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# Transitive eggs

John Bamberg

Tim Penttila

## Abstract

We prove that a pseudo-oval or pseudo-ovoid (that is not an oval or ovoid) admitting an insoluble transitive group of collineations is elementary and arises over an extension field from a conic, an elliptic quadric, or a Suzuki-Tits ovoid.

Keywords: pseudo-oval, pseudo-ovoid, egg, translation generalised quadrangle, transitive

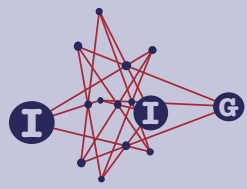
MSC 2000: 51E20

## 1. Introduction

An egg of the projective space  $\text{PG}(2n + m - 1, q)$  is a set  $\mathcal{E}$  of  $q^m + 1$  subspaces of dimension  $(n - 1)$  such that every three are independent (i.e., span a  $(3n - 1)$ -dimensional subspace), and such that each element of  $\mathcal{E}$  is contained in a common complement to the other elements of  $\mathcal{E}$  (i.e., each element of  $\mathcal{E}$  is contained in an  $(n + m - 1)$ -dimensional subspace having no point in common with any other element of  $\mathcal{E}$ ). The theory of eggs is equivalent to the theory of translation generalised quadrangles (see [20, Chapter 8]). If  $q$  is even, then  $m = n$  or  $m = 2n$  (see [20, 8.7.2]), and for  $q$  odd, the only known examples of eggs have  $m = n$  or  $m = 2n$ . Now an ovoid of  $\text{PG}(3, q)$  is an example of an egg where  $m = 2n = 1$ ; hence an egg having  $m = 2n$  is called a *pseudo-ovoid*. Likewise, an oval of  $\text{PG}(2, q)$  is an egg where  $m = n = 2$ , and henceforth, a *pseudo-oval* is an egg with  $m = n$ . If  $\mathcal{O}$  is an oval of  $\text{PG}(2, q^n)$ , then by field reduction from  $\text{GF}(q^n)$  to  $\text{GF}(q)$ , one obtains a pseudo-oval of  $\text{PG}(3n - 1, q)$ . Such pseudo-ovals are called *elementary*. Likewise, field reduction of an ovoid of  $\text{PG}(3, q^n)$  yields an *elementary* pseudo-ovoid of  $\text{PG}(4n - 1, q)$ . All known pseudo-ovals are elementary, and in even characteristic, every known example of a pseudo-ovoid is elementary. There is some conflict over the definition of a

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classical pseudo-ovoid. In [6] and [24], a classical pseudo-ovoid is one which arises by field reduction from an elliptic quadric. However, some authors (e.g., Cossidente and King [9]) also include the Suzuki-Tits ovoids in their definition of a classical ovoid. Such confusion will be avoided in this paper by not using the term classical at all; so we will take the perhaps cumbersome approach of stating our results explicitly.

By Segre's Theorem [22], every oval of  $PG(2, q)$ ,  $q$  odd, is a conic. Similarly, every ovoid of  $PG(3, q)$ , for  $q$  odd, is an elliptic quadric, and this was proved independently by Barlotti [5] and Panella [19]. In the case where  $q$  is even, there also exist the *Suzuki-Tits ovoids* which are inequivalent to elliptic quadrics. The second author and O'Keefe, building on the work of Abatangelo and Larato, showed that the ovals of  $PG(2, q)$ ,  $q$  even, which admit a transitive subgroup of  $PGL_3(q)$  are conics (see [1] and [18]). Similarly, Bagchi and Sastry [2] showed that the ovoids of  $PG(3, q)$ ,  $q$  even, which admit a transitive subgroup of  $PGL_4(q)$  are elliptic quadrics or Suzuki-Tits ovoids. Brown and Lavrauw [6] have shown that an egg of  $PG(4n - 1, q)$ ,  $q$  even, contains a pseudo-conic if and only if it is elementary and arises from an elliptic quadric. Recently, J. A. Thas and K. Thas [24] have shown that every 2-transitive pseudo-oval in even characteristic is elementary and arises from a conic. In this paper, we prove the following result:

**Main Theorem.** Suppose  $\mathcal{E}$  is a pseudo-oval or pseudo-ovoid (that is not an oval or ovoid) admitting an insoluble transitive group of collineations. Then  $\mathcal{E}$  is elementary and arises from a conic, an elliptic quadric, or a Suzuki-Tits ovoid.

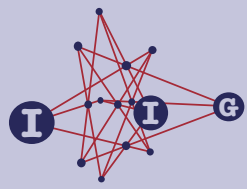
## 2. The approach

A divisor  $x$  of  $q^d - 1$  (where  $d \geq 3$ ) is *primitive* if  $x$  does not divide  $q^i - 1$  for each positive integer  $i < d$ . By a result of Zsigmondy [25], such divisors exist if  $(q, d) \neq (2, 6)$ . Therefore, if  $G$  acts transitively on a set of size  $q^m + 1$  (and  $(q, m) \neq (2, 3)$ ), then a primitive prime divisor of  $q^{2m} - 1$  divides the order of  $G$ . Such groups have an irreducible Sylow subgroup, and from this information, the structure of  $G$  can be described in great detail (see [12]). The authors have used this argument to classify  $m$ -systems of polar spaces which admit an insoluble transitive group (see [3]). From the definitions of a pseudo-ovoid and pseudo-oval, we can apply a similar argument here; which is dependent on the Classification of Finite Simple Groups.

**Note.** Suppose  $\mathcal{E}$  is a pseudo-oval (resp. pseudo-ovoid) of  $PG(2n + m - 1, q)$  where  $q = p^f$  for some prime  $p$ . Under field reduction from  $GF(q)$  to  $GF(p)$ ,

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there arises a pseudo-oval (resp. pseudo-ovoid)  $\tilde{\mathcal{E}}$  of  $\text{PG}((2n + m)f - 1, p)$ . If  $\mathcal{E}$  admits an insoluble transitive subgroup of  $\text{P}\Gamma\text{L}_{2n+m}(q)$ , then  $\tilde{\mathcal{E}}$  admits an insoluble transitive subgroup of  $\text{P}\Gamma\text{L}_{(2n+m)f}(p) = \text{PGL}_{(2n+m)f}(p)$ . We then apply the main result of this paper to  $\tilde{\mathcal{E}}$  to establish that it is elementary, from which it follows that  $\mathcal{E}$  is elementary provided that it is not an oval or ovoid. Hence throughout this paper, we will assume without loss of generality that our given pseudo-oval or pseudo-ovoid admits an insoluble transitive subgroup of the homography group  $\text{PGL}_{2m+n}(q)$ .

### 3. The pseudo-oval case

A pseudo-oval of  $\text{PG}(d - 1, q)$  (where  $d$  is a multiple of 3) is a set of  $q^{e/2} + 1$  subspaces of dimension  $d/3 - 1$ , where  $e = \frac{2}{3}d$ . This phrasing makes it clear how we apply the results of [4].

#### 3.1. Even characteristic

If  $q$  is even, then the tangent spaces of a pseudo-oval  $\mathcal{E}$  all have a  $(d/3 - 1)$ -space in common; the *nucleus* of  $\mathcal{E}$  (see [20, pp. 182]). Since  $G$  must fix the nucleus, we have that  $G$  acts reducibly in this case. Let  $\mathcal{N}$  be the nucleus of  $\mathcal{E}$  and consider the quotient map  $\pi$  from  $\text{PG}(d - 1, q)$  to  $\text{PG}(d - 1, q)/\mathcal{N}$ , and note that the codomain can be identified with  $\text{PG}(2d/3 - 1, q)$ . The image of  $\mathcal{E}$  under  $\pi$  is a spread  $\mathcal{S}$  of  $\text{PG}(2d/3 - 1, q)$  (see [20, pp. 182]). Moreover, we have that  $G$  acts transitively on this spread, and by the Andre/Bruck-Bose construction, we obtain a flag-transitive affine plane admitting an insoluble group. By [7], this affine plane is Desarguesian or a Lüneburg plane, so in particular, it follows that  $\mathcal{E}$  admits a 2-transitive group. So by [24, §8], we have that  $\mathcal{E}$  is an elementary pseudo-oval arising from a conic of  $\text{PG}(2, q^{d/3})$ .

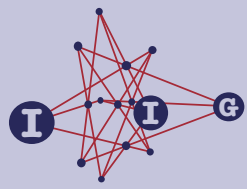
#### 3.2. Odd characteristic

Let  $\mathcal{E}$  be a pseudo-oval of  $\text{PG}(d - 1, q)$ , where  $q$  is odd. Then each element  $E$  of  $\mathcal{E}$  is contained in a unique  $2d/3 - 1$ -subspace  $T_E$  of  $\text{PG}(d - 1, q)$  which is called the *tangent space* at  $E$ . By [20, pp. 182], each point of  $\text{PG}(d - 1, q)$  is contained in 0 or 2 tangent spaces of  $\mathcal{E}$ .

**Theorem 3.1.** *Let  $q = p^f$  where  $p$  is an odd prime, let  $d$  be an integer divisible by 3. If an insoluble subgroup  $G$  of  $\text{PGL}_d(q)$  acts transitively on a pseudo-oval  $\mathcal{E}$  of  $\text{PG}(d - 1, q)$ , then  $\mathcal{E}$  is elementary and is obtained by field reduction of a conic of  $\text{PG}(2, q^{d/3})$ .*

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*Proof.* Let  $\mathcal{E}$  be a pseudo-oval of  $\text{PG}(d-1, q)$  admitting a group  $G \leq \text{PGL}_d(q)$  that is insoluble and acts transitively on  $\mathcal{E}$ , and let  $H$  be the stabiliser in  $G$  of an element of  $\mathcal{E}$ . Note that the number of elements of  $\mathcal{E}$  is  $q^{e/2} + 1$  where  $e = 2/3d$ . We may assume that  $q^{d/3} > 16$  as it was shown by the second author in [21] that if  $q^{d/3} \leq 16$ , then  $\mathcal{E}$  is elementary and is obtained by field reduction of a conic of  $\text{PG}(2, q^{d/3})$ . Let  $\hat{G}$  be a preimage of  $G$  in  $\text{GL}_d(q)$ . Then there exists a subgroup  $\hat{H}$  of  $\hat{G}$  of index  $q^{e/2} + 1$  such that the image of  $\hat{H}$  in  $\text{PGL}_d(q)$  is  $H$ . So we can apply [4, Theorem 3.1] to  $\hat{G}$ . There are six cases to consider from this theorem: the Classical, Imprimitve, Reducible, Extension Field (case (b)), Symplectic Type, and Nearly Simple examples. Straight away, we have that the Symplectic examples do not occur as  $d$  is a multiple of 3. By [4, Lemma 13],  $\hat{G}$  is not in the Classical examples case. So we are left with four families to consider: the Reducible, Imprimitve, Extension Field, and the Nearly simple examples.

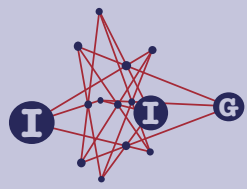
Let us first suppose we are in the Imprimitve examples case. So by [4, Theorem 3.1], we have that  $d = 9$ ,  $q \in \{3, 5\}$ , and  $\hat{G}$  preserves a decomposition of  $V_9(q)$  into 1-spaces. So in particular,  $\hat{G} \leq \text{GL}_1(q) \wr S_9$ . We treat both cases,  $q = 3$  and  $q = 5$ , simultaneously. Let  $\mu$  be the natural projection map from  $\text{GL}_1(q) \wr S_9$  onto  $S_9$ . Now  $\mu(\hat{G})$  is insoluble and primitive (of degree 9), and hence  $\mu(\hat{G}) \in \{\text{PSL}_2(8), \text{P}\Omega_2(8), A_9, S_9\}$  (see [10, Appendix B]). Moreover,  $\mu(\hat{G})$  is 3-transitive in its degree 9 action. Let  $B$  be the kernel of  $\mu$ . So  $|B| = (q-1)^9 \in \{2^9, 2^{18}\}$ . Now  $G \cap B$  is a nontrivial normal subgroup of  $G$  and hence  $G \cap B$  contains the subgroup  $K$  of  $B$  consisting of diagonal matrices with entries  $\pm 1$ . Since  $|\hat{G} : \hat{H}| \in \{28, 126\}$ , we see that a subgroup  $J$  of  $K$  with index at most 2, is contained in  $\hat{H}$ . The only  $J$ -invariant subspaces of  $V_9(q)$  are the spans of vectors from the canonical basis; coordinate subspaces. Let  $E$  be an element of the pseudo-oval. We may assume (up to conjugacy) that  $E$  is  $J$ -invariant and so it is a coordinate plane. Now the action of  $\mu(\hat{G})$  is 3-transitive, and so the orbit of  $E$  under  $\hat{G}$  on planes is  $\binom{9}{3} = 84$ . So the Imprimitve examples case does not arise.

Let us now suppose we are in the Nearly simple case. So  $S \leq G \leq \text{Aut}(S)$  where  $S$  is a finite nonabelian simple group, and  $\hat{G}$  is irreducible. By using the fact that  $q^{d/3} \geq 16$ , we have only two subcases to consider: the Alternating group case and the Natural-characteristic case. In the former, we have  $S = A_{10}$ ,  $d = 9$ ,  $q = 3$ , and the vector space  $V_9(3)$  can be identified with the fully deleted permutation module for  $S_{10}$  over  $\text{GF}(3)$ . It can be readily checked that  $G$  does not have a subgroup of index  $3^3 + 1$ , and so this case does not arise. In the Natural-characteristic case, we have that  $d = 9$  and  $S = \text{PSL}_3(q^2)$  (by [4, Theorem 2.1]). Now by [8], the minimum degree of a nontrivial representation of  $S$  is  $(q^6 - 1)/(q^2 - 1)$ . However

$$q^3 + 1 = (q^6 - 1)/(q^3 - 1) < (q^6 - 1)/(q^2 - 1)$$

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and so  $\hat{G}$  does not have a transitive action of degree  $q^3 + 1$ . Therefore, we have that  $\hat{G}$  is not in the Nearly Simple examples case.

Now suppose we are in the Field Extension examples case. We have that  $\hat{G}$  is irreducible and there is a divisor  $b$  of  $2d/3$  (where  $b \neq 1$ ) such that  $\hat{G}$  preserves a field extension structure  $V_{d/b}(q^b)$  on  $V_d(q)$ . Moreover,  $G \cap \text{GL}_{d/b}(q^b)$  has a subgroup of index  $(q^{e/2} + 1)/x$ , for some  $x$ , and so if  $d/b > 3$ , then we can apply [4, Theorem 3.2] to  $G \cap \text{GL}_{d/b}(q^b)$  with parameters  $q^b$ ,  $d/b$ , and  $e/b$  playing the roles of  $q$ ,  $d$ , and  $e$  respectively. So let us assume that  $d/b > 3$ . Since  $d/b \neq e/b$ , we do not have the Classical examples case. Note that if  $\hat{G}$  fixes a subspace over the field extension  $q^b$ , then it also fixes a subspace that is written over the field  $\text{GF}(q)$ . Hence  $\hat{G} \cap \text{GL}_{d/b}(q^b)$  is irreducible in its action on  $\text{PG}(d/b - 1, q^b)$ . We can also assume that  $G \cap \text{GL}_{d/b}(q^b)$  does not preserve a field extension structure by choosing  $b$  to be maximal. Since  $q^b$  is not prime, we can eliminate the Imprimitive examples, Symplectic Type examples, and the Nearly Simple examples. Therefore  $d/b = 3$  and  $e/b = 2$ . By some old work of Mitchell [17], the only absolutely irreducible insoluble maximal subgroups of  $\text{PSL}_3(q^b)$  are

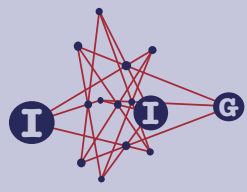
- (i)  $\text{PSL}_2(q^b)$ ;
- (ii)  $\text{PSU}_3(q^b)$  when  $q^b$  is a square;
- (iii)  $A_6$  when  $p \equiv 1, 2, 4, 7, 8, 13 \pmod{15}$  (and  $\text{GF}(q^b)$  contains the squares of 5 and  $-3$ );
- (iv)  $\text{PSL}_2(7)$  when  $p \equiv 1, 2, 4 \pmod{7}$ .

In the case that  $\text{PSU}_3(q^{d/3}) \leq G \cap \text{PGL}_3(q^{d/3}) \leq \text{P}\Gamma\text{L}_3(q^{d/3})$ , we have  $q^{d/3} + 1$  divides  $q^{d/2}(q^{d/3} - 1)(q^{d/2} + 1)$ . This is a contradiction as  $q^{d/3} + 1$  is coprime to  $q^{d/2}$  and  $q^{d/3} - 1$  (note that  $q$  is odd). So this case does not arise. In the case that  $A_6 \leq G \cap \text{PGL}_3(q^{d/3}) \leq S_6$ , we have  $q^{d/3} + 1$  divides  $6!$  (note that  $q^{d/3} + 1$  is coprime to  $|G : G \cap \text{PGL}_3(q^{d/3})|$ ). However,  $q^{d/3} + 1$  divides  $6!$  only if  $q = 3$  and  $d = 6$  (so  $b = 2$ ). So this case does not arise as  $A_6$  does not embed in  $\text{P}\Gamma\text{L}_3(q^b)$  in characteristic 3. In the case that  $\text{PSL}_2(7) \leq G \cap \text{PGL}_3(q^{d/3}) \leq \text{PGL}_2(7)$ , we have  $q^{d/3} + 1$  divides  $336$ . However,  $q^{d/3} + 1$  divides  $336$  only if  $q = 3$  and  $d = 9$  (so  $b = 3$ ). So this case does not arise as  $\text{PSL}_2(7)$  does not embed in  $\text{P}\Gamma\text{L}_3(q^b)$  in characteristic 3. Hence  $\text{PSL}_2(q^b) \leq G$ .

Let  $J = \text{PSL}_2(q^{d/3})$ . It is a classical result, but can also be found in [8], that  $\text{PSL}_2(q^{d/3})$  (where  $d > 2$ ) has a unique conjugacy class of subgroups of index  $q^{d/3} + 1$ . It follows from [14, Proposition 4.3.17], that there is a unique characteristic class of subgroups of  $\text{PGL}_d(q)$  isomorphic to  $J$  (it is not true in general that there is a unique conjugacy class of such subgroups). Let

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$\varphi : V_3(q^{d/3}) \rightarrow V_d(q)$  denote the natural vector space isomorphism here, and let  $\mathcal{C}$  be a conic of  $V_3(q^{d/3})$  admitting  $J$ . Let  $\alpha$  and  $\beta$  be two distinct points of  $\mathcal{C}$ . Then  $\varphi(\alpha)$  and  $\varphi(\beta)$  are  $d/3$ -dimensional vector subspaces of  $V_d(q)$ . Note that  $J$  has a unique conjugacy class of subgroups of index  $q^{d/3} + 1$ , and hence we can assume that the stabiliser of an element  $E$  of  $\mathcal{E}$  is identical to the stabiliser  $J_\alpha$ . Now suppose we have a third vector  $v$  which is neither  $\alpha$  nor  $\beta$ . Then

$$|v^{J_\alpha}| = |J_\alpha : J_{\alpha,v}| = |J_\alpha : J_{\alpha,\beta}| |J_{\alpha,\beta} : J_{\alpha,\beta,v}| = q^{d/3} |J_{\alpha,\beta} : J_{\alpha,\beta,v}|.$$

Now  $J$  is a Zassenhaus group (i.e., a 2-transitive group such that the stabiliser of any three points is trivial) and so  $J_{\alpha,\beta,v} = 1$ . Therefore

$$|v^{J_\alpha}| = q^{d/3} \frac{q^{d/3} - 1}{\gcd(2, q^{d/3} - 1)}$$

which is not a prime power. Now any  $J_\alpha$ -invariant  $d/3$ -subspace of  $V_d(q)$  is a union of orbits of  $J_\alpha$ . Therefore, it follows that the only  $J_\alpha$ -invariant subspace of  $V_d(q)$  is  $\varphi(\alpha)$ . Since  $W$  is  $J_\alpha$ -invariant, we have that  $W = \varphi(\alpha)$  and hence  $\mathcal{E}$  is the image of  $\mathcal{C}$  under  $\varphi$ . Therefore,  $\mathcal{E}$  is elementary and is obtained by field reduction of a conic of  $\text{PG}(2, q^{d/3})$ .

### Reducible examples

We have that  $\hat{G}$  fixes a subspace/quotient space  $U$  of  $V_d(q)$  and  $\dim(U) = u \geq \frac{2}{3}d$ . In fact, it follows that  $u = 2/3d$  by noting that a primitive divisor of  $q^{(2/3)d} - 1$  also divides  $|\hat{G}|$ . So  $\hat{G} \leq q^{u(d-u)} \cdot (\text{GL}_u(q) \times \text{GL}_{d-u}(q))$ . We may assume that  $U$  is a subspace, as for  $q$  odd, each point of  $U$  is in 0 or 2 tangent spaces of  $\mathcal{E}$ . Consider the set of intersections

$$\mathcal{M} = \{T_E \cap U : E \in \mathcal{E}\}.$$

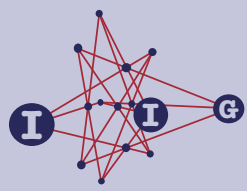
Note that each element of  $\mathcal{M}$  has a common dimension as  $G$  acts transitively on  $\mathcal{M}$ , and thus  $\dim(T_E \cap U) = d/3$  for all  $E \in \mathcal{E}$ . Therefore  $\hat{G}^U$  acts transitively on a set of  $(q^{d/3} + 1)/\delta$  subspaces of dimension  $d/3$  where  $\delta = 1, 2$ . This implies that  $\hat{G}^U$  has a subgroup of index  $(q^{d/3} + 1)/\delta$ , and so we can apply [4, Theorem 3.2] with  $q$ ,  $\frac{2}{3}d$ , and  $\frac{2}{3}d$  playing the roles of  $q$ ,  $d$ , and  $e$  respectively. In the following subcases, we have that  $G$  has a normal insoluble subgroup  $S$ , which is given explicitly. Moreover,  $S$  must have a union of orbits on  $(d/3)$ -spaces of  $U$  of size  $(q^{d/3} + 1)/\delta$  where  $\delta = 1, 2$ .

### Reducible/nearly simple examples

In this case,  $S \leq G^U \cap \text{PGL}_d(q) \leq \text{Aut}(S)$  where  $S$  is a finite nonabelian simple group. Here we have four subcases.

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### Alternating group case

Here  $S = A_r$  and the vector space  $V_u(q)$  can be identified with the fully deleted permutation module for  $S_r$  over  $\text{GF}(q)$ . We have that  $u$  is  $r-1$  or  $r-2$  (according to whether  $p$  does not or does divide  $n$  respectively), and  $q^u = p^u = 3^6, 5^6$ . Suppose  $S = A_7$ ,  $u = 6$ , and  $q = 3$ . Then  $S$  stabilises  $\mathcal{M}$  and hence  $S$  has a union of orbits on planes of  $\text{PG}(5, 3)$  of size 14 or 28. Now  $A_7$ , in its unique irreducible representation in  $\text{PG}(5, 3)$  has the following orbit lengths on planes (n.b., the exponents denote multiplicities):

$$[35^2, 105^4, 140^3, 210^4, 315^6, 420^{10}, 630^6, 840^4, 1260^{15}].$$

Therefore this case does not arise. Now suppose  $q = 5$ . It can be shown using GAP [11] that the  $S$ -invariant sets of planes of size 63 or 126 do not cover every point either 0 or 2 times. Therefore this case does not arise.

### Cross-characteristic case

The table below lists the possibilities for this case.

$S$	$d$	$q$	$u$
$\text{PSL}_2(7)$	9	3	6
$\text{PSL}_2(13)$	9	3	6
$\text{PSU}_3(3^2)$	9	5	6

Now  $\text{PSL}_2(13)$  acts transitively on the points of  $\text{PG}(5, 3)$ , and so this case does not arise. Suppose  $S = \text{PSL}_2(7)$ ,  $u = 6$ , and  $q = 3$ . Then  $S$  stabilises  $\mathcal{M}$  and hence  $S$  has a union of orbits on planes of  $\text{PG}(5, 3)$  of size 14 or 28. Now by using GAP [11] and the unique irreducible representation for  $S$  in  $\text{PG}(5, 3)$ , we have that  $S$  has the following orbit lengths on planes:

$$[7^4, 21^8, 28^{12}, 42^{18}, 56^{12}, 84^{100}, 168^{140}].$$

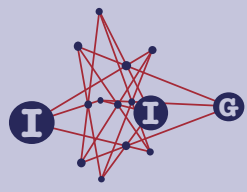
None of the thirteen  $S$ -invariant sets of planes of size 28 have each point of  $\text{PG}(5, 3)$  contained in a constant number (0 or 2) of elements of the set. Likewise, of all the six  $S$ -invariant sets of size 14, none have each point of  $\text{PG}(5, 3)$  contained in a constant number of elements of the set. Therefore, this case does not arise.

Now suppose  $S = \text{PSU}_3(3^2)$ ,  $u = 6$ , and  $q = 5$ . Then  $S$  stabilises  $\mathcal{M}$  and hence  $S$  stabilises a set of points of size  $(q^u - 1)/(2(q - 1)) = 1953$ . However, by using GAP [11] one can calculate that  $S$  has the following orbit lengths on points of  $\text{PG}(5, 5)$ :

$$[189^2, 1008^2, 1512].$$

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Since 1953 cannot be partitioned into these numbers, this case does not arise.

So we are left now with just two more cases: the “Classical examples” and the “Extension field” examples, which can be unified naturally.

### Reducible/classical and extension field examples

We have that  $\hat{G}^U$  preserves a (possibly trivial) field extension structure on  $U$  as a  $u/b$ -dimensional subspace over  $\text{GF}(b)$  where  $b$  is a proper divisor of  $u = (2/3)d$ . So  $\hat{G}^U \leq \text{GL}_{(2/3)d/b}(q^b)$  and we can apply [4, Theorem 3.2] to  $\hat{G}^U \cap \text{GL}_{(2/3)d/b}(q^b)$  where  $q^b$ ,  $u/b$ , and  $u/b$  play the roles of  $q$ ,  $d$ , and  $e$  respectively. We simply have  $d/b = 6$  and  $\text{PSL}_2(q^{d/3}) \leq \hat{G}^U$ . Let  $S = \text{PSL}_2(q^{d/3})$  and note that the preimage of  $S$  acts transitively on the non-zero vectors of  $V_2(q^{d/3})$ . However, we have here that  $S$  stabilises a set of  $q^{d/3} + 1$  subspaces, each of dimension  $d/3 - 1$ , which is impossible for  $d/3 > 1$ . So we conclude that  $G$  is irreducible.  $\square$

## 4. The pseudo-ovoid case

A pseudo-ovoid of  $\text{PG}(d - 1, q)$  (where  $d$  is a multiple of 4) is a set of  $q^{d/2} + 1$  subspaces of dimension  $d/4 - 1$ . Here we can also apply the results of [4], as we did in the pseudo-oval case.

**Theorem 4.1.** *Let  $q = p^f$  where  $p$  is a prime and let  $d$  be an integer divisible by 4. If an insoluble subgroup  $G$  of  $\text{PGL}_d(q)$  acts transitively on a pseudo-ovoid  $\mathcal{E}$  of  $\text{PG}(d - 1, q)$ , then  $\mathcal{E}$  is elementary and arises from an elliptic quadric or Suzuki-Tits ovoid.*

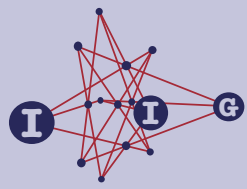
*Proof.* Let  $H$  be the stabiliser of an element of  $\mathcal{E}$  in  $G$ , and let  $\hat{G}$  be a preimage of  $G$  in  $\text{GL}_d(q)$ . Note that the number of elements of a pseudo-ovoid of  $\text{PG}(d - 1, q)$  is  $q^{e/2} + 1$  where  $e = d$ . So there exists a subgroup  $\hat{H}$  of  $\hat{G}$  of index  $q^{d/2} + 1$  such that the image of  $\hat{H}$  in  $\text{PGL}_d(q)$  is  $H$ . Therefore we can apply [4, Theorem 3.2] to  $\hat{G}$ . First note that we can rule out the Reducible examples, Imprimitive examples, and case (a) of the Extension field examples. Recall that by [18], we can assume that  $d > 4$ . Hence we have ruled out the Classical and Symplectic Type examples. Also note that  $d$  is a multiple of 4, and so in the Nearly simple case, we have the following:  $q = 2$ ,  $d = 12$ , and either

- (a)  $A_{13} \leq G \leq S_{13}$ , or
- (b)  $S = \text{PSL}_2(25) \leq G \leq \text{P}\Gamma\text{L}_2(25)$ , and  $S \cap H$  is isomorphic to  $S_5$  (there are two such conjugacy classes of  $S$ ).

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However in the first case, it is clear that  $G$  does not have a subgroup of index 65. In the second case, we know by [13] that  $\text{PSL}_2(25)$  has a unique 12-dimensional irreducible representation (up to quasi-equivalence) over  $\text{GF}(2)$  and it has the following orbit lengths on points:

$$[65, 325^2, 650, 780, 1950].$$

Let  $\mathcal{B}$  be the set of points covered by the pseudo-ovoid  $\mathcal{E}$  of  $\text{PG}(11, 2)$ . Then  $\mathcal{B}$  has size  $(q^{d/4} - 1)(q^{d/2} + 1) = (2^3 - 1)(2^6 + 1) = 455$  and it must be a union of orbits of  $S$  as  $G$  acts transitively on  $\mathcal{E}$ . However, 455 cannot be partitioned into the orbit lengths displayed above, and hence this case does not arise.

That leaves us with the Extension field examples. Here we have that  $\hat{G} \leq \Gamma\text{L}_{d/b}(q^b)$  where  $b$  is a divisor of  $d$  (where  $b \neq 1$ ). If  $d/b > 2$ , We can apply [4, Theorem 3.2] (for  $e/b$  even) and [4, Theorem 3.1] (for  $e/b$  odd) to  $\hat{G} \cap \text{GL}_{d/b}(q^b)$  with parameters  $d/b$ ,  $e/b$ , and  $q^b$  playing the roles of  $d$ ,  $e$ , and  $q$  respectively. We have the following subcases:

- (i)  $d/b = 4$  and  $\Omega_4^-(q^{d/4}) \leq \hat{G} \cap \text{GL}_{d/b}(q^b)$ ;
- (ii)  $d/b = 4$ ,  $q$  is even, and  $\text{Sz}(q^{d/4}) \leq \hat{G} \cap \text{GL}_{d/b}(q^b)$ ;
- (iii)  $d/b = 3$ ,  $q^{d/3}$  is a square, and  $\text{SU}_3(q^{d/3}) \leq \hat{G} \cap \text{GL}_{d/b}(q^b)$ .

- (i) Let us suppose we have the first case above, where  $d/b = 4$  and  $\mathcal{E}$  admits  $\text{P}\Omega_4^-(q^{d/4})$ . Let  $J = \text{P}\Omega_4^-(q^{d/4})$ . It is a classical result, but can also be found in [8], that  $\text{PSL}_2(q^{d/2})$  (where  $d > 2$ ) has a unique conjugacy class of subgroups of index  $q^{d/2} + 1$ . Note that  $\text{P}\Omega_4^-(q^{d/4})$  is isomorphic to  $\text{PSL}_2(q^{d/2})$ , and by [14, Proposition 4.3.6], there is a unique conjugacy class of subgroups of  $\text{PGL}_d(q)$  isomorphic to  $\text{PSL}_2(q^{d/2})$ . Therefore, there is a unique conjugacy class of subgroups of  $\text{PGL}_d(q)$  isomorphic to  $J$ .

Let  $\varphi : V_4(q^{d/4}) \rightarrow V_d(q)$  denote the natural vector space isomorphism here, and let  $\mathcal{Q}$  be an elliptic quadric of  $V_4(q^{d/4})$  admitting  $J$ . Let  $\alpha$  and  $\beta$  be two distinct points of  $\mathcal{Q}$ . Then  $\varphi(\alpha)$  and  $\varphi(\beta)$  are  $d/4$ -dimensional subspaces of  $V_d(q)$ . Note that  $J$  has a unique conjugacy class of subgroups of index  $q^2 + 1$  (see [8]), and hence we can assume that the stabiliser of an element  $E$  of  $\mathcal{E}$  is identical to the stabiliser  $J_\alpha$ . Now suppose we have a third vector  $v$  which is neither  $\alpha$  nor  $\beta$ . Then

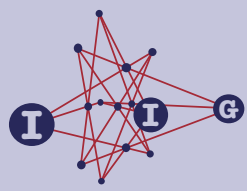
$$|v^{J_\alpha}| = |J_\alpha : J_{\alpha,v}| = |J_\alpha : J_{\alpha,\beta}| |J_{\alpha,\beta} : J_{\alpha,\beta,v}| = q^{d/2} |J_{\alpha,\beta} : J_{\alpha,\beta,v}|.$$

Now  $J$  is a Zassenhaus group and so  $J_{\alpha,\beta,v} = 1$ . Therefore

$$|v^{J_\alpha}| = q^{d/2} \frac{q^{d/2} - 1}{\text{gcd}(2, q^{d/2} - 1)}$$

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which is not a prime power. Now any  $J_\alpha$ -invariant  $d/4$ -subspace of  $V_d(q)$  is a union of orbits of  $J_\alpha$ . Therefore, it follows that the only  $J_\alpha$ -invariant subspace of  $V_d(q)$  is  $\varphi(\alpha)$ . Since  $W$  is  $J_\alpha$ -invariant, we have that  $W = \varphi(\alpha)$  and hence  $\mathcal{E}$  is the image of  $\mathcal{Q}$  under  $\varphi$ . Therefore,  $\mathcal{E}$  is elementary and arises from an elliptic quadric.

(ii) By a similar argument to that above, it is not difficult to show that  $\mathcal{E}$  is the image of a Suzuki-Tits ovoid under field reduction. The key steps to note are that  $Sz(q^{d/4})$  is a Zassenhaus group, there is a unique conjugacy class of subgroups of  $PGL_d(q)$  isomorphic to  $Sz(q^{d/4})$ , and  $Sz(q^{d/4})$  has a unique conjugacy class of subgroups of index  $q^2 + 1$ . In the seminal paper of Suzuki [23, §15], it was shown that  $Sz(q^{d/4})$  is a Zassenhaus group and has a unique conjugacy class of subgroups of index  $q^{d/2} + 1$  and this is the minimum non-trivial degree of  $Sz(q^{d/4})$ . The uniqueness of its representation in  $PGL_d(q)$  needs more work. By a result of Lüneburg (see [16, 27.3 Theorem] or [15]), there is a unique conjugacy class of subgroups of  $PGL_4(q^{d/4})$  isomorphic to  $Sz(q^{d/4})$ . Now by [14, Proposition 4.3.6], there is a unique conjugacy class of subgroups of  $PGL_d(q)$  isomorphic to  $PGL_4(q^{d/4})$ . Therefore, there is a unique conjugacy class of subgroups of  $PGL_d(q)$  isomorphic to  $Sz(q^{d/4})$ . Therefore,  $\mathcal{E}$  is elementary and arises from a Suzuki-Tits ovoid.

(iii) Now suppose we have the third case;  $d/b = 3$ ,  $q^{d/3}$  is a square, and  $\mathcal{E}$  admits  $PSU_3(q^{d/3})$ . Now the smallest orbit of  $PSU_3(q^{d/3})$  on nonzero vectors consists of the non-singular vectors and has size  $(q^{d/3} - 1)(q^{d/2} + 1)$ . Since  $\mathcal{E}$  covers  $(q^{d/4} - 1)(q^{d/2} + 1)$  vectors of  $V_d(q)$ , and this number is strictly smaller than the size of the smallest orbit of  $PSU_3(q^{d/3})$ , we see that this case does not arise.

Suppose now that  $d/b = 2$ . Since  $\hat{G}$  is an insoluble subgroup of  $\Gamma L_2(q^{d/2})$ , it follows from [4, Lemma 5] that  $\hat{G}$  contains  $SL_2(q^{d/2})$ . However,  $SL_2(q^{d/2})$  is transitive on nonzero vectors and hence does not stabilise a set of  $d/4$  vector subspaces of size  $q^{d/2} + 1$ . Hence this case does not arise.  $\square$

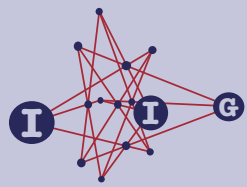
*Remark 4.2.* If a (presently unknown) pseudo-oval or pseudo-ovoid over  $GF(q)$  admitting a soluble transitive group  $G$  exists, then  $G$  is meta-cyclic; indeed  $G$  is a subgroup of  $\Gamma L_1(q^b)$ , for an appropriate positive integer  $b$ .

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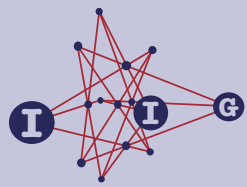
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John Bamberg

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY W.A. 6009, AUSTRALIA

DEPARTMENT OF PURE MATHEMATICS, GHENT UNIVERSITY, GALGLAAN 2, 9000 GENT, BELGIUM

*e-mail*: bamberg@cage.ugent.be

Tim Penttila

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY W.A. 6009, AUSTRALIA

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523, USA

*e-mail*: penttila@math.colostate.edu

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