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About maximal partial 2-spreads in $\mathrm{PG}(3m-1,q)$

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Abstract

In this article we construct maximal partial 2-spreads in $\mathrm{PG}(8,q)$ of deficiency $\delta = (k-1)\cdot q^2$, where $k \le q^2+q+1$ and $\delta = k\cdot q^2 + l\cdot(q^2-1)+1$, where $k+l \le q^2$ and $\delta = (k+1)\cdot q^2 + l\cdot(q^2-1) + m\cdot(q^2-2)+1$, where $k+l+m \le q^2$. Using these results, we also construct maximal partial 2-spreads in $\mathrm{PG}(3m-1,q)$ of various deficiencies for $m \ge 4$.

Keywords: spread, projective space

MSC 2000: 51E23

1 Introduction

A set of t-dimensional subspaces partitioning the points of $\mathrm{PG}(m,q)$ is called a *t-spread*. A $(t+1)$-spread of a linear space is a set of $(t+1)$-dimensional linear subspaces such that projectively they constitute a t-spread. If $t=1$ or $t=2$ one may call it a line-spread or plane-spread, resp. A *partial t-spread* in $\mathrm{PG}(m,q)$ is a set of pairwise disjoint t-dimensional subspaces. A partial t-spread is *maximal* if it is not contained in a larger partial t-spread.

There exists a t-spread in $\mathrm{PG}(m,q)$ if and only if $t+1$ is a divisor of $m+1$. In this case one can define the *deficiency* of a partial t-spread of $\mathrm{PG}(m,q)$ which is the difference of the cardinalities of a t-spread and the partial t-spread considered. For more on spreads, reguli and Segre-varieties the reader can consult [2].

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1.1 Two aspects of a 2-regular Desarguesian 2-spread

A t-spread is called r-regular if whenever some spread-elements together define an (r,t)-regulus then each element of this regulus belongs to the t-spread.

Lavrauw describes in [3] how one can construct r-regular t-spreads by using the $S_{r,t}$ Segre-variety.

We interpret the elements of the finite field $\mathrm{GF}(q^r)$ as r-dimensional vectors over $\mathrm{GF}(q)$. Similarly, $\mathrm{GF}(q^t)$ can be viewed as $\mathrm{GF}(q)^t$. Consider the vector space $V = \mathrm{GF}(q)^r \otimes \mathrm{GF}(q)^t$ of rank rt over $\mathrm{GF}(q)$. For a pure tensor $v \otimes w$ and for $\lambda \in \mathrm{GF}(q^r)$ let $\lambda \cdot (v \otimes w) := (\lambda \cdot v) \otimes w$. For a pure tensor $v \otimes w$ and for $\mu \in \mathrm{GF}(q^t)$ let $(v \otimes w) \cdot \mu := v \otimes (\mu \cdot w)$. The left-multiplication makes V a $\mathrm{GF}(q^r)$-vector space, and the right-multiplication makes V a $\mathrm{GF}(q^t)$-vector space.

For $v \in V$ we define the following subspaces of V:

$$S_r(v) = \{\alpha \cdot v \parallel \alpha \in \mathrm{GF}(q^r)\}\,;$$
$$S_t(v) = \{v \cdot \beta \parallel \beta \in \mathrm{GF}(q^t)\}\,.$$

Lavrauw proves the following

Theorem 1.1. [3]. *The set $S_r = \{S_r(v) \parallel v \in V\}$ is a Desarguesian r-spread of V. The set $S_t = \{S_t(v) \parallel v \in V\}$ is a Desarguesian t-spread of V.* □

So the elements of the 2-regular 2-spread of $\mathrm{PG}(8,q)$ can be considered as points of $\mathrm{PG}(2,q^3)$, this will be called the *plane-representation* of the 2-spread (or briefly *the plane of the spread-elements*) and denoted by $\widehat{\mathcal{S}}$.

Let $\mathcal{S}$ denote the set of the spread-elements as planes in $\mathrm{PG}(8,q)$ (i.e. the conventional representation).

Lemma 1.2. *The $(1,2)$-reguli of $\mathcal{S}$ are the sublines $\simeq \mathrm{PG}(1,q)$ of $\widehat{\mathcal{S}} = \mathrm{PG}(2,q^3)$. The $(2,2)$-reguli in $\mathcal{S}$ are the subplanes $\simeq \mathrm{PG}(2,q)$ of $\widehat{\mathcal{S}} = \mathrm{PG}(2,q^3)$.*

Proof. A $(1,2)$-regulus of $\mathcal{S}$ is a set of $q+1$ spread-elements that constitute the plane-class of an $S_{1,2}$ Segre-variety in $\mathrm{PG}(8,q)$.

A $(2,2)$-regulus of $\mathcal{S}$ is a set of q^2+q+1 spread-elements that constitute either of the two plane-classes of an $S_{2,2}$ Segre-variety in $\mathrm{PG}(8,q)$. □

1.2 Intersections

Two different lines in $\mathrm{PG}(n,q)$ either meet or not. But there are three different ways in which two different planes can intersect each other, which makes the situation more complicated than in the case of line-spreads. Their intersection can be the empty set or a point or a line.

Opposite and irregular planes. Therefore a plane Π not contained in the 2-regular 2-spread can intersect the spread in two ways. Either q^2+q+1 spread-elements meet this plane (each of these spread-elements has only one point in common with Π) or q^2+1 spread-elements intersect this plane (there is one spread-element that has a common line with Π and there are q^2 spread-elements each of which has one common point with Π).

Remark 1.3. A plane that intersects q^2+q+1 spread-elements is an *opposite plane* of one of the reguli of $\mathcal{S}$, so the q^2+q+1 spread-elements which intersect such a plane constitute (in the plane-representation of the 2-spread) a subplane of order q of $\widehat{\mathcal{S}}$. □

Definition 1.4. Let a plane that intersects only q^2+1 spread-elements be called an *irregular plane*. The set of the spread-elements which intersect an irregular plane will be called (in the plane representation) a *club* and the spread-element that meets the irregular plane in a line will be called (in the plane representation) the *head (of this club)*.

Lemma 1.5. *In the plane-representation of the 2-spread, a club constitutes a* $\mathrm{GF}(q)$*-linear set. This linear set is the union of the pointsets of* $q+1$ *different sublines of order* q *of a particular line of* $\widehat{\mathcal{S}}$*.*

Proof. Let Σ denote the irregular plane and let Π denote the spread-element that meet Σ in a line ℓ. For the sake of simplicity, let ℓ be the ideal line. The affine points of Σ represent q^2 distinct spread-elements. (The point P represents the spread-element Γ if and only if $\Gamma\cap\Sigma = P$.)

Let P and Q be two arbitrary affine points in Σ. The affine points of the line PQ represent q spread-elements and the ideal point of PQ belongs to Π. Because of the property of 2-regularity, the spread-elements represented by the affine points of PQ and Π together constitute a $(1,2)$-regulus of $\mathcal{S}$ (i.e. an $S_{1,2}$ Segre-variety in $\mathrm{PG}(8,q)$) that is (in the plane-representation) a subline of order q of $\widehat{\mathcal{S}}$. So the spread-elements intersecting Σ constitute some sublines of order q in $\widehat{\mathcal{S}} = \mathrm{PG}(2,q^3)$.

Choose two lines in Σ which have a common affine point P. The points of these two lines represent two sublines of order q in $\widehat{\mathcal{S}} = \mathrm{PG}(2,q^3)$ which have two common points (one represents the spread-element Π the other one represents the spread-element represented by P). So any two sublines of order q among the above sublines are in the same line of order q^3.

(Note that the plane-spread is Desarguesian, the spread-elements intersecting the affine part of Σ together constitute an affine plane; thus the points of the club, except the head, is an affine pointset.) □

Remark 1.6. A club in $\mathrm{PG}(1,q^3)$ is projectively equivalent to the set of points $\{x \in \mathrm{GF}(q^3) \parallel \mathrm{Tr}(x) = x + x^q + x^{q^2} = 0\} \cup \{\infty\}$, where $\{\infty\}$ denotes the head of the club.

Lemma 1.7. *There are q^2+q+1 irregular planes that belong to the same club.*

Proof. In the plane-representation a club is a subset of a line ℓ (of order q^3); in the conventional representation this line ℓ is a $\mathrm{PG}(5,q)$. In the $\mathrm{PG}(5,q)$ of the club, there are $(q^3+q^2+q+1-1)(q^2+q+1) = q(q^2+q+1)^2$ irregular planes that meet the head in a line (of order q). In the line ℓ (of order q^3) there are $q(q^2+q+1)$ clubs having the same head. □

Definition 1.8. In general, let Δ be a subplane $\mathrm{PG}(2,q)$ in $\mathrm{PG}(2,q^h), h \geq 2$, and let L is a line $\mathrm{PG}(1,q^h)$. In $L = \mathrm{PG}(1,q^h)$, a *club* $\mathcal{C}$ is a projected image of Δ a onto L from a center C which is on an extended line of Δ but not in Δ. Hence $|\mathcal{C}| = q^2+1$ and $\mathcal{C}$ has a special ('multiple') point H called the *head* of the club. By a subline we always mean a $\mathrm{PG}(1,q)$ contained in L.

Obviously the lines of Δ are projected to sublines contained in the club. Each of these sublines will contain the head since their preimages in Δ intersected the line which is the preimage of H.

Lemma 1.9. *If h is odd then there are no other sublines contained in the club. If h is even then there can be other sublines, each of which is the projected image of some conic of Δ:*

- *In this case the club, with the (ordinary and extra) sublines it contains, form a Moebius plane. I.e. for any 3 points of $\mathcal{C}$ there is a unique subline containing them, and this subline is contained in $\mathcal{C}$ as well.*
- *Also in this case every point of the club is equivalent geometrically, any point can play the role of the head.*

Corollary 1.10. *A club and a subline intersect in $0,1,2,3$ or $q+1$ points. In the Moebius case the intersection size 3 does not occur.*

Note that for $h=2$, so for Baer sublines, it is well-known. We could not find a reference for this statement so we give the proof for completeness.

Also note that even if h is even and at least 4, there are clubs which *do not* contain extra sublines, this depends on the center C of projection, see the proof below.

Proof. We use (homogeneous) coordinates. Let $\Delta = \{(x_1,x_2,x_3) \parallel x_1,x_2,x_3 \in \mathrm{GF}(q)\} \setminus \{(0,0,0)\}$, $L = [0,1,0]$, the center of projection $C = (-\omega,1,0)$, so the head is $H=(1,0,0)$ and the projected image of any point $(a,b,1)$ is $(a+b\omega,0,1)$. So the club consists of the head plus a two dimensional vectorspace over $\mathrm{GF}(q)$ contained in $\mathrm{AG}(1,q^h)$. Lets identify $L\setminus H=\mathrm{AG}(1,q^h)$ and $\mathrm{GF}(q^h)$ with $(x,0,1)\mapsto x$. Then this two dimensional vectorspace contains $\mathrm{GF}(q)$ and it is generated over $\mathrm{GF}(q)$ by the 'vectors' 1 and ω.

Consider four non-head points of the club, w.l.o.g. they can be chosen as $0,1,a=a_1+a_2\omega,b=b_1+b_2\omega$. Suppose that they are contained in a subline, it means that their cross-ratio, $y=\dfrac{0-a}{0-b}\cdot\dfrac{1-b}{1-a}$ is in $\mathrm{GF}(q)$. It gives $0=(1-y)ab+yb-a$, substituting $a=a_1+a_2\omega,b=b_1+b_2\omega$ we get a quadratic equation for ω, with coefficients from $\mathrm{GF}(q)$. The coefficient of ω^2 is $(1-y)a_2b_2$. Here $y=1$ would mean $a=b$.

If h is odd then the minimal polynomial of ω cannot be quadratic, hence, using $\omega\notin\mathrm{GF}(q)$, one of a_2 and b_2 is 0, but $0=(1-y)ab+yb-a$ implies that if one of a and b is in $\mathrm{GF}(q)$ then the other is as well. So the four points were contained in the subline $\mathrm{GF}(q)\cup\{H\}$.

So only for $h=$ even can it happen that $a_2b_2\neq 0$. In this case ω gives a quadratic extension of $\mathrm{GF}(q)$ and in fact our two dimensional vectorspace $\{x+y\omega \parallel x,y\in\mathrm{GF}(q)\}$ is the unique $\mathrm{GF}(q^2)$ itself. Now consider the points of the subline through $0,1,a,b$. This consists of the points c for which the cross-ratio y of $0,1,a$ and c is in $\mathrm{GF}(q)\cup\{\infty\}$. We have $y=\dfrac{0-a}{0-c}\cdot\dfrac{1-c}{1-a}$, and the values $y=0,1,\infty$ give $c=1,a,0$. Note that $c=\infty=H$ is impossible as it would imply $y=\dfrac{a}{a-1}$ which is not in $\mathrm{GF}(q)$ unless $a\in\mathrm{GF}(q)$ and the subline is $\mathrm{GF}(q)\cup\{\infty\}$.

So for the subline we have $\left\{c=\dfrac{a}{(1-y)a+y}\;\middle\|\; y\in\mathrm{GF}(q)\cup\{\infty\}\right\}$. But when calculating $c=\dfrac{a}{(1-y)a+y}$, we work within $\mathrm{GF}(q^2)$, so within the club, hence the subline through $0,1,a,b$ will be contained in the club completely.

The previous two paragraphs can be substituted by the argument, that if we know that ω is in the quadratic extension then already the projection can be done within the canonical $\mathrm{PG}(2,q^2)$ and we can use the remark about Baer sublines of $\mathrm{PG}(1,q^2)$.

Finally we remark that the case when the head is one of the four points (Q_1,Q_2,Q_3 and H) is easy to verify: consider $Q_2',Q_3'\in\Delta$, the preimages of Q_2 and Q_3 before the projection, and let Q_4' be that preimage of H collinear with Q_2' and Q_3'. Then suppose that there is a subline in L containing Q_1,Q_2,Q_3 and H. As there is another subline (which is the projected image of the line $Q_2'Q_3'$ of Δ) containing Q_2,Q_3 and H, and the subline through 3 points of L is unique, they must coincide. $\square$

2 Constructions

Let us first consider the set of elements of the 2-regular 2-spread of $\mathrm{PG}(8, q)$ that can be viewed as the pointset of $\widehat{\mathcal{S}} = \mathrm{PG}(2, q^3)$. We drop out some elements of the spread, and then substitute them with a fewer number of opposite elements. When we drop out the elements of one of the $(2, 2)$-reguli of the 2-spread and substitute them with an opposite element, then the elements dropped out cannot be taken back. This procedure can be performed several times. If we do it in a proper way then we can achieve that the partial spread constructed this way cannot even be completed with other planes not contained in the original 2-spread.

Construction 1. Let ℓ_∞ denote the ideal line of the 'plane of the spread-elements' ($\widehat{\mathcal{S}} = \mathrm{PG}(2, q^3)$) and let ℓ' denote an arbitrary subline of order q of ℓ_∞. Let Π denote a subplane of order q of the affine part of $\widehat{\mathcal{S}}$ so that the ideal line of Π is ℓ'. Π and its proper q^4 translates are partitioning the affine part of $\widehat{\mathcal{S}}$. From now on Π and these translates will be called 'the tiles'.

In the conventional aspect, the points of ℓ_∞ are the spread-elements (of $\mathcal{S}$) which constitute a 2-spread in an ideal $\mathrm{PG}(5, q)$-subspace of $\mathrm{PG}(8, q)$. This 2-spread is a 1-regular subspread of the original 2-regular 2-spread and let this subspread be called the 'ideal subspread'. The points of ℓ' constitute a $(1, 2)$-regulus, i.e. the plane-class of an $S_{1,2}$ Segre-variety. Π and its translates together with ℓ' are $(2, 2)$-reguli of the original 2-spread which meet the ideal sub-spread in a common $(1, 2)$-regulus that is ℓ'.

Let us choose k arbitrary translates ('tiles') and drop out their points (which are in fact spread-elements) out of the original 2-spread. Let us also drop the points (i.e. spread-elements) of ℓ' out of the original 2-spread.

The opposite objects of the subspread-elements which are determined by the points of ℓ' are $q^2 + q + 1$ lines of $\mathrm{PG}(5, q)$. (Remember, the subspread is a 1-regular 2-spread of the subspace $\mathrm{PG}(5, q)$.) An opposite object of the spread-elements which are determined by the points of a tile (and ℓ') is a plane of $\mathrm{PG}(8, q)$ that meets $\mathrm{PG}(5, q)$ in one of the opposite-lines.

Choose opposite-planes for each of the $q^2 + q + 1$ opposite-lines in such a way that there should be at least one chosen opposite plane in each chosen translate of Π (in each chosen tile). #

Construction 1 produces a partial 2-spread of deficiency

$$\delta = kq^2 + (q + 1) - (q^2 + q + 1) = (k - 1)q^2 \,.$$

Lemma 2.1. *As each point of this certain ideal* $\mathrm{PG}(5, q)$ *is covered by the above* *constructed partial 2-spread, therefore this partial spread cannot be completed with other opposite planes of the abovementioned reguli.*

Proof. Each opposite plane of these reguli intersects the above $\mathrm{PG}(5,q)$. □

The partial 2-spread cannot be completed with either opposite or irregular planes if there is at least one element of the partial 2-spread that originates from the original regular total 2-spread which meets it. An element of the original 2-spread meets an opposite or irregular plane if and only if the point of $\mathrm{PG}(2,q^3)$ which represents this spread-element belongs to the set of points of $\mathrm{PG}(2,q^3)$ which represents the spread-elements that intersect the opposite or irregular plane. (It is only a tautology but it is worth mentioning.)

Theorem 2.2. *There are certain $48 \cdot q^2 \cdot (\log q + 1)$ tiles so that each projective subplane of order q that does not meet the ideal line and each club that does not meet the ideal line intersect at least one of these tiles.*

We need three lemmas; the first one is the key lemma of Lovász's τ^*-method [4].

Lemma 2.3 (Lovász). *Let G be a bipartite graph with bipartition $A \cup B$. Suppose that the degree of the points in B is at least d. Then there is a set $A' \subset A$, $|A'| \le |A| \cdot \dfrac{1+\log|B|}{d}$, such that any $b \in B$ is adjacent to a point of A'.*

We need two more lemmas.

Lemma 2.4. *If a projective subplane Σ of order q does not meet the ideal line ℓ_∞ then Σ has at most three common points with each tile.*

So a subplane Σ of order q that does not meet the ideal line intersects at least $\dfrac{q^2+q+1}{3}$ tiles.

Lemma 2.5. *If a club Ω does not meet the ideal line then Ω has at most three common points with each tile.*

So a club Ω that does not meet the ideal line ℓ_∞ intersects at least $\dfrac{q^2+1}{3}$ tiles.

Proof of Lemma 2.4. If a subplane Σ of order q has 4 common points with one of the tiles such that no three among them are collinear then the affine part of Σ is the tile itself so Σ contains some points of the ideal line ℓ_∞.

If Σ has 4 common points with a tile Π' and there are at least three points collinear then there exists a unique line ℓ of order q that contains these three common points and ℓ is a common line of Σ and Π'. A tile is an affine subplane of order q (a translate of Π) which has (projectively) an ideal line ℓ' that is a subline of the ideal line ℓ_∞, so ℓ intersects the ideal line ℓ_∞. $\square$

Proof of Lemma 2.5. The corollary of Lemma 1.9 proposed that a club Ω and a subline ℓ intersect in $0, 1, 2, 3$ or $q+1$ points. Suppose that Ω has at least 4 common points with a tile Π'. These common points are collinear, so Ω has 4 common points with a line ℓ (of order q) of the projective subplane $\Pi' \cup \ell_\infty$ and so each point of ℓ (also $\ell \cap \ell_\infty$) belongs to Ω. $\square$

Proof of Theorem 2.2. Let $(A \cup B; E)$ be a bipartite graph, where A is the set of the tiles, $B = B_1 \cup B_2$ is the union of the set B_2 of all subplanes Σ of order q that do not meet the ideal line ℓ_∞ and the set B_1 of all clubs Ω that do not meet the ideal line ℓ_∞. A tile is connected with Σ (or with Ω) if and only if the tile intersects Σ (or Ω). In this graph each point in B is of degree at least $D = (q^2+1)/3$, so because of Lemma 2.3 there exists a subset $A' \subseteq A$ such that any $b \in B$ is adjacent to a point $a \in A'$ and $|A'| \le |A| \cdot (1 + \log|B|) \,/\, D$.

We know that $|A| = q^6/q^2 = q^4$. By counting the choices of the base points we have

$$|B_2| \le \frac{q^6(q^6-1)(q^6-q^3)(q^6-3q^3+3)}{q^2(q^2-1)(q^2-q)(q^2-3q+3)} \le \frac{21}{2} q^{16}$$

and by counting the choices of the head of the club, the choices of line of the club and the choices of two other points of the club, we have

$$|B_1| \le q^6 \cdot q^3 \cdot \binom{q^3-1}{2} \le q^{15} \le \frac{1}{2} q^{16}\,.$$

Thus, $|B| \le 11 \cdot q^{16}$, and so $\log|B| \le 16 \cdot \log q + \log 11 \le 16 \cdot \log q + 2.4$, and hence

$$|A'| \le 3 \cdot q^4 \cdot \frac{1 + 16 \cdot \log q + 2.4}{q^2+1} \le 48 \cdot q^2 \cdot \left(\log q + \frac{1}{4}\right). \qquad \square$$

Theorem 2.6. *If $q \ge 16$ then there exist maximal partial 2-spreads in $\mathrm{PG}(8,q)$ of deficiency $\delta = (k-1)q^2$ where*

$$1 \le k \le \min\left\{ q^4 - 48 \cdot q^2 \cdot \left(\log q + \frac{1}{4}\right)\ ,\ q^2+q+1 \right\}\,.$$

Proof. Since the ideal line ℓ_∞ represents the ideal $\mathrm{PG}(5,q)$ that is completely covered by the partial 2-spread, we should only prove that every subplane of order q of $\mathrm{PG}(2, q^3)$ and every club contains at least one point that either represents a spread-element not dropped out in the construction or is a point of the ideal line ℓ_∞.

Because of the previous theorem we can choose $48 \cdot q^2 \cdot (\log q + 1/4)$ tiles so that these tiles and the ideal line intersects each subplane of order q and each club. If in the course of Construction 1 we do not drop out the spread-elements that belong to the above chosen tiles then the constructed partial 2-spread will be maximal. □

2.1 MPPS's in $\mathrm{PG}(8, q)$ of some other deficiencies

Let us consider the set of elements of the 2-regular 2-spread of $\mathrm{PG}(8, q)$ that can be considered as the pointset of $\mathrm{PG}(2, q^3)$. Again we drop out some elements of the spread, and then substitute them with some planes. But now the construction is based on sublines of order q in $\mathrm{PG}(2, q^3)$, so the basic step of this construction is substituting some spread-elements with irregular planes.

Construction 2. Choose $k + l$ lines through the point P in such a way that neither subplane of order q is covered by these lines.

Choose l clubs (one in each of l lines above) in such a way that these clubs contain the point P but P is not the head of either of these clubs. Choose k disjoint clubs (one in each of the other k lines above) in such a way that these clubs do not contain the point P.

Drop the elements of the above $k+l$ clubs and substitute them by $k+l$ disjoint irregular planes (one in each club) in such a way that the l irregular planes in the first l (not disjoint) clubs intersect the plane represented by P in distinct points. #

Construction 2 produces a partial 2-spread of deficiency

$$\delta = 1 + l \cdot q^2 + k \cdot (q^2 + 1) - k - l = k \cdot q^2 + l \cdot (q^2 - 1) + 1\,.$$

Theorem 2.7. *The above constructed partial 2-spread is maximal if $k + l \leq q^2$.*

Proof. There is no opposite plane which can be added to the above constructed partial 2-spread because neither subplane of order q is covered by the $k+l$ lines.

There is no irregular plane which can be added to the partial spread because its club has $q^2 + 1$ points. □

Construction 3. Choose an arbitrary club in $\mathrm{PG}(2, q^3)$ and let this club be called the 'Cross Club'. Drop the elements of the Cross Club out of the spread and substitute them by one of the $q^2 + q + 1$ irregular planes which meet the spread-elements dropped out. Then choose a point P not in the line of the Cross Club.

We will use four types of clubs in the lines through the point P. A club of first type does not intersect the Cross Club and does not contain the point P. A club of second type intersects the Cross Club and contains the point P. A club of third type intersects the Cross Club but does not contain the point P. A club of fourth type does not intersect the Cross Club but contains the point P. P is not the head of these clubs and the intersection-points of the Cross Club with these clubs are not the head of either of these clubs and are not the head of the Cross Club.

Choose $k + l + m_1 + m_2$ lines through the point P and choose k clubs of first type, l clubs of second type, m_1 clubs of third type and m_2 clubs of fourth type (one in each above line).

Drop the elements of the above clubs and substitute them by irregular planes (one in each club) in such a way that these irregular planes do not meet the irregular plane in the Cross Club and in such a way that the irregular planes in the (not disjoint) clubs of second and fourth type intersect the plane represented by P in disjoint points. #

Construction 3 produces a partial 2-spread of deficiency

$$\delta = (k + 1) \cdot q^2 + (m_1 + m_2) \cdot (q^2 - 1) + l \cdot (q^2 - 2) + 1 \,.$$

Theorem 2.8. *Let $m = m_1 + m_2$. The above constructed partial 2-spread is maximal if $k + m + l \leq q^2$.*

Proof. Construction 3 has been done in such a way that there is no subplane of order q of $\mathcal{S}$ that contains only dropped spread-elements. So there is no opposite plane that can extend the constructed partial 2-spread.

As $k + m + l \leq q^2$, there is no club that contains only dropped spread planes and not contains an irregular plane of the constructed partial 2-spread. □

Results in $\mathbf{PG}(8, q)$. The first three constructions yield maximal partial plane-spreads in $\mathrm{PG}(8, q)$ of deficiency $\delta = (k - 1) \cdot q^2$, where $k \leq q^2 + q + 1$ and $\delta = k \cdot q^2 + l \cdot (q^2 - 1) + 1$, where $k + l \leq q^2$ and $\delta = (k+1) \cdot q^2 + l \cdot (q^2 - 1) + m \cdot (q^2 - 2) + 1$, where $k + l + m \leq q^2$.

2.2 MPPS's in $\mathrm{PG}(3m-1,q)$

If we can do the abovementioned constructions in such a way that the original spread-elements that cover a particular $\mathrm{PG}(5,q)$ are not affected by the construction then we can generalize these constructions for $\mathrm{PG}(3m-1,q)$, $m \geq 4$, in the following way.

Construction 4. Let us choose a $\mathrm{PG}(5,q)$ in $\mathrm{PG}(3m-1,q)$ and let us factorize $\mathrm{PG}(3m-1,q)$ with this chosen space. The factor geometry is a $\mathrm{PG}(3(m-2)-1,q)$, that contains total 2-spreads. Construct a (total) plane spread in the $\mathrm{PG}(3(m-2)-1,q)$ factor geometry. The elements of this spread are 8-dimensional spaces of $\mathrm{PG}(3m-1,q)$ intersecting each other in the chosen $\mathrm{PG}(5,q)$. The number of these 8-dimensional spaces is $\dfrac{q^{3(m-2)}-1}{q^3-1}$.

Construct 2-regular 2-spreads in these 8-dimensional spaces in such a way that each 2-regular 2-spread generates a sub-spread in the chosen $\mathrm{PG}(5,q)$ and these generated sub-spreads are the same.

The abovementioned constructions can be done in some 8-dimensional spaces in such a way that the spread-elements in the chosen $\mathrm{PG}(5,q)$ are not dropped out from either original 2-spreads. #

Construction 4 produces a partial 2-spread of deficiency

$$\begin{aligned}\delta &= \Big(x\cdot(k_x-1)+y\cdot k_y+z\cdot(k_z+1)\Big)\cdot q^2+(y\cdot l_y+z\cdot l_z)\cdot(q^2-1)\\&\qquad\qquad + z\cdot m_z\cdot(q^2-2)+y+z\\&=(y+z)\cdot(q^2+1)+\Big(x\cdot(k_x-1)+y\cdot(k_y-1)+z\cdot k_z\Big)\cdot q^2\\&\qquad\qquad +(y\cdot l_y+z\cdot l_z)\cdot(q^2-1)+z\cdot m_z\cdot(q^2-2)\,.\end{aligned}$$

Notation. In a bipartite graph $(A\cup B;E)$ let A be the set of the elements of the 2-spread of the factor geometry (i.e. the set of the abovementioned 8-dimensional spaces). Let B contain all the other planes of the factor geometry. A' will be the set of such spread-elements in the factor geometry, which are those 8-dimensional spaces in which the (total) 2-spreads remain (total) 2-spreads after the construction. Let $N=3(m-2)-1=3m-7$.

Lemma 2.9. *There exists a subset $A' \subset A$, such that any $b\in B$ is adjacent to a point $a\in A'$ and*

$$|A\setminus A'| \geq \max\left\{q^2\ ,\ \frac{q^{N+1}-1}{q^3-1}\cdot\frac{q^2-3N\log q}{q^2+1}\right\}\,.$$

Proof. We know that $\deg(b) \geq q^2 + 1 \ \ \forall b \in B$ because a plane b intersects either $q^2 + q + 1$ or $q^2 + 1$ elements of A. (See the paragraph 'Opposite and irregular planes' in subsection 'Intersections'.) So for arbitrary A', if $|A \setminus A'| \leq q^2$ then any $b \in B$ is adjacent to a point $a \in A'$.

Because of Lemma 2.3 $\exists A' \subset A$ such that $|A'| \leq |A|\dfrac{1 + \log|B|}{q^2+1}$; and any $b \in B$ is adjacent to a point $a \in A'$.

Since $|A| = \dfrac{q^{N+1}-1}{q^3-1}$, the number of all planes of $\mathrm{PG}(N, q)$ is

$$|A| + |B| = \frac{q^{N-1}-1}{q-1} \cdot \frac{q^N-1}{q^2-1} \cdot \frac{q^{N+1}-1}{q^3-1} = \frac{q^{3m-8}-1}{q-1} \cdot \frac{q^{3m-7}-1}{q^2-1} \cdot |A| \, .$$

So

$$|B| = |A| \cdot \left(\frac{(q^{N-1}-1)(q^N-1)}{(q-1)(q^2-1)} - 1 \right)$$

and hence

$$\log|B| \leq \log|A| + (N-1) \cdot \log q + N \log q \leq 3N \log q.$$

Therefore

$$\begin{aligned} |A \setminus A'| \ \geq \ |A| \left(1 - \frac{1 + \log|B|}{q^2+1} \right) \ &\geq \ \frac{q^{N+1}-1}{q^3-1} \left(1 - \frac{1 + 3N \log q}{q^2+1} \right) \\ &= \frac{q^{N+1}-1}{q^3-1} \cdot \frac{q^2 - 3N \log q}{q^2+1} \, . \end{aligned}$$

□

Theorem 2.10. *In* $\mathrm{PG}(3m-1, q)$*,* $m \geq 4$*, there are maximal partial 2-spreads of deficiency*

$$\begin{aligned} \delta = \big(x \cdot (k_x - 1) + y \cdot k_y + z \cdot (k_z + 1) \big) \cdot q^2 + \big(y \cdot l_y + z \cdot l_z \big) \cdot (q^2 - 1) \\ + z \cdot m_z \cdot (q^2 - 2) + y + z \, , \end{aligned}$$

where

$$k_x \leq q^2 + q + 1, \quad k_y + l_y \leq q^2, \quad k_z + l_z + m_z \leq q^2,$$

and

$$x + y + z \leq \max \left\{ q^2 \ , \ \frac{q^{3(m-2)}-1}{q^3-1} \cdot \frac{q^2 - 3 \cdot (3m-7) \cdot \log q}{q^2+1} \right\} \, .$$

Proof. We know that the above constructed partial 2-spread cannot be extended with planes that are completely contained in either of the 8-dimensional spaces above because the partial 2-spreads in these 8-dimensional spaces are maximal.

So, we should only prove that the above constructed partial 2-spread cannot be extended with planes that are *not* completely contained in either of the 8-dimensional spaces above.

If a plane of $\mathrm{PG}(3m-1,q)$ intersects the chosen $\mathrm{PG}(5,q)$ then this plane cannot be added to the partial spread.

A plane of $\mathrm{PG}(3m-1,q)$ which does not meet the chosen $\mathrm{PG}(5,q)$, together with the $\mathrm{PG}(5,q)$ generates one of the planes of the factor geometry. If this generated plane of the factor geometry meets a plane of the factor geometry which is an 8-dimensional space totally covered by the partial 2-spread then the plane of $\mathrm{PG}(3m-1,q)$ (which generates the above plane of the factor geometry) cannot be added to the partial 2-spread. Because of Lemma 2.9 there exists a set A' of the abovementioned 8-dimensional spaces such that every plane of $\mathrm{PG}(3m-1,q)$ which does not meet the chosen $\mathrm{PG}(5,q)$ meets at least one element of A' and

$$|A \setminus A'| \geq \max\left\{q^2\ ,\ \frac{q^{N+1}-1}{q^3-1}\cdot\frac{q^2-3N\log q}{q^2+1}\right\}\ .$$

If we construct partial 2-spreads only in the elements of $A \setminus A'$ then the constructed partial 2-spread in the $\mathrm{PG}(3m-1,q)$ cannot be completed with such planes that are *not* completely contained in either of the above 8-dimensional space. And if these above constructed partial 2-spreads are maximal then the partial 2-spread in $\mathrm{PG}(3m-1,q)$ is also maximal. □

3 Remarks

These constructions are generalizations of the Gács–Szőnyi construction [1]. It seems to be hard to continue this way of generalization to t-spreads ($t \geq 3$) since this method is based on the examination of the possible intersection-configurations which becomes more and more complicated as t is increasing.

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