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Stable planes with a point transitive abelian group

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Abstract

We consider stable planes $E = (P, \mathcal{L})$ admitting an abelian group Δ which acts (sharply) transitively on the point set P. Known examples are the so-called arc planes with $\Delta = \mathbb{R}^2$, see [1], affine translation planes and shift planes. We examine the possible actions of Δ on the line space $\mathcal{L}$ and use the results in order to characterize affine shift planes and translation planes.

Keywords: stable planes, abelian group action, arc planes

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1 Introduction

Let $E = (P, \mathcal{L})$ be a stable plane with P locally compact, connected and of positive finite dimension. This implies, according to [5] (1.11) and [7] Theorem 1, that each line has dimension l and P has dimension $2l$. Throughout this article we assume that the automorphism group Σ of the stable plane E contains an abelian subgroup $\Delta \leq \Sigma$ which acts transitively (and hence sharply transitively) on the point space P.

Well-known affine examples are translation planes and shift planes (that is, planes with point set $\mathbb{R}^{2l}$ and lines of the form $\{c\} \times \mathbb{R}^l$, $c \in \mathbb{R}^l$, plus all translates of a graph of a suitable function $f : \mathbb{R}^l \to \mathbb{R}^l$). We prove that there are no other affine planes satisfying our assumptions. In particular, if E is affine, P is isomorphic to $\mathbb{R}^{2l}$ and each line is homeomorphic to $\mathbb{R}^l$. For $l = 2$ this agrees with

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a result of N. Knarr, see [3], who determines all types of 4-dimensional compact projective planes with a 4-dimensional abelian closed connected subgroup of the automorphism group. The only two types with a sharply transitive action are translation planes and shift planes. See e.g. [10] for more information about translation planes (chapters 73 and 81) and shift planes (chapter 74).

In [11] M. Stroppel considers a stable plane with a connected locally compact abelian group Ξ of automorphisms. He shows that either Ξ is quasi-perspective, or some point orbits generate (not necessarily proper) subplanes. The quasi-perspective action leads to shear planes as considered by H. Löwe, who characterizes shear planes (see [4]) as stable planes of dimension $2l$ admitting a quasi-perspective collineation group isomorphic to $\mathbb{R}^{2l}$ without fixed points. If, on the other hand, some point orbit under the action of Ξ is the entire point space, this action is sharply transitive and thus belongs to the case considered here.

Our assumptions are chosen in order to study arc planes in arbitrary dimensions. In fact, the arc planes considered by H. Groh in [1] with an abelian group action (i.e. if $\Delta = \mathbb{R}^2$) satisfy our assumptions. Based on the present results, we shall engage in a systematic search for higher-dimensional arc planes in a subsequent paper.

2 Line types

Definition 2.1. A *stable plane* is a pair $E = (P, \mathcal{L})$ consisting of a set P of *points* and a set $\mathcal{L}$ of *lines*, where each line is a subset of P. For any two different points $p_1, p_2 \in P$ there exists a unique line $L \in \mathcal{L}$ with $p_1, p_2 \in L =: p_1 \vee p_2$. Thus, we have the mapping of joining two points $\vee : (P \times P) \setminus \Delta P \to \mathcal{L}$, where ΔP denotes the diagonal in $P \times P$, and also the mapping of intersecting two lines $\wedge : \mathcal{L} \times \mathcal{L} \supseteq \mathrm{dom}(\wedge) \to P$. In addition we have Hausdorff topologies on P and $\mathcal{L}$ such that $\vee$ and $\wedge$ are continuous and $\mathrm{dom}(\wedge) \subseteq \mathcal{L} \times \mathcal{L}$ is open.

Let $E = (P, \mathcal{L})$ always be a stable plane satisfying the general assumptions stated in the introduction.

Proposition 2.2. *The subgroup Δ is closed in Σ and is a Lie group isomorphic to $\mathbb{R}^n \times \Phi$, where Φ is a maximal compact subgroup of Δ. The point space P is homeomorphic to Δ; in particular, Δ is connected.*

Proof. Since Δ is abelian, this also holds for the topological closure $\bar{\Delta}$ in Σ. Acting transitively on P, the closure $\bar{\Delta}$ acts sharply transitively, hence $\Delta = \bar{\Delta} \leq \Sigma$ is closed.

According to [5, (2.3)], the group Σ is a topological transformation group of P. Furthermore, P is locally contractible. Σ is second countable and is locally compact according to [5, (2.9)]. Therefore both properties also hold for the closed subgroup Δ. According to [10, (96.14)], the group Δ is a Lie group. For a fixed $p \in P$, the mapping $\Psi_p : \Delta \to P\,,\ \delta \mapsto p^\delta$ is continuous and bijective, because Δ acts sharply transitively on P. With [10, (96.8)] it follows that the mapping is also open, hence a homeomorphism.

In particular, $\Delta \approx P$ is connected. From the Theorem of Malcew-Iwasawa (e.g. [10, (93.10)]) it follows that $\Delta \cong \mathbb{R}^n \times \Phi$. □

We distinguish two kinds of lines according to their stabilizer in Δ.

Definition 2.3. Let $L \in \mathcal{L}$. If the stabilizer Δ_L is non-trivial, we call L a *straight line*. In the other case $\Delta_L = \{1\}$, we call L an *arc*. Since the type of L is invariant under the action of Δ, we also call the orbit L^Δ *straight orbit* and *arc orbit*, respectively.

First we prove that it is impossible for our stable plane to contain only arcs.

Theorem 2.4. *The line space $\mathcal{L}$ of a stable plane $E = (P, \mathcal{L})$ satisfying our general assumptions contains at least one straight orbit.*

For the proof we use the following lemma; here, $\mathcal{L}_p$ denotes the set of all lines passing through p.

Lemma 2.5. *Let $p \in P$ and let L be an arc. Then L is homeomorphic to the subset $L^\Delta \cap \mathcal{L}_p \subseteq \mathcal{L}_p$, and this subset is open.*

Proof. We have $L^\Delta \cap \mathcal{L}_p = L^\Gamma$, where $\Gamma := \{\delta \in \Delta \mid L^\delta \in \mathcal{L}_p\}$. First we show that $L \approx \Gamma$. The mapping $\Psi_p : \Delta \to P\,,\ \delta \mapsto p^\delta$ is a homeomorphism. By definition we have

$$\gamma \in \Gamma \iff L^\gamma \in \mathcal{L}_p \iff \exists\, q \in L :\ q^\gamma = p \iff p^{\gamma^{-1}} \in L,$$

and therefore $\Psi_p(\Gamma^{-1}) = L$ with $\Gamma^{-1} := \{\gamma^{-1} \mid \gamma \in \Gamma\}$. Thus we have $L \approx \Gamma^{-1} \approx \Gamma$. In particular, Γ is locally compact and closed in Δ; note that, according to [5, (1.3)], each line $L \subseteq P$ is closed.

The mapping $\hat{\Phi} : \Delta \to \mathcal{L}\,,\ \delta \mapsto L^\delta$ is continuous and injective, and so is $\Phi := \hat{\Phi}\big|_\Gamma^{\mathcal{L}_p} : \Gamma \to \mathcal{L}_p$. According to [7, (11b)], the line pencil $\mathcal{L}_p$ has the domain invariance property. In addition, according to [6, (1.2)], the line pencil $\mathcal{L}_p$ and the line L are locally homeomorphic. Thus, it follows (see e.g. [10, (51.19)]) that Φ is an open mapping and therefore $L \approx \Gamma \approx \Phi(\Gamma) = L^\Gamma = L^\Delta \cap \mathcal{L}_p$. In particular, $L^\Gamma = \Phi(\Gamma) \subseteq \mathcal{L}_p$ is open. □

Now we prove Theorem 2.4.

Proof of Theorem 2.4. Assume, by way of contradiction, that all lines are arcs. Let $\{L_i \mid i \in I\}$ be a representing system of orbits and $p \in P$. According to Lemma 2.5, each set $L_i^\Delta \cap \mathcal{L}_p$ is open in $\mathcal{L}_p$. The union of these sets is a disjoint open cover of $\mathcal{L}_p$, because there are no straight lines. Since $\mathcal{L}_p$ is connected according to [5, (1.14)], it follows that $|I| = 1$; so there is only one line orbit, say $L^\Delta = \mathcal{L}$. According to Lemma 2.5, we have $L \approx L^\Delta \cap \mathcal{L}_p = \mathcal{L}_p$. Since $\mathcal{L}_p$ is compact according to [2, (3.8)], the line L and hence each line is compact. Now [2, (3.10)] and (3.11) show that E is a projective plane and the point space P is compact. Thus, Δ is a compact connected abelian Lie group and therefore (see e.g. [10, (94.38)]) a torus. In particular, the fundamental group of P is isomorphic to $\mathbb{Z}^{2l}$, in contradiction to [10, (51.28)] for $l \geq 2$ or to [10, (52.14)] for $l = 1$. □

The presence of straight lines in our stable plane allows us to prove that all lines are manifolds. As a first step we have the following lemma.

Lemma 2.6. *Let K be a straight line. Then K is a manifold and the Lie group Δ_K acts sharply transitively on K.*

Proof. Let $o \in K$. By definition we have $o^{\Delta_K} \subseteq K$. Assume, by way of contradiction, that there is a point $p \in K \setminus o^{\Delta_K}$, say $p = o^\delta$ with $\delta \in \Delta \setminus \Delta_K$. Since Δ is abelian and $p \in K$, we obtain that $K \supseteq p^{\Delta_K} = o^{\delta \cdot \Delta_K} = o^{\Delta_K \cdot \delta}$. Hence $\left(o^{\Delta_K}\right)^\delta \subseteq K \cap K^\delta$, and $K = K^\delta$ since $|o^{\Delta_K}| \geq 2$. This contradicts $\delta \notin \Delta_K$.

We have shown that Δ_K is (sharply) transitive on K. The mapping $\Psi_o : \Delta \to P$ (defined as in the proof of Proposition 2.2) restricts to a homeomorphism $\Delta_K \approx K$. Since $K \subseteq P$ is closed ([5, (1.3)]), the stabilizer Δ_K is a closed subgroup of the Lie group Δ, hence a Lie group. In particular, Δ_K and K are manifolds. □

As an immediate consequence we can prove for later use

Lemma 2.7. *For a straight line K and $p \in P$ we have that $|K^\Delta \cap \mathcal{L}_p| = 1$.*

Proof. Without loss of generality we may assume that $p \in K$. Let $\delta \in \Delta$, then

$$K^\delta \in \mathcal{L}_p \iff \exists\, k \in K : \ k^\delta = p \iff \delta \in \Delta_K,$$

where the equivalence on the right hand side follows from Lemma 2.6 (existence of a $\gamma \in \Delta_K$ with $k^\gamma = p$) together with the sharply transitive action of Δ (uniqueness of this γ). Hence $K^\delta = K$, which proves the assertion. □

Proposition 2.8. *All lines of the stable plane E are manifolds and each line pencil $\mathcal{L}_p$ is homeomorphic to the l-sphere $\mathbb{S}_l$.*

Proof. By Theorem 2.4, there is always a straight line and by Lemma 2.6, this line is a manifold. According to [2, (3.2)], any two lines are locally homeomorphic, hence all lines are manifolds. Now [2,] gives the assertion about the line pencils. □

3 Types of planes

Under certain additional assumptions about the stable plane $E = (P, \mathcal{L})$ and the action of Δ we can characterize some special types of planes.

Lemma 3.1. *Let the stable plane E contain at least two straight orbits. Then each straight line is connected. For straight lines K and L satisfying $K^\Delta \neq L^\Delta$ the intersection $\Delta_K \cap \Delta_L$ is trivial.*

Proof. Let $K, L \in \mathcal{L}$ be straight satisfying $K^\Delta \neq L^\Delta$. Then we have $\Delta_K \cap \Delta_L = \{1\}$. Indeed, assume, by way of contradiction, that there is a $\delta \in \Delta_K \cap \Delta_L$ with $\delta \neq 1$. Since Δ is abelian, Δ_K fixes each element of the orbit K^Δ; so we can assume without loss of generality that K and L intersect in $K \wedge L =: p \in P$. But then $p \neq p^\delta \in K \cap L$, hence we obtain the contradiction $K = L$.

Thus, we have the isomorphism $\Delta_K \times \Delta_L \cong \Delta_K \cdot \Delta_L \leq \Delta$ of Lie groups (see Lemma 2.6). As $\Delta_K \approx K$, the line stabilizers have dimension l, and therefore $\dim(\Delta_K \cdot \Delta_L) = 2l$. Since Δ is abelian and connected (Proposition 2.2), it follows according to [10, (93.12)] that $\Delta_K \cdot \Delta_L = \Delta$. Hence Δ_K and Δ_L are connected and so are K and L. □

Theorem 3.2. *If a stable plane $E = (P, \mathcal{L})$ satisfying our general assumptions contains only straight lines, then it is an affine translation plane with point space isomorphic to $\mathbb{R}^{2l}$.*

Proof. Let $\{K_i \mid i \in I\}$ be a representing system of line orbits. First we show that $\{\Delta_{K_i} \mid i \in I\}$ is a fibration of the Lie group Δ (that is, each $\delta \in \Delta$, $\delta \neq 1$ is contained in exactly one Δ_{K_i}).

From Lemma 3.1 we know that $\Delta_{K_i} \cap \Delta_{K_j} = \{1\}$ holds for all $i \neq j$. On the other hand, let $\delta \in \Delta$. Choose an arbitrary $p \in P$ and set $K := p \vee p^\delta$. Since there are only straight lines, there exists an $i \in I$ satisfying $\Delta_K = \Delta_{K_i}$. Moreover, $\delta \in \Delta_K$, because we have $p, p^\delta \in K$, and according to Lemma 2.6, there exists a $\gamma \in \Delta_K$ such that $p^\gamma = p^\delta$. The sharply transitive action of Δ gives us that $\gamma = \delta$.

From the existence of a non-trivial fibration of the connected Lie group Δ it follows according to [9] that the center of Δ is compact-free. Thus, Δ being abelian is compact-free and with Proposition 2.2 we get $\Delta \cong \mathbb{R}^n$ with $n = \dim P = 2l$. In particular, $P \approx \Delta$ can be made into a real vector space, and each line (being connected according to Lemma 3.1) is an affine subspace of dimension l.

Let $G \in \mathcal{L}$ and $q \notin G$. By Lemma 2.7, there exists exactly one element G^δ in the orbit G^Δ with $q \in G^\delta$. If we show that any other line passing through q intersects G, we have identified G^δ as the unique parallel of G through q and hence obtained an affine plane. Let $L \in \mathcal{L} \setminus G^\Delta$. All lines being affine subspaces of a vector space, we can write lines as solutions of linear systems of equations $G = \{x \in \mathbb{R}^{2l} \mid Ax = a\}$ and $L = \{x \in \mathbb{R}^{2l} \mid Bx = b\}$, where $A, B \in \mathbb{R}^{l \times 2l}$ are matrices of rank l and $a, b \in \mathbb{R}^l$. A point of intersection of G and L can be found by solving the system $G \cap L = \left\{ x \in \mathbb{R}^{2l} \;\middle|\; \begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} a \\ b \end{pmatrix} \right\}$. The associated homogeneous system has only the trivial solution because of $G^\Delta \neq L^\Delta$. Therefore we have $\operatorname{rg}\begin{pmatrix} A \\ B \end{pmatrix} = 2l$ and each inhomogeneous system has a solution, hence G and L intersect.

Thus we have an affine translation plane as desired. □

Remark 3.3. In the proof of the parallel axiom above only basic arguments of linear algebra are used. This is possible because of the invariance of the line system under vector space translation. Another approach to affine planes is given in [8], where this invariance is not part of the assumptions (which only require that the point set is a vector space and the lines are affine subspaces), but turns out to be equivalent to the axiom of parallels (see [8, (1.2)]).

In view of the result from Theorem 3.2, in the following we concentrate on the case that there is at least one arc orbit in our stable plane.

Lemma 3.4. *Let K be a straight line and L an arc. For $\delta \in \Delta_K$, $\delta \neq 1$ we have $L \cap L^\delta = \emptyset$.*

Proof. Assume, by way of contradiction, that there exists a $q \in L^\delta \cap L$. This means that there is an $r \in L$ such that $r^\delta = q$. Since $\delta \neq 1$, the line L is uniquely determined by its points q and $r \neq q$. But according to Lemma 2.6, the set r^{Δ_K} is a line of the orbit K^Δ (hence $r^{\Delta_K} \neq L$) also joining q and r, a contradiction. □

Lemma 3.5. *If there is exactly one straight orbit, then there is also exactly one arc orbit and each arc is homeomorphic to $\mathbb{R}^l$.*

Proof. Let K be a straight line and let $p \in K$. It follows from Proposition 2.8 that $\mathcal{L}_p \setminus \{K\} \approx \mathbb{R}^l$ is connected. According to Lemma 2.7, we have $\{K\} = K^\Delta \cap \mathcal{L}_p$, thus $\mathcal{L}_p \setminus \{K\}$ contains only arcs. By Lemma 2.5, the set $L^\Delta \cap \mathcal{L}_p$ is open in $\mathcal{L}_p \setminus \{K\}$ for each arc L. We have obtained an open disjoint cover of the connected set $\mathcal{L}_p \setminus \{K\}$; this shows that there exists exactly one arc orbit, say L^Δ. Again by Lemma 2.5, we have $L \approx L^\Delta \cap \mathcal{L}_p = \mathcal{L}_p \setminus \{K\} \approx \mathbb{R}^l$. □

In the special case of affine planes containing arcs, we obtain the following result.

Theorem 3.6. *Let $E = (P, \mathcal{L})$ be a stable plane satisfying our general assumptions which is affine and contains at least one arc orbit. Then E is a shift plane. In particular, the following conditions hold.*

1. *There is exactly one straight orbit K^Δ.*
2. *There is exactly one arc orbit L^Δ.*
3. *Each line is homeomorphic to $\mathbb{R}^l$.*
4. *For all $\delta \in \Delta$, we have $K^\delta \cap L \neq \emptyset$.*
5. *The point space is homeomorphic to $\mathbb{R}^{2l}$.*

Proof. We begin by proving the conditions 1 – 5. Let L be an arc and let K_1 be a straight line (existence from Theorem 2.4). We assume, by way of contradiction, that there is a shift (that is, an image under the action of an element of Δ) of K_1 which does not intersect L. For simplicity we assume that $K_1 \cap L = \emptyset$ already holds. Choose $s \in L$ (which implies $s \notin K_1$) and $\gamma \in \Delta$ such that $s \in K_1^\gamma$. Since $s \notin K_1$, we have $K_1^\gamma \neq K_1$ and therefore (see Lemma 2.7) $K_1^\gamma \cap K_1 = \emptyset$. Thus, K_1^γ and L are two different parallels of K_1 passing through s, which contradicts the axiom of parallels.

We assume, again by way of contradiction, that there is another straight line K_2 such that $K_1^\Delta \neq K_2^\Delta$. We know from above that $K_i^\delta \cap L \neq \emptyset$ holds for each $\delta \in \Delta$ and $i = 1, 2$. We choose a point $p \notin L$ and we shift K_1 and K_2 such that the shifted lines contain p. For simplicity we assume that $p \in K_1, K_2$ already holds. As both lines intersect L, let $K_i \cap L =: k_i$. Further, let δ_i denote the element of Δ for which $k_i^{\delta_i} = p$ holds for $i = 1, 2$. According to Lemma 2.6, we have $\delta_i \in \Delta_{K_i}$ and $\delta_i \neq 1$, because of $k_i \neq p$, $i = 1, 2$. Lemma 3.4 shows that the lines L^{δ_1} and L^{δ_2} are two parallels of L passing through p contrary to the axiom of parallels; they are different because otherwise we would obtain that $\delta_1 = \delta_2$, $\Delta_{K_1} = \Delta_{K_2}$ (by Lemma 3.1) and $K_1^\Delta = K_2^\Delta$, contrary to our assumptions.

Thus, conditions 1 and 4 are proved; we set $K := K_1$. By applying Lemma 3.5, we get 2 and $L \approx \mathbb{R}^l$. Now choose a point $q \notin K$. We define the mapping $\Upsilon : K \to \mathcal{L}_q\,,\, x \mapsto x \vee q$. Then Υ is continuous and injective, hence also open (as K and $\mathcal{L}_q$ are l-manifolds). Since K^Δ is the only straight orbit, the image $\Upsilon(K)$ contains only arcs by Lemma 2.7, thus $\Upsilon(K) \subseteq L^\Delta \cap \mathcal{L}_q = \mathcal{L}_q \setminus \{K_q\}$, where K_q is the shift of K passing through q. On the other hand, by condition 4, each element of $\mathcal{L}_q \setminus \{K_q\}$ intersects K. This shows $\Upsilon(K) = \mathcal{L}_q \setminus \{K_q\}$ and therefore $K \approx \mathbb{R}^l$. This proves 3.

From [7, Corollary 10] we get that $P \approx \mathbb{R}^{2l}$. We can coordinatize P such that $K = \{0\} \times \mathbb{R}^l$ becomes the y-axis and a vector space complement W of K becomes the x-axis $W = \mathbb{R}^l \times \{0\}$. Thus, L can be considered as the graph of a function $f : W \supseteq U \to K$. Condition 4 shows $U = W$, hence f becomes a shift function of a shift plane (i.e., the lines are $\{c\} \times \mathbb{R}^l$ for all $c \in \mathbb{R}^l$, together with $s + \mathrm{graph} f$ for all $s \in \mathbb{R}^{2l}$ and a shift function $f : \mathbb{R}^l \to \mathbb{R}^l$) as desired. □

Remark 3.7. It is also possible to prove 5 more directly using 3 and 4 as follows.

For $p \in P$, let K_p be the unique shift of K passing through p. The mapping $p \mapsto K_p$ is injective and continuous by [6, (1.1)]. For the arc L we define L_p as the shift L^δ passing through p with $\delta \in \Delta_K$. The mapping $p \mapsto L_p$ is well-defined by 4 together with Lemma 2.6 and also injective and continuous. Again according to 4, the mapping $P \to K \times L\,,\, p \mapsto (L_p \wedge K, K_p \wedge L)$ is well-defined. It is also continuous and bijective (with the inverse $(a,b) \mapsto L_a \wedge K_b$), hence a homeomorphism. Thus, by 3, we get that $P \approx \mathbb{R}^{2l}$.

Corollary 3.8. *A stable plane satisfying our general assumptions is affine if and only if it is either an affine translation plane or a shift plane. For $l > 2$ only the case of translation planes is left.*

Proof. Shift planes and translation planes are affine. The converse follows from the Theorems 3.2 and 3.6. According to [10, (74.6)], shift planes only exist for $l \in \{1, 2\}$. □

Non-affine examples of stable planes satisfying our assumptions include the arc planes studied by H. Groh in [1] with abelian group action ($\Delta = \mathbb{R}^2$); e.g., take $P = \mathbb{R}^2$ as point space, and as lines take all shifts $s + L\,,\, s \in \mathbb{R}^2$, with $L = \{(x, e^x) \mid x \in \mathbb{R}\}$ or $L = \{(0, y) \mid y \in \mathbb{R}\}$ or $L = \{(x, ax) \mid x \in \mathbb{R}\}$ for some $a \le 0$. Groh gives a complete description of the possibilities. Therefore it would be interesting to see if examples can also be found for $P = \mathbb{R}^{2l}$ with $l \in \{2, 4, 8\}$. However, this turns out to be very hard, and up to now no such examples are known. On the other hand, if one could prove their non-existence, each stable plane with point set $\mathbb{R}^4, \mathbb{R}^8$ or $\mathbb{R}^{16}$, satisfying our assumptions about the automorphism group would be affine.

The existence problem for arc planes of higher dimension will be discussed in detail in a subsequent paper.

References

[1] **H. Groh**, $\mathbb{R}^2$-planes with 2-dimensional point transitive automorphism group, *Abh. Math. Sem. Univ. Hamburg* **48** (1979), 171–202.

[2] **T. Grundhöfer** and **R. Löwen**, Linear topological geometries, **in** *Handbook of incidence geometry*, F. Buekenhout (ed.), Amsterdam, North Holland (1995).

[3] **N. Knarr**, 4-dimensionale projektive Ebenen mit großer abelscher Kollineationsgruppe, *J. Geom.* **31** (1988), 114–124.

[4] **H. Löwe**, Shear planes, *Geom. Dedicata* **52** (1994), 87–104.

[5] **R. Löwen**, Vierdimensionale stabile Ebenen, *Geom. Dedicata* **5** (1976), 239–294.

[6] ______, Halbeinfache Automorphismengruppen von vierdimensionalen stabilen Ebenen sind quasi-einfach, *Math. Ann.* **236** (1978), 15–28.

[7] ______, Topology and dimension of stable planes: On a conjecture of H. Freudenthal, *J. Reine Angew. Math.* **343** (1983), 108–122.

[8] **R. Löwen**, **G. Steinke** and **H. Van Maldeghem**, Affine line systems in real vector spaces, *Adv. Geom.*, special issue dedicated to Adriano Barlotti (2003), 59–74.

[9] **P. Plaumann** and **K. Strambach**, Partitionen Liescher und algebraischer Gruppen, *Forum Math.* **2** (1990), 523–578.

[10] **H. Salzmann**, **D. Betten**, **T. Grundhöfer**, **H. Hähl**, **R. Löwen** and **M. Stroppel**, Compact Projective Planes, Walter de Gruyter, Berlin, New York, 1995.

[11] **M. Stroppel**, Quaternion Hermitian planes, *Results Math.* **23** (1993), 387–397.

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