Base subsets of the Hilbert Grassmannian

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Abstract

Let $H$ be a separable Hilbert space. We consider the Hilbert Grassmannian $G_\infty(H)$ consisting of closed subspaces having infinite dimension and codimension and show that every bijective transformation of $G_\infty(H)$ preserving the class of base subsets is induced by an element of $\text{GL}(H)$ or it is the composition of the transformation induced by an element of $\text{GL}(H)$ and the bijection sending a subspace to its orthogonal complement.

Keywords: Hilbert Grassmannian, base subset, infinite-dimensional topological projective space

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1. Introduction

We start from the classical Grassmannian $G_k(V)$ consisting of all $k$-dimensional subspaces of an $n$-dimensional vector space $V$. A base subset of $G_k(V)$ is the set of $\binom{n}{k}$ distinct $k$-dimensional subspaces spanned by vectors of a certain base of $V$. This construction is closely related with Tits buildings [5]: the Grassmannians $G_k(V)$ ($k = 1, \ldots, n - 1$) are the shadow spaces of the building associated with the group $\text{GL}(V)$ and their base subsets are the shadows of the corresponding apartments. It was proved in [2] that transformations of $G_k(V)$ preserving the class of base subsets are induced by a semilinear isomorphism of $V$ to itself or to $V^*$ (the second possibility can be realized only for the case when $n = 2k$); a more general result can be found in [3].

Suppose that $V$ is an infinite-dimensional vector space and $\dim V = \aleph$. The group $\text{GL}(V)$ acts on the set $S$ of all subspaces of $V$ and the orbits of this action are called Grassmannians. There are the following three types of Grassmannians:

$$G_\alpha(V) := \{ S \in S \mid \dim S = \alpha, \ \text{codim} S = \aleph \}$$
As in the finite-dimensional case, we define base subsets of Grassmannians. It was shown in [4] that every bijective transformation of $G_\alpha(V)$, $\alpha < \aleph$ is induced by a semilinear automorphism of $V$. The methods of [4] cannot be applied to other Grassmannians; this is related with the fact that $V$ and $V^*$ have different dimensions and the duality principles do not hold for infinite-dimensional vector spaces.

Now suppose that $V$ is a topological vector space. Denote by $\Pi_V$ the projective space associated with $V$. The topology of $V$ induces a topology on $\Pi_V$ and we talk about an infinity-dimensional topological projective space. A collineation of $\Pi_V$ to itself will be called closed if it preserves the class of closed subspaces. It follows from Mackey’s results [1] that every closed collineation of $\Pi_V$ to itself is induced by an invertible bounded linear operator if $V$ is a real normed space. In the general case, closed collineations are not determined.

In the present note we give a geometrical characterization of closed collineations of the projective space over a separable Hilbert space.

2. Result

Let $H$ be a separable Hilbert space (real or complex). We write $G_\infty(H)$ for the Hilbert Grassmannian consisting of all closed subspaces with infinite dimension and codimension. Let $B = \{x_i\}_{i \in \mathbb{N}}$ be a base of $H$ (possibly non-orthogonal). The set of all elements of $G_\infty(H)$ spanned by subsets of $B$ is called the base subset of $G_\infty(H)$ associated with (or defined by) the base $B$.

Every closed collineation of $\Pi_H$ to itself induces a bijective transformation of $G_\infty(H)$ preserving the class of base subsets. The bijection

$$\circ : G_\infty(H) \to G_\infty(H)$$

sending subspaces to their orthogonal complements also preserves the class of base subsets.

**Theorem 2.1.** If $f$ is a bijective transformation of $G_\infty(H)$ preserving the class of base subsets, then $f$ is induced by a closed collineation of $\Pi_H$ to itself or it is the composition of the transformation $\circ$ and the transformation induced by a closed collineation of $\Pi_H$ to itself.
3. Preliminaries

Let \( B = \{ x_i \}_{i \in \mathbb{N}} \) be base of \( H \), and let \( \mathcal{B} \) be the associated base subset of \( G_{\infty}(H) \). We denote by \( P_i \) the 1-dimensional subspace containing \( x_i \) and write \( \mathcal{B}_{+i} \) and \( \mathcal{B}_{-i} \) for the sets of all elements of \( \mathcal{B} \) which contain \( P_i \) or do not contain \( P_i \), respectively. Then \( \mathcal{B}_{-i} \) consists of all elements of \( \mathcal{B} \) contained in the hyperplane

\[
S_i := \overline{B \setminus \{x_i\}}.
\]

We say that \( X \subset \mathcal{B} \) is an exact subset if there is only one base subset of \( G_{\infty}(H) \) containing \( X \); otherwise, \( X \) is said to be inexact.

Remark 3.1. It is trivial that \( X \) is exact if and only if for each \( i \in \mathbb{N} \) the intersection of all \( U \in X \) containing \( P_i \) coincides with \( P_i \). For example,

\[
\mathcal{R}_{ij} := (\mathcal{B}_{+i} \cap \mathcal{B}_{+j}) \cup \mathcal{B}_{-i}, \quad i \neq j
\]

is an inexact subset. Indeed, the intersection of all elements containing \( P_k \) coincides with \( P_k \) if \( k \neq i \), however for \( k = i \) this intersection is \( P_i + P_j \). Every element of \( \mathcal{B} \setminus \mathcal{R}_{ij} \) intersects \( P_i + P_j \) by \( P_i \). This means that

\[
\mathcal{R}_{ij} \cup \{U\}
\]

is exact and the inexact subset \( \mathcal{R}_{ij} \) is maximal. Conversely, every maximal inexact subset of \( \mathcal{B} \) coincides with certain \( \mathcal{R}_{ij} \) (the proof of this fact is similar to the proof of Lemma 1 in [4]).

Let \( \mathcal{B}' \) be the base subset of \( G_{\infty}(H) \) defined by a base \( \mathcal{B}' = \{ x'_i \}_{i \in \mathbb{N}} \). We write \( \mathcal{B}'_{+i} \) and \( \mathcal{B}'_{-i} \) for the sets of all elements of \( \mathcal{B}' \) which contain \( P'_i \) or do not contain \( P'_i \) (respectively); here \( P'_i \) is the 1-dimensional subspace containing \( x'_i \). We also define

\[
S'_i := \overline{B' \setminus \{x'_i\}}.
\]

A bijection \( g : \mathcal{B} \to \mathcal{B}' \) is called special if \( g \) and \( g^{-1} \) map inexact subsets to inexact subsets.

**Lemma 3.2.** If \( g : \mathcal{B} \to \mathcal{B}' \) is a special bijection then there exists a bijective transformation \( \delta : \mathbb{N} \to \mathbb{N} \) such that

\[
g(\mathcal{B}_{+i}) = \mathcal{B}'_{+\delta(i)}, \quad g(\mathcal{B}_{-i}) = \mathcal{B}'_{-\delta(i)} \quad \forall \ i \in \mathbb{N}
\]

or

\[
g(\mathcal{B}_{+i}) = \mathcal{B}'_{-\delta(i)}, \quad g(\mathcal{B}_{-i}) = \mathcal{B}'_{+\delta(i)} \quad \forall \ i \in \mathbb{N}.
\]

**Proof.** This is similar to the proof of Lemma 3 in [4].
We say that a special bijection $g : B \to B'$ is of first type if the first equality of Lemma 3.2 holds; otherwise, $g$ is said to be of second type.

**Lemma 3.3.** Let $g$ and $\delta$ be as in the previous lemma. Let also $S, U \in B$. If $g$ is of first type then

$$S \cap U = P_i \Leftrightarrow g(S) \cap g(U) = P'_{\delta(i)} \quad \text{and} \quad S + U = S_i \Leftrightarrow g(S) + g(U) = S'_{\delta(i)}.$$ 

If $g$ is of second type then

$$S \cap U = P_i \Leftrightarrow g(S) + g(U) = S'_{\delta(i)} \quad \text{and} \quad S + U = S_i \Leftrightarrow g(S) \cap g(U) = P'_{\delta(i)}.$$ 

**Proof.** The equality $S \cap U = P_i$ holds if and only if $S, U$ belong to $B_{+i}$ and there is no $j \neq i$ such that $B_{+j}$ contains both $S, U$. Similarly, we have $S + U = S_i$ if and only if $S, U$ belong to $B_{-i}$ and there is no $j \neq i$ such that $B_{-j}$ contains both $S, U$. Lemma 3.2 gives the claim. 

For every 1-dimensional subspace $P$ we denote by $[P]$ the set of all elements of $G_\infty(H)$ containing $P$. If $S$ is a closed hyperplane then we write $[S]$ for the set of all elements of $G_\infty(H)$ contained in $S$.

**Lemma 3.4.** For any $i \in \mathbb{N}$ and any $U \in [P_i] \setminus \mathcal{B}$ there exist $M, N \in B_{+i}$ such that

$$M \cap N = P_i, \quad \text{codim} M + N > 1,$$

and $M, N, U$ are contained in a certain base subset of $G_\infty(H)$.

**Proof.** Let $\{U_j\}_{j \in X} \subset \mathbb{N}$ be a countable collection of elements of $\mathcal{B}$ such that

$$U_k \cap U_m = 0 \quad \text{if} \quad k \neq n.$$ 

For every $j \in X$ we choose $P_j \in U_j \setminus U$ (this is possible, since $U \neq U_j$) and denote by $T$ the element of $\mathcal{B}$ spanned by $P_i$ and all $P_j, j \in X$. Then $U \cap T = P_i$. We take any $M, N \in B_{+i}$ contained in $T$ and such that $M \cap N = P_i$. These subspaces are as required. 

Since $\circ|_B : B \to \circ(B)$ is a special bijection of second type, we have the following dual version of Lemma 3.4.

**Lemma 3.5.** For any $i \in \mathbb{N}$ and any $U \in [S_i] \setminus \mathcal{B}$ there exist $M, N \in B_{-i}$ such that

$$M + N = S_i, \quad \text{dim} M \cap N > 1,$$

and $M, N, U$ are contained in a certain base subset of $G_\infty(H)$. 


4. Proof of the theorem

Let \( f \) be a bijective transformation of \( G_\infty(H) \) preserving the class of base subsets. We consider an arbitrary base subset \( B \subset G_\infty(H) \). Then \( B' := f(B) \) is a base subset and \( f|_B : B \to B' \) is a special bijection. Suppose that \( B = \{x_i\}_{i \in \mathbb{N}} \) and \( B' = \{x'_i\}_{i \in \mathbb{N}} \) are bases associated with \( B \) and \( B' \), respectively. Let \( P_i, P'_i, S_i, S'_i \) and \( \delta : \mathbb{N} \to \mathbb{N} \) be as in the previous section. It is clear that we can assume that \( \delta \) is identical. We have to consider the following possibilities:

(A) \( f|_B \) is a special bijection of first type,

(B) \( f|_B \) is a special bijection of second type.

Case (A). First we claim that

\[
f([P_i]) = [P'_i] \quad \forall \, i \in \mathbb{N}.
\]

Let \( U, M, N \) be as in Lemma 3.4. By Lemma 3.3,

\[
f(M) \cap f(N) = P'_i, \quad \operatorname{codim} f(M) + \operatorname{codim} f(N) > 1. \tag{1}
\]

If \( \tilde{B} \) is a base subset containing \( U, M, N \) then (1) together with Lemma 3.3 show that the restriction of \( f \) to \( \tilde{B} \) is a special bijection of first type. Bases associated with \( \tilde{B} \) and its \( f \)-image contain vectors lying in \( P_i \) and \( P'_i \) (respectively), and Lemma 3.2 implies that \( f(U) \) belongs to \( [P'_i] \). Thus \( f([P_i]) \subset [P'_i] \). Since \( f^{-1} \) preserves the class of base subsets, we can also prove the inverse inclusion. The claim follows.

Using Lemma 3.5 we show that

\[
f([S_i]) = [S'_i] \quad \forall \, i \in \mathbb{N}.
\]

For any 1-dimensional subspace \( P \) there exists \( S_i \) which do not contain \( P \). We consider a base \( \tilde{B} \) of \( H \) such that each vector of \( \tilde{B} \) is contained in \( S_i \) or \( P \). The latter equality guarantees that the restriction of \( f \) to the associated base subset is a special bijection of first type, and arguments given above imply the existence of a 1-dimensional subspace \( P' \) such that \( f([P]) = [P'] \).

Similarly, for every closed hyperplane \( S \) we choose \( P_i \notin S \) and establish the existence of a closed hyperplane \( S' \) such that \( f([S]) = [S'] \).

Therefore, \( f \) gives a closed collineation of \( \Pi_H \). This collineation induces \( f \).

Case (B). Since \( \circ f|_B \) is a special bijection of first type, \( \circ f \) is induced by a closed collineation.
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