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Transitive eggs

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Abstract

We prove that a pseudo-oval or pseudo-ovoid (that is not an oval or ovoid) admitting an insoluble transitive group of collineations is elementary and arises over an extension field from a conic, an elliptic quadric, or a Suzuki-Tits ovoid.

Keywords: pseudo-oval, pseudo-ovoid, egg, translation generalised quadrangle, transitive
MSC 2000: 51E20

1 Introduction

An *egg* of the projective space $\mathrm{PG}(2n+m-1,q)$ is a set $\mathcal{E}$ of q^m+1 subspaces of dimension $(n-1)$ such that every three are independent (i.e., span a $(3n-1)$-dimensional subspace), and such that each element of $\mathcal{E}$ is contained in a common complement to the other elements of $\mathcal{E}$ (i.e., each element of $\mathcal{E}$ is contained in an $(n+m-1)$-dimensional subspace having no point in common with any other element of $\mathcal{E}$). The theory of eggs is equivalent to the theory of translation generalised quadrangles (see [20, Chapter 8]). If q is even, then $m=n$ or $m=2n$ (see [20, 8.7.2]), and for q odd, the only known examples of eggs have $m=n$ or $m=2n$. Now an ovoid of $\mathrm{PG}(3,q)$ is an example of an egg where $m=2n=1$; hence an egg having $m=2n$ is called a *pseudo-ovoid*. Likewise, an oval of $\mathrm{PG}(2,q)$ is an egg where $m=n=2$, and henceforth, a *pseudo-oval* is an egg with $m=n$. If $\mathcal{O}$ is an oval of $\mathrm{PG}(2,q^n)$, then by field reduction from $\mathrm{GF}(q^n)$ to $\mathrm{GF}(q)$, one obtains a pseudo-oval of $\mathrm{PG}(3n-1,q)$. Such pseudo-ovals are called *elementary*. Likewise, field reduction of an ovoid of $\mathrm{PG}(3,q^n)$ yields an *elementary* pseudo-ovoid of $\mathrm{PG}(4n-1,q)$. All known pseudo-ovals are elementary, and in even characteristic, every known example of a pseudo-ovoid is elementary. There is some conflict over the definition of a *classical* pseudo-ovoid. In [6] and [24], a classical pseudo-ovoid is one which arises by field reduction from an elliptic quadric. However, some authors (e.g., Cossidente and King [9]) also include the Suzuki-Tits ovoids in their definition of a classical ovoid. Such confusion will be avoided in this paper by not using the term classical at all; so we will take the perhaps cumbersome approach of stating our results explicitly.

By Segre's Theorem [22], every oval of $\mathrm{PG}(2, q)$, q odd, is a conic. Similarly, every ovoid of $\mathrm{PG}(3, q)$, for q odd, is an elliptic quadric, and this was proved independently by Barlotti [5] and Panella [19]. In the case where q is even, there also exist the *Suzuki-Tits ovoids* which are inequivalent to elliptic quadrics. The second author and O'Keefe, building on the work of Abatangelo and Larato, showed that the ovals of $\mathrm{PG}(2, q)$, q even, which admit a transitive subgroup of $\mathrm{PGL}_3(q)$ are conics (see [1] and [18]). Similarly, Bagchi and Sastry [2] showed that the ovoids of $\mathrm{PG}(3, q)$, q even, which admit a transitive subgroup of $\mathrm{PGL}_4(q)$ are elliptic quadrics or Suzuki-Tits ovoids. Brown and Lavrauw [6] have shown that an egg of $\mathrm{PG}(4n - 1, q)$, q even, contains a pseudo-conic if and only if it is elementary and arises from an elliptic quadric. Recently, J. A. Thas and K. Thas [24] have shown that every 2-transitive pseudo-oval in even characteristic is elementary and arises from a conic. In this paper, we prove the following result:

> **Main Theorem.** Suppose $\mathcal{E}$ is a pseudo-oval or pseudo-ovoid (that is not an oval or ovoid) admitting an insoluble transitive group of collineations. Then $\mathcal{E}$ is elementary and arises from a conic, an elliptic quadric, or a Suzuki-Tits ovoid.

2 The approach

A divisor x of $q^d - 1$ (where $d \geqslant 3$) is *primitive* if x does not divide $q^i - 1$ for each positive integer $i < d$. By a result of Zsigmondy [25], such divisors exist if $(q, d) \neq (2, 6)$. Therefore, if G acts transitively on a set of size $q^m + 1$ (and $(q, m) \neq (2, 3)$), then a primitive prime divisor of $q^{2m} - 1$ divides the order of G. Such groups have an irreducible Sylow subgroup, and from this information, the structure of G can be described in great detail (see [12]). The authors have used this argument to classify m-systems of polar spaces which admit an insoluble transitive group (see [3]). From the definitions of a pseudo-ovoid and pseudo-oval, we can apply a similar argument here; which is dependent on the Classification of Finite Simple Groups.

Note. Suppose $\mathcal{E}$ is a pseudo-oval (resp. pseudo-ovoid) of $\mathrm{PG}(2n + m - 1, q)$ where $q = p^f$ for some prime p. Under field reduction from $\mathrm{GF}(q)$ to $\mathrm{GF}(p)$, there arises a pseudo-oval (resp. pseudo-ovoid) $\tilde{\mathcal{E}}$ of $\mathrm{PG}((2n+m)f-1,p)$. If $\mathcal{E}$ admits an insoluble transitive subgroup of $\mathrm{P\Gamma L}_{2n+m}(q)$, then $\tilde{\mathcal{E}}$ admits an insoluble transitive subgroup of $\mathrm{P\Gamma L}_{(2n+m)f}(p) = \mathrm{PGL}_{(2n+m)f}(p)$. We then apply the main result of this paper to $\tilde{\mathcal{E}}$ to establish that it is elementary, from which it follows that $\mathcal{E}$ is elementary provided that it is not an oval or ovoid. Hence throughout this paper, we will assume without loss of generality that our given pseudo-oval or pseudo-ovoid admits an insoluble transitive subgroup of the homography group $\mathrm{PGL}_{2m+n}(q)$.

3 The pseudo-oval case

A pseudo-oval of $\mathrm{PG}(d-1,q)$ (where d is a multiple of 3) is a set of $q^{e/2}+1$ subspaces of dimension $d/3-1$, where $e=\frac{2}{3}d$. This phrasing makes it clear how we apply the results of [4].

3.1 Even characteristic

If q is even, then the tangent spaces of a pseudo-oval $\mathcal{E}$ all have a $(d/3-1)$-space in common; the *nucleus* of $\mathcal{E}$ (see [20, pp. 182]). Since G must fix the nucleus, we have that G acts reducibly in this case. Let $\mathcal{N}$ be the the nucleus of $\mathcal{E}$ and consider the quotient map π from $\mathrm{PG}(d-1,q)$ to $\mathrm{PG}(d-1,q)/\mathcal{N}$, and note that the codomain can be identified with $\mathrm{PG}(2d/3-1,q)$. The image of $\mathcal{E}$ under π is a spread $\mathcal{S}$ of $\mathrm{PG}(2d/3-1,q)$ (see [20, pp. 182]). Moreover, we have that G acts transitively on this spread, and by the Andre/Bruck-Bose construction, we obtain a flag-transitive affine plane admitting an insoluble group. By [7], this affine plane is Desarguesian or a Lüneburg plane, so in particular, it follows that $\mathcal{E}$ admits a 2-transitive group. So by [24, §8], we have that $\mathcal{E}$ is an elementary pseudo-oval arising from a conic of $\mathrm{PG}(2,q^{d/3})$.

3.2 Odd characteristic

Let $\mathcal{E}$ be a pseudo-oval of $\mathrm{PG}(d-1,q)$, where q is odd. Then each element E of $\mathcal{E}$ is contained in a unique $2d/3-1$-subspace T_E of $\mathrm{PG}(d-1,q)$ which is called the *tangent space* at E. By [20, pp. 182], each point of $\mathrm{PG}(d-1,q)$ is contained in 0 or 2 tangent spaces of $\mathcal{E}$.

Theorem 3.1. *Let $q=p^f$ where p is an odd prime, let d be an integer divisible by 3. If an insoluble subgroup G of $\mathrm{PGL}_d(q)$ acts transitively on a pseudo-oval $\mathcal{E}$ of $\mathrm{PG}(d-1,q)$, then $\mathcal{E}$ is elementary and is obtained by field reduction of a conic of $\mathrm{PG}(2,q^{d/3})$.*

Proof. Let $\mathcal{E}$ be a pseudo-oval of $\mathrm{PG}(d-1,q)$ admitting a group $G \leqslant \mathrm{PGL}_d(q)$ that is insoluble and acts transitively on $\mathcal{E}$, and let H be the stabiliser in G of an element of $\mathcal{E}$. Note that the number of elements of $\mathcal{E}$ is $q^{e/2}+1$ where $e = 2/3d$. We may assume that $q^{d/3} > 16$ as it was shown by the second author in [21] that if $q^{d/3} \leqslant 16$, then $\mathcal{E}$ is elementary and is obtained by field reduction of a conic of $\mathrm{PG}(2, q^{d/3})$. Let $\hat{G}$ be a preimage of G in $\mathrm{GL}_d(q)$. Then there exists a subgroup $\hat{H}$ of $\hat{G}$ of index $q^{e/2}+1$ such that the image of $\hat{H}$ in $\mathrm{PGL}_d(q)$ is H. So we can apply [4, Theorem 3.1] to $\hat{G}$. There are six cases to consider from this theorem: the Classical, Imprimitive, Reducible, Extension Field (case (b)), Symplectic Type, and Nearly Simple examples. Straight away, we have that the Symplectic examples do not occur as d is a multiple of 3. By [4, Lemma 13], $\hat{G}$ is not in the Classical examples case. So we are left with four families to consider: the Reducible, Imprimitive, Extension Field, and the Nearly simple examples.

Let us first suppose we are in the Imprimitive examples case. So by [4, Theorem 3.1], we have that $d = 9$, $q \in \{3, 5\}$, and $\hat{G}$ preserves a decomposition of $V_9(q)$ into 1-spaces. So in particular, $\hat{G} \leqslant \mathrm{GL}_1(q) \wr S_9$. We treat both cases, $q = 3$ and $q = 5$, simultaneously. Let μ be the natural projection map from $\mathrm{GL}_1(q) \wr S_9$ onto S_9. Now $\mu(\hat{G})$ is insoluble and primitive (of degree 9), and hence $\mu(\hat{G}) \in \{\mathrm{PSL}_2(8), \mathrm{P\Gamma L}_2(8), A_9, S_9\}$ (see [10, Appendix B]). Moreover, $\mu(\hat{G})$ is 3-transitive in its degree 9 action. Let B be the kernel of μ. So $|B| = (q-1)^9 \in \{2^9, 2^{18}\}$. Now $G \cap B$ is a nontrivial normal subgroup of G and hence $G \cap B$ contains the subgroup K of B consisting of diagonal matrices with entries ± 1. Since $|\hat{G} : \hat{H}| \in \{28, 126\}$, we see that a subgroup J of K with index at most 2, is contained in $\hat{H}$. The only J-invariant subspaces of $V_9(q)$ are the spans of vectors from the canonical basis; coordinate subspaces. Let E be an element of the pseudo-oval. We may assume (up to conjugacy) that E is J-invariant and so it is a coordinate plane. Now the action of $\mu(\hat{G})$ is 3-transitive, and so the orbit of E under $\hat{G}$ on planes is $\binom{9}{3} = 84$. So the Imprimitive examples case does not arise.

Let us now suppose we are in the Nearly simple case. So $S \leqslant G \leqslant \mathrm{Aut}(S)$ where S is a finite nonabelian simple group, and $\hat{G}$ is irreducible. By using the fact that $q^{d/3} \geqslant 16$, we have only two subcases to consider: the Alternating group case and the Natural-characteristic case. In the former, we have $S = A_{10}$, $d = 9$, $q = 3$, and the vector space $V_9(3)$ can be identified with the fully deleted permutation module for S_{10} over $\mathrm{GF}(3)$. It can be readily checked that G does not have a subgroup of index $3^3 + 1$, and so this case does not arise. In the Natural-characteristic case, we have that $d = 9$ and $S = \mathrm{PSL}_3(q^2)$ (by [4, Theorem 2.1]). Now by [8], the minimum degree of a nontrivial representation of S is $(q^6-1)/(q^2-1)$. However

$$q^3 + 1 = (q^6-1)/(q^3-1) < (q^6-1)/(q^2-1)$$

and so $\hat{G}$ does not have a transitive action of degree q^3+1. Therefore, we have that $\hat{G}$ is not in the Nearly Simple examples case.

Now suppose we are in the Field Extension examples case. We have that $\hat{G}$ is irreducible and there is a divisor b of $2d/3$ (where $b \neq 1$) such that $\hat{G}$ preserves a field extension structure $V_{d/b}(q^b)$ on $V_d(q)$. Moreover, $G \cap \mathrm{GL}_{d/b}(q^b)$ has a subgroup of index $(q^{e/2}+1)/x$, for some x, and so if $d/b > 3$, then we can apply [4, Theorem 3.2] to $G \cap \mathrm{GL}_{d/b}(q^b)$ with parameters q^b, d/b, and e/b playing the roles of q, d, and e respectively. So let us assume that $d/b > 3$. Since $d/b \neq e/b$, we do not have the Classical examples case. Note that if $\hat{G}$ fixes a subspace over the field extension q^b, then it also fixes a subspace that is written over the field $\mathrm{GF}(q)$. Hence $\hat{G} \cap \mathrm{GL}_{d/b}(q^b)$ is irreducible in its action on $\mathrm{PG}(d/b-1, q^b)$. We can also assume that $G \cap \mathrm{GL}_{d/b}(q^b)$ does not preserve a field extension structure by choosing b to be maximal. Since q^b is not prime, we can eliminate the Imprimitive examples, Symplectic Type examples, and the Nearly Simple examples. Therefore $d/b = 3$ and $e/b = 2$. By some old work of Mitchell [17], the only absolutely irreducible insoluble maximal subgroups of $\mathrm{PSL}_3(q^b)$ are

(i) $\mathrm{PSL}_2(q^b)$;

(ii) $\mathrm{PSU}_3(q^b)$ when q^b is a square;

(iii) A_6 when $p \equiv 1, 2, 4, 7, 8, 13 \mod 15$ (and $\mathrm{GF}(q^b)$ contains the squares of 5 and -3);

(iv) $\mathrm{PSL}_2(7)$ when $p \equiv 1, 2, 4 \mod 7$.

In the case that $\mathrm{PSU}_3(q^{d/3}) \leqslant G \cap \mathrm{PGL}_3(q^{d/3}) \leqslant \mathrm{P\Gamma L}_3(q^{d/3})$, we have $q^{d/3}+1$ divides $q^{d/2}(q^{d/3}-1)(q^{d/2}+1)$. This is a contradiction as $q^{d/3}+1$ is coprime to $q^{d/2}$ and $q^{d/3}-1$ (note that q is odd). So this case does not arise. In the case that $A_6 \leqslant G \cap \mathrm{PGL}_3(q^{d/3}) \leqslant S_6$, we have $q^{d/3}+1$ divides $6!$ (note that $q^{d/3}+1$ is coprime to $|G : G \cap \mathrm{PGL}_3(q^{d/3})|$). However, $q^{d/3}+1$ divides $6!$ only if $q = 3$ and $d = 6$ (so $b = 2$). So this case does not arise as A_6 does not embed in $\mathrm{P\Gamma L}_3(q^b)$ in characteristic 3. In the case that $\mathrm{PSL}_2(7) \leqslant G \cap \mathrm{PGL}_3(q^{d/3}) \leqslant \mathrm{PGL}_2(7)$, we have $q^{d/3}+1$ divides 336. However, $q^{d/3}+1$ divides 336 only if $q = 3$ and $d = 9$ (so $b = 3$). So this case does not arise as $\mathrm{PSL}_2(7)$ does not embed in $\mathrm{P\Gamma L}_3(q^b)$ in characteristic 3. Hence $\mathrm{PSL}_2(q^b) \leqslant G$.

Let $J = \mathrm{PSL}_2(q^{d/3})$. It is a classical result, but can also be found in [8], that $\mathrm{PSL}_2(q^{d/3})$ (where $d > 2$) has a unique conjugacy class of subgroups of index $q^{d/3}+1$. It follows from [14, Proposition 4.3.17], that there is a unique characteristic class of subgroups of $\mathrm{PGL}_d(q)$ isomorphic to J (it is not true in general that there is a unique conjugacy class of such subgroups). Let

$\varphi : V_3(q^{d/3}) \to V_d(q)$ denote the natural vector space isomorphism here, and let $\mathcal{C}$ be a conic of $V_3(q^{d/3})$ admitting J. Let α and β be two distinct points of $\mathcal{C}$. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are $d/3$-dimensional vector subspaces of $V_d(q)$. Note that J has a unique conjugacy class of subgroups of index $q^{d/3}+1$, and hence we can assume that the stabiliser of an element E of $\mathcal{E}$ is identical to the stabiliser J_α. Now suppose we have a third vector v which is neither α nor β. Then

$$|v^{J_\alpha}| = |J_\alpha : J_{\alpha,v}| = |J_\alpha : J_{\alpha,\beta}||J_{\alpha,\beta} : J_{\alpha,\beta,v}| = q^{d/3}|J_{\alpha,\beta} : J_{\alpha,\beta,v}|.$$

Now J is a Zassenhaus group (i.e., a 2-transitive group such that the stabiliser of any three points is trivial) and so $J_{\alpha,\beta,v} = 1$. Therefore

$$|v^{J_\alpha}| = q^{d/3}\frac{q^{d/3}-1}{\gcd(2, q^{d/3}-1)}$$

which is not a prime power. Now any J_α-invariant $d/3$-subspace of $V_d(q)$ is a union of orbits of J_α. Therefore, it follows that the only J_α-invariant subspace of $V_d(q)$ is $\varphi(\alpha)$. Since W is J_α-invariant, we have that $W = \varphi(\alpha)$ and hence $\mathcal{E}$ is the image of $\mathcal{C}$ under φ. Therefore, $\mathcal{E}$ is elementary and is obtained by field reduction of a conic of $\mathrm{PG}(2, q^{d/3})$.

Reducible examples

We have that $\hat{G}$ fixes a subspace/quotient space U of $V_d(q)$ and $\dim(U) = u \geqslant \frac{2}{3}d$. In fact, it follows that $u = 2/3d$ by noting that a primitive divisor of $q^{(2/3)d} - 1$ also divides $|\hat{G}|$. So $\hat{G} \leqslant q^{u(d-u)} \cdot (\mathrm{GL}_u(q) \times \mathrm{GL}_{d-u}(q))$. We may assume that U is a subspace, as for q odd, each point of U is in 0 or 2 tangent spaces of $\mathcal{E}$. Consider the set of intersections

$$\mathcal{M} = \{T_E \cap U : E \in \mathcal{E}\}.$$

Note that each element of $\mathcal{M}$ has a common dimension as G acts transitively on $\mathcal{M}$, and thus $\dim(T_E \cap U) = d/3$ for all $E \in \mathcal{E}$. Therefore $\hat{G}^U$ acts transitively on a set of $(q^{d/3}+1)/\delta$ subspaces of dimension $d/3$ where $\delta = 1, 2$. This implies that $\hat{G}^U$ has a subgroup of index $(q^{d/3}+1)/\delta$, and so we can apply [4, Theorem 3.2] with q, $\frac{2}{3}d$, and $\frac{2}{3}d$ playing the roles of q, d, and e respectively. In the following subcases, we have that G has a normal insoluble subgroup S, which is given explicitly. Moreover, S must have a union of orbits on $(d/3)$-spaces of U of size $(q^{d/3}+1)/\delta$ where $\delta = 1, 2$.

Reducible/nearly simple examples

In this case, $S \leqslant G^U \cap \mathrm{PGL}_d(q) \leqslant \mathrm{Aut}(S)$ where S is a finite nonabelian simple group. Here we have four subcases.

Alternating group case

Here $S = A_r$ and the vector space $V_u(q)$ can be identified with the fully deleted permutation module for S_r over $\mathrm{GF}(q)$. We have that u is $r-1$ or $r-2$ (according to whether p does not or does divide n respectively), and $q^u = p^u = 3^6, 5^6$. Suppose $S = A_7$, $u = 6$, and $q = 3$. Then S stabilises $\mathcal{M}$ and hence S has a union of orbits on planes of $\mathrm{PG}(5,3)$ of size 14 or 28. Now A_7, in its unique irreducible representation in $\mathrm{PG}(5,3)$ has the following orbit lengths on planes (n.b., the exponents denote multiplicities):

$$[35^2, 105^4, 140^3, 210^4, 315^6, 420^{10}, 630^6, 840^4, 1260^{15}].$$

Therefore this case does not arise. Now suppose $q = 5$. It can be shown using GAP [11] that the S-invariant sets of planes of size 63 or 126 do not cover every point either 0 or 2 times. Therefore this case does not arise.

Cross-characteristic case

The table below lists the possibilities for this case.

S	d	q	u
$\mathrm{PSL}_2(7)$	9	3	6
$\mathrm{PSL}_2(13)$	9	3	6
$\mathrm{PSU}_3(3^2)$	9	5	6

Now $\mathrm{PSL}_2(13)$ acts transitively on the points of $\mathrm{PG}(5,3)$, and so this case does not arise. Suppose $S = \mathrm{PSL}_2(7)$, $u = 6$, and $q = 3$. Then S stabilises $\mathcal{M}$ and hence S has a union of orbits on planes of $\mathrm{PG}(5,3)$ of size 14 or 28. Now by using GAP [11] and the unique irreducible representation for S in $\mathrm{PG}(5,3)$, we have that S has the following orbit lengths on planes:

$$[7^4, 21^8, 28^{12}, 42^{18}, 56^{12}, 84^{100}, 168^{140}].$$

None of the thirteen S-invariant sets of planes of size 28 have each point of $\mathrm{PG}(5,3)$ contained in a constant number (0 or 2) of elements of the set. Likewise, of all the six S-invariant sets of size 14, none have each point of $\mathrm{PG}(5,3)$ contained in a constant number of elements of the set. Therefore, this case does not arise.

Now suppose $S = \mathrm{PSU}_3(3^2)$, $u = 6$, and $q = 5$. Then S stabilises $\mathcal{M}$ and hence S stabilises a set of points of size $(q^u - 1)/(2(q-1)) = 1953$. However, by using GAP [11] one can calculate that S has the following orbit lengths on points of $\mathrm{PG}(5,5)$:

$$[189^2, 1008^2, 1512].$$

Since 1953 cannot be partitioned into these numbers, this case does not arise.

So we are left now with just two more cases: the "Classical examples" and the "Extension field" examples, which can be unified naturally.

Reducible/classical and extension field examples

We have that $\hat{G}^U$ preserves a (possibly trivial) field extension structure on U as a u/b-dimensional subspace over $\mathrm{GF}(b)$ where b is a proper divisor of $u = (2/3)d$. So $\hat{G}^U \leqslant \Gamma\mathrm{L}_{(2/3)d/b}(q^b)$ and we can apply [4, Theorem 3.2] to $\hat{G}^U \cap \mathrm{GL}_{(2/3)d/b}(q^b)$ where q^b, u/b, and u/b play the roles of q, d, and e respectively. We simply have $d/b = 6$ and $\mathrm{PSL}_2(q^{d/3}) \leqslant \hat{G}^U$. Let $S = \mathrm{PSL}_2(q^{d/3})$ and note that the preimage of S acts transitively on the non-zero vectors of $V_2(q^{d/3})$. However, we have here that S stabilises a set of $q^{d/3}+1$ subspaces, each of dimension $d/3-1$, which is impossible for $d/3 > 1$. So we conclude that G is irreducible. □

4 The pseudo-ovoid case

A pseudo-ovoid of $\mathrm{PG}(d-1, q)$ (where d is a multiple of 4) is a set of $q^{d/2}+1$ subspaces of dimension $d/4-1$. Here we can also apply the results of [4], as we did in the pseudo-oval case.

Theorem 4.1. *Let $q = p^f$ where p is a prime and let d be an integer divisible by 4. If an insoluble subgroup G of $\mathrm{PGL}_d(q)$ acts transitively on a pseudo-ovoid $\mathcal{E}$ of $\mathrm{PG}(d-1, q)$, then $\mathcal{E}$ is elementary and arises from an elliptic quadric or Suzuki-Tits ovoid.*

Proof. Let H be the stabiliser of an element of $\mathcal{E}$ in G, and let $\hat{G}$ be a preimage of G in $\mathrm{GL}_d(q)$. Note that the number of elements of a pseudo-ovoid of $\mathrm{PG}(d-1, q)$ is $q^{e/2}+1$ where $e = d$. So there exists a subgroup $\hat{H}$ of $\hat{G}$ of index $q^{d/2}+1$ such that the image of $\hat{H}$ in $\mathrm{PGL}_d(q)$ is H. Therefore we can apply [4, Theorem 3.2] to $\hat{G}$. First note that we can rule out the Reducible examples, Imprimitive examples, and case (a) of the Extension field examples. Recall that by [18], we can assume that $d > 4$. Hence we have ruled out the Classical and Symplectic Type examples. Also note that d is a multiple of 4, and so in the Nearly simple case, we have the following: $q = 2$, $d = 12$, and either

(a) $A_{13} \leqslant G \leqslant S_{13}$, or

(b) $S = \mathrm{PSL}_2(25) \leqslant G \leqslant \mathrm{P}\Gamma\mathrm{L}_2(25)$, and $S \cap H$ is isomorphic to S_5 (there are two such conjugacy classes of S).

However in the first case, it is clear that G does not have a subgroup of index 65. In the second case, we know by [13] that $\mathrm{PSL}_2(25)$ has a unique 12-dimensional irreducible representation (up to quasi-equivalence) over $\mathrm{GF}(2)$ and it has the following orbit lengths on points:

$$[65, 325^2, 650, 780, 1950].$$

Let $\mathcal{B}$ be the set of points covered by the pseudo-ovoid $\mathcal{E}$ of $\mathrm{PG}(11, 2)$. Then $\mathcal{B}$ has size $(q^{d/4} - 1)(q^{d/2} + 1) = (2^3 - 1)(2^6 + 1) = 455$ and it must be a union of orbits of S as G acts transitively on $\mathcal{E}$. However, 455 cannot be partitioned into the orbit lengths displayed above, and hence this case does not arise.

That leaves us with the Extension field examples. Here we have that $\hat{G} \leqslant \Gamma\mathrm{L}_{d/b}(q^b)$ where b is a divisor of d (where $b \neq 1$). If $d/b > 2$, We can apply [4, Theorem 3.2] (for e/b even) and [4, Theorem 3.1] (for e/b odd) to $\hat{G}\cap\mathrm{GL}_{d/b}(q^b)$ with parameters d/b, e/b, and q^b playing the roles of d, e, and q respectively. We have the following subcases:

(i) $d/b = 4$ and $\Omega_4^-(q^{d/4}) \trianglelefteqslant \hat{G} \cap \mathrm{GL}_{d/b}(q^b)$;

(ii) $d/b = 4$, q is even, and $\mathrm{Sz}(q^{d/4}) \trianglelefteqslant \hat{G} \cap \mathrm{GL}_{d/b}(q^b)$;

(iii) $d/b = 3$, $q^{d/3}$ is a square, and $\mathrm{SU}_3(q^{d/3}) \trianglelefteqslant \hat{G} \cap \mathrm{GL}_{d/b}(q^b)$.

(i) Let us suppose we have the first case above, where $d/b = 4$ and $\mathcal{E}$ admits $\mathrm{P}\Omega_4^-(q^{d/4})$. Let $J = \mathrm{P}\Omega_4^-(q^{d/4})$. It is a classical result, but can also be found in [8], that $\mathrm{PSL}_2(q^{d/2})$ (where $d > 2$) has a unique conjugacy class of subgroups of index $q^{d/2} + 1$. Note that $\mathrm{P}\Omega_4^-(q^{d/4})$ is isomorphic to $\mathrm{PSL}_2(q^{d/2})$, and by [14, Proposition 4.3.6], there is a unique conjugacy class of subgroups of $\mathrm{PGL}_d(q)$ isomorphic to $\mathrm{PSL}_2(q^{d/2})$. Therefore, there is a unique conjugacy class of subgroups of $\mathrm{PGL}_d(q)$ isomorphic to J.

Let $\varphi : V_4(q^{d/4}) \to V_d(q)$ denote the natural vector space isomorphism here, and let $\mathcal{Q}$ be an elliptic quadric of $V_4(q^{d/4})$ admitting J. Let α and β be two distinct points of $\mathcal{Q}$. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are $d/4$-dimensional subspaces of $V_d(q)$. Note that J has a unique conjugacy class of subgroups of index $q^2 + 1$ (see [8]), and hence we can assume that the stabiliser of an element E of $\mathcal{E}$ is identical to the stabiliser J_α. Now suppose we have a third vector v which is neither α nor β. Then

$$|v^{J_\alpha}| = |J_\alpha : J_{\alpha,v}| = |J_\alpha : J_{\alpha,\beta}||J_{\alpha,\beta} : J_{\alpha,\beta,v}| = q^{d/2}|J_{\alpha,\beta} : J_{\alpha,\beta,v}|.$$

Now J is a Zassenhaus group and so $J_{\alpha,\beta,v} = 1$. Therefore

$$|v^{J_\alpha}| = q^{d/2}\frac{q^{d/2} - 1}{\gcd(2, q^{d/2} - 1)}$$

which is not a prime power. Now any J_α-invariant $d/4$-subspace of $V_d(q)$ is a union of orbits of J_α. Therefore, it follows that the only J_α-invariant subspace of $V_d(q)$ is $\varphi(\alpha)$. Since W is J_α-invariant, we have that $W = \varphi(\alpha)$ and hence $\mathcal{E}$ is the image of $\mathcal{Q}$ under φ. Therefore, $\mathcal{E}$ is elementary and arises from an elliptic quadric.

(ii) By a similar argument to that above, it is not difficult to show that $\mathcal{E}$ is the image of a Suzuki-Tits ovoid under field reduction. The key steps to note are that $\mathrm{Sz}(q^{d/4})$ is a Zassenhaus group, there is a unique conjugacy class of subgroups of $\mathrm{PGL}_d(q)$ isomorphic to $\mathrm{Sz}(q^{d/4})$, and $\mathrm{Sz}(q^{d/4})$ has a unique conjugacy class of subgroups of index q^2+1. In the seminal paper of Suzuki [23, §15], it was shown that $\mathrm{Sz}(q^{d/4})$ is a Zassenhaus group and has a unique conjugacy class of subgroups of index $q^{d/2}+1$ and this is the minimum non-trivial degree of $\mathrm{Sz}(q^{d/4})$. The uniqueness of its representation in $\mathrm{PGL}_d(q)$ needs more work. By a result of Lüneburg (see [16, 27.3 Theorem] or [15]), there is a unique conjugacy class of subgroups of $\mathrm{PGL}_4(q^{d/4})$ isomorphic to $\mathrm{Sz}(q^{d/4})$. Now by [14, Proposition 4.3.6], there is a unique conjugacy class of subgroups of $\mathrm{PGL}_d(q)$ isomorphic to $\mathrm{PGL}_4(q^{d/4})$. Therefore, there is a unique conjugacy class of subgroups of $\mathrm{PGL}_d(q)$ isomorphic to $\mathrm{Sz}(q^{d/4})$. Therefore, $\mathcal{E}$ is elementary and arises from a Suzuki-Tits ovoid.

(iii) Now suppose we have the third case; $d/b = 3$, $q^{d/3}$ is a square, and $\mathcal{E}$ admits $\mathrm{PSU}_3(q^{d/3})$. Now the smallest orbit of $\mathrm{PSU}_3(q^{d/3})$ on nonzero vectors consists of the non-singular vectors and has size $(q^{d/3}-1)(q^{d/2}+1)$. Since $\mathcal{E}$ covers $(q^{d/4}-1)(q^{d/2}+1)$ vectors of $V_d(q)$, and this number is strictly smaller than the size of the smallest orbit of $\mathrm{PSU}_3(q^{d/3})$, we see that this case does not arise.

Suppose now that $d/b = 2$. Since $\hat{G}$ is an insoluble subgroup of $\Gamma\mathrm{L}_2(q^{d/2})$, it follows from [4, Lemma 5] that $\hat{G}$ contains $\mathrm{SL}_2(q^{d/2})$. However, $\mathrm{SL}_2(q^{d/2})$ is transitive on nonzero vectors and hence does not stabilise a set of $d/4$ vector subspaces of size $q^{d/2}+1$. Hence this case does not arise. □

Remark 4.2. If a (presently unknown) pseudo-oval or pseudo-ovoid over $\mathrm{GF}(q)$ admitting a soluble transitive group G exists, then G is meta-cyclic; indeed G is a subgroup of $\Gamma\mathrm{L}_1(q^b)$, for an appropriate positive integer b.

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