Base subsets of the Hilbert Grassmannian

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Abstract

Let H be a separable Hilbert space. We consider the Hilbert Grassmannian $\mathcal{G}_\infty(H)$ consisting of closed subspaces having infinite dimension and codimension and show that every bijective transformation of $\mathcal{G}_\infty(H)$ preserving the class of base subsets is induced by an element of $\mathrm{GL}(H)$ or it is the composition of the transformation induced by an element of $\mathrm{GL}(H)$ and the bijection sending a subspace to its orthogonal complement.

`Keywords:` Hilbert Grassmannian, base subset, infinite-dimensional topological projective space
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1. Introduction

We start from the classical Grassmannian $\mathcal{G}_k(V)$ consisting of all k-dimensional subspaces of an n-dimensional vector space V. A base subset of $\mathcal{G}_k(V)$ is the set of $\binom{n}{k}$ distinct k-dimensional subspaces spanned by vectors of a certain base of V. This construction is closely related with Tits buildings [5]: the Grassmannians $\mathcal{G}_k(V)$ ($k = 1, \dots, n-1$) are the shadow spaces of the building associated with the group $\mathrm{GL}(V)$ and their base subsets are the shadows of the corresponding apartments. It was proved in [2] that transformations of $\mathcal{G}_k(V)$ preserving the class of base subsets are induced by a semilinear isomorphism of V to itself or to V^* (the second possibility can be realized only for the case when $n = 2k$); a more general result can be found in [3].

Suppose that V is an infinite-dimensional vector space and $\dim V = \aleph$. The group $\mathrm{GL}(V)$ acts on the set $\mathcal{S}$ of all subspaces of V and the orbits of this action are called *Grassmannians*. There are the following three types of Grassmannians:

$$\mathcal{G}_\alpha(V) := \{\, S \in \mathcal{S} \mid \dim S = \alpha, \;\; \operatorname{codim} S = \aleph \,\}$$

$$\mathcal{G}^{\alpha}(V) := \{\, S \in \mathcal{S} \mid \dim S = \aleph, \;\; \operatorname{codim} S = \alpha \,\}$$

for any cardinality $\alpha < \aleph$ and

$$\mathcal{G}_{\aleph}(V) := \{\, S \in \mathcal{S} \mid \dim S = \operatorname{codim} S = \aleph \,\}.$$

As in the finite-dimensional case, we define base subsets of Grassmannians. It was shown in [4] that every bijective transformation of $\mathcal{G}_{\alpha}(V)$, $\alpha < \aleph$ is induced by a semilinear automorphism of V. The methods of [4] can not be applied to other Grassmannians; this is related with the fact that V and V^* have different dimensions and the duality principles do not hold for infinite-dimensional vector spaces.

Now suppose that V is a topological vector space. Denote by Π_V the projective space associated with V. The topology of V induces a topology on Π_V and we talk about an *infinity-dimensional topological projective space*. A collineation of Π_V to itself will be called *closed* if it preserves the class of closed subspaces. It follows from Mackey's results [1] that every closed collineation of Π_V to itself is induced by an invertible bounded linear operator if V is a real normed space. In the general case, closed collineations are not determined.

In the present note we give a geometrical characterization of closed collineations of the projective space over a separable Hilbert space.

2. Result

Let H be a separable Hilbert space (real or complex). We write $\mathcal{G}_{\infty}(H)$ for the Hilbert Grassmannian consisting of all closed subspaces with infinite dimension and codimension. Let $B = \{x_i\}_{i\in\mathbb{N}}$ be a base of H (possibly non-orthogonal). The set of all elements of $\mathcal{G}_{\infty}(H)$ spanned by subsets of B is called the *base subset* of $\mathcal{G}_{\infty}(H)$ associated with (or defined by) the base B.

Every closed collineation of Π_H to itself induces a bijective transformation of $\mathcal{G}_{\infty}(H)$ preserving the class of base subsets. The bijection

$$\circ : \mathcal{G}_{\infty}(H) \to \mathcal{G}_{\infty}(H)$$

sending subspaces to their orthogonal complements also preserves the class of base subsets.

Theorem 2.1. *If f is a bijective transformation of $\mathcal{G}_{\infty}(H)$ preserving the class of base subsets, then f is induced by a closed collineation of Π_H to itself or it is the composition of the transformation $\circ$ and the transformation induced by a closed collineation of Π_H to itself.*

3. Preliminaries

Let $B = \{x_i\}_{i\in\mathbb{N}}$ be base of H, and let $\mathcal{B}$ be the associated base subset of $\mathcal{G}_\infty(H)$. We denote by P_i the 1-dimensional subspace containing x_i and write $\mathcal{B}_{+i}$ and $\mathcal{B}_{-i}$ for the sets of all elements of $\mathcal{B}$ which contain P_i or do not contain P_i, respectively. Then $\mathcal{B}_{-i}$ consists of all elements of $\mathcal{B}$ contained in the hyperplane

$$S_i := \overline{B \setminus \{x_i\}}.$$

We say that $\mathcal{X} \subset \mathcal{B}$ is an *exact* subset if there is only one base subset of $\mathcal{G}_\infty(H)$ containing $\mathcal{X}$; otherwise, $\mathcal{X}$ is said to be *inexact*.

Remark 3.1. It is trivial that $\mathcal{X}$ is exact if and only if for each $i \in \mathbb{N}$ the intersection of all $U \in \mathcal{X}$ containing P_i coincides with P_i. For example,

$$\mathcal{R}_{ij} := (\mathcal{B}_{+i} \cap \mathcal{B}_{+j}) \cup \mathcal{B}_{-i} \quad i \neq j$$

is an inexact subset. Indeed, the intersection of all elements containing P_k coincides with P_k if $k \neq i$, however for $k = i$ this intersection is $P_i + P_j$. Every element of $\mathcal{B} \setminus \mathcal{R}_{ij}$ intersects $P_i + P_j$ by P_i. This means that

$$\mathcal{R}_{ij} \cup \{U\}$$

is exact and the inexact subset $\mathcal{R}_{ij}$ is maximal. Conversely, every maximal inexact subset of $\mathcal{B}$ coincides with certain $\mathcal{R}_{ij}$ (the proof of this fact is similar to the proof of Lemma 1 in [4]).

Let $\mathcal{B}'$ be the base subset of $\mathcal{G}_\infty(H)$ defined by a base $B' = \{x'_i\}_{i\in\mathbb{N}}$. We write $\mathcal{B}'_{+i}$ and $\mathcal{B}'_{-i}$ for the sets of all elements of $\mathcal{B}'$ which contain P'_i or do not contain P'_i (respectively); here P'_i is the 1-dimensional subspace containing x'_i. We also define

$$S'_i := \overline{B' \setminus \{x'_i\}}.$$

A bijection $g : \mathcal{B} \to \mathcal{B}'$ is called *special* if g and g^{-1} map inexact subsets to inexact subsets.

Lemma 3.2. *If $g : \mathcal{B} \to \mathcal{B}'$ is a special bijection then there exists a bijective transformation $\delta : \mathbb{N} \to \mathbb{N}$ such that*

$$g(\mathcal{B}_{+i}) = \mathcal{B}'_{+\delta(i)}, \quad g(\mathcal{B}_{-i}) = \mathcal{B}'_{-\delta(i)} \quad \forall\, i \in \mathbb{N}$$

or

$$g(\mathcal{B}_{+i}) = \mathcal{B}'_{-\delta(i)}, \quad g(\mathcal{B}_{-i}) = \mathcal{B}'_{+\delta(i)} \quad \forall\, i \in \mathbb{N}.$$

Proof. This is similar to the proof of Lemma 3 in [4]. □

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We say that a special bijection $g : \mathcal{B} \to \mathcal{B}'$ is *of first type* if the first equality of Lemma 3.2 holds; otherwise, g is said to be *of second type*.

Lemma 3.3. *Let g and δ be as in the previous lemma. Let also $S, U \in \mathcal{B}$. If g is of first type then*

$$S \cap U = P_i \Leftrightarrow g(S) \cap g(U) = P'_{\delta(i)} \ \text{ and } \ \overline{S+U} = S_i \Leftrightarrow \overline{g(S)+g(U)} = S'_{\delta(i)}.$$

If g is of second type then

$$S \cap U = P_i \Leftrightarrow \overline{g(S)+g(U)} = S'_{\delta(i)} \ \text{ and } \ \overline{S+U} = S_i \Leftrightarrow g(S) \cap g(U) = P'_{\delta(i)}.$$

Proof. The equality $S \cap U = P_i$ holds if and only if S, U belong to $\mathcal{B}_{+i}$ and there is no $j \neq i$ such that $\mathcal{B}_{+j}$ contains both S, U. Similarly, we have $\overline{S+U} = S_i$ if and only if S, U belong to $\mathcal{B}_{-i}$ and there is no $j \neq i$ such that $\mathcal{B}_{-j}$ contains both S, U. Lemma 3.2 gives the claim. □

For every 1-dimensional subspace P we denote by $[P]$ the set of all elements of $\mathcal{G}_\infty(H)$ containing P. If S is a closed hyperplane then we write $[S]$ for the set of all elements of $\mathcal{G}_\infty(H)$ contained in S.

Lemma 3.4. *For any $i \in \mathbb{N}$ and any $U \in [P_i] \setminus \mathcal{B}$ there exist $M, N \in \mathcal{B}_{+i}$ such that*

$$M \cap N = P_i, \ \operatorname{codim} \overline{M+N} > 1,$$

and M, N, U are contained in a certain base subset of $\mathcal{G}_\infty(H)$.

Proof. Let $\{U_j\}_{j \in X \subset \mathbb{N}}$ be a countable collection of elements of $\mathcal{B}$ such that

$$U_k \cap U_m = 0 \ \text{ if } \ k \neq n.$$

For every $j \in X$ we choose $P_j \in U_j \setminus U$ (this is possible, since $U \neq U_j$) and denote by T the element of $\mathcal{B}$ spanned by P_i and all P_j, $j \in X$. Then $U \cap T = P_i$. We take any $M, N \in \mathcal{B}_{+i}$ contained in T and such that $M \cap N = P_i$. These subspaces are as required. □

Since $\circ|_{\mathcal{B}} : \mathcal{B} \to \circ(\mathcal{B})$ is a special bijection of second type, we have the following dual version of Lemma 3.4.

Lemma 3.5. *For any $i \in \mathbb{N}$ and any $U \in [S_i] \setminus \mathcal{B}$ there exist $M, N \in \mathcal{B}_{-i}$ such that*

$$\overline{M+N} = S_i, \ \dim M \cap N > 1,$$

and M, N, U are contained in a certain base subset of $\mathcal{G}_\infty(H)$.

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4. Proof of the theorem

Let f be a bijective transformation of $\mathcal{G}_\infty(H)$ preserving the class of base subsets. We consider an arbitrary base subset $\mathcal{B} \subset \mathcal{G}_\infty(H)$. Then $\mathcal{B}' := f(\mathcal{B})$ is a base subset and $f|_\mathcal{B} : \mathcal{B} \to \mathcal{B}'$ is a special bijection. Suppose that $B = \{x_i\}_{i\in\mathbb{N}}$ and $B' = \{x'_i\}_{i\in\mathbb{N}}$ are bases associated with $\mathcal{B}$ and $\mathcal{B}'$, respectively. Let P_i, P'_i, S_i, S'_i and $\delta : \mathbb{N} \to \mathbb{N}$ be as in the previous section. It is clear that we can assume that δ is identical. We have to consider the following possibilities:

(A) $f|_\mathcal{B}$ is a special bijection of first type,

(B) $f|_\mathcal{B}$ is a special bijection of second type.

Case (A). First we claim that

$$f([P_i]) = [P'_i] \quad \forall\, i \in \mathbb{N}.$$

Let U, M, N be as in Lemma 3.4. By Lemma 3.3,

$$f(M) \cap f(N) = P'_i,\ \operatorname{codim} \overline{f(M) + f(N)} > 1. \tag{1}$$

If $\widetilde{\mathcal{B}}$ is a base subset containing U, M, N then (1) together with Lemma 3.3 show that the restriction of f to $\widetilde{\mathcal{B}}$ is a special bijection of first type. Bases associated with $\widetilde{\mathcal{B}}$ and its f-image contain vectors lying in P_i and P'_i (respectively), and Lemma 3.2 implies that $f(U)$ belongs to $[P'_i]$. Thus $f([P_i]) \subset [P'_i]$. Since f^{-1} preserves the class of base subsets, we can also prove the inverse inclusion. The claim follows.

Using Lemma 3.5 we show that

$$f([S_i]) = [S'_i] \quad \forall\, i \in \mathbb{N}.$$

For any 1-dimensional subspace P there exists S_i which do not contain P. We consider a base $\widetilde{B}$ of H such that each vector of $\widetilde{B}$ is contained in S_i or P. The latter equality guarantees that the restriction of f to the associated base subset is a special bijection of first type, and arguments given above imply the existence of a 1-dimensional subspace P' such that $f([P]) = [P']$.

Similarly, for every closed hyperplane S we choose $P_i \not\in S$ and establish the existence of a closed hyperplane S' such that $f([S]) = [S']$.

Therefore, f gives a closed collineation of Π_H. This collineation induces f.

Case (B). Since $\circ f|_\mathcal{B}$ is a special bijection of first type, $\circ f$ is induced by a closed collineation.

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