from the set P of points of Σ to the set of 1-dimensional linear subspaces of V such that:

(PE1) ρ is injective;

(PE2) for every block B of Σ, the set $\rho(B) := \bigcup_{p\in B} \rho(p)$ is a $(d+1)$-dimensional linear subspace of V;

(PE3) $\rho(P)$ spans V.

Clearly, $\mathcal{S} = \rho(\mathcal{B}) = \{\rho(B)\}_{B\in\mathcal{B}}$, where $\mathcal{B}$ is the block-set of Σ, but we prefer to say that $\mathcal{S}$ is the *image* of Σ, thus writing $\mathcal{S} = \rho(\Sigma)$ even if these conventions are slightly abusive.

The point of view we have thus chosen makes it easier to define morphisms of dimensional dual hyperovals, by exploiting the usual machinery set up for embeddings (see Pasini [13]; compare Yoshiara [20, section 2.5]). So, if $\rho' : \Sigma \to \mathsf{PG}(V')$ is another d-dimensional projective embedding of Σ, a *morphism* from $\mathcal{S}' = \rho'(\Sigma)$ to $\mathcal{S}$ is a morphism from ρ' to ρ, namely a semi-linear mapping $f : V' \to V$ such that $f\rho' = \rho$. Needless to say, an *isomorphism* is an invertible morphism. In spite of the above, $\mathrm{Aut}(\mathcal{S})$ is not defined as the same as the group of all automorphisms of ρ. Every automorphism of ρ acts trivially on $\mathcal{S}$, whereas one would like to see some action of $\mathrm{Aut}(\mathcal{S})$ on $\mathcal{S}$. The stabilizer of $\mathcal{S}$ in $P\Gamma L(N, q)$ is perhaps the most natural choice for $\mathrm{Aut}(\mathcal{S})$, but it is more convenient to define $\mathrm{Aut}(\mathcal{S})$ as the (setwise) stabilizer of $\mathcal{S}$ in $\Gamma L(N, q)$, which is the same as the group of all isomorphisms from $\rho\alpha$ to $\rho\beta$, where α, β range over $\mathrm{Aut}(\Sigma)$. We shall follow this convention.

It is well known that every d-dimensional projective embedding $\rho : \Sigma \to \mathsf{PG}(V)$ admits a *projective hull*, namely a pair $(\hat{\rho}, \hat{f})$ uniquely determined up to isomorphisms by the following conditions (see Pasini [13]):

(PH1) $\hat{\rho} : \Sigma \to \mathsf{PG}(\widehat{V})$ is a d-dimensional projective embedding of Σ and $\hat{f} : \hat{\rho} \to \rho$ is a morphism from $\hat{\rho}$ to ρ;

(PH2) for any other d-dimensional projective embedding $\rho' : \Sigma \to \mathsf{PG}(V')$, if there is a morphism $f : \rho' \to \rho$ then a morphism $g : \hat{\rho} \to \rho'$ also exists such that $fg = \hat{f}$.

The image $\hat{\mathcal{S}} = \hat{\rho}(\Sigma)$ of Σ by $\hat{\rho}$ will be called the *projective hull* of $\mathcal{S} = \rho(\Sigma)$. We say that $\mathcal{S}$ is *projectively dominant* if $\hat{\rho} \cong \rho$, namely $\hat{f}$ is an isomorphism. In short, $\mathcal{S}$ is projectively dominant if it is its own projective hull. Clearly, if N attains the upper bound of Proposition 1.1, then $\mathcal{S}$ is projectively dominant.

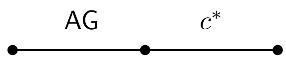
1.3 Expansions

Given $\Sigma = (P, \mathcal{B})$ as above, let $\rho : \Sigma \to \mathsf{PG}(V)$ be a d-dimensional projective embedding and put $\mathcal{S} := \rho(\Sigma)$. The *affine expansion* of Σ by ρ (also called the *expansion* of ρ, for short, or the *expansion* of $\mathcal{S}$) is the geometry $\mathrm{Exp}(\rho)$ of rank 3 defined as follows:

Taken the integers 0, 1, 2 as types for $\mathrm{Exp}(\rho)$, the 0-elements of $\mathrm{Exp}(\rho)$ (also called points) are the vectors of V, the 1-elements (lines) are the cosets of the subspaces $\rho(p)$ of V for $p \in P$ and the 2-elements (blocks) are the cosets of the subspaces $\rho(B)$, for $B \in \mathcal{B}$. The incidence relation is the natural one, namely inclusion.

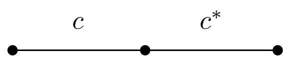
The residues of the points of $\mathrm{Exp}(\rho)$ are isomorphic to Σ whereas the residues of the blocks of $\mathrm{Exp}(\rho)$ are $(d+1)$-dimensional affine spaces over $\mathsf{GF}(q)$. So, $\mathrm{Exp}(\rho)$ belongs to the following diagram, where the label AG denotes the class of affine spaces and c^* the class of dual circular spaces:

(AG.c^*)



In particular, when $q = 2$ then $\mathrm{Exp}(\rho)$ is a semibiplane. In this case its diagram can be depicted as follows:

($c.c^*$)



Let U be the translation group of the affine geometry $\mathsf{AG}(V)$ of V. Then $\mathrm{Aut}(\mathcal{S})$, regarded as a subgroup of the automorphism group $\mathrm{Aut}(\mathsf{AG}(V))$ of $\mathsf{AG}(V)$, normalizes U. The semidirect product $U{:}\mathrm{Aut}(\mathcal{S})$ is the stabilizer of $\mathrm{Exp}(\rho)$ in $\mathrm{Aut}(\mathsf{AG}(V)) = A\Gamma L(N, q)$. We shall denote that stabilizer by $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho))$. Clearly, $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho))$ is point-transitive on $\mathrm{Exp}(\rho)$, and it is flag-transitive if and only if $\mathrm{Aut}(\mathcal{S})$ acts doubly-transitively on $\mathcal{S}$. In principle, $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho))$ might be smaller than the full automorphism group of the geometry $\mathrm{Exp}(\rho)$. Actually, we are not aware of any example like that, but examples of this kind are met frequently when we switch from d-dimensional projective embeddings to (d,q)-embeddings (to be defined in the next subsection).

Every morphism $f : \rho' \to \rho$ induces a covering $\mathrm{Exp}(f)$ from $\mathrm{Exp}(\rho')$ to $\mathrm{Exp}(\rho)$. In particular, if $(\hat{\rho}, \hat{f})$ is the projective hull of ρ, then $\mathrm{Exp}(\hat{f})$ is a covering from $\mathrm{Exp}(\hat{\rho})$ to $\mathrm{Exp}(\rho)$. It also follows from the definition of $\hat{\rho}$ that $\mathrm{Aut}(\mathcal{S})$ lifts to the stabilizer of $\ker(f)$ in $\mathrm{Aut}(\widehat{\mathcal{S}})$, where $\widehat{\mathcal{S}} := \hat{\rho}(\Sigma)$. Accordingly, $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho))$ lifts to a subgroup of $\mathrm{Aut}_{\hat{\rho}}(\mathrm{Exp}(\hat{\rho}))$. However, in general, $\mathrm{Exp}(\hat{\rho})$ is not the universal cover of $\mathrm{Exp}(\rho)$. This unpleasant asimmetry can be repaired by generalizing the notion of embeddings, as we shall do in the next subsection. The definition we shall give is borrowed from [13]. It is not so general as in [13], but it suites our present needs.

1.4 Abstract embeddings and their expansions

An *abstract* (d,q)-*embedding* (a (d,q)-*embedding* for short) of $\Sigma = (P, \mathcal{B})$ in a group U is a mapping ρ from P into the subgroup lattice of U such that:

(AE1) $\rho(p)$ is an elementary abelian subgroup of U of order q, for any $p \in P$;

(AE2) we have $\rho(e_1) \cap \rho(e_2) = 1$ for any two distinct points $e_1, e_2 \in P$;

(AE3) $\rho(B) := \bigcup_{p \in B} \rho(p)$ is an elementary abelian subgroup of U of order q^{d+1}, for every block $B \in \mathcal{B}$;

(AE4) the set $\bigcup_{p \in P} \rho(p)$ spans U.

We call U the *target group* of the (d,q)-embedding ρ. The family $\rho(\Sigma) := \{\rho(p)\}_{p \in P} \cup \{\rho(B)\}_{B \in \mathcal{B}}$ will be called the *image* of Σ by ρ. We write $\rho : \Sigma \to U$ when we want to recall that ρ is a (d,q)-embedding of Σ in U.

Given another (d,q)-embedding $\overline{\rho} : \Sigma \to \overline{U}$, a morphism from $\overline{\rho}$ to ρ is a homomorphism $f : \overline{U} \to U$ such that, for every element (point or block) X of Σ, f induces an isomorphism from $\overline{\rho}(X)$ to $\rho(X)$. The *kernel* of the morphism f is its kernel $\ker(f)$ as a homorphism from $\overline{U}$ to U.

Clearly, d-dimensional projective embeddings are (d,q)-embeddings (with additive groups of vector spaces as target groups) and morphisms of d-dimensional projective embeddings are morphisms in the above sense. Hulls of (d,q)-embeddings are defined in the same way as projective hulls of d-dimensional projective embeddings. We refer to [13] for details. We only recall how the hull $(\tilde{\rho}, \tilde{f})$ of a (d,q)-embedding $\rho : \Sigma \to U$ is constructed. The image $\rho(\Sigma)$ of Σ is actually an amalgam of groups. Let $\widetilde{U}$ be the universal completion of that amalgam and $\tilde{f} : \widetilde{U} \to U$ be the canonical projection of $\widetilde{U}$ onto U. For every element (point or block) X of Σ, let us denote by $\widetilde{\rho(X)}$ the subgroup of

$\widetilde{U}$ corresponding to the member $\rho(X)$ of $\rho(\Sigma)$. Namely, $\widetilde{\rho(X)}$ is $\rho(X)$ itself, but regarded as a subgroup of $\widetilde{U}$ rather than of U. Taking $\widetilde{U}$ as the target group of $\tilde{\rho}$, we define $\tilde{\rho}(X) := \widetilde{\rho(X)}$. Clearly, the canonical projection $\tilde{f} : \widetilde{U} \to U$ is a morphism from $\tilde{\rho}$ to ρ. We call $(\tilde{\rho}, \tilde{f})$ the *abstract hull* of ρ (also the *hull* of ρ, for short). The group $\widetilde{U}$ will be called the *universal representation group* of ρ. (This terminology is motivated by some resemblance with representation groups in the sense of Ivanov and Shpectorov [11].) We say that ρ is *abstractly dominant* (for short, *dominant*) if it is its own abstract hull. By a little abuse, we extend this terminology to $\rho(\Sigma)$, thus calling $\widetilde{U}$ the *universal representation group* of $\rho(\Sigma)$, for instance.

The *expansion* $\mathrm{Exp}(\rho)$ of a (d, q)-embedding $\rho : \Sigma \to U$ is defined in the same way as the expansion of a d-dimensional projective embedding: the elements of U are the points of $\mathrm{Exp}(\rho)$, the lines of $\mathrm{Exp}(\rho)$ are the right cosets $u \cdot \rho(p)$ for $p \in P$ and $u \in U$, the blocks are the right cosets $u \cdot \rho(B)$ for $B \in \mathcal{B}$. The target group U, acting on itself by left multiplication, acts regularly on the point-set of $\mathrm{Exp}(\rho)$. We call it the *translation group* of $\mathrm{Exp}(\rho)$. Let $\mathrm{Aut}(\rho(\Sigma))$ be the set-wise stabilizer of $\rho(\Sigma)$ in $\mathrm{Aut}(U)$. Then $\mathrm{Aut}(\rho(\Sigma))$ can be regarded as a subgroup of $\mathrm{Aut}(\mathrm{Exp}(\rho))$. It stabilizes the point $1 \in U$ of $\mathrm{Exp}(\rho)$ and normalizes the translation group U. The semi-direct product $U{:}\mathrm{Aut}(\rho(\Sigma))$, regarded as a subgroup of $\mathrm{Aut}(\mathrm{Exp}(\rho))$, will be denoted by $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho))$. In general, $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho)) < \mathrm{Aut}(\mathrm{Exp}(\rho))$.

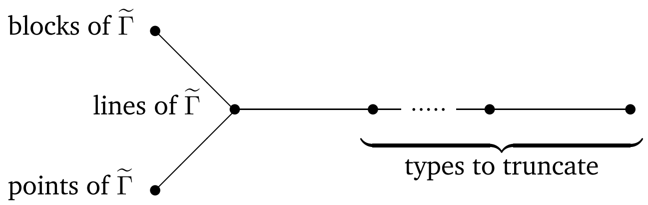
Every morphism $f : \overline{\rho} \to \rho$ induces a covering $\mathrm{Exp}(f)$ from $\mathrm{Exp}(\overline{\rho})$ to $\mathrm{Exp}(\rho)$ and $\ker(f)$, regarded as a subgroup of the translation group of $\mathrm{Exp}(\overline{\rho})$, is the deck group of the covering $\mathrm{Exp}(f)$. The following is proved in [13]:

Proposition 1.2. *The universal cover of the geometry* $\mathrm{Exp}(\rho)$ *is the expansion* $\mathrm{Exp}(\tilde{\rho})$ *of the abstract hull* $\tilde{\rho}$ *of* ρ*. In particular,* $\mathrm{Exp}(\rho)$ *is simply connected if and only if* ρ *is abstractly dominant.*

For quite a few d-dimensional dual hyperovals $\mathcal{S} = \rho(\Sigma)$ the expansion $\mathrm{Exp}(\rho)$ is known to be simply connected (see Pasini and Yoshiara [14] and [15]). In those cases, the d-dimensional projective embedding ρ is abstractly dominant. However, this is not always the case. For instance, let Σ_{22} be the dual of the circular space with 22 points and $\mathcal{S}_{22} = \rho(\Sigma_{22})$ be its realization as a 2-dimensional dual hyperoval of $\mathsf{PG}(5, 4)$ with $\mathrm{Aut}(\mathcal{S}_{22}) = 3^{\cdot}\mathrm{Aut}(\mathsf{M}_{22})$, mentioned in Subsection 1.1 (see [20, 5.1] for more details). As the ambient space of ρ attains the upper bound of Proposition 1.1, ρ is projectively dominant. However, the geometry $\Gamma := \mathrm{Exp}(\rho)$ is not simply connected, but it admits a simply connected double cover $\widetilde{\Gamma}$ (Huybrechts and Pasini [10]). Accordingly, ρ is not abstractly dominant and $\widetilde{\Gamma}$ is the expansion of the abstract hull $\tilde{\rho} : \Sigma_{22} \to \widetilde{U}$ of ρ. The universal representation group $\widetilde{U}$ of ρ is non-commutative. In fact, $\widetilde{U}$ is the extraspecial group 2_+^{1+12}. (We warn the reader that $\widetilde{U}$ is sloppily described as 2^{13} in [10].)

1.5 The problem discussed in this paper

Let $q = 2$ and $\rho : \Sigma \to \mathsf{PG}(V)$ be a d-dimensional projective embedding of Σ, with $V = V(N, 2)$. It may happen that the universal cover $\widetilde{\Gamma}$ of $\Gamma := \mathrm{Exp}(\rho)$ is a truncated Coxeter complex of type D_M, where $M = 2^{d+1}$:



If this is the case then we say that the dimensional dual hyperoval $\mathcal{S} = \rho(\Sigma)$ is *Coxeter-like*, for short. Suppose that $\mathcal{S}$ is indeed Coxeter-like. Then $\widetilde{\Gamma}$ has 2^{M-1} points. The universal representation group $\widetilde{U}$ of $\mathcal{S}$ acts regularly on the point-set of $\widetilde{\Gamma}$. Hence $|\widetilde{U}| = 2^{M-1}$. However, if $d > 2$ then $M - 1 > \binom{d+2}{2} + 2$ and, if $d = 2$ then $M - 1 > \binom{d+2}{2}$. So, if $d > 1$ then $\widetilde{U}$ is non-commutative. Indeed, if $\widetilde{U}$ were commutative, then it would be an elementary abelian 2-group, whence a $\mathsf{GF}(2)$-vector space, contrary to the bounds stated by Proposition 1.1. On the other hand, $\widetilde{U}$ is a subgroup of the Coxeter group $W := W_{\mathsf{D}_M}$ of type D_M, since $\mathrm{Aut}(\widetilde{\Gamma}) = W$. We recall that $W = O_2(W){:}\mathrm{Sym}(M)$ with $O_2(W) = 2^{M-1}$. Thus, $\widetilde{U}$ cannot be a subgroup of $O_2(W)$. So, we should cut $\widetilde{U}$ off of W, but keeping in mind that $\widetilde{U}$, being non-commutative, cannot be entirely contained in $O_2(W)$. This is the problem we shall tackle in this paper.

Before to go on, we get rid of the case $d = 1$. In this case $\Gamma = \mathrm{Exp}(\rho)$ is the geometry obtained from $\mathsf{PG}(3, 2)$ by removing all points and lines of a given plane π and all points and planes through a distinguished point p of π. It is well known (and easy to check) that this geometry is simply connected. Hence ρ is abstractly dominant.

Assume now $d > 1$. For $N = \binom{d+2}{2}$, only two Coxeter-like dimensional dual hyperovals are known with ambient vector space $V(N, 2)$. They are those denoted by $\mathcal{S}(\emptyset)$ and $\mathcal{S}(V \setminus \{0\})$ in [4] and [8]. In this paper, we shall denote them by $\mathcal{S}_H$ and $\mathcal{S}_{BDF}$ respectively, where the subscript H is meant to record C. Huybreths, who has studied this dimensional dual hyperoval in [9], whereas BDF should remind us of Buratti and Del Fra [4]. Several Coxeter-like d-dimensional dual hyperovals are known with ambient vector space of dimension less then $\binom{d+2}{2}$, but all of them turn out to be homorphic images of $\mathcal{S}_H$ (Pasini and Yoshiara [15]). So, as far as our knowledge presently goes, $\mathcal{S}_H$ and $\mathcal{S}_{BDF}$ are the most interesting Coxeter-like examples, and $\mathcal{S}_H$ is perhaps the most interesting of the two.

In this paper we shall focus on $\mathcal{S}_H$. In Section 2 we shall describe $\mathcal{S}_H$ and its expansion. Denoting by ρ_H the d-dimensional projective embedding giving rise to $\mathcal{S}_H$, in Section 3 we shall describe the abstract hull $\tilde{\rho}$ of ρ_H. As a by-product of our construction, we will obtain that ρ_H is projectively dominant (as already proved by Buratti and Del Fra [4]). Finally, in Section 4 we shall show how the universal representation group of $\mathcal{S}_H$ is placed inside the Coxeter group W of type D_M.

A warning, before to finish: in [13, Theorem 8.5] it is claimed that the universal representation group of $\mathcal{S}_H$ is abelian. Clearly, that claim is wrong.

2 $\mathcal{S}_H$ and its expansion

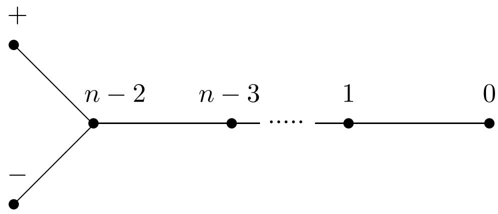
2.1 Huybrechts's construction of $\mathcal{S}_H$

Let $V = V(n+1, 2)$ ($n \geq 2$) and $V_0 \cong V(n, 2)$ be a hyperplane of V. Denoting by $\mathcal{A}$ the affine geometry $\mathsf{PG}(V) \setminus \mathsf{PG}(V_0)$, let $\Sigma = (P, \mathcal{B})$ be the dual of the point-line system of $\mathcal{A}$. Thus, P is the set of lines of $\mathcal{A}$ and $\mathcal{B}$ is the point-set of $\mathcal{A}$. As the point-line system of $\mathcal{A}$ is a circular space with 2^n points, Σ is a dual circular space. Turning to the line-grassmannian $\mathrm{G}r(\mathsf{PG}(V))$ of $\mathsf{PG}(V)$, the members of $\mathcal{B}$ are maximal singular subspaces of $\mathrm{G}r(\mathsf{PG}(V))$, of (projective) dimension $d = n - 1$, whereas P is a set of points of $\mathrm{G}r(\mathsf{PG}(V))$. Let ρ_{gr} be the natural projective embedding of $\mathrm{G}r(\mathsf{PG}(V))$ into $\mathsf{PG}(V \wedge V)$ and ρ_H be the mapping induced by ρ_{gr} on $\Sigma = (P, \mathcal{B})$. Then $\rho_H(P)$ spans $\mathsf{PG}(V \wedge V)$ and the image $\mathcal{S}_H = \rho_H(\mathcal{B})$ of $\mathcal{B}$ by ρ_H is a d-dimensional dual hyperoval of $\mathsf{PG}(V \wedge V)$. This is indeed the dimensional dual hyperoval discovered by Huybrechts [9].

2.2 $\mathrm{Aut}(\mathcal{S}_H)$ and $\mathrm{Aut}_{\rho_H}(\mathrm{Exp}(\rho_H))$

The automorphism group $\mathrm{Aut}(\mathcal{A})$ of the affine geometry $\mathcal{A}$ is the stabilizer of V_0 in $L_{n+1}(2)$, namely $ASL(n, 2) = T{:}L$, where T is the translation group of $\mathcal{A}$ and $L \cong L_n(2)$. Clearly, also $\mathrm{Aut}(\mathcal{S}_H) \cong ASL(n, 2)$. Accordingly, $\mathrm{Aut}_{\rho_H}(\mathrm{Exp}(\rho_H)) = U{:}A_H$, where $U := V \wedge V$ (the latter being regarded as an elementary abelian 2-group) and $A_H := \mathrm{Aut}(\mathcal{S}_H)$. Clearly $A_H = T_H{:}L_H$, where $T_H \cong T$ and $L \cong L_n(2)$. The group A acts on $U = V \wedge V$ as a subgroup of the group $L_{n+1}(2)$ of linear transformations of V, in its natural action on $V \wedge V$. More explicitly, recall that A_H is the stabilizer of V_0 in $L_{n+1}(2)$. Accordingly, the subgroup L_H of A_H is the stabilizer of V_0 and a vector $v_\infty \in V \setminus V_0$. As $V = \langle v_\infty \rangle \oplus V_0$, we can split $V \wedge V$ as $(v_\infty \wedge V_0) \oplus (V_0 \wedge V_0)$. The group L_H, acting on $V \wedge V$ as a subgroup of $L_{n+1}(2)$, stabilizes both $v_\infty \wedge V_0$ and $V_0 \wedge V_0$. Moreover, on each of these two subspaces, the action of L_H is that induced by its action on V_0. The elements of T_H can be identified with the vectors of V_0. The subspace $V_0 \wedge V_0$ is centralized by T_H whereas, if $\tau_z \in T_H$ corresponds to $z \in V_0$, then $\tau_z(v_\infty \wedge x) = v_\infty \wedge x + x \wedge z$ for every $x \in V_0$.

The expansion $\mathrm{Exp}(\rho_H)$ can also be described as follows. Let Δ be the building of type D_{n+1} over $\mathsf{GF}(2)$ and $F = \{M^+, M^-\}$ be a flag of Δ of type $\{+, -\}$, where types are given as below and, for $\epsilon = +$ or $-$, M^ϵ is the element of F of type ϵ.



Then $\mathrm{Exp}(\rho_H)$ is isomorphic to the subgeometry Γ of Δ formed by the elements of type $+, -$ and $n-2$ at maximal distance from F (see Pasini and Yoshiara [14, section 6]). The incidence relation of Γ is inherited from Δ except that two elements X^+, X^- of Γ of type $+$ and $-$, incident in Δ, are declared to be incident in Γ if and only if the flag $\{X^+, X^-\}$ has maximal distance from F. The elements of Γ of type $n-2$ correspond to the lines of $\mathrm{Exp}(\rho_H)$. We can take the elements of type $+$ as points and those of type $-$ as blocks, but we can also interchange the roles of $+$ and $-$, thus taking $(-)$-elements as points and $(+)$-elements as blocks. This makes it clear that Γ admits a duality δ, contributed by a graph automorphism of $\mathrm{Aut}(\Delta)$. The group $\mathrm{Aut}_{\rho_H}(\mathrm{Exp}(\rho_H))$ is just the stabilizer of F in $\mathrm{Aut}(\Delta)$. Its subgroup L_H also stabilizes a unique flag opposite to F, whereas T_H acts trivially on one of the two members of F, say M^+. The subgroup $v_\infty \wedge V_0$ of U acts trivially on M^-, whereas $V_0 \wedge V_0$ acts trivially on both M^+ and M^-. The above mentioned duality δ normalizes $\mathrm{Aut}_{\rho_H}(\mathrm{Exp}(\rho_H))$. It can be chosen in such a way as to normalize both L_H and $V_0 \wedge V_0$ and interchange T_H with $v_\infty \wedge V_0$. Accordingly, δ permutes M^+ and M^- and stabilizes the flag opposite to F stabilized by L_H.

2.3 The universal cover of $\mathrm{Exp}(\rho_H)$

With $\Gamma = \mathrm{Exp}(\rho_H)$ as above, let $\widetilde{\Gamma}$ be its universal cover. Then $\widetilde{\Gamma}$ is a truncated Coxeter complex of type D_{2^n} (Baumeister, Meixner and Pasini [1]; see also Pasini and Yoshiara [14] or Baumeister and Stroth [3]). So, $\mathcal{S}_H$ is Coxeter-like.

Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the covering projection from $\widetilde{\Gamma}$ onto Γ and K be the deck group of π. By comparing the number of chambers of $\widetilde{\Gamma}$ with the number of chambers of Γ we immediately obtain that

$$|K| = 2^{2^n - \binom{n+1}{2} - 1}.$$

Note that if $n = 2$ then $K = 1$, namely $\widetilde{\Gamma} = \Gamma$. So, henceforth we assume $n > 2$.

In order to describe K more precisely we need to recall a few facts on the Coxeter groups of type D_{2^n} and C_{2^n}. Let S be a set of size 2^n and U_S be the $\mathsf{GF}(2)$-vector space with S as a basis. Namely, the vectors of U_S are the functions from S to $\mathsf{GF}(2)$. The *support* of such a vector f is the set $S(f) := \{p \in S \mid f(p) = 1\}$ and the *weight* $w(f)$ of f is the size $|S(f)|$ of $S(f)$. If $X = S(f)$, we shall write $f = f_X$. In particular, if $p \in S$ then $f_{\{p\}}$, also denoted by f_p, is the vector with $S(f) = \{p\}$. The vectors of even weight form a maximal subspace U_S^+ of U_S. Clearly, $U_S = U_S^+ \oplus \langle f_p \rangle$ for any $p \in S$. Note also that, for every $X \subseteq S$, we have $f_X = \sum_{p \in X} f_p$.

The group $\mathrm{Sym}(S)$ of all permutations of S can quite naturally be regarded as a group of linear transformations of U_S. The Coxeter group C of type C_{2^n} is the semidirect product $U_S{:}\mathrm{Sym}(S)$. Clearly, $\mathrm{Sym}(S)$ stabilizes U_S^+. The subgroup $D := U_S^+{:}\mathrm{Sym}(S)$ of C is the Coxeter group of type D_{2^n}.

As $|S| = 2^n$, we can regard S as the point-set of an affine space. Let us choose such an affine space $\mathcal{A}_S$ on S. For every $k = 0, 1, \ldots, n$, let $U_{(k)}$ be the subspace of U_S spanned by the vectors f_X, for X a k-dimensional affine subspace of $\mathcal{A}_S$. Clearly,

$$U_{(0)} = U_S > U_{(1)} = U_S^+ > U_{(2)} > \cdots > U_{(n-1)} > U_{(n)} = \langle f_S \rangle.$$

Note that $U_{(1)}$ and $U_{(n)}$ are normalized by $\mathrm{Sym}(S)$. On the other hand, the following is clear:

Lemma 2.1. *If $1 < k < n$ then the normalizer of $U_{(k)}$ in $\mathrm{Sym}(S)$ is the automorphism group $\mathrm{Aut}(\mathcal{A}_S) \cong AGL(n, 2)$ of $\mathcal{A}_S$.*

So, $\mathrm{Aut}(A_S)$ acts on $U_{(k)}/U_{(k+1)}$. We can say more on this action, but we need a few preliminaries. Pick an independent spanning set $p_0, p_1, \ldots, p_n$ of $\mathcal{A}_S$, take p_0 as the null vector and the family $B := \{\{p_i\}\}_{i=1}^n$ as a basis of a vector space $V_0 \cong V(n, 2)$ associated to $\mathcal{A}_S$, so that $\mathcal{A}_S = \mathsf{AG}(V_0)$. Thus, every point $p \in S$ can be regarded as a vector v_p of V_0. Needless to say, v_{p_0} is the null vector of V_0 and, if p, q are distinct points of $\mathcal{A}_S \setminus \{p_0\}$ and $\{p_0, p, q, r\}$ is the plane of $\mathcal{A}_S$ spanned by $\{p_0, p, q\}$, then $v_p + v_q = v_r$. The k-dimensional linear subspaces of V_0 are the k-dimensional affine subspaces of $\mathcal{A}_S$ that contain p_0. We write v_i for v_{p_i}.

Let $L = Aut(V_0)$ be the stabilizer of p_0 in $\mathrm{Aut}(\mathcal{A}_S) < \mathrm{Sym}(S)$. Clearly, the action of $\mathrm{Sym}(S)$ on U_S induces an action of L on U_S, and L stabilizes $U_{(k)}$ for every $k = 0, 1, \ldots, n$. Hence L acts on the quotient $U_{(k)}/U_{(k+1)} \cong U_{[k]}$. On the other hand $L = \mathrm{Aut}(V_0)$ also acts on $\wedge^k V_0$. The following statement is implicit in the calculations of [1], but we shall give a more perspicuos proof of it in Section 4:

Theorem 2.2. *For every $k = 1, 2, \ldots, n-1$, $\wedge^k V_0$ and $U_{(k)}/U_{(k+1)}$ are isomorphic as L-modules.*

The L-module $\wedge^k V_0$ is irreducible. Therefore:

Corollary 2.3. *The L-module $U_{(k)}/U_{(k+1)}$ is irreducible, for $k = 1, 2, \ldots, n-1$.*

Consequently,

Corollary 2.4. *The sequence $0 < U_{(n)} < \cdots < U_{(2)} < U_{(1)} < U_S$ is a composition series for $\mathrm{Aut}(\mathcal{A}_S)$.*

We shall now analyze the structure of U_S more thoroughly. We firstly state a few conventions. If $J \subseteq \{1, 2, \ldots, n\}$, we put $v_J = \sum_{j \in J} v_j$. In particular, $v_\emptyset = 0$ (the null vector of V_0). It is also clear that, for every $I \subseteq \{1, 2, \ldots, n\}$, the set $S_I := \{v_J\}_{J \subseteq I}$ is a linear subspace of V_0 of dimension $|I|$ (but not all linear subspaces of V_0 have this form). In particular, $S_{1,2,\ldots,n} = V_0$. For $k = 0, 1, \ldots, n$, put $S_k(B) = \{S_J\}_{|J|=k}$. Then the set $\{f_X \mid X \in S_k(B)\}$ spans a linear subspace $U_{[k]}$ of U_S. Clearly, $U_{[0]}$ and $U_{[n]}$ are 1-dimensional. The following statement is also implicit in the calculations of [1], but we shall obtain it in a more clear way:

Lemma 2.5. $\dim(U_{[k]}) = \dim(\wedge^k V_0) = \binom{n}{k}$*, for every $k = 0, 1, \ldots, n$.*

Proof. By definition, the elements f_X, with $X \in S_k(B)$, span $U_{[k]}$. Their number is $\binom{n}{k}$. It remains to prove that they are independent, namely that any $X = \langle v_{i_1}, \ldots, v_{i_k} \rangle$ is not the symmetric difference of other elements in $S_k(B)$. This is evident since $v_{i_1} + \cdots + v_{i_k}$, which belongs to X, does not belong to any other element of $S_k(B)$. $\square$

Theorem 2.6. $U_{(k)} = \bigoplus_{i=k}^{n} U_{[i]}$ *for every $k = 0, 1, 2, \ldots, n$.*

Proof. Every f_X with X a subset of V_0 can be obtained as a sum of suitable members of $\cup_{i=0}^n U_{[i]}$. Indeed, let $X = \{v_{J_1}, \ldots, v_{J_r}\}$. Put $\mathcal{X} = \{J_i\}_{i=1}^r$ and let m be the maximal size of a member of $\mathcal{X}$. We form a set $\mathcal{Y} = \mathcal{Y}_m \cup \mathcal{Y}_{m-1} \cup \cdots \cup \mathcal{Y}_0$ as follows: $\mathcal{Y}_m$ contains all subsets of members of $\mathcal{X}$ of size m and $\mathcal{Y}_{i-1}$ contains all subsets of members of $\overline{\mathcal{Y}}_i := \cup_{j=i}^m \mathcal{Y}_j$ that have size $i-1$ and either are contained in an odd number of members of $\overline{\mathcal{Y}}_i$ but do not belong to $\mathcal{X}$, or are contained in an even number of members of $\overline{\mathcal{Y}}_i$ and belong to $\mathcal{X}$. It is not difficult to see that $f_X = \sum_{Y \in \mathcal{Y}} f_Y \in \cup_{i=0}^m U_{[i]}$.

By the above, $\cup_{k \geq 0} U_{[k]}$ spans U_S. However, $\dim(U_{[k]}) = \binom{n}{k}$ by Lemma 2.5. As $\dim(U_S) = 2^n$ and $\sum_{k=0}^n \binom{n}{k} = 2^n$, we obtain that $U_S = \oplus_{k=0}^n U_{[k]}$.

Clearly, $U_{(k)}$ contains the span of $\cup_{i=k}^n U_{[i]}$. The latter is isomorphic to $\oplus_{i=k}^n U_{[i]}$, by the above. By Theorem 2.2 and Lemma 2.5, $\dim(U_{(k)}/U_{(k+1)}) = \dim(U_{[k]}) = \binom{n}{k}$. So, $\dim(U_{(k)}) = \sum_{i=k}^n \binom{n}{i} = \dim(\oplus_{i=k}^n U_{[i]})$. Therefore $U_{(k)} = \oplus_{i=k}^n U_{[i]}$. □

By the above, $U_{(k)}/U_{(k+1)} \cong U_{[k]}$ (as vector spaces). However, when $1 < k < n$, L does not stabilize the subspace $U_{[k]}$ of U_S (but it stabilizes the 1-dimensional subspaces $U_{[0]}$ and $U_{[n]}$). Indeed, L maps the $S_k(B)$ onto the set of k-dimensional subspaces of V_0, and the latter spans $U_{(k)}$ (see the proof of Theorem 2.6). So, the previous isomorphism is not an isomorphism of L-modules and the decomposition $U_S = \oplus_{k=0}^n U_{[k]}$ is not preserved by L.

We turn to K, now.

Theorem 2.7. $K = U_{(3)} = \bigoplus_{i=3}^n U_{[i]}$.

Proof. This statement is implicit in the computations of Baumeister, Meixner and Pasini [1, section 5], but we can also obtain it as follows. It is known that Γ admits a quotient $\overline{\Gamma}$, which can be described as the canonical gluing of two copies of the affine space $\mathsf{AG}(n, 2)$ (see [1]; also Baumeister and Pasini [2]). Let $\overline{K}$ be the deck group of the covering projection from $\widetilde{\Gamma}$ onto $\overline{\Gamma}$. Then $\overline{K} = U_{(2)}$, as proved by Baumeister and Pasini [2] (see also Baumeister, Meixner and Pasini [1, section 5]). However, $K < \overline{K}$ and, by comparing orders, we see that $[\overline{K} : K] = \binom{n}{2}$. Moreover, K is normalized by $\mathrm{Aut}(\mathcal{A}_S)$. In view of Corollary 2.4, we now see that $K = U_{(3)}$ is the unique possibility. □

Theorem 2.8. *If* $n = 3$ *then* $\mathrm{Aut}(\mathrm{Exp}(\rho_H)) \cong D/K = 2^6{:}\mathrm{Sym}(8)$. *If* $n > 3$ *then* $\mathrm{Aut}(\mathrm{Exp}(\rho_H)) = 2^{\binom{n+1}{2}}{:}\mathsf{AGL}(n, 2) = \mathrm{Aut}_{\rho_H}(\mathrm{Exp}(\rho_H))$.

Proof. This follows from Lemma 2.1 and the fact that $\mathrm{Aut}(\Gamma)$ lifts through π to the normalizer of K in $\mathrm{Aut}(\widetilde{\Gamma})$ (see [12, chapter 12]). □

We warn the reader that the above theorem should not be read as if it claimed that π maps U_S^+ onto $V \wedge V$ and the translation group T_S of $\mathcal{A}_S$ onto T_H. In fact, as we will see later, π indeed maps $U_{(2)}$ onto the subgroup $V_0 \wedge V_0$ of $V \wedge V$ and $U_{(1)}T_S$ onto $(V \wedge V)T_H$, but it maps $U_{(2)}T_S$ onto $V \wedge V$.

2.4 What remains to do

With ρ_H as in Subsection 2.1, let $\tilde{\rho}$ be the hull of ρ_H and $\widetilde{U}$ be the target group of $\tilde{\rho}$. By Proposition 1.2, the universal cover $\widetilde{\Gamma}$ of $\Gamma = \mathrm{Exp}(\rho_H)$ is the expansion $\mathrm{Exp}(\tilde{\rho})$ of $\tilde{\rho}$ and the kernel of the projection of $\widetilde{U}$ onto U is the deck group K of the covering projection $\pi : \widetilde{\Gamma} \to \Gamma$. Therefore $\widetilde{U}$ is a subgroup of the lifting $\widetilde{G}$ of $G := \mathrm{Aut}_{\rho_H}(\mathrm{Exp}(\rho_H))$ to $\widetilde{\Gamma}$ through π, and $K \trianglelefteq \widetilde{U}$. Moreover, $\widetilde{U}/K \cong U = V \wedge V$. On the other hand, $\widetilde{U}$ is a normal subgroup of $\widetilde{G}$, which in its turn is an extension of U_S^+ by $\mathrm{Aut}(\mathcal{A}_S) = \mathsf{AGL}(n, 2)$. Note that $U_S^+ = U_{[1]} \oplus U_{[2]} \oplus K \cong (V \wedge V) \oplus K$ (indeed $V \wedge V \cong V_0 \oplus (V_0 \wedge V_0)$, $V_0 \cong U_{[1]}$ and $V_0 \wedge V_0 \cong U_{[2]}$). However, $\widetilde{U}$ is non-commutative when $n > 2$, as remarked in Subsection 1.5. So, when $n > 2$ the group $\widetilde{U}$ is certainly different from U_S^+ (but $\widetilde{U} \neq U_S^+$ even when $n = 2$, as it will turn out from the computations of Section 4). So, we must still describe $\widetilde{U}$ and determine how it sits inside $\widetilde{G}$. We shall do this in the next two sections.

3 The abstract hull of ρ_H

Throughout this and the next section Σ, ρ_H, $\mathcal{S}_H$, Γ and $\widetilde{\Gamma}$ are as in Section 2, and $n = d + 1 > 2$.

Given a basis $\{v_\infty, v_1, \ldots, v_n\}$ of $V = V(n+1, 2)$, let $V_0 = \langle v_1, \ldots v_n \rangle$ and $V_0^* = V_0 - \{0\}$. We may assume that the basis $\{v_1, \ldots, v_n\}$ coincides with that defined in Subsection 2.3. Every element of V can be written as $\epsilon v_\infty + x$, with $x \in V_0$ and $\epsilon \in \{0, 1\}$. Since $(\epsilon v_\infty + x) \wedge (\epsilon' v_\infty + x') = (\epsilon x' + \epsilon' x) \wedge v_\infty + x \wedge x'$, the exterior product $U = V \wedge V$ is isomorphic to $V_0 \oplus (V_0 \wedge V_0)$ via the isomorphism $v_\infty \wedge V_0 \cong V_0$, as we have already remarked in Section 2, but we warn the reader that, in spite of these isomorphisms, he should resist temptation of regarding $v_\infty \wedge V_0$ as the same thing as V_0 and, accordingly, U as the same thing as $V_0 \oplus (V_0 \wedge V_0)$. Indeed U will turn out to be a quotient of the group

$$\widetilde{U} = [(V_0 \wedge V_0) \oplus (V_0 \wedge V_0 \wedge V_0) \oplus \ldots]\widetilde{T}$$

to be defined later in this section. In the projection of $\widetilde{U}$ onto U, $v_\infty \wedge V_0$ is the image of the subgroup $\widetilde{T} < \widetilde{U}$, which is not the same thing as V_0.

The points of Σ, regarded as lines of $\mathsf{AG}(V_0)$, are classes $[(x, y)]$ of ordered pairs (x, y), with $x \in V_0$, $y \in V_0^*$, via the equivalence relation $(x, y) \sim (x+y, y)$.

Accordingly, a block of Σ is a set described as $\{[(x,y)]\}_{y\in V_0^*}$, for $x \in V_0$. The d-embedding $\rho_H : \Sigma \to U$ is defined as follows:

$$\rho_H[(x,y)] = \{(x+v_\infty)\wedge y, 0\}, \quad x \in V_0,\ y \in V_0^*.$$

From now on ρ_H will be simply written as ρ. The image by ρ of a block B of Σ is a subspace of $\mathsf{PG}(V \wedge V)$ described as $\{(x+v_\infty)\wedge y\}_{y\in V_0^*} \cup \{0\}$. Such a subspace will be denoted by B_x.

Consider the graded algebra

$$E := 1 \oplus V_0 \oplus (V_0 \wedge V_0) \oplus (V_0 \wedge V_0 \wedge V_0) \oplus \cdots \oplus (\underbrace{V_0 \wedge \cdots \wedge V_0}_{n \text{ times}})$$

where $1 := \wedge^0 V_0$ (a 1-dimensional vector space over $\mathsf{GF}(2)$).

For any $x \in V_0$, $x = \sum_{i=1}^{n} a_i v_i$, we call *support* of x the set $S(x) := \{i : a_i \neq 0\}$. So, $x = \sum_{i\in S(x)} v_i$.

For every $x \in V_0$, define the following elements $\widehat{x}, \overline{x} \in E$:

$$\widehat{x} = \sum_{K\subseteq S(x), K\neq\emptyset} \ \bigwedge_{k\in K} v_k, \quad \text{if } x \neq 0 \quad \text{and} \quad \widehat{0} = 0.$$

$$\overline{x} = 1 + \widehat{x} = \sum_{K\subseteq S(x)} \ \bigwedge_{k\in K} v_k,$$

using the convention $\bigwedge_{k\in\emptyset} v_k = 1$.

Lemma 3.1. $\overline{x+y} = \overline{x} \wedge \overline{y}, \quad \forall x, y \in V_0.$

Proof. Let $X = S(x)$ and $Y = S(y)$. Then, denoting by $\triangle$ the symmetric difference of sets, we have $S(x+y) = X \triangle Y$. We get:

$$\overline{x+y} = \sum_{Z\subseteq X\triangle Y} \ \bigwedge_{l\in Z} v_l$$

$$\overline{x} \wedge \overline{y} = (\sum_{K\subseteq X} \ \bigwedge_{k\in K} v_k) \wedge (\sum_{H\subseteq Y} \ \bigwedge_{h\in H} v_h) = \sum_{\substack{K\subseteq X, H\subseteq Y\\ K\cap H=\emptyset}} \ \bigwedge_{i\in K\cup H} v_i$$

A subset Z of $X\triangle Y$ is a union of two disjoint subsets of X and Y, respectively. More precisely $Z = H \cup K$, with $H = Z\cap X$ and $K = Z \cap Y$. Thus the statement follows. $\square$

By induction, the previous result can be extended as follows:

Lemma 3.2. $\overline{\sum_{i=1}^{t} x_i} = \bigwedge_{i=1}^{t} \overline{x_i}, \quad \forall x_1, \dots x_t \in V_0.$

Lemma 3.3. $\widehat{x+y} = \widehat{x} + \widehat{y} + (\widehat{x} \wedge \widehat{y}), \quad \forall x, y \in V_0.$

Proof. By Lemma 3.1, we have:

$$\widehat{x+y} = \overline{x+y} + 1 = \overline{x} \wedge \overline{y} + 1 = (1+\widehat{x}) \wedge (1+\widehat{y}) + 1 = \widehat{x} + \widehat{y} + (\widehat{x} \wedge \widehat{y}). \quad \square$$

Put $\widetilde{T} := \{t_x\}_{x \in V_0}$ and define in $\widetilde{T}$ the following product: $t_x t_y = t_{x+y}$. Thus $\widetilde{T}$ is a group isomorphic to V_0. Define an action of $\widetilde{T}$ on E in the following way: $t_x \bullet u = u + u \wedge \widehat{x}$, $\forall u \in E$, $x \in V_0$. This definition is well posed, since the following equalities hold:

$$t_x \bullet (t_y \bullet u) = t_{x+y} \bullet u, \quad t_x \bullet (u+v) = t_x \bullet u + t_x \bullet v, \quad t_0 \bullet u = u.$$

Consider the semidirect product $E : \widetilde{T}$ with respect to the above action of $\widetilde{T}$ on E.

Denote by I the following sub-algebra of E:

$$I := (V_0 \wedge V_0) \oplus (V_0 \wedge V_0 \wedge V_0) \oplus \cdots \oplus (\underbrace{V_0 \wedge \cdots \wedge V_0}_{n \text{ times}})$$

Restricting to I the action of $\widetilde{T}$ we get the group $I : \widetilde{T}$. Denote it by $\widetilde{U}$. This might look as a notational abuse, as the symbol $\widetilde{U}$ has already been used in Section 2 to denote the target group of the hull $\tilde{\rho}$ of $\rho = \rho_H$. However, this abuse is quite harmless. Indeed we shall soon prove that $\widetilde{U}$, defined as above, is indeed that target group.

Note first that $\widetilde{U}$ is non-commutative. Indeed, given $u \in I$, $t_x \in \widetilde{T}$ and denoting by (u, t_x) their commutator, we have

$$(u, t_x) = u^{-1} t_x^{-1} u t_x = u(t_x \bullet u) = u \wedge \widehat{x}$$

which is non-zero in general.

Define the embedding $\widetilde{\rho} : \Sigma \longrightarrow \widetilde{U}$ in the following way:

$$\widetilde{\rho}[(x, y)] = \langle (\widehat{x}, t_y) t_y \rangle = \{(\widehat{x} \wedge \widehat{y}) t_y, 0\}.$$

This definition makes sense since $\widehat{x+y} \wedge \widehat{y} = (\widehat{x} + \widehat{y} + (\widehat{x} \wedge \widehat{y})) \wedge \widehat{y} = \widehat{x} \wedge \widehat{y}$, by Lemma 3.3. Note that the elements $(\widehat{x}, t_y) t_y$ are involutions. In fact, since $\widehat{x}$

and t_y are involutions, we have $(t_y, \widehat{x}) = t_y\widehat{x}t_y\widehat{x}$, therefore $(t_y, \widehat{x})t_y = t_y\widehat{x}t_y\widehat{x}t_y$. It follows that

$$(t_y, \widehat{x})t_y(t_y, \widehat{x})t_y = t_y\widehat{x}t_y\widehat{x}t_yt_y\widehat{x}t_y\widehat{x}t_y = 1.$$

The image by $\widetilde{\rho}$ of a block B of Σ is a set described as

$$\{(\widehat{x}, t_y)t_y\}_{y \in V_0^*} \cup \{0\} \;=\; \{(\widehat{x} \wedge \widehat{y})t_y\}_{y \in V_0^*} \cup \{0\}.$$

Such a set will be denoted by $\widetilde{B}_x$.

Lemma 3.4. *$\widetilde{\rho}$ is an embedding.*

Proof. Condition (AE1) is obviously satisfied. Let $[(x, y)], [(x', y')]$ such that $\widetilde{\rho}([(x, y)]) \cap \widetilde{\rho}([(x', y')]) \neq \{0\}$. Since these groups have order 2, then $\widetilde{\rho}([(x, y)]) = \widetilde{\rho}([(x', y')])$, namely $(\widehat{x} \wedge \widehat{y})t_y = (\widehat{x'} \wedge \widehat{y'})t'_y$, whence

$$(\widehat{x'} \wedge \widehat{y'}) + (\widehat{x} \wedge \widehat{y}) = t'_y t_y.$$

This implies

$$\begin{cases} \widehat{x'} \wedge \widehat{y'} = \widehat{x} \wedge \widehat{y} \\ t'_y = t_y \end{cases}$$

The second equality implies $y' = y$, then it follows $(\widehat{x} + \widehat{x'}) \wedge \widehat{y} = 0$ from the first equality, namely

$$\left(\sum_{K \subseteq S(x), K \neq \emptyset} \bigwedge_{k \in K} v_k + \sum_{K' \subseteq S(x'), K' \neq \emptyset} \bigwedge_{k' \in K'} v_{k'} \right) \wedge \sum_{H \subseteq S(y), H \neq \emptyset} \bigwedge_{h \in H} v_h = 0.$$

This implies for the lower grade:

$$\left(\sum_{k \in S(x)} v_k + \sum_{k' \in S(x')} v_{k'} \right) \wedge \sum_{h \in S(y)} v_h = 0.$$

i. e. $(x + x') \wedge y = 0$. It follows that either $x = x'$ or $y = x + x'$. Thus $(x', y') = (x, y)$ or $(x', y') = (x', y) = (x + y, y)$. In both cases we have $[(x, y)] = [(x', y')]$. Then (AE2) holds.

Note that

$$(t_y, \widehat{x})t_y = t_y\widehat{x}t_y\widehat{x}t_y = t_y\widehat{x}(t_y \bullet \widehat{x}) = t_y\widehat{x}(\widehat{x} + \widehat{x} \wedge \widehat{y}) = t_y(\widehat{x} \wedge \widehat{y}) = t_y(t_y, \widehat{x})$$

Thus:

$$(t_y, \widehat{x})t_y = t_y(t_y, \widehat{x}) \tag{1}$$

For any $x \in V_0$, consider $(\widehat{x} \wedge \widehat{y})t_y$, $(\widehat{x} \wedge \widehat{y'})t_{y'}$. By (1) and Lemma 3.3, we have:

$$\begin{aligned}(\widehat{x} \wedge \widehat{y})t_y(\widehat{x} \wedge \widehat{y'})t_{y'} &= t_y(\widehat{x} \wedge \widehat{y})(\widehat{x} \wedge \widehat{y'})t_{y'} = t_y(\widehat{x} \wedge (\widehat{y} + \widehat{y'}))t_{y'} \\ &= t_y(\widehat{x} \wedge (\widehat{y} + \widehat{y'}))t_y t_y t_{y'} = t_y(\widehat{x} \wedge (\widehat{y} + \widehat{y'}))t_y t_{y+y'} \\ &= (t_y \bullet (\widehat{x} \wedge (\widehat{y} + \widehat{y'})))t_{y+y'} \\ &= (\widehat{x} \wedge (\widehat{y} + \widehat{y'}) + \widehat{x} \wedge (\widehat{y} + \widehat{y'}) \wedge \widehat{y})t_{y+y'} \\ &= (\widehat{x} \wedge (\widehat{y} + \widehat{y'}) + \widehat{x} \wedge \widehat{y'} \wedge \widehat{y}))t_{y+y'} \\ &= (\widehat{x} \wedge (\widehat{y} + \widehat{y'} + \widehat{y} \wedge \widehat{y'}))t_{y+y'} = (\widehat{x} \wedge \widehat{y + y'})t_{y+y'}\,.\end{aligned}$$

This proves that $\widetilde{B}_x$ is a group isomorphic to V_0, thus (AE3) is satisfied. Clearly, condition (AE4) also holds. □

Consider the map $\tilde{f} : \widetilde{U} \longrightarrow U$ defined in the following way. For any $wt_y \in \widetilde{U}$, with $w = \sum a_{ij}(v_i \wedge v_j) + \sum a_{ijk}(v_i \wedge v_j \wedge v_k) + \cdots \in I$ and $t_y \in \widetilde{T}$, set $w^* = \sum a_{ij}(v_i \wedge v_j)$, the grade 2 part of w, and define

$$\tilde{f}(wt_y) = w^* + v_\infty \wedge y.$$

Theorem 3.5. *$(\widetilde{\rho}, \tilde{f})$ is the abstract hull of ρ.*

Proof. Since the dimension of I equals $\displaystyle\sum_{i=2}^{d+1} \binom{d+1}{i} = 2^{d+1} - d - 2 = M - d - 2$, then $|\widetilde{U}| = |I \cdot \widetilde{T}| = 2^{M-d-2}2^{d+1} = 2^{M-1}$, which is the correct size of the universal representation group of ρ (see Section 2).

It remains to prove that $\tilde{f}\widetilde{\rho} = \rho$. We have, for any $x \in V_0$, $y \in V_0^*$:

$$\tilde{f}\widetilde{\rho}([x, y]) = \tilde{f}((\widehat{x} \wedge \widehat{y})t_y) = x \wedge y + v_\infty \wedge y = \rho([x, y]).$$

Therefore $\tilde{f}\widetilde{\rho} = \rho$. □

The next proposition follows from the previous description of $\widetilde{U}$ and $\tilde{f}$.

Proposition 3.6. *$Ker(\tilde{f})$ is equal to the commutator subgroup of $\widetilde{U}$.*

Corollary 3.7. *The embedding ρ_H is projectively dominant.*

4 Recovering the $L-$module $\widetilde{U}$ inside the Coxeter group

In order to describe the lifting of $\mathrm{Aut}(\mathrm{Exp}(\rho))$, we need to identify E with U_S, preserving their respective decompositions. Consider, for any $k = 1, \ldots, n$, the map $\sigma : \wedge^k V_0 \longrightarrow U_{[k]}$, obtained by linear extension of the following:

$$\sigma(\bigwedge_{j \in K} v_j) = f_{S_K}, \ \forall K \subseteq \{1, \ldots, n\}, \ |K| = k.$$

Since $U_S = \oplus_{k=0}^{n} U_{[k]}$, this provides an isomorphism, also called σ, between E and U_S as graded algebras. In particular σ sends $\bigoplus_{h \geq k} \bigwedge^h V_0$ to $U_{(k)}$.

We have:

Lemma 4.1. $\sigma(\overline{v}) = f_v, \quad \forall v \in V_0.$

Proof. Let $v = v_{i_1} + \cdots + v_{i_k}$ and set $K = S(v) = \{i_1, \ldots i_k\}$. We have:

$$\sigma(\overline{v}) = \sigma(\sum_{J \subseteq K} \bigwedge_{j \in J} v_j) = \sigma(1 + \sum_{j \in K} v_j + \sum_{J \subseteq K, |J|=2} \bigwedge_{j \in J} v_j + \cdots + \bigwedge_{j \in K} v_j) =$$

$$= f_\emptyset + \sum_{J \subseteq K, |J| \leq 1} f_{v_J} + \sum_{J \subseteq K, |J| \leq 2} f_{v_J} + \cdots + \sum_{J \subseteq K} f_{v_J}$$

In the sum above, every $J \subseteq K$ appears $2^{k-|J|}$ times, thus f_{v_K}, namely f_v, is the only surviving summand. □

Lemma 4.2. $\sigma(\widehat{x} \wedge \widehat{y}) = f_{\{v_0, x, y, x+y\}}, \quad \forall x, y \in V_0.$

Proof. We have $\widehat{x} \wedge \widehat{y} = (\overline{x} + 1) \wedge (\overline{y} + 1) = 1 + \overline{x} + \overline{y} + \overline{x + y},$ by Lemma 3.1. The statement follows from Lemma 4.1. □

Extending in obvious way the isomorphism σ to $E : \widetilde{T}$, we define the product $U_S : \widetilde{T}$. Given $wt_x \in \widetilde{U}$, let A be such that $\sigma(w) = f_A$. Often, it will be useful to pass from wt_x to the corresponding $\sigma(w)t_x = f_A t_x$. In that case, whenever it will not cause any ambiguity, we set, by abuse of notation, $wt_x = f_A t_x$.

4.1 $\mathbf{Aut}_\rho(\mathbf{Exp}(\rho))$

Consider $\mathrm{Exp}(\rho)$, affine expansion of ρ.
The elements of type 0 of $\mathrm{Exp}(\rho)$ are the elements of U,

$$v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \, .$$

The elements of type 1 of $\mathrm{Exp}(\rho)$ are the right cosets

$$\langle (v_\infty + x) \wedge y \rangle + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \, .$$

The elements of type 2 of $\mathrm{Exp}(\rho)$ are the right cosets

$$\{(v_\infty + x) \wedge y\}_{y \in V_0} + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \, .$$

Recall that $\mathrm{Aut}(\mathrm{Exp}(\rho)) = U : (T_H : L_H)$ (Section 2.2). We describe the actions of these subgroups on $\mathrm{Exp}(\rho)$ in detail.

Action of U. The group U acts on the elements of $\mathrm{Exp}(\rho)$ by sum.

Action of T_H. Denoting by τ_z the element of T_H, corresponding to the translation by the vector z, we get:

$$\tau_z \left(v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right) = (v_\infty + z) \wedge q + \sum a_{ij} v_i \wedge v_j$$

for the elements of type 0;

$$\begin{aligned} \tau_z \left(\langle (v_\infty + x) \wedge y \rangle + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right) \\ = \langle (v_\infty + x + z) \wedge y \rangle + (v_\infty + z) \wedge q + \sum a_{ij} v_i \wedge v_j \end{aligned}$$

for the elements of type 1;

$$\begin{aligned} \tau_z \left(\{(v_\infty + x) \wedge y\}_{y \in V_0} + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right) \\ = \{(v_\infty + x + z) \wedge y\}_{y \in V_0} + (v_\infty + z) \wedge q + \sum a_{ij} v_i \wedge v_j \end{aligned}$$

for the elements of type 2.

Action of L_H. Given $\alpha \in L$ (see Section 2.2), we denote by $\alpha[x]$ the image of a vector x in V_0. By abuse of notation we also call α the element of L_H corresponding to α in the isomorphism $L \cong L_H$. The element $\alpha \in L_H$ acts as follows:

$$\alpha\left(v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j\right) = v_\infty \wedge \alpha[q] + \sum a_{ij}\alpha[v_i] \wedge \alpha[v_j]$$

for the elements of type 0;

$$\begin{aligned}\alpha\left(\langle (v_\infty + x) \wedge y\rangle + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j\right) \\ = \langle (v_\infty + \alpha[x]) \wedge \alpha[y]\rangle + v_\infty \wedge \alpha[q] + \sum a_{ij}\alpha[v_i] \wedge \alpha[v_j]\end{aligned}$$

for the elements of type 1;

$$\begin{aligned}\alpha\left(\{(v_\infty + x) \wedge y\}_{y\in V_0} + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j\right) \\ = \{(v_\infty + \alpha[x]) \wedge \alpha[y]\}_{y\in V_0} + v_\infty \wedge \alpha[q] + \sum a_{ij}\alpha[v_i] \wedge \alpha[v_j]\end{aligned}$$

for the elements of type 2.

The structure of $U : (T_H : L_H)$. The actions of L_H on T_H, of T_H on U and of L_H on U are as follows:

$$\begin{aligned}&\alpha \bullet \tau_z = \tau_{\alpha[z]}\,;\\ &\tau_z \bullet \left(v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j\right) = (v_\infty + z) \wedge q + \sum a_{ij} v_i \wedge v_j\,;\\ &\alpha \bullet \left(v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j\right) = v_\infty \wedge \alpha[q] + \sum a_{ij}\alpha[v_i] \wedge \alpha[v_j]\,.\end{aligned}$$

4.2 $\mathrm{Aut}_{\widetilde{\rho}}(\mathrm{Exp}(\widetilde{\rho}))$

Consider $Exp(\widetilde{\rho})$, affine expansion of $\widetilde{\rho}$.
The elements of type 0 of $\mathrm{Exp}(\widetilde{\rho})$ are the elements wt_x of $\widetilde{U} = I : \widetilde{T}$;
the elements of type 1 of $\mathrm{Exp}(\widetilde{\rho})$ are the right cosets$\langle (\widehat{p} \wedge \widehat{y})t_y\rangle wt_x$;
the elements of type 2 of $\mathrm{Exp}(\widetilde{\rho})$ are the right cosets $\{(\widehat{p} \wedge \widehat{y})t_y\}_{y\in V_0} wt_x$.

Action of $\widetilde{U}$. The group $\widetilde{U}$ is trivially a subgroup of $\mathrm{Aut}(\mathrm{Exp}(\widetilde{\rho}))$, its action being defined by right product:

$$ut_z(wt_x) = wt_x ut_z = (w + u + u \wedge \widehat{z})t_{x+z}$$

for the elements of type 0;

$$ut_z(\langle(\widehat{p}\wedge\widehat{y})t_y\rangle wt_x) = (\langle\widehat{p}\wedge\widehat{y})t_y\rangle wt_x ut_z = \langle(\widehat{p}\wedge\widehat{y})t_y\rangle(w+u+u\wedge\widehat{z})t_{x+z}$$

for the elements of type 1;

$$\begin{aligned} ut_z(\{(\widehat{p}\wedge\widehat{y})t_y\}_{y\in V_0} wt_x) &= \{(\widehat{p}\wedge\widehat{y})t_y\}_{y\in V_0}\, wt_x ut_z \\ &= \{(\widehat{p}\wedge\widehat{y})t_y\}_{y\in V_0}\,(w+u+u\wedge\widehat{z})t_{x+z} \end{aligned}$$

for the elements of type 2.

The group T'. Denote by T' the set $\{\widehat{z}t_z | z\in V_0\}$. T' is a subgroup of $E:\widetilde{T}$, since the elements $\widehat{z}t_z$ are involutions and

$$\widehat{z}t_z\widehat{y}t_y = \widehat{z}t_z\widehat{y}t_z t_z t_y = (\widehat{z}+\widehat{y}+\widehat{y}\wedge\widehat{z})t_{z+y} = (\widehat{z+y})t_{z+y}.$$

T' is isomorphic to the translation groups T and T_H (see Section 2.2). We define the following action of T' on $\mathrm{Exp}(\widetilde{\rho})$:

$$\begin{aligned} \widehat{z}t_z(wt_x) = \widehat{z}t_z wt_x(\widehat{z}t_z)^{-1} = \widehat{z}t_z wt_x t_z\widehat{z} = \widehat{z}t_z wt_z t_x \widehat{z}t_x t_x \\ = (w+w\wedge\widehat{z}+\widehat{z}\wedge\widehat{x})t_x \end{aligned}$$

for the elements of type 0;

$$\begin{aligned} \widehat{z}t_z(\langle(\widehat{p}\wedge\widehat{y})t_y\rangle wt_x) &= \widehat{z}t_z\langle(\widehat{p}\wedge\widehat{y})t_y\rangle wt_x t_z\widehat{z} \\ &= \widehat{z}t_z\langle(\widehat{p}\wedge\widehat{y})t_y\rangle t_z\widehat{z}\widehat{z}t_z wt_x t_z\widehat{z} \\ &= \langle(\widehat{p}\wedge\widehat{y}+\widehat{p}\wedge\widehat{y}\wedge\widehat{z}+\widehat{z}\wedge\widehat{y})t_y\rangle(w+w\wedge\widehat{z}+\widehat{z}\wedge\widehat{x})t_x \\ &= \langle((\widehat{p+z})\wedge\widehat{y})t_y\rangle(w+w\wedge\widehat{z}+\widehat{z}\wedge\widehat{x})t_x \end{aligned}$$

for the elements of type 1;

$$\begin{aligned} \widehat{z}t_z(\{(\widehat{p}\wedge\widehat{y})t_y\}_{y\in V_0} wt_x) &= \widehat{z}t_z\{(\widehat{p}\wedge\widehat{y})t_y\}_{y\in V_0} wt_x t_z\widehat{z} \\ &= \{((\widehat{p+z})\wedge\widehat{y})t_y\}_{y\in V_0}(w+w\wedge\widehat{z}+\widehat{z}\wedge\widehat{x})t_x \end{aligned}$$

for the elements of type 2.

The group L. We shall now define the action of L on E. We shall exploit the isomorphism $\sigma : E \longrightarrow U_S$ in such a way that the action of L on U_S will be reproduced on E. In short we want E and U_S to be isomorphic as L-modules, too. The action of L that we are going to define is based on the equalities

$\alpha(f_p) = f_{\alpha[p]}$, $\alpha(t_x) = t_{\alpha[x]}$, extended to the three types of $\mathrm{Exp}(\widetilde{\rho})$. More explicitly, given wt_x in $\widetilde{U}$:

$$wt_x = (\sum_{\{i,j\}} a_{ij} v_i \wedge v_j + \sum_{\{i,j,k\}} a_{ijk} v_i \wedge v_j \wedge v_k + \cdots) t_x .$$

σ maps this element onto the following:

$$(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots) t_x .$$

With the abuse of notation announced at the beginning of Section 4, we define:

$$\begin{aligned}\alpha(wt_x) &= \alpha\Big(\Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\Big) t_x\Big) \\ &= \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\Big) t_{\alpha[x]}\end{aligned}$$

for the elements of type 0;

$$\begin{aligned}&\alpha\big(\langle(\widehat{p} \wedge \widehat{y})t_y\rangle wt_x\big) \\ &= \alpha\Big(f_{\{v_0,p,y,p+y\}} t_y \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\Big) t_x\Big) \\ &= f_{\{v_0,\alpha[p],\alpha[y],\alpha[p]+\alpha[y]\}} t_{\alpha[y]} \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\Big) t_{\alpha[x]}\end{aligned}$$

for the elements of type 1;

$$\begin{aligned}&\alpha\big(\{(\widehat{p} \wedge \widehat{y})t_y\}_{y \in V_0} wt_x\big) \\ &= \alpha\Big(\{f_{\{v_0,p,y,p+y\}} t_y\}_{y \in V_0} \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\Big) t_x\Big) \\ &= \{f_{\{v_0,\alpha[p],\alpha[y],\alpha[p]+\alpha[y]\}} t_{\alpha[y]}\}_{y \in V_0} \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\Big) t_{\alpha[x]}\end{aligned}$$

for the elements of type 2.

A candidate for $\mathbf{Aut}_{\widetilde{\rho}}(\mathbf{Exp}(\widetilde{\rho}))$. We define in obvious way an action of L on $\widetilde{U}$ and of T' on $\widetilde{U}$:

$$\widehat{z} t_z \bullet wt_x = (w + w \wedge \widehat{z} + \widehat{z} \wedge \widehat{x}) t_x ;$$

$$\begin{aligned}\alpha \bullet wt_x &= \alpha \bullet \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\Big) t_x \\ &= \Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{\{i,j,k\}} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\Big) t_{\alpha[x]} .\end{aligned}$$

In order to define an action of L on T', we need some supplementary results.

Lemma 4.3. $\alpha(u \wedge w) = \alpha(u) \wedge \alpha(w), \quad \forall \alpha \in L,\ \forall u, w \in I.$

Proof. By the linearity of α, it is enough to prove that

$$\alpha(\bigwedge_{j\in J} v_j) = \bigwedge_{j\in J} \alpha(v_j).$$

By the isomorphism between U_S and E, the definition of α and its linearity, we have:

$$\alpha(\bigwedge_{j\in J} v_j) = \alpha(\sum_{K\subseteq J} f_{v_K}) = \sum_{K\subseteq J} f_{\alpha[v_K]}.$$

Note that, by Lemma 4.1:

$$\alpha(v_j) = \alpha(1+\overline{v_j}) = \alpha(f_{v_0} + f_{v_j}) = f_{\alpha[v_0]} + f_{\alpha[v_j]} = 1 + \overline{\alpha[v_j]}.$$

Set $|J| = t$ and denote by $\binom{J}{s}$ the set of s-tuples of distinct elements in J. We have by Lemma 3.2:

$$\begin{aligned}
\bigwedge_{j\in J} \alpha(v_j) = \bigwedge_{j\in J}(1+\overline{\alpha[v_j]}) = \\
&= 1 + \sum_{j\in J}\overline{\alpha[v_j]} + \sum_{\{j_1,j_2\}\in\binom{J}{2}} \overline{\alpha[v_{j_1}]}\wedge\overline{\alpha[v_{j_2}]} + \cdots \\
&\qquad + \sum_{\{j_1,\dots,j_{t-1}\}\in\binom{J}{t-1}} \overline{\alpha[v_{j_1}]}\wedge\ldots\wedge\overline{\alpha[v_{j_{t-1}}]} + \bigwedge_{j\in J}\overline{\alpha[v_j]} \\
&= 1 + \sum_{j\in J}\overline{\alpha[v_j]} + \sum_{\{j_1,j_2\}\in\binom{J}{2}} \overline{\alpha[v_{j_1}]+\alpha[v_{j_2}]} + \cdots \\
&\qquad + \sum_{\{j_1,\dots,j_{t-1}\}\in\binom{J}{t-1}} \overline{\alpha[v_{j_1}]+\cdots+\alpha[v_{j_{t-1}}]} + \overline{\sum_{j\in J}\alpha[v_j]} \\
&= 1 + \sum_{j\in J} f_{\alpha[v_j]} + \sum_{\{j_1,j_2\}\in\binom{J}{2}} f_{\alpha[v_{j_1}]+\alpha[v_{j_2}]} + \cdots \\
&\qquad + \sum_{\{j_1,\dots,j_{t-1}\}\in\binom{J}{t-1}} f_{\alpha[v_{j_1}]+\cdots+\alpha[v_{j_{t-1}}]} + f_{\alpha[v_J]} \\
&= \sum_{K\subseteq J} f_{\alpha[v_K]}\,. \qquad \square
\end{aligned}$$

Lemma 4.4. $\alpha(\overline{u}) = \overline{\alpha[u]}, \quad \forall \alpha \in L,\ \forall u \in V_0.$

Proof. $\alpha(\overline{u}) = \alpha(f_u) = f_{\alpha[u]} = \overline{\alpha[u]}.$ $\square$

It immediately follows:

Lemma 4.5. $\alpha(\widehat{u}) = \widehat{\alpha[u]}, \quad \forall\alpha \in L,\ \forall u \in V_0$.

We warn the reader that Lemma 4.3 does not imply that $\bigwedge^k V_0$ is stabilized by α. In fact $\alpha(\bigwedge^k V_0) \neq \bigwedge^k V_0$, because $\alpha(\bigwedge^k V_0)$ in general involves contributions from $\bigwedge^h V_0$, with $h > k$. However, $\bigoplus_{h\geq k} \bigwedge^k V_0 / \bigoplus_{h>k} \bigwedge^h V_0$ is stable under the action of α. Actually, on that quotient, that action is the natural one. Theorem 2.2 immediately follows from this remark.

Now, we define the following action of L on T':

$$\alpha \bullet \widehat{z} t_z = \widehat{\alpha[z]} t_{\alpha[z]}, \qquad \forall\alpha \in L,\ \forall z \in V_0 .$$

The actions defined above allow to construct the product $\widetilde{U}\cdot(T'\cdot L)$. All products involved here are semi-direct. Accordingly we will write $\widetilde{U} : (T' : L)$ instead of $\widetilde{U} \cdot (T' \cdot L)$.

Proposition 4.6. $\widetilde{U} : (T' : L)$ *is an automorphism group of* $\mathrm{Exp}(\widetilde{\rho})$.

Proof. We have to prove that the structure of the product of $\widetilde{U} : (T' : L)$ is compatible with the actions defined in every single factor. The only non-trivial check concerns the factor $T' : L$. For any wt_x in $\widetilde{U}$, we have, by Lemmas 4.3 and 4.5:

$$\begin{aligned}
\alpha\widehat{z}t_z\alpha^{-1}(wt_x) &= \alpha\widehat{z}t_z \left(\alpha^{-1}(w)t_{\alpha^{-1}[x]}\right) \\
&= \alpha\Big(\big(\alpha^{-1}(w) + \alpha^{-1}(w) \wedge \widehat{z} + \widehat{z} \wedge \widehat{\alpha^{-1}[x]}\big)t_{\alpha^{-1}[x]}\Big) \\
&= \big(w + w \wedge \alpha(\widehat{z}) + \alpha(\widehat{z}) \wedge \alpha\alpha^{-1}(\widehat{x})\big)t_x \\
&= \big(w + w \wedge \widehat{\alpha[z]} + \widehat{\alpha[z]} \wedge \widehat{x}\big)t_x
\end{aligned}$$

that coincides with the action of $\widehat{\alpha[z]}t_{\alpha[z]}$ on wt_x. This proves the compatibility with the action of $T' : L$ on the elements of type 0. The control on the elements of type 1 and 2 is similar. $\square$

We claim that the group defined above is the lifting of $\mathrm{Aut}_\rho(\mathrm{Exp}(\rho))$, namely:

Theorem 4.7. $\mathrm{Aut}_{\widetilde{\rho}}(\mathrm{Exp}(\widetilde{\rho})) = \widetilde{U} : (T' : L)$

In the sequel we shall prove Theorem 4.7 by defining a suitable projection of $\widetilde{U} : (T' : L)$ onto $U : (T_H : L_H)$.

The projection. We recall the projection $\tilde{f} : \widetilde{U} \longrightarrow U$ defined in Section 3:

$$\tilde{f}(wt_x) = v_\infty \wedge x + w^*, \quad \forall w \in I, \forall x \in V_0 .$$

Note that, given $v_i \wedge v_j \in I$, since $v_i \wedge v_j = 1 + f_{v_i} + f_{v_j} + f_{v_i+v_j}$ (with the usual abuse of notation), we have $\tilde{f}(1 + f_{v_i} + f_{v_j} + f_{v_i+v_j}) = v_i \wedge v_j \in U$. This equality can be generalized in the following way:

Proposition 4.8. *$\tilde{f}(1 + f_p + f_q + f_{p+q}) = p \wedge q$, for all $p, q \in V_0$.*

Proof. Set $p = \sum_{i \in S(p)} v_i$ and $q = \sum_{i \in S(q)} v_i$. We get:

$$\begin{aligned} 1 + f_p + f_q + f_{p+q} &= 1 + \overline{p} + \overline{q} + \overline{p+q} \\ &= 1 + 1 + \sum_{i \in S(p)} v_i + \sum_{\{i,j\} \in \binom{S(p)}{2}} v_i \wedge v_j + \cdots \\ &\qquad + 1 + \sum_{i \in S(q)} v_i + \sum_{\{i,j\} \in \binom{S(q)}{2}} v_i \wedge v_j + \cdots \\ &\qquad + 1 + \sum_{i \in S(p+q)} v_i + \sum_{\{i,j\} \in \binom{S(p+q)}{2}} v_i \wedge v_j + \cdots \end{aligned}$$

It follows:

$$\begin{aligned} &\tilde{f}(1 + f_p + f_q + f_{p+q}) \\ &\qquad = \sum_{\{i,j\} \in \binom{S(p)}{2}} v_i \wedge v_j + \sum_{\{i,j\} \in \binom{S(q)}{2}} v_i \wedge v_j + \sum_{\{i,j\} \in \binom{S(p+q)}{2}} v_i \wedge v_j . \end{aligned}$$

Since $S(p+q) = S(p) \triangle S(q)$, we can delete the summands $v_i \wedge v_j$ with:
i, j belonging to $S(p) \cap S(q)$, as common to the first and the second sum,
i, j belonging to $S(p) - S(q)$, as common to the first and the third sum,
i, j belonging to $S(q) - S(p)$, as common to the second and the third sum.

Thus:

$$\begin{aligned} &\tilde{f}(1 + f_p + f_q + f_{p+q}) \\ &\qquad = \sum_{\substack{i \in S(p) \cap S(q) \\ j \in S(p) - S(q)}} v_i \wedge v_j + \sum_{\substack{i \in S(p) \cap S(q) \\ j \in S(q) - S(p)}} v_i \wedge v_j + \sum_{\substack{i \in S(p) - S(q) \\ j \in S(q) - S(p)}} v_i \wedge v_j . \end{aligned}$$

On the other hand

$$
\begin{aligned}
p \wedge q &= \sum_{i \in S(p)} v_i \wedge \sum_{j \in S(q)} v_j \\
&= \Bigg(\sum_{i \in S(p) \cap S(q)} v_i + \sum_{i \in S(p) - S(q)} v_i \Bigg) \wedge \Bigg(\sum_{j \in S(p) \cap S(q)} v_j + \sum_{j \in S(q) - S(p)} v_j \Bigg) \\
&= \sum_{\substack{i \in S(p) \cap S(q) \\ j \in S(p) - S(q)}} v_i \wedge v_j + \sum_{\substack{i \in S(p) \cap S(q) \\ j \in S(q) - S(p)}} v_i \wedge v_j + \sum_{\substack{i \in S(p) - S(q) \\ j \in S(q) - S(p)}} v_i \wedge v_j \,. \qquad \square
\end{aligned}
$$

Now, we introduce a projection π from $\widetilde{U} : (T' : L)$ to $U : (T_H : L_H)$, which extends $\tilde{f}$. It is enough to define the images by π of the elements of T' and L. We do it in the following natural way:

$$\pi(\widehat{z} t_z) = \tau_z, \quad \pi(\alpha) = \alpha \,.$$

We have to check that this projection reproduces the actions defined in the semi-direct products.

The action of T' on $\widetilde{U}$ is projected on the action of T_H on U. In fact, since $(w + w \wedge \widehat{z} + \widehat{z} \wedge \widehat{x})^*$, the grade 2 part of $w + w \wedge \widehat{z} + \widehat{z} \wedge \widehat{x}$, equals $w^* + z \wedge x$, we have:

$$
\begin{aligned}
\pi(\widehat{z} t_z) \bullet \pi(w t_x) = \tau_z \bullet (v_\infty \wedge x + w^*) &= (v_\infty + z) \wedge x + w^* \\
&= \pi((w + w \wedge \widehat{z} + \widehat{z} \wedge \widehat{x}) t_x) = \pi(\widehat{z} t_z \bullet w t_x) \,.
\end{aligned}
$$

The action of L on $\widetilde{U}$ is projected on the action of L_H on U. Indeed, using the notation on the isomorphism between U_S and E and Proposition 4.8:

$$
\begin{aligned}
\pi(\alpha) \bullet \pi(w t_x) &= \alpha \bullet \Big(v_\infty \wedge x + \sum_{\{i,j\}} a_{ij} v_i \wedge v_j \Big) \\
&= v_\infty \wedge \alpha[x] + \sum_{\{i,j\}} a_{ij} \alpha[v_i] \wedge \alpha[v_j] \\
&= \pi\bigg(\Big(\sum_{\{i,j\}} a_{ij} \big(f_{v_0} + f_{\alpha[v_i]} + f_{\alpha[v_j]} + f_{\alpha[v_i] + \alpha[v_j]} \big) + \cdots \Big) t_{\alpha[x]} \bigg) \\
&= \pi\bigg(\Big(\sum_{\{i,j\}} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \cdots \Big) t_{\alpha[x]} \bigg) = \pi(\alpha \bullet w t_x) \,.
\end{aligned}
$$

The action of L on T' is projected on the action of L_H on T_H. In fact:

$$\pi(\alpha) \bullet \pi(\widehat{z} t_z) = \alpha \bullet \tau_z = \tau_{\alpha[z]} = \pi(\widehat{\alpha[z]} t_{\alpha[z]}) = \pi(\alpha \bullet \widehat{z} t_z) \,.$$

Finally, a straightforward computation shows that the actions of the groups $\widetilde{U}$, T', L on the three types of $\mathrm{Exp}(\widetilde{\rho})$ are respectively projected by π on the actions of U, T_H, L_H on the three types of $\mathrm{Exp}(\rho)$. So, Theorem 4.7 is proved.

References

[1] **B. Baumeister**, **T. Meixner** and **A. Pasini**, GF(2)-expansions, *Geom. Dedicata* **67** (1997), 163–180.

[2] **B. Baumeister** and **A. Pasini**, On flat flag-transitive $c.c^*$-geometries, *J. Algebraic Combin.* **6** (1997), 2–26.

[3] **B. Baumeister** and **G. Stroth**, The non-canonical gluings of two affine spaces, **in** *Groups and Geometries* (L. of Martino et al. eds.), Birkhäuser, Basel (1998), 9–28.

[4] **M. Buratti** and **A. Del Fra**, Semi-Boolean Steiner quadruple systems and dimensional dual hyperovals, *Adv. Geom.*, special issue in honour of Adriano Barlotti (2003), S254–S270.

[5] **J. H. Conway**, **R. T. Curtis**, **S. P. Norton**, **R. A. Parker** and **R. A. Wilson**, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[6] **B. N. Cooperstein** and **J. A. Thas**, On generalized k-arcs in PG$(2n, q)$, *Ann. Comb.* **5** (2001), 141–152.

[7] **A. Del Fra**, On d-dimensional dual hyperovals, *Geom. Dedicata* **79** (2000), 157–178.

[8] **A. Del Fra** and **S. Yoshiara**, Dimensional dual hyperovals associated with Steiner systems, *European J. Combin.* **26** (2005), 173–194.

[9] **C. Huybrechts**, Dimensional dual hyperovals in projective spaces and $c.$AG*-geometries, *Discrete Math.* **255** (2002), 193–223.

[10] **C. Huybrechts** and **A. Pasini**, Flag-transitive extensions of dual affine spaces, *Beitrăge zur Algebra und Geometrie*, **40** (1999), 503–531.

[11] **A. V. Ivanov** and **S. V. Shpectorov**, *Geometries of Sporadic Groups II. Representations and Amalgams*, Cambridge University Press, Cambridge 2002.

[12] **A. Pasini**, *Diagram Geometries*, Oxford University Press, Oxford 1994.

[13] **A. Pasini**, Embeddings and expansions, *Bull. Belg. Math. Soc. Simon Stevin* **10** (2003), 585–626.

[14] **A. Pasini** and **S. Yoshiara**, On a new family of flag-transitive semibiplanes, *European J. Combin.* **22** (2001), 529–545.

[15] **A. Pasini** and **S. Yoshiara**, New distance regular graphs arising from dimensional dual hyperovals, *European J. Combin.* **22** (2001), 547–560.

[16] **J. A. Thas** and **H. Van Maldeghem**, Characterizations of quadric and Hermitian Veroneseans, *J. Geom.* **76** (2003), 282–293.

[17] **J. A. Thas** and **H. Van Maldeghem**, Characterizations of the finite quadric Veroneseans $\mathcal{V}_n^{2^n}$, *Quart. J. Oxford* **55** (2004), 99–113.

[18] **S. Yoshiara**, A family of d-dimensional dual hyperovals in $\mathsf{PG}(2d+1,2)$, *European J. Combin.* **20** (1999), 489–503.

[19] **S. Yoshiara**, Ambient spaces of dimensional dual arcs, *J. Alg. Combin.* **19** (2004), 5–23.

[20] **S. Yoshiara**, Dimensional dual arcs. A survey, *to appear*.

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