The universal representation group of Huybrechts’s dimensional dual hyperoval

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Abstract

A $d$-dimensional dual hyperoval can be regarded as the image $S = \rho(\Sigma)$ of a full $d$-dimensional projective embedding $\rho$ of a dual circular space $\Sigma$. The affine expansion $\text{Exp}(\rho)$ of $\rho$ is a semiplane and its universal cover is the expansion of the abstract hull $\tilde{\rho}$ of $\rho$.

In this paper we consider Huybrechts’s dual hyperoval, namely $\rho(\Sigma)$ where $\Sigma$ is the dual of the affine space $\text{AG}(n, 2) \subset \text{PG}(n, 2)$ and $\rho$ is induced by the embedding of the line grassmannian of $\text{PG}(n, 2)$ in $\text{PG}(\binom{n+1}{2} - 1, 2)$.

It is known that the universal cover of $\text{Exp}(\rho)$ is a truncation of a Coxeter complex of type $D_{2^n}$ and that, if $\bar{U}$ is the codomain of the abstract hull $\tilde{\rho}$ of $\rho$, then $\bar{U}$ is a subgroup of the Coxeter group $D$ of type $D_{2^n}$, $|\bar{U}| = 2^{2^n - 1}$ but $\bar{U}$ is non-commutative. This information does not explain what the structure of $\bar{U}$ is and how $\bar{U}$ is placed inside $D$. These questions will be answered in this paper.

Keywords: dimensional dual hyperovals, semiplanes, embeddings, Coxeter groups, exterior algebras

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1. Introduction

Throughout this paper we adopt the notation of [5] for groups, thus using the colon “:” for split extensions, the symbol $2^n$ to denote an elementary abelian 2-group of order $2^n$, and so on. We refer to [12] for basics on diagram geometry.
1.1. Dimensional dual hyperovals

Dimensional dual hyperovals have been introduced by Yoshiara [18] and Huybrechts and Pasini [10], mainly in connection with the investigation of AG.c*-geometries (in particular c.c*-geometries with (IP), namely semibiplanes; see Pasini and Yoshiara [14]). In particular, a 2-dimensional dual hyperoval of size 22, living inside the hermitian variety \( \mathcal{H}(5, 4) \) and admitting \( 3 \cdot \text{Aut}(M_{22}) \) as its automorphism group, naturally gives rise to an AG.c*-geometry for \( 2^{12}:3 \cdot \text{Aut}(M_{22}) \), thoroughly investigated in [10]. However, dimensional dual hyperovals have soon appeared to be relevant in other respects than strict diagram geometry or the investigation of semibiplanes. For instance, they play a role in the characterization of Veronesean varieties by Thas and Van Maldeghem [16], [17]. They are quite naturally linked with distance regular graphs, semi-partial geometries and codes (Cooperstein and Thas [6], Pasini and Yoshiara [15]). In particular, new distance regular graphs have been discovered with the aid of dimensional dual hyperovals (Pasini and Yoshiara [15]). Interesting connection also exist between certain dimensional dual hyperovals and Steiner systems (Huybrechts [9], Buratti and Del Fra [4], Del Fra and Yoshiara [8]).

In view of the above, it is not surprising that dimensional dual hyperovals have soon been regarded as objects interesting in themselves. The earliest thoroughful investigation of their geometrical properties is due to Del Fra [7], but many progresses have been done since then. We refer the reader to Yoshiara [20] for a survey of this topic, updated at 2005. We shall only recall a few essentials here.

Let \( V := V(N, q) \), the \( N \)-dimensional vector space over \( \text{GF}(q) \). According to the usual definition, for a positive integer \( d < N \), a \( d \)-dimensional dual hyperoval of \( \text{PG}(V) \) is a family \( S \) of \( d \)-dimensional subspaces of \( \text{PG}(V) \) such that:

\[(DH1) \ |X \cap Y| = 1 \text{ for any two distinct members } X, Y \in S;\]

\[(DH2) \text{ every point of } \text{PG}(V) \text{ belongs to either none or just two members of } S;\]

\[(DH3) \langle \bigcup_{X \in S} X \rangle = \text{PG}(V).\]

Clearly, (DH3) implies \( N \geq 2d + 1 \). An upper bound for \( N \) has been discovered by Yoshiara [19] (see also Del Fra [7] for the case of \( d = 2 \)).

**Proposition 1.1.** If either \( q > 2 \) or \( d \leq 2 \) then \( N \leq \binom{d+2}{2} \). If \( q = 2 \) and \( d > 2 \) then \( N \leq \binom{d+2}{2} + 2 \).

Actually, \( N \leq \binom{d+2}{2} \) in all known examples. So, one might conjecture that \( N \leq \binom{d+2}{2} \) even when \( q = 2 \).
1.2. Dimensional projective embeddings

It readily follows from (DH1) and (DH2) that $S$, regarded as an abstract combinatorial structure, is nothing but the dual $\Sigma$ of the circular space on $\theta := q^d + q^{d-1} + \cdots + q + 2$ points. (We recall that $\Sigma$ is the point-block geometry with $\theta(\theta - 1)/2$ points and $\theta$ blocks, where every point belongs to precisely two blocks, every block has $\theta - 1$ points and any two blocks have just one point in common.) Accordingly, we may regard $S$ as the image of a (full) $d$-dimensional projective embedding $\rho : \Sigma \to \text{PG}(V)$ of $\Sigma$ in $\text{PG}(V)$. Explicitly, $\rho$ is a mapping from the set $P$ of points of $\Sigma$ to the set of 1-dimensional linear subspaces of $V$ such that:

(PE1) $\rho$ is injective;

(PE2) for every block $B$ of $\Sigma$, the set $\rho(B) := \bigcup_{p \in B} \rho(p)$ is a $(d + 1)$-dimensional linear subspace of $V$;

(PE3) $\rho(P)$ spans $V$.

Clearly, $S = \rho(B) = \{\rho(B)\}_{B \in B}$, where $B$ is the block-set of $\Sigma$, but we prefer to say that $S$ is the image of $\Sigma$, thus writing $S = \rho(\Sigma)$ even if these conventions are slightly abusive.

The point of view we have thus chosen makes it easier to define morphisms of dimensional dual hyperovals, by exploiting the usual machinery set up for embeddings (see Pasini [13]; compare Yoshiara [20, section 2.5]). So, if $\rho' : \Sigma \to \text{PG}(V)$ is another $d$-dimensional projective embedding of $\Sigma$, a morphism from $S' = \rho'(\Sigma)$ to $S$ is a morphism from $\rho'$ to $\rho$, namely a semi-linear mapping $f : V' \to V$ such that $f\rho' = \rho$. Needless to say, an isomorphism is an invertible morphism. In spite of the above, $\text{Aut}(S)$ is not defined as the same as the group of all automorphisms of $\rho$. Every automorphism of $\rho$ acts trivially on $\Sigma$, whereas one would like to see some action of $\text{Aut}(S)$ on $S$. The stabilizer of $S$ in $\text{PGL}(N, q)$ is perhaps the most natural choice for $\text{Aut}(S)$, but it is more convenient to define $\text{Aut}(S)$ as the (setwise) stabilizer of $S$ in $\text{GL}(N, q)$, which is the same as the group of all isomorphisms from $\rho\alpha$ to $\rho\beta$, where $\alpha, \beta$ range over $\text{Aut}(\Sigma)$. We shall follow this convention.

It is well known that every $d$-dimensional projective embedding $\rho : \Sigma \to \text{PG}(V)$ admits a projective hull, namely a pair $(\hat{\rho}, \hat{f})$ uniquely determined up to isomorphisms by the following conditions (see Pasini [13]):

(PH1) $\hat{\rho} : \Sigma \to \text{PG}(\hat{V})$ is a $d$-dimensional projective embedding of $\Sigma$ and $\hat{f} : \hat{\rho} \to \rho$ is a morphism from $\hat{\rho}$ to $\rho$;

(PH2) for any other $d$-dimensional projective embedding $\rho' : \Sigma \to \text{PG}(V')$, if
there is a morphism $f : \rho' \to \rho$ then a morphism $g : \hat{\rho} \to \rho'$ also exists such that $fg = \hat{f}$.

The image $\hat{S} = \hat{\rho}(\Sigma)$ of $\Sigma$ by $\hat{\rho}$ will be called the projective hull of $S = \rho(\Sigma)$. We say that $S$ is projectively dominant if $\hat{\rho} \cong \rho$, namely $\hat{f}$ is an isomorphism. In short, $S$ is projectively dominant if it is its own projective hull. Clearly, if $N$ attains the upper bound of Proposition 1.1, then $S$ is projectively dominant.

1.3. Expansions

Given $\Sigma = (P, B)$ as above, let $\rho : \Sigma \to \text{PG}(V)$ be a $d$-dimensional projective embedding and put $S := \rho(\Sigma)$. The affine expansion of $\Sigma$ by $\rho$ (also called the expansion of $\rho$, for short, or the expansion of $S$) is the geometry $\text{Exp}(\rho)$ of rank 3 defined as follows:

Taken the integers 0, 1, 2 as types for $\text{Exp}(\rho)$, the 0-elements of $\text{Exp}(\rho)$ (also called points) are the vectors of $V$, the 1-elements (lines) are the cosets of the subspaces $\rho(p)$ of $V$ for $p \in P$ and the 2-elements (blocks) are the cosets of the subspaces $\rho(B)$, for $B \in B$. The incidence relation is the natural one, namely inclusion.

The residues of the points of $\text{Exp}(\rho)$ are isomorphic to $\text{AG}$ whereas the residues of the blocks of $\text{Exp}(\rho)$ are $(d + 1)$-dimensional affine spaces over $\text{GF}(q)$. So, $\text{Exp}(\rho)$ belongs to the following diagram, where the label $\text{AG}$ denotes the class of affine spaces and $c^*$ the class of dual circular spaces:

\[
\text{(AG, c*)} \quad \text{AG} \quad c^*
\]

In particular, when $q = 2$ then $\text{Exp}(\rho)$ is a semibiplane. In this case its diagram can be depicted as follows:

\[
\text{(c, c*)} \quad c \quad c^*
\]

Let $U$ be the translation group of the affine geometry AG$(V)$ of $V$. Then $\text{Aut}(S)$, regarded as a subgroup of the automorphism group $\text{Aut}(\text{AG}(V))$ of AG$(V)$, normalizes $U$. The semidirect product $U: \text{Aut}(S)$ is the stabilizer of $\text{Exp}(\rho)$ in $\text{Aut}(\text{AG}(V)) = A\Gamma L(N, q)$. We shall denote that stabilizer by $\text{Aut}_\rho(\text{Exp}(\rho))$. Clearly, $\text{Aut}_\rho(\text{Exp}(\rho))$ is point-transitive on $\text{Exp}(\rho)$, and it is flag-transitive if and only if $\text{Aut}(S)$ acts doubly-transitively on $S$. In principle, $\text{Aut}_\rho(\text{Exp}(\rho))$ might be smaller than the full automorphism group of the geometry $\text{Exp}(\rho)$. Actually, we are not aware of any example like that, but examples of this kind
are met frequently when we switch from $d$-dimensional projective embeddings to $(d, q)$-embeddings (to be defined in the next subsection).

Every morphism $f : \rho' \to \rho$ induces a covering $\text{Exp}(f)$ from $\text{Exp}(\rho')$ to $\text{Exp}(\rho)$. In particular, if $(\tilde{\rho}, \tilde{f})$ is the projective hull of $\rho$, then $\text{Exp}(f)$ is a covering from $\text{Exp}(\tilde{\rho})$ to $\text{Exp}(\rho)$. It also follows from the definition of $\tilde{\rho}$ that $\text{Aut}(\tilde{S})$ lifts to the stabilizer of $\ker(f)$ in $\text{Aut}(\tilde{S})$, where $\tilde{S} := \tilde{\rho}(\Sigma)$. Accordingly, $\text{Aut}_\rho(\text{Exp}(\rho))$ lifts to a subgroup of $\text{Aut}_\rho(\text{Exp}(\tilde{\rho}))$. However, in general, $\text{Exp}(\tilde{\rho})$ is not the universal cover of $\text{Exp}(\rho)$. This unpleasant asymmetry can be repaired by generalizing the notion of embeddings, as we shall do in the next subsection. The definition we shall give is borrowed from [13]. It is not so general as in [13], but it suites our present needs.

### 1.4. Abstract embeddings and their expansions

An abstract $(d, q)$-embedding (a $(d, q)$-embedding for short) of $\Sigma = (P, B)$ in a group $U$ is a mapping $\rho : P \to U$ into the subgroup lattice of $U$ such that:

1. $(\text{AE}1)$ $\rho(p)$ is an elementary abelian subgroup of $U$ of order $q$, for any $p \in P$;
2. $(\text{AE}2)$ we have $\rho(e_1) \cap \rho(e_2) = 1$ for any two distinct points $e_1, e_2 \in P$;
3. $(\text{AE}3)$ $\rho(B) := \bigcup_{p \in B} \rho(p)$ is an elementary abelian subgroup of $U$ of order $q^{d+1}$, for every block $B \in B$;
4. $(\text{AE}4)$ the set $\bigcup_{p \in P} \rho(p)$ spans $U$.

We call $U$ the target group of the $(d, q)$-embedding $\rho$. The family $\rho(\Sigma) := \{\rho(p)\}_{p \in P} \cup \{\rho(B)\}_{B \in B}$ will be called the image of $\Sigma$ by $\rho$. We write $\rho : \Sigma \to U$ when we want to recall that $\rho$ is a $(d, q)$-embedding of $\Sigma$ in $U$.

Given another $(d, q)$-embedding $\overline{\rho} : \Sigma \to \overline{U}$, a morphism from $\overline{\rho}$ to $\rho$ is a homomorphism $f : \overline{U} \to U$ such that, for every element (point or block) $X$ of $\Sigma$, $f$ induces an isomorphism from $\overline{\rho}(X)$ to $\rho(X)$. The kernel of the morphism $f$ is its kernel $\ker(f)$ as a homomorphism from $\overline{U}$ to $U$.

Clearly, $d$-dimensional projective embeddings are $(d, q)$-embeddings (with additive groups of vector spaces as target groups) and morphisms of $d$-dimensional projective embeddings are morphisms in the above sense. Hulls of $(d, q)$-embeddings are defined in the same way as projective hulls of $d$-dimensional projective embeddings. We refer to [13] for details. We only recall how the hull $(\tilde{\rho}, \tilde{f})$ of a $(d, q)$-embedding $\rho : \Sigma \to U$ is constructed. The image $\rho(\Sigma)$ of $\Sigma$ is actually an amalgam of groups. Let $\tilde{U}$ be the universal completion of that amalgam and $\tilde{f} : \tilde{U} \to U$ be the canonical projection of $\tilde{U}$ onto $U$. For every element (point or block) $X$ of $\Sigma$, let us denote by $\rho(\tilde{X})$ the subgroup of
The expansion $\text{Exp}(\rho)$ of a $(d, q)$-embedding $\rho : \Sigma \to U$ is defined in the same way as the expansion of a $d$-dimensional projective embedding: the elements of $U$ are the points of $\text{Exp}(\rho)$, the lines of $\text{Exp}(\rho)$ are the right cosets $u \cdot \rho(p)$ for $p \in P$ and $u \in U$, the blocks are the right cosets $u \cdot \rho(B)$ for $B \in B$. The target group $U$, acting on itself by left multiplication, acts regularly on the point-set of $\text{Exp}(\rho)$. We call it the translation group of $\text{Exp}(\rho)$. Let $\text{Aut}(\rho(\Sigma))$ be the set-wise stabilizer of $\rho(\Sigma)$ in $\text{Aut}(U)$. Then $\text{Aut}(\rho(\Sigma))$ can be regarded as a subgroup of $\text{Aut}(\text{Exp}(\rho))$. It stabilizes the point $1 \in U$ of $\text{Exp}(\rho)$ and normalizes the translation group $U$. The semi-direct product $U: \text{Aut}(\rho(\Sigma))$, regarded as a subgroup of $\text{Aut}(\text{Exp}(\rho))$, will be denoted by $\text{Aut}_\rho(\text{Exp}(\rho))$. In general, $\text{Aut}_\rho(\text{Exp}(\rho)) < \text{Aut}(\text{Exp}(\rho))$.

Every morphism $f : \overline{\rho} \to \rho$ induces a covering $\text{Exp}(f)$ from $\text{Exp}(\overline{\rho})$ to $\text{Exp}(\rho)$ and $\ker(f)$, regarded as a subgroup of the translation group of $\text{Exp}(\overline{\rho})$, is the deck group of the covering $\text{Exp}(f)$. The following is proved in [13]:

**Proposition 1.2.** The universal cover of the geometry $\text{Exp}(\rho)$ is the expansion $\text{Exp}(\overline{\rho})$ of the abstract hull $\overline{\rho}$ of $\rho$. In particular, $\text{Exp}(\rho)$ is simply connected if and only if $\rho$ is abstractly dominant.

For quite a few $d$-dimensional dual hyperovals $S = \rho(\Sigma)$ the expansion $\text{Exp}(\rho)$ is known to be simply connected (see Pasini and Yoshiara [14] and [15]). In those cases, the $d$-dimensional projective embedding $\rho$ is abstractly dominant. However, this is not always the case. For instance, let $\Sigma_{22}$ be the dual of the circular space with 22 points and $S_{22} = \rho(\Sigma_{22})$ be its realization as a 2-dimensional dual hyperoval of $\text{PG}(5, 4)$ with $\text{Aut}(S_{22}) = 3 \text{Aut}(M_{22})$, mentioned in Subsection 1.1 (see [20, 5.1] for more details). As the ambient space of $\rho$ attains the upper bound of Proposition 1.1, $\rho$ is projectively dominant. However, the geometry $\Gamma := \text{Exp}(\rho)$ is not simply connected, but it admits a simply connected double cover $\tilde{\Gamma}$ (Huybrechts and Pasini [10]). Accordingly, $\rho$ is not abstractly dominant and $\tilde{\Gamma}$ is the expansion of the abstract hull $\overline{\rho} : \Sigma_{22} \to \tilde{U}$ of $\rho$. The universal representation group $\tilde{U}$ of $\rho$ is non-commutative. In fact, $\tilde{U}$ is
the extraspecial group $2^{1+12}_+$. (We warn the reader that $\tilde{U}$ is sloppily described as $2^{13}$ in [10].)

1.5. The problem discussed in this paper

Let $q = 2$ and $\rho : \Sigma \to \text{PG}(V)$ be a $d$-dimensional projective embedding of $\Sigma$, with $V = V(N, 2)$. It may happen that the universal cover $\tilde{\Gamma}$ of $\Gamma := \text{Exp}(\rho)$ is a truncated Coxeter complex of type $D_M$, where $M = 2^{d+1}$:

- blocks of $\tilde{\Gamma}$
- lines of $\tilde{\Gamma}$
- points of $\tilde{\Gamma}$

If this is the case then we say that the dimensional dual hyperoval $S = \rho(\Sigma)$ is \textit{Coxeter-like}, for short. Suppose that $S$ is indeed Coxeter-like. Then $\tilde{\Gamma}$ has $2^{M-1}$ points. The universal representation group $\tilde{U}$ of $S$ acts regularly on the point-set of $\tilde{\Gamma}$. Hence $|\tilde{U}| = 2^{M-1}$. However, if $d > 2$ then $M - 1 > (d+2)/2 + 2$ and, if $d = 2$ then $M - 1 > (d+2)/2$. So, if $d > 1$ then $\tilde{U}$ is non-commutative. Indeed, if $\tilde{U}$ were commutative, then it would be an elementary abelian 2-group, whence a $\text{GF}(2)$-vector space, contrary to the bounds stated by Proposition 1.1. On the other hand, $\tilde{U}$ is a subgroup of the Coxeter group $W := W_{D_M}$ of type $D_M$, since $\text{Aut}(\tilde{\Gamma}) = W$. We recall that $W = O_2(W):\text{Sym}(M)$ with $O_2(W) = 2^{M-1}$. Thus, $\tilde{U}$ cannot be a subgroup of $O_2(W)$. So, we should cut $\tilde{U}$ off of $W$, but keeping in mind that $\tilde{U}$, being non-commutative, cannot be entirely contained in $O_2(W)$. This is the problem we shall tackle in this paper.

Before to go on, we get rid of the case $d = 1$. In this case $\Gamma = \text{Exp}(\rho)$ is the geometry obtained from $\text{PG}(3, 2)$ by removing all points and lines of a given plane $\pi$ and all points and planes through a distinguished point $p$ of $\pi$. It is well known (and easy to check) that this geometry is simply connected. Hence $\rho$ is abstractly dominant.

Assume now $d > 1$. For $N = (d+2)/2$, only two Coxeter-like dimensional dual hyperovals are known with ambient vector space $V(N, 2)$. They are those denoted by $S(\emptyset)$ and $S(V \setminus \{0\})$ in [4] and [8]. In this paper, we shall denote them by $S_H$ and $S_{BDF}$ respectively, where the subscript $H$ is meant to record C. Huybrechts, who has studied this dimensional dual hyperoval in [9], whereas $BDF$ should remind us of Buratti and Del Fra [4]. Several Coxeter-like $d$-dimensional dual hyperovals are known with ambient vector space of dimen-
sion less then \((d+2)/2\), but all of them turn out to be homorphic images of \(S_H\) (Pasini and Yoshiara [15]). So, as far as our knowledge presently goes, \(S_H\) and \(S_{BDF}\) are the most interesting Coxeter-like examples, and \(S_H\) is perhaps the most interesting of the two.

In this paper we shall focus on \(S_H\). In Section 2 we shall describe \(S_H\) and its expansion. Denoting by \(\rho_H\) the \(d\)-dimensional projective embedding giving rise to \(S_H\), in Section 3 we shall describe the abstract hull \(\tilde{\rho}\) of \(\rho_H\). As a by-product of our construction, we will obtain that \(\rho_H\) is projectively dominant (as already proved by Buratti and Del Fra [4]). Finally, in Section 4 we shall show how the universal representation group of \(S_H\) is placed inside the Coxeter group \(W\) of type \(D_M\).

A warning, before to finish: in [13, Theorem 8.5] it is claimed that the universal representation group of \(S_H\) is abelian. Clearly, that claim is wrong.

\section{\(S_H\) and its expansion}

\subsection{Huybrechts’s construction of \(S_H\)}

Let \(V = V(n+1,2)\) \((n \geq 2)\) and \(V_0 \cong V(n,2)\) be a hyperplane of \(V\). Denoting by \(\mathcal{A}\) the affine geometry \(PG(V) \setminus PG(V_0)\), let \(\Sigma = (P,B)\) be the dual of the point-line system of \(\mathcal{A}\). Thus, \(P\) is the set of lines of \(\mathcal{A}\) and \(B\) is the point-set of \(\mathcal{A}\). As the point-line system of \(\mathcal{A}\) is a circular space with \(2^n\) points, \(\Sigma\) is a dual circular space. Turning to the line-grassmannian \(Gr(PG(V))\) of \(PG(V)\), the members of \(B\) are maximal singular subspaces of \(Gr(PG(V))\), of (projective) dimension \(d = n - 1\), whereas \(P\) is a set of points of \(Gr(PG(V))\). Let \(\rho_{gr}\) be the natural projective embedding of \(Gr(PG(V))\) into \(PG(V \wedge V)\) and \(\rho_H\) be the mapping induced by \(\rho_{gr}\) on \(\Sigma = (P,B)\). Then \(\rho_H(P)\) spans \(PG(V \wedge V)\) and the image \(S_H = \rho_H(B)\) of \(B\) by \(\rho_H\) is a \(d\)-dimensional dual hyperoval of \(PG(V \wedge V)\). This is indeed the dimensional dual hyperoval discovered by Huybrechts [9].

\subsection{\text{Aut}(\(S_H\)) and \text{Aut}_{\rho_H}(\text{Exp}(\rho_H))\)}

The automorphism group \(\text{Aut}(\mathcal{A})\) of the affine geometry \(\mathcal{A}\) is the stabilizer of \(V_0\) in \(L_{n+1}(2)\), namely \(ASL(n,2) = T:L\), where \(T\) is the translation group of \(\mathcal{A}\) and \(L \cong L_n(2)\). Clearly, also \(\text{Aut}(S_H) \cong ASL(n,2)\). Accordingly, \(\text{Aut}_{\rho_H}(\text{Exp}(\rho_H)) = U:A_H\), where \(U := V \wedge V\) (the latter being regarded as an elementary abelian 2-group) and \(A_H := \text{Aut}(S_H)\). Clearly \(A_H = T_H:L_H\), where \(T_H \cong T\) and \(L \cong L_n(2)\). The group \(A\) acts on \(U = V \wedge V\) as a subgroup of the group \(L_{n+1}(2)\) of linear transformations of \(V\), in its natural action on \(V \wedge V\). More explicitly,
recall that $A_H$ is the stabilizer of $V_0$ in $L_{n+1}(2)$. Accordingly, the subgroup $L_H$ of $A_H$ is the stabilizer of $V_0$ and a vector $v_\infty \in V \setminus V_0$. As $V = \langle v_\infty \rangle \oplus V_0$, we can split $V \setminus V$ as $(v_\infty \setminus V_0) \oplus (V_0 \setminus V_0)$. The group $L_H$, acting on $V \setminus V$ as a subgroup of $L_{n+1}(2)$, stabilizes both $v_\infty \setminus V_0$ and $V_0 \setminus V_0$. Moreover, on each of these two subspaces, the action of $L_H$ is that induced by its action on $V_0$. The elements of $T_H$ can be identified with the vectors of $V_0$. The subspace $V_0 \setminus V_0$ is centralized by $T_H$ whereas, if $z \in T_H$ corresponds to $z \in V_0$, then $\tau_z(v_\infty \setminus x) = v_\infty \setminus x + x \setminus z$ for every $x \in V_0$.

The expansion $\text{Exp}(\rho_H)$ can also be described as follows. Let $\Delta$ be the building of type $D_{n+1}$ over $\text{GF}(2)$ and $F = \{M^+, M^-\}$ be a flag of $\Delta$ of type $\{+,-\}$, where types are given as below and, for $\epsilon = +$ or $-\, M^\epsilon$ is the element of $F$ of type $\epsilon$.

\[
\begin{array}{cccccc}
 & + & & & & \\
 & n-2 & n-3 & \cdots & 1 & 0 \\
- & & & & & \\
\end{array}
\]

Then $\text{Exp}(\rho_H)$ is isomorphic to the subgeometry $\Gamma$ of $\Delta$ formed by the elements of type $+,-$ and $n-2$ at maximal distance from $F$ (see Pasini and Yoshiara [14, section 6]). The incidence relation of $\Gamma$ is inherited from $\Delta$ except that two elements $X^+, X^-$ of $\Gamma$ of type $+$ and $-$, incident in $\Delta$, are declared to be incident in $\Gamma$ if and only if the flag $\{X^+, X^+\}$ has maximal distance from $F$. The elements of $\Gamma$ of type $n-2$ correspond to the lines of $\text{Exp}(\rho_H)$. We can take the elements of type $+$ as points and those of type $-$ as blocks, but we can also interchange the roles of $+$ and $-$, thus taking $(-)$-elements as points and $(+)$-elements as blocks. This makes it clear that $\Gamma$ admits a duality $\delta$, contributed by a graph automorphism of $\text{Aut}(\Delta)$. The group $\text{Aut}_{\rho_H}(\text{Exp}(\rho_H))$ is just the stabilizer of $F$ in $\text{Aut}(\Delta)$. Its subgroup $L_H$ also stabilizes a unique flag opposite to $F$, whereas $T_H$ acts trivially on one of the two members of $F$, say $M^+$. The subgroup $v_\infty \setminus V_0$ of $U$ acts trivially on $M^-$, whereas $V_0 \setminus V_0$ acts trivially on both $M^+$ and $M^-$. The above mentioned duality $\delta$ normalizes $\text{Aut}_{\rho_H}(\text{Exp}(\rho_H))$. It can be chosen in such a way as to normalize both $L_H$ and $V_0 \setminus V_0$ and interchange $T_H$ with $v_\infty \setminus V_0$. Accordingly, $\delta$ permutes $M^+$ and $M^-$ and stabilizes the flag opposite to $F$ stabilized by $L_H$. 
2.3. The universal cover of \( \text{Exp}(\rho_H) \)

With \( \Gamma = \text{Exp}(\rho_H) \) as above, let \( \tilde{\Gamma} \) be its universal cover. Then \( \tilde{\Gamma} \) is a truncated Coxeter complex of type \( D_{2n} \) (Baumeister, Meixner and Pasini [1]; see also Pasini and Yoshiara [14] or Baumeister and Stroth [3]). So, \( S_H \) is Coxeter-like.

Let \( \pi : \tilde{\Gamma} \to \Gamma \) be the covering projection from \( \tilde{\Gamma} \) onto \( \Gamma \) and \( K \) be the deck group of \( \pi \). By comparing the number of chambers of \( \tilde{\Gamma} \) with the number of chambers of \( \Gamma \) we immediately obtain that

\[
|K| = 2^{2n} - \binom{n+1}{2} - 1.
\]

Note that if \( n = 2 \) then \( K = 1 \), namely \( \tilde{\Gamma} = \Gamma \). So, henceforth we assume \( n > 2 \).

In order to describe \( K \) more precisely we need to recall a few facts on the Coxeter groups of type \( D_{2n} \) and \( C_{2n} \). Let \( S \) be a set of size \( 2^n \) and \( U_S \) be the GF(2)-vector space with \( S \) as a basis. Namely, the vectors of \( U_S \) are the functions from \( S \) to \( \text{GF}(2) \). The support of such a vector \( f \) is the set \( S(f):= \{ p \in S \mid f(p) = 1 \} \) and the weight \( w(f) \) of \( f \) is the size \( |S(f)| \) of \( S(f) \). If \( X = S(f) \), we shall write \( f = f_X \). In particular, if \( p \in S \) then \( f(p) \), also denoted by \( f_p \), is the vector with \( S(f) = \{ p \} \). The vectors of even weight form a maximal subspace \( U^+_S \) of \( U_S \). Clearly, \( U_S = U^+_S \oplus \langle f_p \rangle \) for any \( p \in S \). Note also that, for every \( X \subseteq S \), we have \( f_X = \sum_{p \in X} f_p \).

The group \( \text{Sym}(S) \) of all permutations of \( S \) can quite naturally be regarded as a group of linear transformations of \( U_S \). The Coxeter group \( C \) of type \( C_{2n} \) is the semidirect product \( U_S: \text{Sym}(S) \). Clearly, \( \text{Sym}(S) \) stabilizes \( U^+_S \). The subgroup \( D := U^+_S: \text{Sym}(S) \) of \( C \) is the Coxeter group of type \( D_{2n} \).

As \( |S| = 2^n \), we can regard \( S \) as the point-set of an affine space. Let us choose such an affine space \( A_S \) on \( S \). For every \( k = 0, 1, \ldots, n \), let \( U^{(k)} \) be the subspace of \( U_S \) spanned by the vectors \( f_X \), for \( X \) a \( k \)-dimensional affine subspace of \( A_S \). Clearly,

\[
U^{(0)} = U_S > U^{(1)} = U^+_S > U^{(2)} > \cdots > U^{(n-1)} > U^{(n)} = \langle f_S \rangle.
\]

Note that \( U^{(1)} \) and \( U^{(n)} \) are normalized by \( \text{Sym}(S) \). On the other hand, the following is clear:

**Lemma 2.1.** If \( 1 < k < n \) then the normalizer of \( U^{(k)} \) in \( \text{Sym}(S) \) is the automorphism group \( \text{Aut}(A_S) \cong \text{AGL}(n, 2) \) of \( A_S \).

So, \( \text{Aut}(A_S) \) acts on \( U^{(k)}/U^{(k+1)} \). We can say more on this action, but we need a few preliminaries. Pick an independent spanning set \( p_0, p_1, \ldots, p_n \) of \( A_S \), take \( p_0 \) as the null vector and the family \( B := \{ \{ p_i \} \}_i \) as a basis of a vector space \( V_0 \cong V(n, 2) \) associated to \( A_S \), so that \( A_S = \text{AG}(V_0) \). Thus, every
point \( p \in S \) can be regarded as a vector \( v_p \) of \( V_0 \). Needless to say, \( v_{p_0} \) is the null vector of \( V_0 \) and, if \( p, q \) are distinct points of \( \mathcal{A}_S \setminus \{p_0\} \) and \( \{p_0, p, q, r\} \) is the plane of \( \mathcal{A}_S \) spanned by \( \{p_0, p, q\} \), then \( v_p + v_q = v_r \). The \( k \)-dimensional linear subspaces of \( V_0 \) are the \( k \)-dimensional affine subspaces of \( \mathcal{A}_S \) that contain \( p_0 \). We write \( v_i \) for \( v_{p_i} \).

Let \( L = Aut(V_0) \) be the stabilizer of \( p_0 \) in \( Aut(\mathcal{A}_S) < \text{Sym}(S) \). Clearly, the action of \( \text{Sym}(S) \) on \( U_S \) induces an action of \( L \) on \( U_S \), and \( L \) stabilizes \( U(k) \) for every \( k = 0, 1, \ldots, n \). Hence \( L \) acts on the quotient \( U(k)/U(k+1) \cong U[k] \). On the other hand \( L = Aut(V_0) \) also acts on \( \wedge^k V_0 \). The following statement is implicit in the calculations of [1], but we shall give a more perspicuous proof of it in Section 4:

**Theorem 2.2.** For every \( k = 1, 2, \ldots, n-1, \wedge^k V_0 \) and \( U(k)/U(k+1) \) are isomorphic as \( L \)-modules.

The \( L \)-module \( \wedge^k V_0 \) is irreducible. Therefore:

**Corollary 2.3.** The \( L \)-module \( U(k)/U(k+1) \) is irreducible, for \( k = 1, 2, \ldots, n-1 \).

Consequently,

**Corollary 2.4.** The sequence \( 0 < U(n) < \cdots < U(2) < U(1) < U_S \) is a composition series for \( Aut(\mathcal{A}_S) \).

We shall now analyze the structure of \( U_S \) more thoroughly. We firstly state a few conventions. If \( I \subseteq \{1, 2, \ldots, n\} \), we put \( v_I = \sum_{j \in I} v_j \). In particular, \( v_{\emptyset} = 0 \) (the null vector of \( V_0 \)). It is also clear that, for every \( I \subseteq \{1, 2, \ldots, n\} \), the set \( S_I := \{v_I\}_{J \subseteq I} \) is a linear subspace of \( V_0 \) of dimension \( \left| I \right| \) (but not all linear subspaces of \( V_0 \) have this form). In particular, \( S_{\emptyset} = 0 \). For \( k = 0, 1, \ldots, n \), put \( S_k(B) = \{S_J\}_{J \subseteq I} \). Then the set \( \{f_X \mid X \in S_k(B)\} \) spans a linear subspace \( U[k] \) of \( U_S \). Clearly, \( U[0] \) and \( U[n] \) are 1-dimensional. The following statement is also implicit in the calculations of [1], but we shall obtain it in a more clear way:

**Lemma 2.5.** \( \dim(U[k]) = \dim(\wedge^k V_0) = \binom{n}{k} \), for every \( k = 0, 1, \ldots, n \).

**Proof.** By definition, the elements \( f_X \), with \( X \in S_k(B) \), span \( U[k] \). Their number is \( \binom{n}{k} \). It remains to prove that they are independent, namely that any \( X = \langle v_{i_1}, \ldots, v_{i_k} \rangle \) is not the symmetric difference of other elements in \( S_k(B) \). This is evident since \( v_{i_1} + \cdots + v_{i_k} \), which belongs to \( X \), does not belong to any other element of \( S_k(B) \). \( \square \)

**Theorem 2.6.** \( U(k) = \bigoplus_{i=k}^n U[i] \) for every \( k = 0, 1, 2, \ldots, n \).
Proof. Every \( f_X \) with \( X \) a subset of \( V_0 \) can be obtained as a sum of suitable members of \( \bigcup_{i=0}^{n} U_{[i]} \). Indeed, let \( X = \{ v_{J_1}, \ldots, v_{J_r} \} \). Put \( \mathcal{X} = \{ J_i \}_{i=1}^{r} \) and let \( m \) be the maximal size of a member of \( \mathcal{X} \). We form a set \( \mathcal{Y} = \mathcal{Y}_m \cup \mathcal{Y}_{m-1} \cup \cdots \cup \mathcal{Y}_0 \) as follows: \( \mathcal{Y}_m \) contains all subsets of members of \( \mathcal{X} \) of size \( m \) and \( \mathcal{Y}_{i-1} \) contains all subsets of members of \( \mathcal{Y}_i := \bigcup_{j=i}^{m} \mathcal{Y}_j \) that have size \( i-1 \) and either are contained in an odd number of members of \( \mathcal{Y}_i \) but do not belong to \( \mathcal{X} \), or are contained in an even number of members of \( \mathcal{Y}_i \) and belong to \( \mathcal{X} \). It is not difficult to see that \( f_X = \sum_{Y \in \mathcal{Y}} f_Y \in \bigcup_{i=0}^{m} U_{[i]} \).

By the above, \( U_{k=0} U_{[k]} \) spans \( U_S \). However, \( \dim(U_{[k]}) = \binom{n}{k} \) by Lemma 2.5. As \( \dim(U_S) = 2^n \) and \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \), we obtain that \( U_S = \bigoplus_{k=0}^{n} U_{[k]} \).

Clearly, \( U_{(k)} \) contains the span of \( U_{i=k}^{n} U_{[i]} \). The latter is isomorphic to \( \bigoplus_{i=k}^{n} U_{[i]} \), by the above. By Theorem 2.2 and Lemma 2.5, \( \dim(U_{(k)})/U_{(k+1)} = \dim(U_{[k]}) = \binom{n}{k} \). So, \( \dim(U_{(k)}) = \sum_{i=k}^{n} \binom{n}{i} = \dim(\bigoplus_{i=k}^{n} U_{[i]}) \). Therefore \( U_{(k)} = \bigoplus_{i=k}^{n} U_{[i]} \).

By the above, \( U_{(k)}/U_{(k+1)} \cong U_{[k]} \) (as vector spaces). However, when \( 1 < k < n \), \( L \) does not stabilize the subspace \( U_{[k]} \) of \( U_S \) (but it stabilizes the 1-dimensional subspaces \( U_{[0]} \) and \( U_{[n]} \)). Indeed, \( L \) maps the \( S_k(B) \) onto the set of \( k \)-dimensional subspaces of \( V_0 \), and the latter spans \( U_{(k)} \) (see the proof of Theorem 2.6). So, the previous isomorphism is not an isomorphism of \( L \)-modules and the decomposition \( U_S = \bigoplus_{k=0}^{n} U_{[k]} \) is not preserved by \( L \).

We turn to \( K \), now.

**Theorem 2.7.** \( K = U_{(3)} = \bigoplus_{i=3}^{n} U_{[i]} \).

**Proof.** This statement is implicit in the computations of Baumeister, Meixner and Pasini [1, section 5], but we can also obtain it as follows. It is known that \( \Gamma \) admits a quotient \( \overline{\Gamma} \), which can be described as the canonical gluing of two copies of the affine space \( \text{AG}(n, 2) \) (see [1]; also Baumeister and Pasini [2]). Let \( \overline{K} \) be the deck group of the covering projection from \( \overline{\Gamma} \) onto \( \Gamma \). Then \( \overline{K} = U_{(2)} \), as proved by Baumeister and Pasini [2] (see also Baumeister, Meixner and Pasini [1, section 5]). However, \( K < \overline{K} \) and, by comparing orders, we see that \( [\overline{K} : K] = \binom{n}{3} \). Moreover, \( K \) is normalized by \( \text{Aut}(\mathcal{A}_S) \). In view of Corollary 2.4, we now see that \( K = U_{(3)} \) is the unique possibility. \( \square \)

**Theorem 2.8.** If \( n = 3 \) then \( \text{Aut}(\text{Exp}(\rho_H)) \cong D/K = 2^6 : \text{Sym}(8) \). If \( n > 3 \) then \( \text{Aut}(\text{Exp}(\rho_H)) = 2^{n+1} : \text{AGL}(n, 2) = \text{Aut}_{\rho_H}(\text{Exp}(\rho_H)) \).

**Proof.** This follows from Lemma 2.1 and the fact that \( \text{Aut}(\Gamma) \) lifts through \( \pi \) to the normalizer of \( K \) in \( \text{Aut}(\overline{\Gamma}) \) (see [12, chapter 12]). \( \square \)
We warn the reader that the above theorem should not be read as if it claimed that \( \pi \) maps \( U_S^+ \) onto \( V \wedge V \) and the translation group \( T_S \) of \( A_S \) onto \( T_H \). In fact, as we will see later, \( \pi \) indeed maps \( U_{(2)} \) onto the subgroup \( V_0 \wedge V_0 \) of \( V \wedge V \) and \( U_{(1)}T_S \) onto \( (V \wedge V)T_H \), but it maps \( U_{(2)}T_S \) onto \( V \wedge V \).

2.4. What remains to do

With \( \rho_H \) as in Subsection 2.1, let \( \tilde{\rho} \) be the hull of \( \rho_H \) and \( \tilde{U} \) be the target group of \( \tilde{\rho} \). By Proposition 1.2, the universal cover \( \tilde{\Gamma} \) of \( \Gamma = \text{Exp}(\rho_H) \) is the expansion \( \text{Exp}(\tilde{\rho}) \) of \( \tilde{\rho} \) and the kernel of the projection of \( \tilde{U} \) onto \( U \) is the deck group \( K \) of the covering projection \( \pi : \tilde{\Gamma} \to \Gamma \). Therefore \( \tilde{U} \) is a subgroup of the lifting \( \tilde{G} \) of \( G := \text{Aut}_\rho(\text{Exp}(\rho_H)) \) to \( \tilde{\Gamma} \) through \( \pi \), and \( K \leq \tilde{U} \). Moreover, \( \tilde{U}/K \cong U = V \wedge V \).

On the other hand, \( U \) is a normal subgroup of \( G \), which in its turn is an extension of \( U_S^+ \) by \( \text{Aut}(A_S) = AGL(n, 2) \). Note that \( U_S^+ = U_{[1]} \oplus U_{[2]} \oplus K \cong (V \wedge V) \oplus K \) (indeed \( V \wedge V \cong V_0 \oplus (V_0 \wedge V_0) \), \( V_0 \cong U_{[1]} \) and \( V_0 \wedge V_0 \cong U_{[2]} \)). However, \( \tilde{U} \) is non-commutative when \( n > 2 \), as remarked in Subsection 1.5. So, when \( n > 2 \) the group \( \tilde{U} \) is certainly different from \( U_S^+ \) (but \( \tilde{U} \neq U_S^+ \) even when \( n = 2 \), as it will turn out from the computations of Section 4). So, we must still describe \( \tilde{U} \) and determine how it sits inside \( \tilde{G} \). We shall do this in the next two sections.

3. The abstract hull of \( \rho_H \)

Throughout this and the next section \( \Sigma \), \( \rho_H \), \( S_H \), \( \Gamma \) and \( \tilde{\Gamma} \) are as in Section 2, and \( n = d + 1 > 2 \).

Given a basis \( \{v_\infty, v_1, \ldots, v_n\} \) of \( V = V(n + 1, 2) \), let \( V_0 = \langle v_1, \ldots, v_n \rangle \) and \( V_0^* = V_0 - \{0\} \). We may assume that the basis \( \{v_1, \ldots, v_n\} \) coincides with that defined in Subsection 2.3. Every element of \( V \) can be written as \( e v_\infty + x \), with \( x \in V_0 \) and \( \epsilon \in \{0, 1\} \). Since \((e v_\infty + x) \wedge (e' v_\infty + x') = (e x' + e' x) \wedge v_\infty + x \wedge x' \), the exterior product \( U = V \wedge V \) is isomorphic to \( V_0 \oplus (V_0 \wedge V_0) \) via the isomorphism \( v_\infty \wedge V_0 \cong V_0 \), as we have already remarked in Section 2, but we warn the reader that, in spite of these isomorphisms, he should resist temptation of regarding \( v_\infty \wedge V_0 \) as the same thing as \( V_0 \) and, accordingly, \( U \) as the same thing as \( V_0 \oplus (V_0 \wedge V_0) \). Indeed \( U \) will turn out to be a quotient of the group

\[
\tilde{U} = [(V_0 \wedge V_0) \oplus (V_0 \wedge V_0 \wedge V_0) \oplus \ldots] \tilde{T}
\]

to be defined later in this section. In the projection of \( \tilde{U} \) onto \( U \), \( v_\infty \wedge V_0 \) is the image of the subgroup \( \tilde{T} < \tilde{U} \), which is not the same thing as \( V_0 \).

The points of \( \Sigma \), regarded as lines of \( AG(V_0) \), are classes \( [(x, y)] \) of ordered pairs \((x, y)\), with \( x \in V_0 \), \( y \in V_0^* \), via the equivalence relation \( (x, y) \sim (x + y, y) \).
Accordingly, a block of $\Sigma$ is a set described as $\{(x, y)\}_{y \in V_0^*}$, for $x \in V_0$. The $d$-embedding $\rho_H : \Sigma \to U$ is defined as follows:

$$\rho_H[(x, y)] = \{(x + v_\infty) \land y, 0\}, \quad x \in V_0, \quad y \in V_0^*.$$ 

From now on $\rho_H$ will be simply written as $\rho$. The image by $\rho$ of a block $B$ of $\Sigma$ is a subspace of $\text{PG}(V \land V)$ described as $\{(x + v_\infty) \land y\}_{y \in V_0^*} \cup \{0\}$. Such a subspace will be denoted by $B_x$.

Consider the graded algebra

$$E := 1 \oplus V_0 \oplus (V_0 \land V_0) \oplus (V_0 \land V_0 \land V_0) \oplus \cdots \oplus (V_0 \land \cdots \land V_0)$$

where $1 := \land^0 V_0$ (a 1-dimensional vector space over GF(2)).

For any $x \in V_0$, $x = \sum_{i=1}^{n} a_i v_i$, we call support of $x$ the set $S(x) := \{i : a_i \neq 0\}$. So, $x = \sum_{i \in S(x)} v_i$.

For every $x \in V_0$, define the following elements $\widehat{x}, \overline{x} \in E$:

$$\widehat{x} = \sum_{K \subseteq S(x), K \not= \emptyset} \land_{k \in K} v_k,$$

if $x \neq 0$ and $\widehat{0} = 0$.

$$\overline{x} = 1 + \widehat{x} = \sum_{K \subseteq S(x)} \land_{k \in K} v_k,$$

using the convention $\land_{k \in \emptyset} v_k = 1$.

**Lemma 3.1.** $\overline{x + y} = \overline{x} \land \overline{y}$, $\forall x, y \in V_0$.

**Proof.** Let $X = S(x)$ and $Y = S(y)$. Then, denoting by $\triangle$ the symmetric difference of sets, we have $S(x + y) = X \triangle Y$. We get:

$$\overline{x + y} = \sum_{Z \subseteq X \triangle Y} \land_{l \in Z} v_l$$

$$\overline{x} \land \overline{y} = \left( \sum_{K \subseteq X} \land_{k \in K} v_k \right) \land \left( \sum_{H \subseteq Y} \land_{h \in H} v_h \right) = \sum_{K \subseteq X, H \subseteq Y} \land_{i \in K \cup H} v_i$$

A subset $Z$ of $X \triangle Y$ is a union of two disjoint subsets of $X$ and $Y$, respectively. More precisely $Z = H \cup K$, with $H = Z \cap X$ and $K = Z \cap Y$. Thus the statement follows. $\blacksquare$
By induction, the previous result can be extended as follows:

**Lemma 3.2.** \[\sum_{i=1}^{t} x_i = \sum_{i=1}^{t} x_i; \quad \forall x_1, \ldots, x_t \in V_0.\]

**Lemma 3.3.** \[\bar{x} + \bar{y} = \hat{x} + \hat{y} + (\hat{x} \wedge \hat{y}), \quad \forall x, y \in V_0.\]

**Proof.** By Lemma 3.1, we have:

\[\bar{x} + \bar{y} = \bar{x} + \bar{y} + 1 = \bar{x} \wedge \bar{y} + 1 = (1 + \bar{x}) \wedge (1 + \bar{y}) + 1 = \hat{x} + \hat{y} + (\hat{x} \wedge \hat{y}).\] □

Put \(\tilde{T} := \{t_x\}_{x \in V_0}\) and define in \(\tilde{T}\) the following product: \(t_x t_y = t_{x+y}\). Thus \(\tilde{T}\) is a group isomorphic to \(V_0\). Define an action of \(\tilde{T}\) on \(E\) in the following way:

\[t_x \cdot u = u + u \wedge \hat{x}, \quad \forall u \in E, \quad x \in V_0.\] This definition is well posed, since the following equalities hold:

\[t_x \cdot (t_y \cdot u) = t_{x+y} \cdot u, \quad t_x \cdot (u + v) = t_x \cdot u + t_x \cdot v, \quad t_0 \cdot u = u.\]

Consider the semidirect product \(E : \tilde{T}\) with respect to the above action of \(\tilde{T}\) on \(E\).

Denote by \(I\) the following sub-algebra of \(E\):

\[I := (V_0 \wedge V_0) \oplus (V_0 \wedge V_0 \wedge V_0) \oplus \cdots \oplus (V_0 \wedge \cdots \wedge V_0)\]

Restricting to \(I\) the action of \(\tilde{T}\) we get the group \(I : \tilde{T}\). Denote it by \(\tilde{U}\). This might look as a notational abuse, as the symbol \(\tilde{U}\) has already been used in Section 2 to denote the target group of the hull \(\tilde{\rho} = \rho_H\). However, this abuse is quite harmless. Indeed we shall soon prove that \(\tilde{U}\), defined as above, is indeed that target group.

Note first that \(\tilde{U}\) is non-commutative. Indeed, given \(u \in I\), \(t_x \in \tilde{T}\) and denoting by \((u, t_x)\) their commutator, we have

\[(u, t_x) = u^{-1} t_x^{-1} u t_x = u (t_x \cdot u) = u \wedge \hat{x}\]

which is non-zero in general.

Define the embedding \(\tilde{\rho} : \Sigma \longrightarrow \tilde{U}\) in the following way:

\[\tilde{\rho}((x, y)) = ((\hat{x}, t_y) t_y) = \{(\hat{x} \wedge \hat{y}) t_y, 0\}.\]

This definition makes sense since \(\hat{x} + \hat{y} \wedge \hat{y} = (\hat{x} + \hat{y} + (\hat{x} \wedge \hat{y}) \wedge \hat{y} = \hat{x} \wedge \hat{y},\) by Lemma 3.3. Note that the elements \((\hat{x}, t_y) t_y\) are involutions. In fact, since \(\hat{x}\)
and $t_y$ are involutions, we have $(t_y, \widehat{x}) = t_y\widehat{x}t_y\widehat{x}$, therefore $(t_y, \widehat{x})t_y = t_y\widehat{x}t_y\widehat{x}t_y$. It follows that

$$(t_y, \widehat{x})t_y(t_y, \widehat{x})t_y = t_y\widehat{x}t_y\widehat{x}t_y\widehat{x}t_y\widehat{x}t_y = 1.$$ 

The image by $\tilde{\rho}$ of a block $B$ of $\Sigma$ is a set described as

$$\{(\widehat{x}, t_y) t_y \in V_0 \cup \{0\} = \{(\widehat{x} \wedge \widehat{y}) t_y \mid y \in V_0 \cup \{0\}\}.$$

Such a set will be denoted by $\tilde{B}_x$.

**Lemma 3.4.** $\tilde{\rho}$ is an embedding.

**Proof.** Condition (AE1) is obviously satisfied. Let $[(x, y)], [(x', y')]$ such that $\tilde{\rho}([(x, y)]) \cap \tilde{\rho}([(x', y')]) \neq \emptyset$. Since these groups have order 2, then $\tilde{\rho}([(x, y)]) = \tilde{\rho}([(x', y')])$, namely $(\widehat{x} \wedge \widehat{y}) t_y = (\widehat{x'} \wedge \widehat{y'}) t'_y$, whence

$$(\widehat{x'} \wedge \widehat{y'}) + (\widehat{x} \wedge \widehat{y}) = t'_y t_y.$$ 

This implies

$$\begin{cases} \widehat{x'} \wedge \widehat{y'} = \widehat{x} \wedge \widehat{y} \\ t'_y = t_y \end{cases}.$$ 

The second equality implies $y' = y$, then it follows $(\widehat{x} + \widehat{x'}) \wedge \widehat{y} = 0$ from the first equality, namely

$$\left(\sum_{K \subseteq S(x), K \neq \emptyset} v_k + \sum_{K' \subseteq S(x'), K' \neq \emptyset} v_{k'} \right) \wedge \left(\sum_{H \subseteq S(y), H \neq \emptyset} v_h \right) = 0.$$ 

This implies for the lower grade:

$$\left(\sum_{k \in S(x)} v_k + \sum_{k' \in S(x')} v_{k'} \right) \wedge \sum_{h \in S(y)} v_h = 0.$$ 

i. e. $(x + x') \wedge y = 0$. It follows that either $x = x'$ or $y = x + x'$. Thus $(x', y') = (x, y)$ or $(x', y') = (x', y) = (x + y, y)$. In both cases we have $[(x, y)] = [(x', y')]$. Then (AE2) holds.

Note that

$$(t_y, \widehat{x}) t_y = t_y \widehat{x} t_y \widehat{x} t_y = t_y \widehat{x} (t_y \bullet \widehat{x}) = t_y \widehat{x} (\widehat{x} + \widehat{y}) = t_y (\widehat{x} \wedge \widehat{y}) = t_y (t_y, \widehat{x})$$

Thus:

$$(t_y, \widehat{x}) t_y = t_y (t_y, \widehat{x}) \tag{1}$$
For any \( x \in V_0 \), consider \((\hat{x} \wedge \hat{y})t_y, (\hat{x} \wedge \hat{y}')t_{y'}\). By (1) and Lemma 3.3, we have:

\[
(\hat{x} \wedge \hat{y})t_y(\hat{x} \wedge \hat{y}')t_{y'} = t_y(\hat{x} \wedge \hat{y})(\hat{x} \wedge \hat{y}')t_{y'} = t_y(\hat{x} \wedge (\hat{y} + \hat{y}'))t_{y'} \\
= t_y(\hat{x})t_y = t_y(\hat{x} + \hat{y}')t_{y'} \\
= (t_y \cdot (\hat{x} \wedge (\hat{y} + \hat{y}')))t_{y+y'} \\
= (\hat{x} \wedge (\hat{y} + \hat{y}') + \hat{x} \wedge (\hat{y} + \hat{y}') \wedge \hat{y})t_{y+y'} \\
= (\hat{x} \wedge (\hat{y} + \hat{y}') + \hat{x} \wedge \hat{y} \wedge \hat{y})t_{y+y'} \\
= (\hat{x} \wedge (\hat{y} + \hat{y}') + \hat{x} \wedge \hat{y} \wedge \hat{y})t_{y+y'}.
\]

This proves that \( \tilde{B}_x \) is a group isomorphic to \( V_0 \), thus (AE3) is satisfied. Clearly, condition (AE4) also holds.

Consider the map \( \tilde{f} : \tilde{U} \rightarrow U \) defined in the following way. For any \( wt_y \in \tilde{U} \), with \( w = \sum a_{ij}(v_i \wedge v_j) + \sum a_{ijk}(v_i \wedge v_j \wedge v_k) + \cdots \in I \) and \( t_y \in T \), set \( w^* = \sum a_{ij}(v_i \wedge v_j) \), the grade 2 part of \( w \), and define

\[
\tilde{f}(wt_y) = w^* + v_\infty \wedge y.
\]

**Theorem 3.5.** \( (\tilde{\rho}, \tilde{f}) \) is the abstract hull of \( \rho \).

**Proof.** Since the dimension of \( I \) equals \( \sum_{i=2}^{d+1} \binom{d+1}{i} = 2^{d+1} - d - 2 = M - d - 2 \), then \( |\tilde{U}| = |I \cdot \tilde{T}| = 2^{M-d-2}2^{d+1} = 2^{M-1} \), which is the correct size of the universal representation group of \( \rho \) (see Section 2).

It remains to prove that \( \tilde{f}\tilde{\rho} = \rho \). We have, for any \( x \in V_0, y \in V_0^* \):

\[
\tilde{f}\tilde{\rho}([x, y]) = \tilde{f}((\hat{x} \wedge \hat{y})t_y) = x \wedge y + v_\infty \wedge y = \rho([x, y]).
\]

Therefore \( \tilde{f}\tilde{\rho} = \rho \). ☐

The next proposition follows from the previous description of \( \tilde{U} \) and \( \tilde{f} \).

**Proposition 3.6.** \( \text{Ker}(\tilde{f}) \) is equal to the commutator subgroup of \( \tilde{U} \).

**Corollary 3.7.** The embedding \( \rho_H \) is projectively dominant.
4. Recovering the \( L \)-module \( \widetilde{U} \) inside the Coxeter group

In order to describe the lifting of \( \text{Aut}(\text{Exp}(\rho)) \), we need to identify \( E \) with \( U_S \), preserving their respective decompositions. Consider, for any \( k = 1, \ldots, n \), the map \( \sigma : \bigwedge^k V_0 \rightarrow U_{[k]} \), obtained by linear extension of the following:

\[
\sigma(\bigwedge_{j \in K} v_j) = f_{S_K}, \quad \forall K \subseteq \{1, \ldots, n\}, |K| = k.
\]

Since \( U_S = \bigoplus_{k=0}^n U_{[k]} \), this provides an isomorphism, also called \( \sigma \), between \( E \) and \( U_S \) as graded algebras. In particular \( \sigma \) sends \( \bigwedge_{h \geq k} \bigwedge^h V_0 \) to \( U_{(k)} \).

We have:

**Lemma 4.1.** \( \sigma(\bar{v}) = f_v, \quad \forall v \in V_0. \)

_Proof._ Let \( v = v_{i_1} + \cdots + v_{i_k} \) and set \( K = S(v) = \{i_1, \ldots, i_k\} \). We have:

\[
\sigma(\bar{v}) = \sigma(\sum_{J \subseteq K} \bigwedge_{j \in J} v_j) = \sigma(1 + \sum_{j \in K} v_j + \sum_{J \subseteq K, |J| = 2} \bigwedge_{j \in J} v_j + \cdots + \bigwedge_{j \in K} v_j) =
\]

\[
= f_0 + \sum_{J \subseteq K, |J| \leq 1} f_{v_J} + \sum_{J \subseteq K, |J| \leq 2} f_{v_J} + \cdots + \sum_{J \subseteq K} f_{v_J}
\]

In the sum above, every \( J \subseteq K \) appears \( 2^k - |J| \) times, thus \( f_{v_K} \), namely \( f_v \), is the only surviving summand.

**Lemma 4.2.** \( \sigma(\bar{x} \wedge \bar{y}) = f_{(v_0, x, y, x+y)}, \quad \forall x, y \in V_0. \)

_Proof._ We have \( \bar{x} \wedge \bar{y} = (x+1) \wedge (y+1) = 1 + x + y + x + y \), by Lemma 3.1. The statement follows from Lemma 4.1.

Extending in obvious way the isomorphism \( \sigma \) to \( E : \widetilde{T} \), we define the product \( U_S : \widetilde{T} \). Given \( wt_x \in \widetilde{U} \), let \( A \) be such that \( \sigma(w) = f_A \). Often, it will be useful to pass from \( wt_x \) to the corresponding \( \sigma(w)t_x = f_A t_x \). In that case, whenever it will not cause any ambiguity, we set, by abuse of notation, \( wt_x = f_A t_x \).
4.1. \( \text{Aut}_p(\text{Exp}(\rho)) \)

Consider \( \text{Exp}(\rho) \), affine expansion of \( \rho \).

The elements of type 0 of \( \text{Exp}(\rho) \) are the elements of \( U \),

\[
v_\infty \land q + \sum a_{ij} v_i \land v_j .
\]

The elements of type 1 of \( \text{Exp}(\rho) \) are the right cosets

\[
\langle (v_\infty + x) \land y \rangle + v_\infty \land q + \sum a_{ij} v_i \land v_j .
\]

The elements of type 2 of \( \text{Exp}(\rho) \) are the right cosets

\[
\{ (v_\infty + x) \land y \}_{y \in V_0} + v_\infty \land q + \sum a_{ij} v_i \land v_j .
\]

Recall that \( \text{Aut}(\text{Exp}(\rho)) = U : (T_H : L_H) \) (Section 2.2). We describe the actions of these subgroups on \( \text{Exp}(\rho) \) in detail.

**Action of \( U \).** The group \( U \) acts on the elements of \( \text{Exp}(\rho) \) by sum.

**Action of \( T_H \).** Denoting by \( \tau_z \) the element of \( T_H \), corresponding to the translation by the vector \( z \), we get:

\[
\tau_z \left( v_\infty \land q + \sum a_{ij} v_i \land v_j \right) = (v_\infty + z) \land q + \sum a_{ij} v_i \land v_j
\]

for the elements of type 0;

\[
\tau_z \left( \langle (v_\infty + x) \land y \rangle + v_\infty \land q + \sum a_{ij} v_i \land v_j \right)
= \langle (v_\infty + x + z) \land y \rangle + (v_\infty + z) \land q + \sum a_{ij} v_i \land v_j
\]

for the elements of type 1;

\[
\tau_z \left( \{ (v_\infty + x) \land y \}_{y \in V_0} + v_\infty \land q + \sum a_{ij} v_i \land v_j \right)
= \{ (v_\infty + x + z) \land y \}_{y \in V_0} + (v_\infty + z) \land q + \sum a_{ij} v_i \land v_j
\]

for the elements of type 2.
Action of \( L_H \). Given \( \alpha \in L \) (see Section 2.2), we denote by \( \alpha[x] \) the image of a vector \( x \) in \( V_0 \). By abuse of notation we also call \( \alpha \) the element of \( L_H \) corresponding to \( \alpha \) in the isomorphism \( L \cong L_H \). The element \( \alpha \in L_H \) acts as follows:

\[
\alpha \left( v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right) = v_\infty \wedge \alpha[q] + \sum a_{ij} \alpha[v_i] \wedge \alpha[v_j]
\]
for the elements of type 0;

\[
\alpha \left( (v_\infty + x) \wedge y + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right)
= \left( (v_\infty + \alpha[x]) \wedge \alpha[y] \right) + v_\infty \wedge \alpha[q] + \sum a_{ij} \alpha[v_i] \wedge \alpha[v_j]
\]
for the elements of type 1;

\[
\alpha \left( \{v_\infty + x\} \wedge y \} y \in V_0 + v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right)
= \{v_\infty + \alpha[x] \} \wedge \alpha[y] \} y \in V_0 + v_\infty \wedge \alpha[q] + \sum a_{ij} \alpha[v_i] \wedge \alpha[v_j]
\]
for the elements of type 2.

The structure of \( U : (T_H : L_H) \). The actions of \( L_H \) on \( T_H \), of \( T_H \) on \( U \) and of \( L_H \) on \( U \) are as follows:

\[
\alpha \bullet \tau_z = \tau_{\alpha[z]} ;
\]

\[
\tau_z \bullet \left( v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right) = (v_\infty + z) \wedge q + \sum a_{ij} v_i \wedge v_j ;
\]

\[
\alpha \bullet \left( v_\infty \wedge q + \sum a_{ij} v_i \wedge v_j \right) = v_\infty \wedge \alpha[q] + \sum a_{ij} \alpha[v_i] \wedge \alpha[v_j] .
\]

4.2. \( \text{Aut}_\rho(\text{Exp}(\tilde{\rho})) \)

Consider \( \text{Exp}(\tilde{\rho}) \), affine expansion of \( \tilde{\rho} \).

The elements of type 0 of \( \text{Exp}(\tilde{\rho}) \) are the elements \( wt_x \) of \( \tilde{U} = I : \tilde{T} \); the elements of type 1 of \( \text{Exp}(\tilde{\rho}) \) are the right cosets \( (\tilde{\rho} \wedge \tilde{y}) t_y \) \( wt_x \); the elements of type 2 of \( \text{Exp}(\tilde{\rho}) \) are the right cosets \( \{ (\tilde{\rho} \wedge \tilde{y}) t_y \} y \in V_0 wt_x \).

Action of \( \tilde{U} \). The group \( \tilde{U} \) is trivially a subgroup of \( \text{Aut}(\text{Exp}(\tilde{\rho})) \), its action being defined by right product:

\[
wt_z(wt_x) = wt_x wt_z = (w + u + u \wedge \tilde{z}) t_{x+z}
\]
for the elements of type 0;
\[
zt_x(((\hat{p} \land \hat{y}) t_y) w t_x) = ((\hat{p} \land \hat{y}) t_y) w t_x \cdot z = (\hat{p} \land \hat{y}) t_y (w + u + u \land \hat{z}) t_{x+z}
\]
for the elements of type 1;
\[
zt_x((\hat{p} \land \hat{y}) t_y)_y \in V_0 = ((\hat{p} \land \hat{y}) t_y) w t_x u t_x
\]
\[
= (\hat{p} \land \hat{y}) t_y (w + u + u \land \hat{z}) t_{x+z}
\]
for the elements of type 2.

The group $T'$. Denote by $T'$ the set $\{z_t | z \in V_0\}$. $T'$ is a subgroup of $E : \hat{T}$, since the elements $\hat{z}_t$ are involutions and
\[
\hat{z}_t \cdot y_t = \hat{z}_t \cdot \hat{y} \cdot z_t \cdot y_t = (\hat{z} + \hat{y} \land \hat{z}) t_{z+y} = (\hat{z} + \hat{y}) t_{z+y}.
\]
$T'$ is isomorphic to the translation groups $T$ and $T_H$ (see Section 2.2). We define the following action of $T'$ on $\text{Exp}(\hat{p})$:
\[
\hat{z}_t (w t_x) = \hat{z}_t w t_x (\hat{z}_t)^{-1} = \hat{z}_t w t_x z_t = \hat{z}_t w t_x \cdot \hat{z} t_x t_x
\]
\[
= (w + \hat{z} + \hat{z} \land \hat{x}) t_x
\]
for the elements of type 0;
\[
\hat{z}_t ((\hat{p} \land \hat{y}) t_y) w t_x = \hat{z}_t ((\hat{p} \land \hat{y}) t_y) w t_x t_x
\]
\[
= \hat{z}_t ((\hat{p} \land \hat{y}) t_y) t_z \cdot \hat{z} t_x t_x
\]
\[
= ((\hat{p} \land \hat{y} + \hat{p} \land \hat{y} \land \hat{z} + \hat{y} \land \hat{y}) t_y) (w + \hat{z} + \hat{z} \land \hat{x}) t_x
\]
\[
= ((\hat{p} + z) \land \hat{y} t_y) (w + \hat{z} + \hat{z} \land \hat{x}) t_x
\]
for the elements of type 1;
\[
\hat{z}_t ((\hat{p} \land \hat{y}) t_y)_y \in V_0 = \hat{z}_t ((\hat{p} \land \hat{y}) t_y)_y \in V_0 w t_x \cdot z
\]
\[
= \{((\hat{p} + z) \land \hat{y}) t_y\}_y \in V_0 (w + \hat{z} + \hat{z} \land \hat{x}) t_x
\]
for the elements of type 2.

The group $L$. We shall now define the action of $L$ on $E$. We shall exploit the isomorphism $\sigma : E \longrightarrow U_S$ in such a way that the action of $L$ on $U_S$ will be reproduced on $E$. In short we want $E$ and $U_S$ to be isomorphic as $L$-modules, too. The action of $L$ that we are going to define is based on the equalities
\[ \alpha(f_p) = f_{\alpha[p]}, \quad \alpha(t_x) = t_{\alpha[x]}, \] extended to the three types of \( \text{Exp}(\bar{\rho}) \). More explicitly, given \( wt_x \) in \( \hat{U} \):

\[ wt_x = (\sum_{i,j} a_{ij} v_i \land v_j + \sum_{i,j,k} a_{ijk} v_i \land v_j \land v_k + \cdots) t_x. \]

\( \sigma \) maps this element onto the following:

\[ (\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots) t_x. \]

With the abuse of notation announced at the beginning of Section 4, we define:

\[ \alpha(wt_x) = \alpha\left(\left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\right) t_x\right) \]

\[ = \left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\right) t_{\alpha[x]} \]

for the elements of type 0;

\[ \alpha(\{(\bar{\rho} \land \bar{\gamma})t_y\}wt_x) \]

\[ = \alpha\left(\{f_{v_0,p,y,p+y}\}t_y\left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\right) t_x\right) \]

\[ = f_{v_0,\alpha[p],\alpha[y],\alpha[p]+\alpha[y]}t_{\alpha[y]}\left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\right) t_{\alpha[x]} \]

for the elements of type 1;

\[ \alpha(\{\bar{\rho} \land \bar{\gamma}\}t_y\}y \in v_0 wt_x) \]

\[ = \alpha\left(\{f_{v_0,p,y,p+y}\}t_y\}y \in v_0 \left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\right) t_x\right) \]

\[ = \{f_{v_0,\alpha[p],\alpha[y],\alpha[p]+\alpha[y]}t_{\alpha[y]}\}y \in v_0 \left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\right) t_{\alpha[x]} \]

for the elements of type 2.

**A candidate for \( \text{Aut}_{\bar{\rho}}(\text{Exp}(\bar{\rho})) \).** We define in obvious way an action of \( L \) on \( \hat{U} \) and of \( T' \) on \( \hat{U} \):

\[ \hat{t}_z \bullet wt_x = (w + w \land \hat{t} + \hat{t} \land \hat{t}) t_x; \]

\[ \alpha \bullet wt_x = \alpha \bullet \left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{v_J} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{v_J} + \cdots\right) t_x \]

\[ = \left(\sum_{i,j} a_{ij} \sum_{J \subseteq \{i,j\}} f_{\alpha[v_J]} + \sum_{i,j,k} a_{ijk} \sum_{J \subseteq \{i,j,k\}} f_{\alpha[v_J]} + \cdots\right) t_{\alpha[x]}. \]
In order to define an action of \( L \) on \( T' \), we need some supplementary results.

**Lemma 4.3.** \( \alpha(u \wedge w) = \alpha(u) \wedge \alpha(w) \), \( \forall \alpha \in L, \forall u, w \in I \).

**Proof.** By the linearity of \( \alpha \), it is enough to prove that
\[
\alpha(\bigwedge_{j \in J} v_j) = \bigwedge_{j \in J} \alpha(v_j).
\]

By the isomorphism between \( U_S \) and \( E \), the definition of \( \alpha \) and its linearity, we have:
\[
\alpha(\bigwedge_{j \in J} v_j) = \alpha(\sum_{K \subseteq J} f_{v_K}) = \sum_{K \subseteq J} f_{\alpha[v_K]}.
\]

Note that, by Lemma 4.1:
\[
\alpha(v_j) = \alpha(1 + v_j) = \alpha(f_{v_0} + f_{v_j}) = f_{\alpha[v_0]} + f_{\alpha[v_j]} = 1 + \alpha[v_j].
\]

Set \( |J| = t \) and denote by \( \binom{J}{s} \) the set of \( s \)-tuples of distinct elements in \( J \). We have by Lemma 3.2:
\[
\bigwedge_{j \in J} \alpha(v_j) = \bigwedge_{j \in J} (1 + \alpha[v_j]) =
\]
\[
= 1 + \sum_{j \in J} \alpha[v_j] + \sum_{\{j_1, j_2\} \in \binom{J}{2}} \alpha[v_{j_1}] \alpha[v_{j_2}] + \cdots
\]
\[
+ \sum_{\{j_1, \ldots, j_{t-1}\} \in \binom{J}{t-1}} \alpha[v_{j_1}] \cdots \alpha[v_{j_{t-1}}] + \bigwedge_{j \in J} \alpha[v_j]
\]
\[
= 1 + \sum_{j \in J} \alpha[v_j] + \sum_{\{j_1, j_2\} \in \binom{J}{2}} \alpha[v_{j_1}] + \alpha[v_{j_2}] + \cdots
\]
\[
+ \sum_{\{j_1, \ldots, j_{t-1}\} \in \binom{J}{t-1}} \alpha[v_{j_1}] + \cdots + \alpha[v_{j_{t-1}}] + \sum_{j \in J} \alpha[v_j]
\]
\[
= 1 + \sum_{j \in J} f_{\alpha[v_j]} + \sum_{\{j_1, j_2\} \in \binom{J}{2}} f_{\alpha[v_{j_1}] + \alpha[v_{j_2}]} + \cdots
\]
\[
+ \sum_{\{j_1, \ldots, j_{t-1}\} \in \binom{J}{t-1}} f_{\alpha[v_{j_1}] + \cdots + \alpha[v_{j_{t-1}}]} + f_{\alpha[v_j]}
\]
\[
= \sum_{K \subseteq J} f_{\alpha[v_K]}.
\]

**Lemma 4.4.** \( \alpha(\overline{u}) = \overline{\alpha[u]} \), \( \forall \alpha \in L, \forall u \in V_0 \).

**Proof.** \( \alpha(\overline{u}) = \alpha(f_u) = f_{\alpha[u]} = \overline{\alpha[u]} \).
It immediately follows:

Lemma 4.5. \( \alpha(\hat{u}) = \hat{\alpha[u]}, \ \forall \alpha \in L, \forall u \in V_0. \)

We warn the reader that Lemma 4.3 does not imply that \( \wedge^k V_0 \) is stabilized by \( \alpha \). In fact \( \alpha(\wedge^k V_0) \neq \wedge^k V_0 \), because \( \alpha(\wedge^k V_0) \) in general involves contributions from \( \wedge^h V_0 \), with \( h > k \). However, \( \bigoplus_{h \geq k} \wedge^h V_0 / \bigoplus_{h > k} \wedge^h V_0 \) is stable under the action of \( \alpha \). Actually, on that quotient, that action is the natural one. Theorem 2.2 immediately follows from this remark.

Now, we define the following action of \( L \) on \( T' \):

\[
\alpha \cdot \hat{z} t_x = \hat{\alpha(z)} t_{\alpha[z]}, \quad \forall \alpha \in L, \forall z \in V_0.
\]

The actions defined above allow to construct the product \( \tilde{U} : (T' : L) \). All products involved here are semi-direct. Accordingly we will write \( \tilde{U} : (T' : L) \) instead of \( \tilde{U} : (T' : L) \).

Proposition 4.6. \( \tilde{U} : (T' : L) \) is an automorphism group of \( \text{Exp}(\tilde{\rho}) \).

Proof. We have to prove that the structure of the product of \( \tilde{U} : (T' : L) \) is compatible with the actions defined in every single factor. The only non-trivial check concerns the factor \( T' : L \). For any \( wt_x \) in \( \tilde{U} \), we have, by Lemmas 4.3 and 4.5:

\[
\alpha \hat{\alpha} t_x \alpha^{-1}(wt_x) = \alpha \hat{\alpha} t_x (\alpha^{-1}(w) t_{\alpha^{-1}[z]})
\]

\[
= \alpha \left( (\alpha^{-1}(w) + \alpha^{-1}(w) \wedge \hat{z} + \hat{z} \wedge \alpha^{-1}[x]) t_{\alpha^{-1}[z]} \right)
\]

\[
= (w + w \wedge \alpha(z) + \alpha(z) \wedge \alpha^{-1}(\hat{z})) t_x
\]

\[
= (w + w \wedge \alpha(z) + \alpha(z) \wedge \hat{z}) t_x
\]

that coincides with the action of \( \hat{\alpha(z)} t_{\alpha[z]} \) on \( wt_x \). This proves the compatibility with the action of \( T' : L \) on the elements of type 0. The control on the elements of type 1 and 2 is similar. \( \square \)

We claim that the group defined above is the lifting of \( \text{Aut}_\rho(\text{Exp}(\rho)) \), namely:

Theorem 4.7. \( \text{Aut}_\tilde{\rho}(\text{Exp}(\tilde{\rho})) = \tilde{U} : (T' : L) \)

In the sequel we shall prove Theorem 4.7 by defining a suitable projection of \( \tilde{U} : (T' : L) \) onto \( U : (T_H : L_H) \).
The projection. We recall the projection $\hat{f} : \tilde{U} \rightarrow U$ defined in Section 3:

$$\hat{f}(wt_x) = v_\infty \land x + w^*, \quad \forall w \in I, \forall x \in V_0.$$ 

Note that, given $v_i \land v_j \in I$, since $v_i \land v_j = 1 + f_{v_i} + f_{v_j} + f_{v_i+v_j}$ (with the usual abuse of notation), we have $\hat{f}(1 + f_{v_i} + f_{v_j} + f_{v_i+v_j}) = v_i \land v_j \in U$. This equality can be generalized in the following way:

**Proposition 4.8.** $\hat{f}(1 + f_p + f_q + f_{p+q}) = p \land q$, for all $p, q \in V_0$.

**Proof.** Set $p = \sum_{i \in S(p)} v_i$ and $q = \sum_{i \in S(q)} v_i$. We get:

$$1 + f_p + f_q + f_{p+q} = 1 + \bar{p} + \bar{q} + \bar{p} + \bar{q}$$

$$= 1 + 1 + \sum_{i \in S(p)} v_i + \sum_{\{i,j\} \in (S(p))^2} v_i \land v_j + \cdots$$

$$+ 1 + \sum_{i \in S(q)} v_i + \sum_{\{i,j\} \in (S(q))^2} v_i \land v_j + \cdots$$

$$+ 1 + \sum_{i \in S(p+q)} v_i + \sum_{\{i,j\} \in (S(p+q))^2} v_i \land v_j + \cdots$$

It follows:

$$\hat{f}(1 + f_p + f_q + f_{p+q}) = \sum_{\{i,j\} \in (S(p))^2} v_i \land v_j + \sum_{\{i,j\} \in (S(q))^2} v_i \land v_j + \sum_{\{i,j\} \in (S(p+q))^2} v_i \land v_j.$$ 

Since $S(p+q) = S(p) \triangle S(q)$, we can delete the summands $v_i \land v_j$ with:

- $i, j$ belonging to $S(p) \cap S(q)$, as common to the first and the second sum,
- $i, j$ belonging to $S(p) - S(q)$, as common to the first and the third sum,
- $i, j$ belonging to $S(q) - S(p)$, as common to the second and the third sum.

Thus:

$$\hat{f}(1 + f_p + f_q + f_{p+q})$$

$$= \sum_{i \in S(p) \cap S(q)} v_i \land v_j + \sum_{i \in S(p) - S(q)} v_i \land v_j + \sum_{i \in S(q) - S(p)} v_i \land v_j.$$
On the other hand
\[
p \wedge q = \sum_{i \in S(p)} v_i \wedge \sum_{j \in S(q)} v_j
\]
\[
= \left( \sum_{i \in S(p) \cap S(q)} v_i + \sum_{i \in S(p) - S(q)} v_i \right) \wedge \left( \sum_{j \in S(p) \cap S(q)} v_j + \sum_{j \in S(q) - S(p)} v_j \right)
\]
\[
= \sum_{i \in S(p) \cap S(q)} v_i \wedge v_j + \sum_{i \in S(p) \cap S(q)} v_i \wedge v_j + \sum_{i \in S(p) - S(q)} v_i \wedge v_j . \quad \square
\]

Now, we introduce a projection \( \pi \) from \( \tilde{U} : (T' : L) \) to \( U : (T_H : L_H) \), which extends \( \tilde{f} \). It is enough to define the images by \( \pi \) of the elements of \( T' \) and \( L \). We do it in the following natural way:
\[
\pi(\tilde{t}_z) = \tau_z, \quad \pi(\alpha) = \alpha .
\]

We have to check that this projection reproduces the actions defined in the semi-direct products.

The action of \( T' \) on \( \tilde{U} \) is projected on the action of \( T_H \) on \( U \). In fact, since \((w + w \wedge \tilde{z} + \tilde{z} \wedge \tilde{x})^*\), the grade 2 part of \( w + w \wedge \tilde{z} + \tilde{z} \wedge \tilde{x} \), equals \( w^* + z \wedge x \), we have:
\[
\pi(\tilde{t}_z) \cdot \pi(wt_x) = \tau_z \cdot (v_\infty \wedge x + w^*) = (v_\infty + z) \wedge x + w^*
\]
\[
= \pi((w + w \wedge \tilde{z} + \tilde{z} \wedge \tilde{x})t_x) = \pi(\tilde{t}_z \cdot wt_x) .
\]

The action of \( L \) on \( \tilde{U} \) is projected on the action of \( L_H \) on \( U \). Indeed, using the notation on the isomorphism between \( U_S \) and \( E \) and Proposition 4.8:
\[
\pi(\alpha) \cdot \pi(wt_x) = \alpha \cdot \left( v_\infty \wedge x + \sum_{\{i,j\}} a_{ij}v_i \wedge v_j \right)
\]
\[
= v_\infty \wedge \alpha[x] + \sum_{\{i,j\}} a_{ij} \alpha[v_i] \wedge \alpha[v_j]
\]
\[
= \pi\left( \sum_{\{i,j\}} a_{ij} (f_{v_0} + f_{\alpha[v_i]} + f_{\alpha[v_j]} + f_{\alpha[v_i] + \alpha[v_j]}) + \cdots \right) t_\alpha[x]
\]
\[
= \pi\left( \sum_{\{i,j\}} a_{ij} \sum_{j \leq \{i,j\}} f_{\alpha[v_j]} + \cdots \right) t_\alpha[x] = \pi(\alpha \cdot wt_x) .
\]

The action of \( L \) on \( T' \) is projected on the action of \( L_H \) on \( T_H \). In fact:
\[
\pi(\alpha) \cdot \pi(\tilde{t}_z) = \alpha \cdot \tau_z = \tau_\alpha[z] = \pi(\alpha[z]t_\alpha[z]) = \pi(\alpha \cdot \tilde{t}_z) .
\]
Finally, a straightforward computation shows that the actions of the groups \( \tilde{U}, T', L \) on the three types of \( \text{Exp}(\bar{p}) \) are respectively projected by \( \pi \) on the actions of \( U, T_H, L_H \) on the three types of \( \text{Exp}(\bar{p}) \). So, Theorem 4.7 is proved.

References


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