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Certain generalized quadrangles inside polar spaces of rank 4

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Abstract

Let Δ be the dual of a thick polar space Π of rank 4. The points, lines, quads, and hexes of Δ correspond with the singular 3-spaces, planes, lines, respectively points of Π. Pralle and Shpectorov [14] have investigated *ovoidal hyperplanes* of Δ which intersect every hex in the extension of an ovoid of a quad. With every ovoidal hyperplane there corresponds a unique generalized quadrangle Γ. In the finite case, Γ has been classified combinatorially, and it has been shown that only the symplectic and elliptic dual polar spaces $DSp_8(q)$ and $DO^-_{10}(q)$ of Witt index 4 have ovoidal hyperplanes. For $DSp_8(\mathbb{K})$ over an arbitrary field $\mathbb{K}$, it holds $\Gamma \cong Sp_4(\mathbb{H})$ for some field $\mathbb{H}$.

In this paper, we construct an embedding projective space for the generalized quadrangle Γ arising from an ovoidal hyperplane of the orthogonal dual polar space $DO^-_{10}(\mathbb{K})$ for a field $\mathbb{K}$. Assuming $\mathrm{char}(\mathbb{K}) \neq 2$ when $\mathbb{K}$ is infinite, we prove that Γ is a hermitian generalized quadrangle over some division ring $\mathbb{H}$.

Moreover we show that an ovoidal hyperplane H arises from the universal embedding of Δ, if the ovoids $Q \cap H$ of all ovoidal quads Q are classical. This condition is satisfied for the finite dual polar spaces $DSp_8(q)$ and $DO^-_{10}(q)$ by [14].

Keywords: dual polar space, generalized quadrangle, hyperplane, ovoid, polar space, spread, symplectic spread, embedding
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1 Introduction

Let Δ be the dual of a classical thick polar space Π of rank $n \geq 3$. The points of Δ are the $(n-1)$-dimensional singular subspaces of Π and the lines of Δ the

$(n-2)$-dimensional singular subspaces of Π. More generally, for $i = 1, ..., n$, the elements of type i of Δ are the singular subspaces of Π of dimension $n-i$. If α is an element of type 3 of Δ, then the lower residue $Res^-_\Delta(\alpha)$ of α, i.e. the geometry of points and lines incident with α, is a generalized quadrangle. The elements of type 3 of Δ are called *quads*. The elements of type 4 of Δ are called *hexes*, sometimes in the literature also *symps*.

By $\perp$ and $\perp_\Pi$, we denote the collinearity relations in Δ, respectively Π. For a point p of Δ, its perp $p^\perp$ is the set of points on the lines through p. If E_1 and E_2 are two elements of Δ, then $\langle E_1, E_2\rangle$ denotes the smallest convex subspace of Δ containing E_1 and E_2; e.g. for two points p and r of Δ at distance 2, $\langle p, r\rangle$ is the unique quad containing p and r which corresponds in the polar space Π with the intersection of the two maximal singular subspaces corresponding with p and r. If the classical polar space Π is embedded in the projective space $PG(V)$ for a vector space V, then $\perp_V$ denotes the orthogonality relation in V induced by Π, and $\langle X\rangle_V$ the span in V of the subset X of V.

A hyperplane of a geometry is a proper subspace meeting every line. Let H be a hyperplane of the dual polar space Δ. If E is an arbitrary element of Δ of type ≥ 2 not contained in H, then H meets $Res^-_\Delta(E)$ in a hyperplane of $Res^-_\Delta(E)$. For a quad α not contained in H, there are three possible intersection configurations with H (cf. Payne and Thas [12, 2.3.1]): $\alpha \cap H$ is either the perp $p^\perp$ of a point p, or a subquadrangle, or an ovoid (i.e. a set of pairwise noncollinear points meeting every line). Accordingly, we call α *singular*, *subquadrangular*, respectively *ovoidal*. A hyperplane H is called *locally uniform* if it intersects all quads not contained in H in the same kind of hyperplane, otherwise *locally non-uniform*. We often omit the word 'locally'. We call elements of type > 1 *deep (w.r.t. H)* if they are contained in H, e.g. a line or a quad are deep if all their points belong to H.

A point $p \in H$ is *deep (w.r.t. H)* if $p^\perp \subset H$. Considering only the hyperplane $H \cap Res^-(\Sigma)$ induced by H on a hex Σ, a point $p \in \Sigma \cap H$ is called deep w.r.t. Σ if $p^\perp \cap \Sigma \subset H$. Note that in general, a point deep w.r.t. to a hex is not deep w.r.t. the hyperplane.

Since a dual polar space Δ of rank n is a near $2n$-gon (cf. Cameron [3]), every hyperplane of the lower residue of an element of Δ extends to a hyperplane of Δ as follows. If E is an element of type $1 < t \leq n-1$ and H_E is a hyperplane of $Res^-_\Delta(E)$, then the set $H := \bigcup_{x\in H_E} \Delta_{\leq n-t}(x)$ is a hyperplane of Δ, called the *extension of H_E*. For instance, if $n = 3$ and E is an ovoid of a quad, then H is the *extension of the ovoid E*. It also follows from the near polygon property of Δ that for a point p the set $H_p := \Delta_{\leq n-1}(p)$ is a hyperplane, the *singular hyperplane with deepest point p* consisting of all points at non-maximal distance from p.

The aim of this paper is the construction of a projective space E embedding a generalized quadrangle which is associated with a certain kind of hyperplane of a dual polar space Δ of rank 4. These hyperplanes and the dual polar spaces admitting such hyperplanes will be specified in the following section. If Π denotes the classical polar space dual of Δ, let V be the embedding space of Π. Since the projective space E will be constructed by means of subspaces of V, we often use vector space notations in this paper. In particular, we denote the symplectic dual polar space of rank 4 by $DSp_8(\mathbb{K})$ instead of $DW(7,\mathbb{K})$, the elliptic orthogonal dual polar space by $DO^-_{10}(\mathbb{K})$ instead of $DQ^-(9,\mathbb{K})$ and the classical unital by $H_3(q^2)$ instead of $U(2,q^2)$ although these notations are usually reserved for the corresponding groups.

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1.1 Ovoidal hyperplanes

The hexes of a dual polar space Δ of rank 4 are dual polar spaces of rank 3. A hyperplane H of Δ intersects hexes not contained in H in hyperplanes of dual polar spaces of rank 3. By Shult [17] and Pralle [13], there are only two families of locally singular hyperplanes in a finite thick dual polar space of rank 3. Pasini and Shpectorov [11] have classified the locally subquadrangular hyperplanes of finite thick dual polar spaces of rank 3. They have also proved in the finite case that no locally ovoidal hyperplane exists admitting a flag-transitive complement. Moreover, the finite symplectic dual polar space $DSp_6(q)$ has no ovoid by Cooperstein and Pasini [7]. The existence of ovoids in $DO_7(q)$ and $DH_{2n+1}(q^2)$ with $q \geq 3$, and of locally subquadrangular hyperplanes in infinite dual polar spaces of rank 3 are outstanding problems.

Let Δ be a thick dual polar space of rank 4 such that no subquadrangular quad exists. Then the intersection of a hyperplane H with a hex Σ is one of the following (due to Pralle [13] and the above mentioned results):

- $\Sigma \cap H = \Sigma$.
- $\Sigma \cap H$ is a singular hyperplane of Σ.
- $\Sigma \cap H$ is the extension of an ovoid of a quad.
- There are a point p and a set O of mutually noncollinear points meeting every line of Σ at distance 2 from p such that $\Sigma \cap H = p^{\perp} \cup O$.
- $\Sigma \cap H$ is an ovoid.

- The set $\mathcal{P}$ of deep points with respect to the singular quads of Σ is a locally singular hyperplane of a dual polar space $\Delta_0 \cong DO_7(\mathbb{K})$. The polar space Π_0 dual of Δ_0 is a subspace of the polar space Π' dual of Σ where the lines of Π_0 are lines of Π'. The set $\mathcal{P}$ together with the lines of Σ contained in H form a split Cayley hexagon $H(\mathbb{K})$. The hyperplane $\Sigma \cap H$ contains all points of Σ on lines of $H(\mathbb{K})$.

Pralle and Shpectorov [14] have investigated the hyperplanes H of a dual polar space Δ of rank 4 such that H intersects every hex in the extension of an ovoid. Such a hyperplane is called an *ovoidal hyperplane*. Let H be an ovoidal hyperplane of Δ and denote the set of deep quads by $\mathcal{L}$. Considering the polar space Π dual of Δ, the line set of Π corresponding to $\mathcal{L}$ is a spread of Π with the following *spread property*:

(SP) Let $\mathcal{D}$ be the set of linear 4-spaces of Π in which $\mathcal{L}$ induces spreads. For every point Σ of Π, the linear 4-spaces of $\mathcal{D}$ containing Σ all contain the spread line $\lambda \in \mathcal{L}$ covering Σ and form a spread of the generalized quadrangle $Res^+_\Pi(\lambda)$.

We denote the set of points of Δ corresponding to the 4-spaces of $\mathcal{D}$ by $\mathcal{D}$, too. The following theorem collects the results of [14] about the hyperplane H of Δ.

Theorem 1.1 ([14]). *With the above notations the following hold:*

(a) *The points of $\mathcal{D}$ in a hex Σ form an ovoid $O(\Sigma)$ of the unique quad $\delta(\Sigma) \subset \Sigma$ belonging to $\mathcal{L}$.*

(b) $H = \bigcup_{x \in \mathcal{D}} x^\perp = \bigcup_{\delta \in \mathcal{L}} \delta$.

(c) *The point-line geometry $\Gamma = (\mathcal{L}, \mathcal{D})$ with incidence induced from Δ is a generalized quadrangle.*

(d) *If Δ is finite and its quads are of order (s,t), then Γ is of order (t^2, st).*

(e) *If Δ is finite, then either $\Delta \cong DSp_8(q)$ and $\Gamma \cong DSp_4(q^2)$, or $\Delta \cong DO^-_{10}(q)$ and $\Gamma \cong DH_5(q^2)$. Moreover, the ovoid $Q \cap H$ of an ovoidal quad Q is an elliptic quadric $Q^-_4(q)$, respectively a unital $H_3(q^2)$.*

(f) *If $\Delta \cong DSp_8(\mathbb{K})$ for an arbitrary field $\mathbb{K}$, then $\Gamma \cong DSp_4(\mathbb{H})$ for some field $\mathbb{H}$.*

We observe that the quads through a point $p \in \mathcal{D}$ are either singular or deep, since all lines through p are contained in H. More precisely, the deep quads containing p form a spread of the 3-dimensional projective space $Res^+_\Delta(p)$.

In [14, Section 1.3], three examples are presented for hyperplanes of the dual polar spaces $DSp_8(\mathbb{K})$, $DO^-_{10}(\mathbb{K})$, and $DH_8(\mathbb{C})$ for which the generalized quadrangle Γ is isomorphic to the symplectic generalized quadrangle $Sp_4(\mathbb{H})$, a hermitian generalized quadrangle $H_5(\mathbb{H})$, respectively $H_4(\mathbb{Q})$ where $\mathbb{H}$ is a separable quadratic extension of $\mathbb{K}$ admitting an involutory field automorphism defining the hermitian generalized quadrangle $H_5(\mathbb{H})$, and where $\mathbb{Q}$ denotes the quaternions. The hermitian example over the complex numbers and the quaternions shows that the classification in (e) indeed requires the finiteness assumption.

1.2 Main Theorem

In this paper $\Delta \cong DQ^-_{10}(\mathbb{K})$ for a field $\mathbb{K}$ and H is an ovoidal hyperplane of Δ, i.e. it intersects every hex in the extension of an ovoid. We will construct an embedding for the generalized quadrangle Γ arising from H. Our main result is the following.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, if $\Delta \cong DO^-_{10}(\mathbb{K})$ for an infinite field $\mathbb{K}$, then Γ is embeddable in $PG(4, \mathbb{H})$ where $\mathbb{H}$ is a division ring. If $\mathbb{H}$ has characteristic $\neq 2$, then Γ is a hermitian generalized quadrangle.*

In [14], the generalized quadrangle Γ arising from $\Delta \cong DO^-_{10}(\mathbb{K})$ has been characterized combinatorially for finite fields $\mathbb{K} = \mathbb{F}_q$. If a generalized quadrangle is fully embedded in $PG(4, \mathbb{H})$ for some division ring $\mathbb{H}$ and if $\mathrm{char}(\mathbb{H}) \neq 2$, then it consists of the totally isotropic subspaces of a reflexive sesquilinear form and is either a parabolic quadric or a hermitian generalized quadrangle (for details, see section 2.1). In section 2.4 with $\Delta \cong DO^-_{10}(\mathbb{K})$ for a field $\mathbb{K}$, we construct a projective space $PG(4, \mathbb{H})$ over a suitable division ring $\mathbb{H}$ and show that Γ fully embeds in $PG(4, \mathbb{H})$, thus proving the assertions of Theorem 1.2.

1.3 Finite ovoidal hyperplanes arise from embeddings

The main subject of this paper is the investigation of the generalized quadrangle Γ associated with an ovoidal hyperplane of $DO^-_{10}(\mathbb{K})$. However, a lot of recent research is devoted to the embedding of geometries (see e.g. Kasikova and Shult [10], Cardinali, De Bruyn and Pasini [4], and De Bruyn [8]). In this subsection, we deduce from Theorem 1.1 that ovoidal hyperplanes of finite thick dual polar spaces of rank 4 arise from embeddings.

A *(projective) embedding* of a point-line geometry Γ is an injective mapping ε of the points of Γ onto a spanning set of a projective space $PG(V)$ for a $\mathbb{K}$-vector space V such that the points of a line of Γ are mapped onto the points of one line of $PG(V)$. If $t : V \to W$ is a $\mathbb{K}$-homomorphism with kernel U that intersects trivially each 2-space of V spanned by any pair of embedded points of Γ, then $\varepsilon' := t \circ \varepsilon$ is an embedding $\Gamma \to PG(W)$ which is called a *homomorphic image* of ε. The embedding ε is called *universal* if every embedding of Γ is a homomorphic image of ε.

If $\varepsilon : \Gamma \to PG(V)$ is an embedding of a geometry Γ and h is a hyperplane of $PG(V)$, then the set of points $H = \varepsilon^{-1}(h \cap \varepsilon(\Gamma))$ of Γ is a hyperplane of Γ. If H is a hyperplane of Γ such that $\langle \varepsilon(H) \rangle_{PG(V)}$ is a hyperplane of $PG(V)$, then H is said *to arise from the embedding* ε.

Lemma 1.3. *If H is the extension of a classical ovoid of a quad of $\Delta \cong DSp_6(\mathbb{K})$, then H arises from the Grassmann embedding of Δ.*

Proof. The Grassmann embedding $e_{gr} : DSp_{2n}(\mathbb{K}) \to PG\left(\binom{2n}{n} - \binom{2n}{n-2} - 1, \mathbb{K}\right)$ is induced by the embedding $Gr_n(\mathbb{K}^{2n}) \to PG\left(\bigwedge^n \mathbb{K}^{2n}\right)$ of the Grassmannian $Gr_n(\mathbb{K}^{2n})$ of n-spaces of $\mathbb{K}^{2n}$ (cf. Cooperstein [6]). Since $DSp_6(\mathbb{K})$ is transitive on the classical ovoids of quads, the assertion follows if the extension of any classical ovoid of a quad arises from e_{gr}. Let the polar space $Sp_6(\mathbb{K})$ be defined by the form

$$f(x, y) = x_1y_4 - x_4y_1 + x_2y_5 - x_5y_2 + x_3y_6 - x_6y_3$$

for all $x = (x_1, ..., x_6),\ y = (y_1, ..., y_6) \in \mathbb{K}^6$. Describing a totally isotropic plane z of $\mathbb{K}^6$ by a matrix $Z := (z_{lm})_{1\le l\le 3,\ 1\le m\le 6}$ with rows the vectors of a basis of z, the Grassmann coordinate z_{ijk} for $1 \le i < j < k \le 6$ is $\det\left((z_{lm})_{1\le l\le 3,\ m\in\{ijk\}}\right)$. The Grassmann coordinates of a totally isotropic plane satisfy the six equations

$$\begin{aligned} 0 &= z_{124} - z_{236} = z_{125} + z_{136} = z_{134} + z_{235} \\ &= z_{145} - z_{356} = z_{146} + z_{256} = z_{245} + z_{346}\ . \end{aligned}$$

Let Q be the quad of the dual polar space $\Delta \cong DSp_{\mathbf{6}}(\mathbb{K})$ which is the point $(1, 0, 0, 0, 0, 0)$ of $Sp_6(\mathbb{K})$. The points of Δ in Q are the totally isotropic planes spanned by the rows of a matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_5 & a_6 \\ 0 & 0 & b_3 & 0 & b_5 & b_6 \end{pmatrix}$ with Grassmann coordinates satisfying $p_{123}p_{156} + p_{135}p_{126} + p_{125}^2 = 0$. Since we suppose the existence of elliptic quadrics, there exists an irreducible polynomial $t^2 + t + \alpha \in \mathbb{K}[t]$. The points of Q with $p_{135} = \alpha p_{126} + p_{125}$ form a classical ovoid $\Omega \cong O_4^-(\mathbb{K})$ of Q. They are the planes with matrices

$$P(a, b) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -a - \alpha b & a \\ 0 & 0 & 1 & 0 & a & b \end{pmatrix} \text{ and } P(\infty) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for $a, b \in \mathbb{K}$. The points of $P(a,b)^{\perp}$ are

$$\begin{pmatrix} 1 & s & 0 & 0 & -s(a+\alpha b) & as \\ 0 & 1 & t & 0 & a(t-1)-\alpha b & a+bt \\ r & 0 & 0 & st & -t & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & s & 0 & 0 & -s(a+\alpha b) & as \\ 0 & 0 & 1 & 0 & a & b \\ r & 0 & 0 & s & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & -a-\alpha b & a \\ 0 & 0 & 1 & 0 & a & b \\ r & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

for all $r, s, t \in \mathbb{K}$. The points of $P(\infty)^{\perp}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ r & -t & 1 & -st & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ r & 1 & 0 & s & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ r & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence the points of the extension H of the ovoid Ω of the quad Q belong to the hyperplane $y_{245} + y_{346} = 0$, whence H arises from e_{gr}. □

Theorem 1.4. *Suppose $\Delta \cong DSp_8(\mathbb{K})$ or $DO^-_{10}(\mathbb{K})$ for a field $\mathbb{K}$, and let H be an ovoidal hyperplane of Δ. Then H arises from the universal embedding of Δ if the ovoid $Q \cap H$ for every ovoidal quad Q arises from an embedding. In particular, if Δ is finite, then the ovoidal hyperplanes arise from an embedding.*

Proof. For $DO^-_{10}(q)$, the assertion of the theorem is stated firstly in De Bruyn [8, Theorem 1.4]. For the general case with $\Delta \cong DSp_8(\mathbb{K})$ or $DO^-_{10}(\mathbb{K})$ where the intersection of H with every hex Σ is the extension of a classical ovoid of a quad contained in Σ, the proof is an application of Corollary 4 of Section 1 of Ronan [15] and of the simple connectedness of the hyperplane complement $\Delta - H$ due to Cardinali, De Bruyn and Pasini [4] and Shpectorov [16].

We follow the notations of Corollary 1.5 of [4] and handle $DSp_8(\mathbb{K})$ and $DO^-_{10}(\mathbb{K})$ separately. For $DO^-_{10}(\mathbb{K})$, let $n_0 = 2$ and the class $\mathbf{D}$ consist of the quads and hexes of Δ and of Δ itself. For a quad Q of Δ, the set $\mathcal{H}(Q)$ consists of the classical ovoids and singular hyperplanes of Q, for a hex $\Sigma \in \mathbf{D}$, $\mathcal{H}(\Sigma)$ consists of the extensions of classical ovoids of any quad of Σ, and $\mathcal{H}(\Delta)$ contains the ovoidal hyperplanes of Δ. Then (1) and (2) of Corollary 1.5 of [4] are satisfied. By assumption, the ovoids of ovoidal quads arise from an embedding, whence (4) of Corollary 1.5 of [4] is fulfilled. According to Theorem 1 of Shpectorov [16], the complement of a hyperplane of a dual polar space of rank 3 with at least five points on each line is simply connected. Since $DO^-_8(\mathbb{K})$ has at least five points on every line, condition (3) of Corollary 1.5 of [4] holds, too. We conclude that the ovoidal hyperplanes of Δ arise from an embedding.

The same argument applies for $DSp_8(\mathbb{K})$ if $|\mathbb{K}| \geq 4$. For $\mathbb{K} = \mathbb{F}_2$, the theorem holds by Corollary 2 of Section 1 of Ronan [15]. Since $DSp_8(3)$ has only four points on each line, condition (3) of Corollary 1.5 of [4], i.e. the simple connectedness of the hyperplane complement of the extension of a classical ovoid of a quad in a hex, is not guaranteed by [16]. So, we either have to show (3), or we choose $n_0 = 3$ and prove condition (4). The latter has been done in Lemma 1.3, whence the ovoidal hyperplanes of $DSp_8(3)$ arise from an embedding. We remark that this proof for $DSp_8(\mathbb{K})$ works for every field $\mathbb{K}$ and does not use [16].

If Δ is finite, then the ovoid $Q \cap H$ of an ovoidal quad Q is classical by Theorem 1.1(e). Hence the condition on the ovoidal quads stated in the theorem is trivially satisfied in the finite case and the assertion follows as above if every line has at least 4 points. If Δ has only three points on every line, then every hyperplane of Δ arises from the universal embedding of Δ by Corollary 2 of Section 1 of Ronan [15]. □

2 Proof of Theorem 1.2

2.1 Embedding generalized quadrangles

Let Δ be the orthogonal dual polar space $DO^-_{10}(\mathbb{K})$ for some division ring $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}_q$, then Γ has been identified in [14] as the hermitian generalized quadrangle $H_5(q^2)$ using a combinatorial characterization: A generalized quadrangle $\mathcal{Q}$ is a hermitian generalized quadrangle $H_5(q^2)$ if and only if it has order (q^2, q^3) and every hyperbolic line has at least $q+1$ points (Payne and Thas [12, 5.5.1]) where the *hyperbolic line* $\{x, y\}^{\perp_{\mathcal{Q}}\perp_{\mathcal{Q}}}$ through two noncollinear points x and y of $\mathcal{Q}$ consists of the points collinear with all points of the trace $\{x, y\}^{\perp_{\mathcal{Q}}} = x^{\perp_{\mathcal{Q}}} \cap y^{\perp_{\mathcal{Q}}}$, whence $\{x, y\}^{\perp_{\mathcal{Q}}\perp_{\mathcal{Q}}} = \{z \in \mathcal{Q} \mid \{x, y\}^{\perp_{\mathcal{Q}}} \subseteq z^{\perp_{\mathcal{Q}}}\}$).

For the infinite case, we use the classification results by Tits [18], and Buekenhout and Lefèvre [2] and Dienst [9] of embeddable generalized quadrangles to determine the structure of Γ. We follow [18] for notation. For a survey, see Cohen [5]. Let $\varphi : \Gamma \to PG(W)$ be an embedding of Γ into a projective space $PG(W)$ over a division ring $\mathbb{H}$, and let π be a non-degenerate polarity of $PG(W)$ such that $\varphi(L)$ is a totally isotropic line for every line L of Γ. Then π is represented by a non-degenerate (σ, ε)-hermitian form $f : W \times W \to \mathbb{H}$. The form f is called *trace-valued* if $f(x, x) \in \{t + \sigma(t)\varepsilon \mid t \in \mathbb{H}\}$ for all $x \in W$ ([18, 8.1.4]). If $\mathrm{char}(\mathbb{H}) \neq 2$, or if $\sigma|_{Z(\mathbb{H})} \neq id$ where $Z(\mathbb{H})$ is the center of $\mathbb{H}$, then f is trace-valued and we apply Theorem 8.6 of [18]:

(i) If every (σ, ε)-hermitian form where ε and σ belong to f, is trace-valued, then the embedding is dominant, and $\varphi(\Gamma)$ is the generalized quadrangle of totally isotropic subspaces of π in $PG(W)$.

(ii) If f is not trace-valued, then there exist an embedding $\varphi' : \Gamma \to P'$ into a projective space P' with a polarity π', a pseudoquadratic form κ' of P', and a morphism $\mu : P' \to PG(W)$ such that π' is the polarity associated with κ' and such that $\varphi'(\Gamma)$ is the generalized quadrangle of totally singular subspaces of P' with respect to κ'.

In case (i), the generalized quadrangle $\varphi(\Gamma)$ is isomorphic to the generalized quadrangle of totally isotropic points and lines of a non-degenerate polarity defined by a reflexive sesquilinear form, or of the singular points and lines of a quadratic form. These forms are classified and well-known (cf. [5, 3.14]): In $\mathbb{H}^5$, a non-degenerate reflexive sesquilinear form is (anti)-hermitian or symmetric. Hence a generalized quadrangle fully embedded in $\mathcal{P} := PG(4, \mathbb{H})$ is either a hermitian generalized quadrangle or a parabolic quadric $O_5(\mathbb{H})$. In both cases, two opposite lines determine full subquadrangles uniquely. In $O_5(\mathbb{H})$, such a subquadrangle is a hyperbolic quadric $O_4^+(\mathbb{H})$, i.e. a grid. In a hermitian generalized quadrangle embedded in $\mathcal{P}$, it is a non-degenerate hermitian generalized quadrangle embedded in a hyperplane of $\mathcal{P}$. Hence the isomorphism class of the subquadrangles spanned by two opposite lines of Γ determine the isomorphism class of Γ. We study them in section 2.3.

In case (ii), the embedding φ' is dominant and the embedding φ which we will construct in section 2.4 may be only a quotient of φ'. So, we cannot use the classification of sesquilinear and quadratic forms to characterize $\varphi(\Gamma)$ uniquely. The generalized quadrangle $\varphi'(\Gamma)$ is defined by a pseudoquadratic form (the theory of pseudoquadratic forms has been developed by Tits in [18]). However, we keep in mind that this happens only for a division ring $\mathbb{H}$ of characteristic 2 with $\sigma|_{Z(\mathbb{H})} = id$. In particular, if $\mathbb{H}$ is a commutative field and σ is not the identity as for instance for hermitian forms over finite fields, then we are back in case (i).

In section 2.4, we construct a projective space $PG(4, \mathbb{H})$ for a division ring $\mathbb{H}$ which fully embeds the generalized quadrangle $\Gamma = (\mathcal{L}, \mathcal{D})$. Its points are the points of the generalized quadrangle and the subquadrangles defined by pairs of opposite lines of Γ. In section 2.3, we investigate the subquadrangle defined by two opposite lines of Γ and prove that it is not only a grid. Then by the above, Γ is a hermitian generalized quadrangle provided that the characteristic of $\mathbb{H}$ is distinct from 2 when $\mathbb{H}$ is infinite.

2.2 Spreads of subspaces of $PG(V)$ induced by $\mathcal{L}$

Let Δ be the dual of the classical thick dual polar space $\Pi \cong O_{10}^-(\mathbb{K})$ of rank 4 admitting a spread $\mathcal{L}$ with the spread property (SP). In this section, we investigate the intersections of the spread $\mathcal{L}$ with totally singular subspaces of Π which are important for the construction of an embedding space of the generalized quadrangle Γ. Three of the resulting propositions stem from [14] where we refere to for proofs.

Let $PG(V)$ be the embedding projective space of $\Pi \cong O^{-}_{10}(\mathbb{K})$ with $V = \mathbb{K}^{10}$. By the spread property (SP) a totally singular projective 3-space of Π either contains no line of $\mathcal{L}$ or contains exactly one line of $\mathcal{L}$, or $\mathcal{L}$ induces a spread in it. Moreover, a point p of Δ belongs to $\mathcal{D}$ if and only if $\mathcal{L}$ induces a spread in the projective 3-space p of Π.

Proposition 2.1. *If $p = \langle \alpha, \beta \rangle_V$ for $\alpha, \beta \in \mathcal{L}$ with $\beta \subset \alpha^{\perp_V}$, then $p \in \mathcal{D}$.*

Proof. Let q be a projective 3-space of $\mathcal{D}$ on β not containing α. In the generalized quadrangle Γ, α is a point and q a line not through α. Hence there exists a unique point on q collinear with α. Since $\beta \subset \alpha^{\perp_V}$, this point is β. Since the line of Γ through α and β is a projective 3-space of $\mathcal{D}$, it follows $p = \langle \alpha, \beta \rangle_V \in \mathcal{D}$. □

Proposition 2.2 ([14, Proposition 22]). *For $\alpha \in \mathcal{L}$, $\mathcal{L}$ induces a spread in the subspace $\alpha^{\perp_V} \cap \Pi$.*

Proposition 2.3 ([14, Proposition 23]). *For two non-orthogonal spread lines $\alpha, \beta \in \mathcal{L}$, $\mathcal{L}$ induces a spread in $\langle \alpha, \beta \rangle_V \cap \Pi$.*

In particular, if $W \cong O^{+}_{4}(\mathbb{K})$ contains two members of $\mathcal{L}$, then the lines of $\mathcal{L}$ contained in W form one of the two reguli of the quadric W. Hence the spread $\mathcal{L}$ of Π is regular.

Proposition 2.4 ([14, Corollary 8]). *Every point $p \in (\bigcup_{\delta \in \mathcal{L}} \delta) \setminus \mathcal{D}$ is contained in exactly one deep quad $\delta(p) \in \mathcal{L}$.*

2.3 The subquadrangle for two opposite lines of Γ

Denote the embedding projective space of $\Pi \cong O^{-}_{10}(\mathbb{K})$ by $\mathcal{P} = PG(V)$ with the vector space $V := \mathbb{K}^{10}$. Let L, M be two opposite lines of Γ. They are disjoint totally singular linear 4-spaces of Π spanning a non-degenerate polar subspace $\langle L, M \rangle_V \cap \Pi \cong O^{+}_{8}(\mathbb{K})$. We define a geometry $\Sigma(L, M)$ as follows: The points of $\Sigma(L, M)$ are the lines of $\mathcal{L}$ contained in $\langle L, M \rangle_V$, and the lines of $\Sigma(L, M)$ are the linear 4-spaces of $\mathcal{D}$ contained in $\langle L, M \rangle_V$.

Proposition 2.5. *For two opposite lines L, M of Γ, the geometry $\Sigma(L, M)$ is a full non-degenerate subquadrangle of Γ.*

Proof. Let N be a line of $\Sigma(L, M)$ and let α be a point of $\Sigma(L, M)$ not on N. Since Γ is a generalized quadrangle, there is a unique line J of Γ through α

intersecting N (Theorem 1.1 (c)). In Π, the linear 4-space J is contained in $\langle L, M\rangle_V$ since it contains the disjoint lines α and $J \cap N$ both being contained in $\langle L, M\rangle_V$. Thus J belongs to $\Sigma(L, M)$ and $\Sigma(L, M)$ is a generalized quadrangle.

The points of Γ on a line N of $\Sigma(L, M)$ are the lines of the spread $\mathcal{L}$ contained in N. Since N is contained in $\langle L, M\rangle_V$, the lines of $\mathcal{L}$ in N are contained in $\langle L, M\rangle_V$, too. Hence they are points of $\Sigma(L, M)$ proving that the subquadrangle $\Sigma(L, M)$ of Γ is full. $\square$

In the following proposition, we investigate the lines through a point of $\Sigma(L, M)$. From now on, by d-spaces we mean d-dimensional linear subspaces of V.

Proposition 2.6. *If α is a point of $\Sigma(L, M)$, i.e. a line of $\mathcal{L}$ contained in $\langle L, M\rangle_V \cap \Pi \cong O_8^+(\mathbb{K})$, then the lines of $\Sigma(L, M)$ through α are the 4-spaces $\langle \alpha, \beta\rangle_V$ of $\langle L, M\rangle_V$ where β runs through the lines of a regulus of a hyperbolic quadric $W \cong O_4^+(\mathbb{K})$ which is complementary to α in the 6-space $\alpha^{\perp_V} \cap \langle L, M\rangle_V$ such that the regulus is contained in $\mathcal{L}$.*

Proof. Let α be a point of the subquadrangle $\Sigma(L, M)$ of Γ, i.e. α is a line of the spread $\mathcal{L}$ of $\Pi \cong O_{10}^-(\mathbb{K})$ contained in $\langle L, M\rangle_V \cap \Pi \cong O_8^+(\mathbb{K})$. Since $\Sigma(L, M)$ is a generalized quadrangle by Proposition 2.5, there are distinct totally singular 4-spaces $B, B' \in \mathcal{D}$ through α, which are contained in $\alpha^{\perp_V} \cap \langle L, M\rangle_V$. Take any two lines $\beta, \beta' \in \mathcal{L} \setminus \{\alpha\}$ contained in B, respectively B'. Then $\beta' \cap \beta^{\perp_V} = 0$ and by Proposition 2.3, $\mathcal{L}$ induces a spread in the quadric $W = \langle \beta, \beta'\rangle_V \cong O_4^+(\mathbb{K})$ which consists of the regulus through β and β'.

Since $\langle \beta, \beta'\rangle_V = W \subset \alpha^{\perp_V} \cap \langle L, M\rangle_V \cong \alpha \oplus O_4^+(\mathbb{K})$ and $\alpha \cap W = 0$, it follows $\alpha \oplus W = \alpha^{\perp_V} \cap \langle L, M\rangle_V$. Hence the lines of $\Sigma(L, M)$ through α are the 4-spaces spanned by α and the lines of the regulus of W through β and β' which all belong to $\mathcal{L}$. $\square$

Suppose Γ is a generalized quadrangle fully embedded in $PG(4, \mathbb{H})$ for a division ring $\mathbb{H}$ of characteristic $\neq 2$ if $\mathbb{H}$ is infinite. Then, as mentioned in section 2.1, the generalized quadrangle Γ is a hermitian generalized quadrangle if and only if for any two opposite lines L and M, the subquadrangle $\Sigma(L, M)$ is not only a grid. The latter condition is established by Proposition 2.6. In the following section, we construct an embedding space $PG(4, \mathbb{H})$ for $\Gamma = (\mathcal{L}, \mathcal{D})$.

2.4 The embedding of Γ

Let $\mathcal{P} = PG(9, \mathbb{K})$ be the embedding projective space of the polar space Π dual of Δ, and let V be the vector space $\mathbb{K}^{10}$ such that $\mathcal{P} = PG(V)$. We define a point-line geometry E by means of subspaces of V and show that it is a $PG(4,\mathbb{H})$ embedding Γ. There are two types of POINTS and three types of LINES in E, and INCIDENCE is symmetrized containment.

1. POINTS of type 1 are the 8-spaces $\alpha^{\perp_V}$ of V for $\alpha \in \mathcal{L}$.
2. POINTS of type 2 are the 8-spaces $\langle L, M\rangle_V$ of V for two disjoint 4-spaces L, M of $\mathcal{D}$. Note $\langle L, M\rangle_V \cap \Pi \cong O_8^+(\mathbb{K})$.

So, the POINTS of the first type correspond with the points of the generalized quadrangle Γ. A POINT $\langle L, M\rangle_V$ of type 2 corresponds with the subquadrangle of Γ spanned by the two opposite lines L and M of Γ. Next we define the LINES of the embedding space E as 6-spaces of V:

1. The 6-spaces $L^{\perp_V}$ for $L \in \mathcal{D}$ (note $L^{\perp_V} \cong L \oplus X$ for some exterior line X of $\mathcal{P}$).
2. The 6-spaces $\alpha \oplus W$ for $\alpha \in \mathcal{L}$ with $W \subset \alpha^{\perp_V}$ and $W \cap \Pi \cong O_4^+(\mathbb{K})$ such that $\mathcal{L}$ induces a spread in $W \cap \Pi$.
3. The 6-spaces W with $W \cap \Pi \cong O_6^-(\mathbb{K})$ in which $\mathcal{L}$ induces a spread.

To prove that E is a projective space embedding the generalized quadrangle $\Gamma = (\mathcal{L}, \mathcal{D})$, we use the following Theorem by Anne Parmentier (see Buekenhout [1]). A linear space S is a projective space if it is endowed with a polarity π (i.e. a symmetric relation on the points of S) with the following properties where we set $p^\pi := \{x \in S \mid x\pi p\}$ for a point p of S and $L^\pi := \bigcap_{x\in L} x^\pi$ for a point set L of S:

(P1) For every line L and point p of S, either $L \subseteq p^\pi$ or $L \cap p^\pi$ is a point.

(P2) $L = L^{\pi\pi}$ for every line L.

(P3) $p^\pi \neq S$ for every point p.

In the remainder of this section, we show that E is a linear space and construct an appropriate polarity π in E. The following propositions and corollary provide the main technical tools.

Proposition 2.7. *If* $\mathbf{L} = \alpha \oplus W$ *is a* LINE *of type* 2*, then* $\mathcal{L}$ *induces a spread on* $\mathbf{L} \cap \Pi$.

Proof. The totally singular points in the 6-space $\mathbf{L}$ are the points in the totally singular 4-spaces $U_\beta := \langle \alpha, \beta\rangle_V$ where β is a totally singular line of the quadric $W \cap \Pi \cong O_4^+(\mathbb{K})$. Since $W \cap \Pi$ is a grid, the totally singular points of $\mathbf{L}$ are covered already by the subspaces U_β for β running through the lines of one regulus of $W \cap \Pi$. Let R be the regulus of $W \cap \Pi$ which belongs to $\mathcal{L}$. For the lines $\beta \in R$ it holds $U_\beta \in \mathcal{D}$ by Proposition 2.1 and $\mathcal{L}$ induces a spread in U_β by Proposition 2.3. Now the singular points of $\mathbf{L}$ are the disjoint union $\bigcup_{\beta \in R}(U_\beta \setminus \alpha) \cup \alpha$. Since $\mathcal{L}$ induces spreads in $U_\beta \setminus \alpha$ for all $\beta \in R$ and since $\alpha \in \mathcal{L}$, the assertion follows. □

Proposition 2.8. *Every* POINT *of type* 2 *lies on a* LINE *of type* 2*.*

Proof. Let $\mathbf{p} = \langle L, M\rangle_V$ with $\mathbf{p} \cap \Pi \cong O_8^+(\mathbb{K})$ be a POINT of type 2 with $L, M \in \mathcal{D}$. For a line $\alpha \in \mathcal{L}$ in L, the space $\mathbf{L} := \alpha^{\perp_V} \cap \mathbf{p}$ is a 6-dimensional space with radical α and $(\mathbf{L} \cap \Pi)/\alpha \cong O_4^+(\mathbb{K})$. For the line $\beta := \alpha^{\perp_V} \cap M \in \mathcal{L}$ and a line $\gamma \in \mathcal{L} \setminus \{\alpha\}$ in L, $\mathcal{L}$ induces a spread in $\langle \beta, \gamma\rangle_V$ by Proposition 2.3. Hence the 4-spaces through α in $\mathbf{L} = \alpha \oplus \langle \beta, \gamma\rangle_V$ which intersect $\langle \beta, \gamma\rangle_V$ in a line of the regulus through β and γ, belong to $\mathcal{D}$. So $\mathbf{L}$ is a LINE of type 2. □

Proposition 2.9. *The line set* $\mathcal{L}$ *induces a spread on* $\mathbf{p} \cap \Pi$ *for every* POINT $\mathbf{p}$ *of* E*.*

Proof. For a POINT $\mathbf{p}$ of type 1, this is Proposition 2.2.

Let $\mathbf{p} = \langle L, M\rangle_V$ be a POINT of type 2 with $L, M \in \mathcal{D}$. Let r be a point of $\langle L, M\rangle_V \cap \Pi$. If r is a point of L or M, it lies on a line of $\mathcal{L}$ contained in $\mathbf{p}$ since $\mathcal{L}$ induces spreads $\mathcal{L}_L$ in L and $\mathcal{L}_M$ in M. Suppose r is neither contained in L nor in M, and let α be the line of $\mathcal{L}$ containing r. If α is not contained in $\mathbf{p}$, then $\alpha^{\perp_V} \cap \mathbf{p}$ is a 6-space of V with radical r and $((\alpha^{\perp_V} \cap \mathbf{p}) \cap \Pi)/r \cong O_5(\mathbb{K})$.

According to Proposition 2.8, let $\mathbf{L} = \zeta^{\perp_V} \cap \mathbf{p}$ be a LINE of type 2 for a line $\zeta \in \mathcal{L}_L$. Then $r \notin \mathbf{L}$ since $\mathcal{L}$ induces a spread in $\mathbf{L} \subset \mathbf{p}$, but the spread line α through r is not contained in $\mathbf{p}$ by assumption. It follows that $r \notin \zeta^{\perp_V}$ and α is complementary to $\zeta^{\perp_V}$ by Proposition 2.2. Putting $Z := \alpha^{\perp_V} \cap \zeta^{\perp_V}$, it holds $Z \cap \Pi \cong O_6^-(\mathbb{K})$. By Proposition 2.2, $\mathcal{L}$ induces a spread in $Z \cap \Pi$. Since ζ is disjoint from $\alpha^{\perp_V}$, it follows $(\alpha^{\perp_V} \cap \mathbf{L}) \cap \Pi \cong O_4^+(\mathbb{K})$.

The line $\xi := \alpha^{\perp_V} \cap L$ belongs to $\mathcal{L}$ since $\mathcal{L}$ induces spreads in both $\alpha^{\perp_V}$ and L. As in the proof of Proposition 2.8, the 6-space $X := \xi^{\perp_V} \cap \mathbf{p}$ is a LINE of type 2, and $\mathcal{L}$ induces a spread in $X \cap \Pi$. On the one hand, since the spread line α through r is not contained in $\mathbf{p}$, it follows $r \notin X$. On the other hand it holds $r = \alpha \cap \mathbf{p} \in \xi^{\perp_V} \cap \mathbf{p} = X$ and we have reached a contradiction. □

Corollary 2.10. *The* 8*-spaces of two* POINTS *of* E *intersect in a* 6*-space* W *of* V *such that* $\mathcal{L}$ *induces a spread in* $W \cap \Pi$*.*

Proof. First, let $\mathbf{p}$ be a POINT of E of type 2 and $L, M \in \mathcal{D}$ maximal singular subspaces of Π such that $\mathbf{p} = \langle L, M\rangle_V$. Suppose $\mathbf{r}$ is a POINT of E with

$\dim(\mathbf{p} \cap \mathbf{r}) \geq 7$. Then $\dim(L \cap \mathbf{r}) \geq 3$. Since, by Proposition 2.9, $\mathcal{L}$ induces a spread in L and in $\mathbf{r} \cap \Pi$, it follows $L \subset \mathbf{r}$. Similarly, $M \subset \mathbf{r}$, whence $\mathbf{p} = \mathbf{r}$.

Secondly, suppose $\mathbf{p}$ and $\mathbf{r}$ are POINTS of E of type 1 and let $\alpha, \beta \in \mathcal{L}$ be lines such that $\mathbf{p} = \alpha^{\perp_V}$ and $\mathbf{r} = \beta^{\perp_V}$. Then $\mathbf{p} \cap \mathbf{r} = \{\alpha, \beta\}^{\perp_V}$. Since $\langle \alpha, \beta \rangle_V$ has dimension 4, it follows $\dim(\mathbf{p} \cap \mathbf{r}) = 6$.

Since by Proposition 2.9, $\mathcal{L}$ induces a spread of the singular points in each 8-space of V being a POINT of E, $\mathcal{L}$ induces a spread of the singular points in the intersection of two such subspaces. □

Proposition 2.11. *Any two* POINTS *of E lie on a unique* LINE.

Proof. By Corollary 2.10, the intersection of any two POINTS of E is a 6-space of V. Thus the uniqueness of a LINE through two POINTS of E is immediate. It remains to show existence, i.e. the intersection of two 8-spaces being POINTS of E is of one of the types defining a LINE of E.

First, if $\mathbf{a}, \mathbf{b}$ are two POINTS of type 1, then there are lines $\alpha, \beta \in \mathcal{L}$ such that $\mathbf{a} = \alpha^{\perp_V}$ and $\mathbf{b} = \beta^{\perp_V}$. Either α and β are orthogonal in Π spanning a singular 4-space belonging to $\mathcal{D}$, or $\langle \alpha, \beta \rangle_V \cap \Pi$ is a hyperbolic quadric $O_4^+(\mathbb{K})$. In the latter case, $\alpha^{\perp_V} \cap \beta^{\perp_V}$ is a 6-space $W := \{\alpha, \beta\}^{\perp_V}$ with $W \cap \Pi \cong O_6^-(\mathbb{K})$ and $\mathcal{L}$ induces a line spread of $W \cap \Pi$ since $\mathcal{L}$ induces line spreads in both $\alpha^{\perp_V} \cap \Pi$ and $\beta^{\perp_V} \cap \Pi$. Hence W is the single LINE of E of type 3 containing the POINTS $\mathbf{a}$ and $\mathbf{b}$.

In the former case, $L := \langle \alpha, \beta \rangle_V$ is a singular 4-space of Π belonging to $\mathcal{D}$, and $L^{\perp_V}$ is the unique LINE of E of type 1 through the POINTS $\mathbf{a}$ and $\mathbf{b}$.

Secondly, let $\mathbf{a}$ be a POINT of type 1 and $\mathbf{b}$ be a POINT of type 2. There is a line $\alpha \in \mathcal{L}$ such that $\mathbf{a}$ is its perp $\alpha^{\perp_V}$ and there are disjoint 4-spaces L, M of $\mathcal{D}$ such that $\mathbf{b}$ is the 8-space $\langle L, M \rangle_V$ with $\langle L, M \rangle_V \cap \Pi \cong O_8^+(\mathbb{K})$. By Proposition 2.9, α is either contained in $\langle L, M \rangle_V$ or disjoint from $\langle L, M \rangle_V$. In the former case by Corollary 2.10, the space $\alpha^{\perp_V}$ intersects $\langle L, M \rangle_V$ in a unique 6-space $W := \alpha \oplus U$ where $U \cap \Pi$ is a hyperbolic quadric $O_4^+(\mathbb{K})$. Hence W is the unique LINE of E through $\mathbf{a}$ and $\mathbf{b}$ and is of type 2.

In the latter case, $\langle L, M \rangle_V$ intersects $\alpha^{\perp_V}$ in a complement W of α with $W \cap \Pi \cong O_6^-(\mathbb{K})$ in which $\mathcal{L}$ induces a spread by Proposition 2.9. Hence W is the unique LINE of E of type 3 containing $\mathbf{a}$ and $\mathbf{b}$.

Thirdly, let $\mathbf{a}$ and $\mathbf{b}$ be two POINTS of type 2. There are opposite 4-spaces $L, M \in \mathcal{D}$ such that $\mathbf{a} = \langle L, M \rangle_V$ with $\langle L, M \rangle_V \cap \Pi \cong O_8^+(\mathbb{K})$, and opposite 4-spaces $K, N \in \mathcal{D}$ such that $\mathbf{b} = \langle K, N \rangle_V$ with $\langle K, N \rangle_V \cap \Pi \cong O_8^+(\mathbb{K})$. Denote the 6-space $\langle L, M \rangle_V \cap \langle K, N \rangle_V$ of V by U. We claim either $U \cap \Pi \cong O_6^-(\mathbb{K})$ or

$U \cong \alpha \oplus W$ where α is a line of $\mathcal{L}$ and $W \cap \Pi$ is a hyperbolic quadric $O_4^+(\mathbb{K})$ contained in $\alpha^{\perp_V}$.

For, first suppose $\langle L, M\rangle_V$ and $\langle K, N\rangle_V$ share a singular 4-space of $\mathcal{D}$, say $K = L$. Since $\langle L, M\rangle_V$ intersects the 8-space $\langle L, N\rangle_V$ in a 6-space, $N \cap \langle L, M\rangle_V$ is a line λ of Π which belongs to $\mathcal{L}$ by Proposition 2.9. Similarly, $\mu = M \cap \langle L, N\rangle_V$ is a line of $\mathcal{L}$. We assume $\lambda \neq \mu$ since otherwise we may choose a different $N \in \mathcal{D}$ defining $\mathbf{b}$ such that $\lambda \neq \mu$. It follows $\langle L, \lambda\rangle_V = \langle L, \mu\rangle_V$ and $W = \langle \lambda, \mu\rangle_V$ with $\langle \lambda, \mu\rangle_V \cap \Pi \cong O_4^+(\mathbb{K})$ or W is singular and belongs to $\mathcal{D}$. Firstly, if $W \cap \Pi \cong O_4^+(\mathbb{K})$, then let α be the unique line of L perpendicular to λ. Then $\alpha^{\perp_V} \supseteq W$, as $W = \langle L, \lambda\rangle_V$. Then the intersection $\langle L, M\rangle_V \cap \langle L, N\rangle_V$ is the 6-space $U = \langle L, \lambda\rangle_V = \alpha \oplus W$. Thus U is a LINE of type 2 of E. Secondly, if $W \in \mathcal{D}$, then $\alpha = W \cap L$ belongs to $\mathcal{L}$. For every $\nu \in \mathcal{L}$ in L different from α, it holds $\langle \mu, \nu\rangle_V \cong \langle \lambda, \nu\rangle_V$ with $\langle \lambda, \nu\rangle_V \cap \Pi \cong O_4^+(\mathbb{K})$. So, in this case $\langle L, M\rangle_V \cap \langle L, N\rangle_V$ is a LINE of type 2, too.

Next, suppose $\langle L, M\rangle_V$ and $\langle K, N\rangle_V$ do not share any singular 4-space of $\mathcal{D}$. In particular, K and N are not contained in $\langle L, M\rangle_V$. Since $\mathcal{L}$ induces spreads in $\langle L, M\rangle_V \cap \Pi$ by Proposition 2.9 and in K and N, and since K and N intersect $\langle L, M\rangle_V$ at least in lines for dimension reasons, it follows that $K \cap \langle L, M\rangle_V$ and $N \cap \langle L, M\rangle_V$ are lines κ and ν, respectively, belonging to $\mathcal{L}$. W.l.o.g. assume $\kappa = K \cap M$, $\nu = N \cap L$, and $K \cap L = \{0\} = M \cap N$. By assumption, κ and ν are not orthogonal since otherwise $\langle \kappa, \nu\rangle_V \in \mathcal{D}$, and it follows $\langle \kappa, \nu\rangle_V \cap \Pi \cong O_4^+(\mathbb{K})$. Now there are two possibilities for the radical of U: If U is non-degenerate, then $U \cap \Pi \cong O_6^-(\mathbb{K})$ and U is a LINE of type 3 of E since $\mathcal{L}$ induces a spread in $U \cap \Pi$ by Corollary 2.10. If the radical of U contains a point p, then the spread line τ covering p is contained in U since $p \in \kappa^{\perp_\Pi} \cap \nu^{\perp_\Pi}$ and since $\mathcal{L}$ induces a spread in $\kappa^{\perp_\Pi} \cap \nu^{\perp_\Pi}$ by Corollary 2.10. On the other hand, it holds $\tau \neq \kappa, \nu$. Consequently $\langle \tau, \kappa\rangle_V$ and $\langle \tau, \nu\rangle_V$ belong to $\mathcal{D}$. So $\mathbf{a}$ and $\mathbf{b}$ share an element of $\mathcal{D}$ in contradiction to the assumption. $\square$

Before defining a polarity π on the linear space E, we describe the LINES of E as sets of POINTS in the following proposition. We remark already here that, when we will have proved that E is a projective space embedding Γ in Proposition 2.14, then the following proposition will enlighten that LINES of type 1 are the embedded lines of Γ, LINES of type 2 are the tangents of E at Γ and LINES of type 3 are the secants of E at Γ.

Proposition 2.12. *All* POINTS *on* LINES *of type* 1 *are of type* 1*, every* LINE *of type* 2 *has exactly one* POINT *of type* 1*, and the* POINTS *of type* 1 *on a* LINE $\mathbf{L}$ *of type* 3 *with* $\mathbf{L} \cap \Pi \cong O_6^-(\mathbb{K})$ *are the* 8*-spaces* $\alpha^{\perp_V}$ *for all* $\alpha \subset \mathbf{L}^{\perp_V} \cap \Pi \cong O_4^+(\mathbb{K})$ *belonging to* $\mathcal{L}$*. Considering the* POINTS *and* LINES *of* E *as subspaces of* V*, the set of* POINTS *on a* LINE $\mathbf{L}$ *of* E *of type* 1 *induces a spread of the dual* $(V/\mathbf{L})^*$ *of the quotient* $V/\mathbf{L}$*, and the* POINTS *on a* LINE $\mathbf{L}$ *of type* 2 *or* 3 *induce spreads of* $V/\mathbf{L}$*.*

Proof. If $\mathbf{L}$ is a LINE of type 1, i.e. $\mathbf{L}^{\perp_V} \in \mathcal{D}$, then the POINTS on $\mathbf{L}$ contain the 6-space $\mathbf{L}$. All POINTS $\alpha^{\perp_V}$ for lines $\alpha \in \mathcal{L}$ contained in $\mathbf{L}^{\perp_V}$ contain $\mathbf{L}$. Since no POINT which is a hyperbolic quadric $O_8^+(\mathbb{K})$, contains $\mathbf{L}$, the LINES of type 1 are the embedded lines of Γ. If X is a 9-space of V containing $\mathbf{L}$, then X is degenerate and its radical is contained in $\mathbf{L}^{\perp_V}$. Hence the unique $\alpha \in \mathcal{L}$ through the radical of X is contained in $\mathbf{L}^{\perp_V}$ and gives the POINT $\alpha^{\perp_V}$ contained in X. For any line $\beta \in \mathcal{L}$ with $\beta^{\perp_V} \subseteq X$, let $p \in X \setminus \beta^{\perp_V}$. Then $p^{\perp_V} \cap \beta$ is the radical of X, whence $\beta = \alpha$. This proves the second assertion for LINES of type 1.

If $\mathbf{L}$ is a LINE of type 2 with radical $\alpha \in \mathcal{L}$ considering $\mathbf{L}$ as 6-space of V, then the only POINT of type 1 on $\mathbf{L}$ is $\alpha^{\perp_V}$. For each line $\beta \in \mathcal{L}$ not in $\alpha^{\perp_V}$, it follows $\beta \cap \alpha^{\perp_V} = \emptyset$ since if $\beta \cap \alpha^{\perp_V} \neq \emptyset$, then $\beta \subseteq \alpha^{\perp_V}$ by Proposition 2.2. Hence $\beta^{\perp_V}$ intersects the 6-space $\mathbf{L}$ in a hyperbolic quadric $O_4^+(\mathbb{K})$ complementary to α in $\mathbf{L}$. Then the 8-space $\langle \mathbf{L}, \beta\rangle_V$ is a hyperbolic quadric $O_8^+(\mathbb{K})$, whence a POINT of type 2.

We now prove that if R is a 7-space containing $\mathbf{L}$, then $R \setminus \mathbf{L}$ contains a singular point r. For, if $R \setminus \mathbf{L}$ does not contain any singular point, then all lines through singular points of $\mathbf{L}$ and points of $R \setminus \mathbf{L}$ are tangents implying $R \setminus \mathbf{L} \subseteq \mathbf{L}^{\perp_V}$. This is impossible, since $R \setminus \mathbf{L}$ is an affine 7-space and $\mathbf{L}^{\perp_V}$ is a 4-space.

Let now r be a singular point of $R \setminus \mathbf{L}$. As we have seen above, the spread line $\rho \in \mathcal{L}$ covering r is disjoint from $\mathbf{L}$ and defines a POINT $\langle \mathbf{L}, \rho\rangle_V$ of type 2 of E. Considering these POINTS of type 2 in the quotient $V/\mathbf{L}$ as subspaces $\langle \mathbf{L}, \rho\rangle_V/\mathbf{L}$, they are disjoint by means of Proposition 2.9 and cover the whole of $V/\mathbf{L}$ by the previous paragraph. Hence we have proved the second assertion of the proposition for LINES of type 2.

Let $\mathbf{L}$ be a LINE of type 3, i.e. $\mathbf{L} \cap \Pi \cong O_6^-(\mathbb{K})$. For each line $\alpha \in \mathcal{L}$ lying in $\mathbf{L}^{\perp_V} \cap \Pi \cong O_4^+(\mathbb{K})$, the POINT $\alpha^{\perp_V}$ lies on $\mathbf{L}$. Since $\mathcal{L}$ induces a spread in $\mathbf{L} \cap \Pi$, it also induces a spread in $\mathbf{L}^{\perp_V} \cap \Pi$ by Proposition 2.2. Hence one of the two reguli of $\mathbf{L}^{\perp_V} \cap \Pi$ belongs to $\mathcal{L}$ and defines POINTS of type 1 on $\mathbf{L}$.

Let $\beta \in \mathcal{L}$ such that $\beta \not\subseteq \langle \mathbf{L}, \alpha\rangle_V$ for any $\alpha \in \mathcal{L}$ with $\alpha \subset \mathbf{L}^{\perp_V} \cap \Pi$. Then $\beta \cap \mathbf{L} = \emptyset$. Let $X := \langle \mathbf{L}, \beta\rangle_V$, $p \in \beta^{\perp_\Pi} \cap \mathbf{L}$ and γ be the unique element of $\mathcal{L}$ through p. Then $\gamma \subset \mathbf{L}$ by the definition of LINES of type 3 and $\gamma \perp_\Pi \beta$ by Proposition 2.2. So, the 4-space $B := \langle \beta, \gamma\rangle_V$ belongs to $\mathcal{D}$. We next show that X contains an $M \in \mathcal{D}$ disjoint from B. For, let $\alpha \in \mathcal{L}$ be contained in $\mathbf{L}$ and disjoint from $Y := \beta^{\perp_V} \cap \mathbf{L}$. Such an α exists since $\langle Y\rangle_V$ is 4-dimensional and $\mathcal{L}$ induces a spread on $\beta^{\perp_V} \cap \Pi$ by Proposition 2.2 which is one of the two reguli of $Y \cap \Pi \cong O_4^+(\mathbb{K})$. Let $\gamma' \in \mathcal{L}$ contained in Y disjoint from γ and consider $B' := \langle \beta, \gamma' \rangle_V$. Then $B' \in \mathcal{D}$ and $\delta := \alpha^{\perp_V} \cap B'$ belongs to $\mathcal{L}$ by Proposition 2.2. Hence $M := \langle \alpha, \delta \rangle_V$ is a member of $\mathcal{D}$ as required. It follows that the 8-space $\langle \mathbf{L}, \beta \rangle_V \cap \Pi$ is a hyperbolic quadric $O_8^+(\mathbb{K})$, whence a POINT of type 2 on the LINE $\mathbf{L}$. Hence all POINTS on $\mathbf{L}$ are defined by lines of $\mathcal{L}$ such that in the quotient $V/\mathbf{L}$, any two of them are disjoint by Proposition 2.9.

It remains to show that there is no point of $V/\mathbf{L}$ which is not covered by the quotient $X/\mathbf{L}$ for any POINT X on $\mathbf{L}$. For suppose R is a 7-space containing $\mathbf{L}$ such that $R \setminus \mathbf{L}$ does not contain any singular point. Similarly to the consideration of LINES of type 2, it follows that every line through a singular point of $\mathbf{L}$ and any point of $R \setminus \mathbf{L}$ is a tangent, whence $R \setminus \mathbf{L} \subseteq \mathbf{L}^{\perp_V}$. Again this is impossible for dimension reasons.

Hence every 7-space of V containing $\mathbf{L}$ contains a singular point not in $\mathbf{L}$ which is covered by a unique 8-space being a POINT on $\mathbf{L}$. Thus the POINTS on $\mathbf{L}$ form a spread of the quotient $V/\mathbf{L}$. □

Now we define a polarity π on the linear space E by means of the polarity $\perp_V$ of $V = \mathbb{K}^{10}$ which is defined through the polar space $\Pi \cong O_{10}^-(\mathbb{K})$.

- For a POINT $\mathbf{p}$ of E, $\mathbf{p}^\pi$ is the set of POINTS of E that contain $\mathbf{p}^{\perp_V}$.
- For a LINE $\mathbf{L}$ of E, we set $\mathbf{L}^\pi := \bigcap_{\mathbf{p} \in \mathbf{L}} \mathbf{p}^\pi$.

We understand the set $\mathbf{L}^{\pi\pi}$ to be $\mathbf{L}^{\pi\pi} = \bigcap_{\mathbf{p} \in \mathbf{L}^\pi} \mathbf{p}^\pi$.

Proposition 2.13. *For a* LINE $\mathbf{L}$*, the set* $\mathbf{L}^\pi$ *consists of the* POINTS *of* E *containing* $\mathbf{L}^{\perp_V}$.

Proof. If $\mathbf{q}$ is a POINT of $\mathbf{L}^\pi = \bigcap_{\mathbf{p} \in \mathbf{L}} \mathbf{p}^\pi$, then $\mathbf{p}^{\perp_V} \subseteq \mathbf{q}$ for all $\mathbf{p} \in \mathbf{L}$. It follows $\mathbf{q}^{\perp_V} \subseteq \mathbf{p}$ for all $\mathbf{p} \in \mathbf{L}$, whence $\mathbf{q}^{\perp_V} \subseteq \bigcap_{\mathbf{p} \in \mathbf{L}} \mathbf{p} = \mathbf{L}$. So $\mathbf{L}^{\perp_V} \subseteq \mathbf{q}$.

Vice versa, let $\mathbf{q}$ be a POINT with $\mathbf{L}^{\perp_V} \subseteq \mathbf{q}$. So $\mathbf{q}^{\perp_V} \subseteq \mathbf{L}$, whence $\mathbf{q}^{\perp_V} \subseteq \mathbf{p}$ for all POINTS $\mathbf{p}$ on $\mathbf{L}$. It follows $\mathbf{p}^{\perp_V} \subseteq \mathbf{q}$ for all $\mathbf{p} \in \mathbf{L}$, thus $\mathbf{q} \in \bigcap_{\mathbf{p} \in \mathbf{L}} \mathbf{p}^\pi$. □

Proposition 2.14. *The mapping* π *is a polarity of* E.

Proof. We show that π has the properties (P1)–(P3) stated at the beginning of this section. (P3) follows straightforward from the corresponding property of the polarity $\perp_V$ of V.

For (P2), by Proposition 2.13 the set $\mathbf{L}^\pi$ consists of all POINTS containing the 4-space $\mathbf{L}^{\perp_V}$. The set $\mathbf{L}^{\pi\pi} = \bigcap_{\mathbf{p} \in \mathbf{L}^\pi} \mathbf{p}^\pi$ is the set of POINTS that contain $\mathbf{p}^{\perp_V}$ for all POINTS $\mathbf{p}$ containing $\mathbf{L}^{\perp_V}$. Clearly, $\mathbf{L}^{\pi\pi}$ contains the POINTS containing the 6-space $\mathbf{L}$, whence $\mathbf{L} \subseteq \mathbf{L}^{\pi\pi}$. It remains to show $\mathbf{L}^{\pi\pi} \subseteq \mathbf{L}$, or equivalently that the intersection of the family of POINTS containing $\mathbf{L}^{\perp_V}$ is not larger than $\mathbf{L}^{\perp_V}$.

We consider the three types of LINES separately. First suppose $\mathbf{L}$ is a line of type 1 with $\mathbf{L}^{\perp_V} \in \mathcal{D}$. Then the POINTS of type 1 containing $\mathbf{L}^{\perp_V}$ are the 8-spaces $\alpha^{\perp_V}$ for the lines $\alpha \in \mathcal{L}$ contained in $\mathbf{L}^{\perp_V}$. Hence the intersection of these points is $\mathbf{L}$. The POINTS of type 2 containing $\mathbf{L}^{\perp_V}$ are the 8-spaces isomorphic to $O_8^+(\mathbb{K})$ containing $\mathbf{L}^{\perp_V}$ as a generator. They intersect $\mathbf{L}$ in $\mathbf{L}^{\perp_V}$ only. Hence the intersection of the POINTS containing $\mathbf{L}^{\perp_V}$ is exactly $\mathbf{L}^{\perp_V}$.

If $\mathbf{L}$ is a LINE of type 2, then $\mathbf{L} \cong \alpha \oplus W$ for a line $\alpha \in \mathcal{L}$ and a 4-space W of V with $W \cap \Pi \cong O_4^+(\mathbb{K})$ contained in $\alpha^{\perp_V}$ and $\mathcal{L}$ inducing on W one of its two reguli. The 4-space $\mathbf{L}^{\perp_V}$ is contained in $\alpha^{\perp_V}$ going through α, its singular points are those on α. The POINTS of type 1 containing $\mathbf{L}^{\perp_V}$ are the POINTS $\beta^{\perp_V}$ for $\beta \in \mathcal{L}$ with $\beta \subset \mathbf{L}$; their intersection is contained in $W^{\perp_V} \cap \alpha^{\perp_V} = \mathbf{L}^{\perp_V}$.

If $\mathbf{L}$ is a LINE of type 3, then $\mathbf{L} \cap \Pi \cong O_6^-(\mathbb{K})$ and $\mathbf{L}^{\perp_V} \cap \Pi$ is a hyperbolic quadric $O_4^+(\mathbb{K})$. The POINTS of type 1 containing $\mathbf{L}^{\perp_V}$ are the POINTS $\alpha^{\perp_V}$ for $\alpha \in \mathcal{L}$ with $\mathbf{L}^{\perp_V} \subset \alpha^{\perp_V}$, whence $\alpha \subset \mathbf{L}$. Since $\mathcal{L}$ induces a spread in $\mathbf{L} \cap \Pi$, the intersection of the POINTS $\alpha^{\perp_V}$ with $\alpha \subset \mathbf{L}$ and $\alpha \in \mathcal{L}$ is precisely $\mathbf{L}^{\perp_V}$.

It remains to show (P1). For, let $\mathbf{p}$ be a POINT and $\mathbf{L}$ be a LINE of E. If $\mathbf{p}$ is a POINT of type 1, then $\mathbf{p}^{\perp_V}$ is a totally singular line belonging to $\mathcal{L}$. Otherwise, $\mathbf{p}^{\perp_V}$ is a line of $PG(V)$ exterior to Π. Clearly, if $\mathbf{p}^{\perp_V} \subseteq \mathbf{L}$, then all POINTS on $\mathbf{L}$ belong to $\mathbf{p}^{\pi}$. So, suppose $\mathbf{p}^{\perp_V} \not\subseteq \mathbf{L}$. Then $\mathbf{p}^{\perp_V}$ and $\mathbf{L}$ either are disjoint or $\mathbf{p}^{\perp_V} \cap \mathbf{L}$ is a non-singular point. For, if $\mathbf{p}^{\perp_V} \cap \mathbf{L}$ would be a singular point, then $\mathbf{p}$ would be a POINT of type 1, whence $\mathbf{p}^{\perp_V} \in \mathcal{L}$ and $\mathbf{p}^{\perp_V} \subset \mathbf{L}$ by the definition of LINES and since $\mathcal{L}$ induces a spread in each member of $\mathcal{D}$ and in each POINT by Proposition 2.9.

We investigate separately the cases $\mathbf{L}$ is either of type 1, or of type 2 or 3. Suppose firstly that $\mathbf{L}$ is a LINE of type 1, namely $\mathbf{L}^{\perp_V} \in \mathcal{D}$. If $\mathbf{p}$ is of type 1, then $\mathbf{p}^{\perp_V} \in \mathcal{L}$, $\mathbf{L}$ and $\mathbf{p}^{\perp_V}$ are disjoint and $\mathbf{q} := \langle \mathbf{L}, \mathbf{p}^{\perp_V} \rangle_V$ is 8-dimensional. If $\alpha := \mathbf{p} \cap \mathbf{L}^{\perp_V}$, then $\alpha \in \mathcal{L}$ by Propositions 2.4 and 2.2 and $\mathbf{q} = \alpha^{\perp_V}$. Thus $\mathbf{q}$ is a POINT of type 1 and the unique POINT on $\mathbf{L}$ belonging to $\mathbf{p}^{\pi}$.

If $\mathbf{p}$ is of type 2, then the line $\mathbf{p}^{\perp_V}$ contains no singular points. Suppose firstly $\mathbf{p}^{\perp_V}$ and $\mathbf{L}$ are disjoint. Then $Q := \langle \mathbf{L}, \mathbf{p}^{\perp_V} \rangle_V$ is 8-dimensional. The line $\alpha := Q^{\perp_V} = \mathbf{p} \cap \mathbf{L}^{\perp_V}$ is a line of Π belonging to $\mathcal{L}$ since $\mathcal{L}$ induces spreads in both $\mathbf{L}^{\perp_V}$ and $\mathbf{p} \cap \Pi$ by Propositions 2.4 and 2.9. Hence $\mathbf{q}$ is a POINT of type 1 and the unique POINT of $\mathbf{p}^{\pi}$ on $\mathbf{L}$.

Suppose now that $\mathbf{p}^{\perp_V} \cap \mathbf{L}$ is a point. Then $\mathbf{p} \cap \mathbf{L}^{\perp_V}$ contains a 3-space X. Since $\mathcal{L}$ induces spreads in $\mathbf{L}^{\perp_V}$ and $\mathbf{p} \cap \Pi$, it follows $\mathbf{L}^{\perp_V} \subset \mathbf{p}$. But then $\mathbf{p}^{\perp_V} \subset \mathbf{L}$ in contradiction to our assumptions on $\mathbf{L}$ and $\mathbf{p}^{\perp_V}$.

It remains to consider the cases where $\mathbf{L}$ has type 2 or 3. If $\mathbf{p}$ is of type 1, then the conclusions follow from the fact that the set of POINTS on $\mathbf{L}$ define a spread of $V/\mathbf{L}$ by Proposition 2.12 and $\mathcal{L}$ induces a spread in $\mathbf{q} \cap \mathbf{p}$ for every POINT $\mathbf{q}$ by Proposition 2.9. So, only the case in which $\mathbf{p}$ is of type 2 remains to consider. Henceforth $\mathbf{p}$ is assumed to be of type 2, $\mathbf{L}$ is of type 2 or 3 and $\mathbf{p}^{\perp_V} \not\subseteq \mathbf{L}$. We first prove the following.

Claim. *For every* POINT $\mathbf{q} \neq \mathbf{p}$*, if* $\mathbf{q} \cap \mathbf{p}^{\perp_V}$ *contains a point, then* $\mathbf{p}^{\perp_V} \subseteq \mathbf{q}$.

Suppose that $\mathbf{q} \cap \mathbf{p}^{\perp_V}$ is a point. Then $X := \langle \mathbf{q}, \mathbf{p}^{\perp_V} \rangle_V$ is 9-dimensional and $(\mathbf{q}^{\perp_V} \cap \mathbf{p})^{\perp_V}$ contains X. As Π is non-degenerate, it follows $(\mathbf{q}^{\perp_V} \cap \mathbf{p})^{\perp_V} = X$ and $\mathbf{q}^{\perp_V} \cap \mathbf{p}$ is a point. If $\mathbf{q}$ would be a POINT of type 1, then $\mathbf{q}^{\perp_V} \in \mathcal{L}$. Since $\mathcal{L}$ induces a spread in $\mathbf{p} \cap \Pi$ by Proposition 2.9, the spread line covering the point $\mathbf{q}^{\perp_V} \cap \mathbf{p}$ is contained in $\mathbf{p}$ and equals the line $\mathbf{q}^{\perp_V}$ – a contradiction. So the point $\mathbf{q}^{\perp_V} \cap \mathbf{p}$ is non-singular, and $\mathbf{q}$ is of type 2.

By Proposition 2.12 the LINE $\mathbf{pq} = \mathbf{p} \cap \mathbf{q}$ is of type 2 or 3. Suppose it is of type 2. Then it contains a 4-space $M \in \mathcal{D}$. The point $p := \mathbf{q} \cap \mathbf{p}^{\perp_V}$ is non-singular since $\mathbf{p}^{\perp_V}$ is an exterior line by the assumption that $\mathbf{p}$ is of type 2. On the other hand, p is orthogonal to M since $M \subset \mathbf{p}$. This is impossible since every maximal singular subspace of $\mathbf{q} \cap \Pi \cong O_8^+(\mathbb{K})$ is its own perp in $\mathbf{q}$. Therefore the LINE $\mathbf{pq}$ is of type 3. Then $(\mathbf{p} \cap \mathbf{q})^{\perp_V} \cap \Pi \cong O_4^+(\mathbb{K})$ and $(\mathbf{p} \cap \mathbf{q})^{\perp_V} \cap \mathbf{q} \cap \Pi$ is a singular line of $\mathcal{L}$. This is a contradiction to $\mathbf{p}^{\perp_V}$ being an exterior line. Hence the claim is proved.

We now go back to the proof of property (P1) and the LINE $\mathbf{L}$. Let p be a point of $PG(V)$ in $\mathbf{p}^{\perp_V} \setminus \mathbf{L}$ and $\mathbf{q}$ be the POINT on $\mathbf{L}$ containing p which is uniquely determined since the POINTS on $\mathbf{L}$ form a spread of $V/\mathbf{L}$ by Proposition 2.12. Then $\mathbf{q}$ contains $\mathbf{p}^{\perp_V}$ by the above lemma. Clearly, $\mathbf{q}$ is the unique POINT of $\mathbf{L}$ containing $\mathbf{p}^{\perp_V}$. Property (P1) and Proposition 2.14 are proved. □

By a Theorem of Anne Parmentier (Buekenhout [1]), the linear space E is a projective space by Propositions 2.11 and 2.14. The following proposition determines its dimension.

Proposition 2.15. $\dim(E) = 4$.

Proof. We first prove $\dim(E) \geq 4$. For a line $\alpha \in \mathcal{L}$ and a 4-space $L \in \mathcal{D}$ through α, let $\mathbf{p}$ be the POINT $\alpha^{\perp_V}$ of type 1, and $\mathbf{L}$ be the LINE $\mathbf{L} := L^{\perp_V}$ of type 1. Then the set $\mathbf{L}^\pi$ of POINTS containing $\mathbf{L}^{\perp_V} = L$ and the set $\mathbf{p}^\pi$ of POINTS containing $\mathbf{p}^{\perp_V} = \alpha$ are subspaces of E, and it holds $E \supset \mathbf{p}^\pi \supset \mathbf{L}^\pi \supset \mathbf{L} \supset \mathbf{p}$. Hence $\{\mathbf{p}, \mathbf{L}, \mathbf{L}^\pi, \mathbf{p}^\pi\}$ is a flag of E, whence $\dim(E) \geq 4$.

It remains to show $\dim(E) \leq 4$. For elements e_1 and e_2 of E, $\langle e_1, e_2\rangle_E$ denotes the subspace of E spanned by e_1 and e_2. Given $\mathbf{p}, \mathbf{L}, \mathbf{L}^\pi$ and $\mathbf{p}^\pi$ as above, it suffices to show

1. $\langle \mathbf{L}, \mathbf{q}\rangle_E = \mathbf{L}^\pi$ for some POINT $\mathbf{q} \in \mathbf{L}^\pi \setminus \mathbf{L}$,
2. $\langle \mathbf{L}^\pi, \mathbf{q}\rangle_E = \mathbf{p}^\pi$ for some POINT $\mathbf{q} \in \mathbf{p}^\pi \setminus \mathbf{L}^\pi$, and
3. $\langle \mathbf{p}^\pi, \mathbf{q}\rangle_E = E$ for some POINT $\mathbf{q} \in E \setminus \mathbf{p}^\pi$.

Proof of Claim 1. Choose a POINT $\mathbf{q} = \langle L, M\rangle_V$ of type 2 with $M \in \mathcal{D}$ disjoint from L. Clearly, $\mathbf{q} \in \mathbf{L}^\pi \setminus \mathbf{L}$. Let $\mathbf{q}'$ be any other POINT in $\mathbf{L}^\pi \setminus \mathbf{L}$. Then $\mathbf{q}'$ is of type 2, say $\mathbf{q}' = \langle L, M'\rangle_V$ for $M' \in \mathcal{D}$ disjoint from L, since otherwise it would contain $L^{\perp_V}$, whence be a POINT on the LINE $\mathbf{L}$. The LINE $\mathbf{L}' := \mathbf{q} \cap \mathbf{q}'$ is of type 2. Hence by Lemma 2.12 the LINE $\mathbf{L}'$ is tangent to Γ in $\mathbf{p}' := \lambda^{\perp_V}$ for a line $\lambda \in \mathcal{L}$ contained in L. So, $\mathbf{p}'$ is a POINT of the LINE $\mathbf{L}$. This proves $\mathbf{q}' \in \langle \mathbf{L}, \mathbf{q}\rangle_E$.

Proof of Claim 2. Choose a POINT $\mathbf{q} = \beta^{\perp_V}$ for $\beta \in \mathcal{L}$ such that $\mathbf{q} \cap L = \alpha$. So, $\mathbf{q} \in \mathbf{p}^\pi \setminus \mathbf{L}^\pi$. Let $\mathbf{q}'$ be any other POINT containing α but not L. So, $\mathbf{q}' \cap L = \alpha$. It remains to show that the LINE $\mathbf{M} := \mathbf{q} \cap \mathbf{q}'$ contains a POINT of $\mathbf{L}^\pi$.

Case 1. $\mathbf{q}'$ is of type 1, say $\mathbf{q}' = \gamma^{\perp_V}$ for a $\gamma \in \mathcal{L}$. Then $\mathbf{M}$ is a LINE of type 1 or 3 by Proposition 2.12. If $\beta \perp_\Pi \gamma$, then $\mathbf{M}$ is of type 1, it holds $\alpha \subset \mathbf{M}^{\perp_V} := \langle \beta, \gamma\rangle_V \in \mathcal{D}$ and $\mathbf{p}$ is a POINT of $\mathbf{M}$. So, $\mathbf{q}' \in \langle \mathbf{q}, \mathbf{L}^\pi\rangle_E$. If $\beta \not\perp_\Pi \gamma$, then $\langle \beta, \gamma\rangle_V \cap \Pi \cong O_4^+(\mathbb{K})$, and $\mathbf{M}$ is of type 3. The space $X := \langle \mathbf{M}, L\rangle_V$ is 8-dimensional of Witt index 4. Hence it is either isomorphic to $O_8^+(\mathbb{K})$ or degenerate. In the latter case, let ρ be its radical, whence $\rho = \mathbf{M}^{\perp_V} \cap L$. As $\mathbf{M}$ is spanned by lines of $\mathcal{L}$, $\mathcal{L}$ induces a spread on $\mathbf{M}^{\perp_V} \cap \Pi$. Hence $\rho \in \mathcal{L}$ and $\rho^{\perp_V}$ is a POINT of type 1 with $\rho^{\perp_V} \in \mathbf{L}^\pi$ and we are done. In the former case, i.e. $X \cap \Pi \cong O_8^+(\mathbb{K})$, pick a point p of Π in $L \setminus \alpha$. By Proposition 2.12, the POINTS on $\mathbf{M}$ which is of type 3, form a spread of $V/\mathbf{M}$. Hence there exists a POINT $\mathbf{y}$ containing $\mathbf{M}$ and the line λ of $\mathcal{L}$ through p which is contained in L. So, $L \subset \mathbf{y}$, since $\mathbf{y}$ contains both α and λ and $L = \langle \alpha, \lambda\rangle_V$. Hence $\mathbf{y} = X$.

Case 2. $\mathbf{q}'$ is of type 2, say $\mathbf{q}' = \langle L', M\rangle_V$ with disjoint 4-spaces $L', M \in \mathcal{D}$ and $L' \cap L = \alpha$. The LINE $\mathbf{M}$ is of type 2 or 3.

Suppose firstly $\mathbf{M} = \gamma \oplus W$ is a LINE of type 2 for a subspace W of V with $W \cap \Pi \cong O_4^+(\mathbb{K})$, $W \leq \gamma^{\perp_V}$ and $\mathcal{L}$ inducing on $W \cap \Pi$ one of its two reguli. If $\gamma = \alpha$, then $\langle L, \mathbf{M}\rangle_V = \alpha^{\perp_V}$ and $\langle L, \mathbf{M}\rangle_V$ is a POINT of type 1 lying on the LINE $\mathbf{M}$ and contained in the set $\mathbf{L}^\pi$. If $\gamma \neq \alpha$, then $\langle L, \mathbf{M}\rangle_V$ is non-degenerate and contains disjoint 4-spaces of $\mathcal{D}$, namely L and any 4-space of $\mathcal{D}$ through γ in $\mathbf{M}$ distinct from $\langle \alpha, \gamma\rangle_V$. So, $\langle L, \mathbf{M}\rangle_V$ is a POINT of type 2 on the LINE $\mathbf{M}$ belonging to $\mathbf{L}^\pi$.

If $\mathbf{M}$ with $\mathbf{M} \cap \Pi \cong O_6^-(\mathbb{K})$ is a LINE of type 3, then the 8-space $\langle L, \mathbf{M}\rangle_V$ has Witt index 4 and is either isomorphic to $O_8^+(\mathbb{K})$ or degenerate. An argument as in Case 1 yields the conclusion.

Proof of Claim 3. Let $\mathbf{q} = \beta^{\perp_V}$ be a POINT of E with $\beta \in \mathcal{L}$ such that α and $\mathbf{q}$ are disjoint. Hence $\langle \alpha, \beta\rangle_V \cap \Pi \cong O_4^+(\mathbb{K})$. Given another POINT $\mathbf{q}'$ such that $\mathbf{q}'$ and α are disjoint, let $\mathbf{M} := \mathbf{q} \cap \mathbf{q}'$. We must prove that the LINE $\mathbf{M}$ contains a POINT of $\mathbf{p}^\pi$.

Case 1. $\mathbf{M}$ is of type 2 or 3. By Proposition 2.12, there exists a unique POINT on $\mathbf{M}$ containing α and we are done.

Case 2. $\mathbf{M}$ is of type 1. Then $\mathbf{q}' = \gamma^{\perp_V}$ for a line $\gamma \in \mathcal{L}$ with $\beta \perp_\Pi \gamma$. Put $\lambda = \mathbf{q} \cap L$ and $\lambda' = \mathbf{q}' \cap L$. Since $\mathcal{L}$ induces spreads on $\mathbf{q} \cap \Pi$, $\mathbf{q}' \cap \Pi$ and L, both λ and λ' are lines of Π in $\mathcal{L}$. Suppose first that $\lambda = \lambda'$. Then $\lambda^{\perp_V}$ is a POINT of $\mathbf{M}$ in $\mathbf{p}^\pi$. Suppose next $\lambda \neq \lambda'$. Since $\mathbf{M}$ is of type 1, it holds $\mathbf{M}^{\perp_V} \in \mathcal{D}$. Put $\mu := \alpha^{\perp_V} \cap \mathbf{M}^{\perp_V}$ $(\in \mathcal{L})$. Then $\mu^{\perp_V}$ is a POINT of $\mathbf{M}$ and belongs to $\mathbf{p}^\pi$. □

By Propositions 2.11–2.15, the generalized quadrangle Γ has an embedding in a 4-dimensional projective space $PG(4, \mathbb{H})$.

By Propositions 2.5 and 2.6, two opposite lines of Γ define a proper full subquadrangle which is not only a grid. As explained in section 2.1, Γ is a hermitian generalized quadrangle if $\mathrm{char}(\mathbb{H}) \neq 2$ or if $\mathbb{H} = \mathbb{F}_{2^h}$ for an $h \in \mathbb{N}$. This proves Theorem 1.2.

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