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On minimum size blocking sets of external lines to a quadric in PG(d, q)

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Abstract

We characterize the minimum size blocking sets with respect to the external lines to a non-singular quadric or a quadric with a point vertex in PG(d, q), $d \geq 4$ and $q \geq 9$. Our results show that these minimum size blocking sets are equal to the sets of points not on the quadric in a suitably chosen hyperplane with respect to the quadric.

Keywords: blocking sets, quadrics

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1 Introduction

A *blocking set* in a projective space $\mathbb{P} = $ PG(d, q) is a subset of $\mathbb{P}$ which meets every line. Blocking sets have been investigated by a great variety of authors, from many points of view [5, 8, 9]. Now, let $\mathcal{G}$ be a set of lines of $\mathbb{P}$. A point set B of $\mathbb{P}$ is a *blocking set with respect to* $\mathcal{G}$ (or a $\mathcal{G}$-*blocking set*) if every line in $\mathcal{G}$ is incident with at least one point of B.

In [1, 6], all minimum size blocking sets with respect to the set $\mathcal{E}$ of the external lines to a non-singular conic in PG$(2, q)$ have been determined. We point out that

$$\text{the minimum size of an } \mathcal{E}\text{-blocking set of PG}(2, q) \text{ is } q - 1. \tag{1.1}$$

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Now, let $d \geq 3$ and let $\mathcal{W}$ be a non-singular quadric or a cone with a point as the vertex. If d is odd, the notations $\mathcal{Q}^+$, $\mathcal{Q}^-$ and $\mathcal{C}$ are used for hyperbolic quadrics, elliptic quadrics and cones, respectively; if d is even, we write $\mathcal{Q}$, $\mathcal{C}^+$ and $\mathcal{C}^-$ for non-singular quadrics and cones with base a hyperbolic or elliptic quadric, respectively. Denote by $\mathcal{F}$ the set of all external lines to $\mathcal{W}$. If Π is a hyperplane, then $\Pi \backslash \mathcal{W}$ is an $\mathcal{F}$-blocking set.

The minimum size of such a blocking set is:

(a) $q^{2n-1} - q^n$, if $d = 2n$ and $\mathcal{W}$ is a $\mathcal{C}^+$ (Π is a tangent hyperplane);
(b) q^{2n-1}, if $d = 2n$ and $\mathcal{W}$ is a $\mathcal{C}^-$ (Π is a non-tangent hyperplane through the vertex);
(c) $q^{2n} - q^n$, if $d = 2n+1$ and $\mathcal{W}$ is a $\mathcal{C}$ (Π is a hyperplane through the vertex intersecting $\mathcal{C}$ in a cone with base a hyperbolic quadric);
(d) $q^{2n-1} - q^{n-1}$, if $d = 2n$ and $\mathcal{W}$ is a $\mathcal{Q}$ (Π is a hyperplane intersecting $\mathcal{Q}$ in a hyperbolic quadric);
(e) $q^{2n} - q^n$, if $d = 2n+1$ and $\mathcal{W}$ is a $\mathcal{Q}^+$ (Π is a tangent hyperplane);
(f) q^{2n}, if $d = 2n+1$ and $\mathcal{W}$ is a $\mathcal{Q}^-$ (Π is a non-tangent hyperplane).

In [2, 3, 4], we proved that, if $d = 3$, the correct sizes for the smallest $\mathcal{F}$-blocking sets are those in (c), (e) and (f), and that a minimum size $\mathcal{F}$-blocking set is always of type $\Pi \backslash \mathcal{W}$, for a suitable hyperplane Π if $q \geq 9$.

In this paper, as a generalization of the previous results, we prove the following result.

Theorem 1.1. *Let $\mathcal{W}$ be a non-singular quadric or a cone, with vertex a point, in PG(d,q), $d \geq 3$ and $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{W}$, then $B = \Pi \backslash \mathcal{W}$ for a suitable hyperplane Π (see the list from (a) to (f)).*

Theorem 1.1, which holds for $d = 3$, will be proved by induction on d; so, from now on, we assume that the statement is true in PG(h,q), $3 \leq h \leq d-1$.

Observe that, if S is a subspace of dimension at least 2 of PG(d,q) and B is an $\mathcal{F}$-blocking set, then $S \cap B$ is a blocking set of S with respect to the lines in S external to $S \cap \mathcal{W}$. Moreover, with B of minimum size, $B \cap \mathcal{W} = \emptyset$. Let Π be a hyperplane. If $p \in \Pi \backslash \mathcal{W}$, a line exists through p external to $\mathcal{W}$ and not in Π, unless $\mathcal{W}$ is a non-singular quadric in PG$(2n,q)$, q even, Π is a tangent hyperplane and p is the nucleus of $\mathcal{W}$. So, a proper subset B' of $\Pi \backslash \mathcal{W}$ is an $\mathcal{F}$-blocking set if and only if $\mathcal{W}$ is a non-singular quadric in PG$(2n,q)$, q even, Π is a tangent hyperplane and $B' = \Pi \backslash (\mathcal{W} \cup \{u\})$, where u is the nucleus of $\mathcal{W}$. By (d), $\Pi \backslash (\mathcal{W} \cup \{u\})$ is not of minimum size. Hence, if B is a minimum size blocking set and $B \subseteq \Pi$, for some hyperplane Π, then $B = \Pi \backslash \mathcal{W}$.

Throughout this paper, we assume the main properties of the quadrics to be known [7]. Here we only introduce some definitions and notations. By an *external* (*tangent* or *secant*) *line*, we mean a line external (unisecant or bisecant) to $\mathcal{W}$. Similarly, by a *tangent hyperplane*, we mean a hyperplane tangent to $\mathcal{W}$. Moreover, a hyperplane meeting $\mathcal{W}$ in a non-singular quadric is sometimes referred to as a *secant hyperplane*. Finally, the notation S_k is often used for a k-dimensional subspace of PG(d, q) and, if the hyperplane tangent to $\mathcal{W}$ at a point p exists, it is denoted by Π_p.

2 Cones in PG(d, q)

Let $\mathcal{W}$ be a cone, with vertex a point, of PG(d, q), $d \geq 4$ and $q \geq 9$. Throughout this section, B denotes a minimum size blocking set with respect to the set $\mathcal{F}$ of all external lines to $\mathcal{W}$.

Proposition 2.1. *Let $d = 2n$. If $\mathcal{W}$ is a cone $\mathcal{C}^+$, then:*

(i) $|B| = q^{2n-1} - q^n$*;*

(ii) *for any secant hyperplane* Π*,* $|B \cap \Pi| = q^{2n-2} - q^{n-1}$*;*

(iii) *for any secant hyperplane* Π*,* $B \cap \Pi = S_{2n-2} \backslash \mathcal{C}^+$ *for a suitable subspace* S_{2n-2} *in* Π.

Proof. Since B is of minimum size, from (a) in Section 1, it follows that

$$|B| \leq q^{2n-1} - q^n \,. \tag{2.1}$$

By the induction hypothesis, Theorem 1.1 (e) implies that

$$|B \cap \Pi| \geq q^{2n-2} - q^{n-1} \,, \tag{2.2}$$

for any secant hyperplane Π; so, counting in two ways the point-hyperplane pairs (p, Π), $p \in B \cap \Pi$ and Π a secant hyperplane, yields

$$|B| q^{2n-1} \geq q^{2n} (q^{2n-2} - q^{n-1}) \,. \tag{2.3}$$

From (2.1) and (2.3), it follows that equality holds in both (2.1) and (2.2); so, (i) and (ii) are proved. Since, by the induction hypothesis (see again Theorem 1.1 (e)), (ii) implies (iii), the statement is completely proved. □

In a similar way, we can prove the following two propositions.

Proposition 2.2. *Let $d = 2n$. If $\mathcal{W}$ is a cone $\mathcal{C}^-$, then:*

(i) $|B| = q^{2n-1}$ *;*

(ii) *for any secant hyperplane Π, $|B \cap \Pi| = q^{2n-2}$;*

(iii) *for any secant hyperplane Π, $B \cap \Pi = S_{2n-2} \backslash \mathcal{C}^-$, for a suitable subspace S_{2n-2} in Π .*

Proposition 2.3. *Let $d = 2n + 1$. If $\mathcal{W}$ is a cone $\mathcal{C}$, then:*

(i) $|B| = q^{2n} - q^n$ *;*

(ii) *for any secant hyperplane Π, $|B \cap \Pi| = q^{2n-1} - q^{n-1}$;*

(iii) *for any secant hyperplane Π, $B \cap \Pi = S_{2n-1} \backslash \mathcal{C}$, for a suitable subspace S_{2n-1} in Π .*

Theorem 2.4. *Let $\mathcal{W}$ be a cone, with vertex a point, in PG(d, q), $d \geq 3$ and $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{W}$, then:*

(i) $|B| = q^{2n-1} - q^n$, *if $d = 2n$ and $\mathcal{W}$ is a $\mathcal{C}^+$;*

(ii) $|B| = q^{2n-1}$, *if $d = 2n$ and $\mathcal{W}$ is a $\mathcal{C}^-$;*

(iii) $|B| = q^{2n} - q^n$, *if $d = 2n + 1$.*

Moreover, $B = \Pi \backslash \mathcal{W}$ for a suitable hyperplane Π (see (a) - (c) in Section 1).

Proof. By Propositions 2.1 – 2.3, (i) – (iii) hold.

Now, let $d = 2n$ and let $\mathcal{W}$ be a cone $\mathcal{C}^+$. Consider a secant hyperplane S. By Proposition 2.1, $q^{2n-1} - q^n = |B| > |B \cap S| = q^{2n-2} - q^{n-1}$ and a subspace S_{2n-2} exists in S such that $B \cap S = S_{2n-2} \backslash \mathcal{C}^+$. Let $p \in B \backslash S$ and let S' be the hyperplane joining S_{2n-2} with p. Since $|B \cap S'| > |B \cap S| = q^{2n-2} - q^{n-1}$, then Proposition 2.1 (ii) implies that S' is not secant; so S' contains the vertex v of $\mathcal{C}^+$. Therefore, S' is the hyperplane Π joining S_{2n-2} with v. Hence, $B \subseteq \Pi$, from which $B = \Pi \backslash \mathcal{C}^+$.

Using Propositions 2.2 and 2.3, the same argument as above can be applied to the cones $\mathcal{C}^-$ and $\mathcal{C}$, respectively. So the statement is completely proved. $\square$

3 Non-singular quadrics in PG$(2n, q)$

Let $\mathcal{Q}$ be a non-singular quadric of PG$(2n, q)$, $n \geq 2$ and $q \geq 9$, and let B be a minimum size blocking set with respect to the set $\mathcal{F}$ of all external lines to $\mathcal{Q}$. By (d) in Section 1,

$$|B| \leq q^{2n-1} - q^{n-1} . \tag{3.1}$$

Proposition 3.1. *There exists a line tangent to $\mathcal{Q}$ and skew to B.*

Proof. Through a point of $\mathcal{Q}$, there pass q^{2n-1} secant lines; so, (3.1) implies that there exists a secant line L skew to B. Set $L \cap \mathcal{Q} = \{p_1, p_2\}$ and denote by $\mathcal{T}$ and $\mathcal{S}$ the sets of planes through L meeting $\mathcal{Q}$ in a singular and a non-singular conic, respectively. Since the number of elements of $\mathcal{T}$ equals the number of lines of $\mathcal{Q}$ through p_1, $|\mathcal{T}| = q^{2n-3} + q^{2n-4} + \cdots + q + 1$. So, $|\mathcal{S}| = q^{2n-2}$.

This implies, by (1.1), that the union of all planes in $\mathcal{S}$ shares with B at least $q^{2n-2}(q-1)$ points. Then, by (3.1), there are at most $q^{2n-2} - q^{n-1}$ points of B in elements of $\mathcal{T}$. Since any plane in $\mathcal{T}$ contains $q-1$ tangent lines and $(q-1)|\mathcal{T}| = q^{2n-2} - 1 > q^{2n-2} - q^{n-1}$, we conclude that there is a tangent line skew to B. So the statement is proved. □

Proposition 3.2. (i) *$|B| = q^{2n-1} - q^{n-1}$;*

(ii) *if there exists a tangent line through a point $p \in \mathcal{Q}$ skew to B, then $|\Pi_p \cap B| = q^{2n-2} - q^{n-1}$ and $\Pi_p \cap B = \Omega \backslash \mathcal{Q}$, for a suitable hyperplane Ω of Π_p intersecting $\mathcal{Q}$ in a cone with vertex p and base a hyperbolic quadric.*

Proof. Let p be a point in $\mathcal{Q}$ such that a tangent line L through p exists skew to B. The q^{2n-2} planes on L not in Π_p all intersect $\mathcal{Q}$ in a non-singular conic; so, by (1.1), each one of them shares at least $q-1$ points with B. Therefore,

$$|B \backslash \Pi_p| \geq q^{2n-1} - q^{2n-2}\,. \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$|B \cap \Pi_p| \leq q^{2n-2} - q^{n-1}\,. \tag{3.3}$$

On the other hand, by the induction hypothesis (see Theorem 1.1 (c)), we have $|B \cap \Pi_p| \geq q^{2n-2} - q^{n-1}$; so, by (3.3),

$$|B \cap \Pi_p| = q^{2n-2} - q^{n-1}\,. \tag{3.4}$$

This implies, by the induction hypothesis (see again Theorem 1.1 (c)), the second part of (ii). Hence (ii) holds.

Finally, (3.1), (3.2) and (3.4) imply (i). The statement is completely proved. □

Now, we can prove the following result.

Theorem 3.3. *Let $\mathcal{Q}$ be a non-singular quadric in PG$(2n, q)$, $n \geq 2$ and $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $|B| = q^{2n-1} - q^{n-1}$ and $B = \Pi \backslash \mathcal{Q}$ for a hyperplane Π intersecting $\mathcal{Q}$ in a hyperbolic quadric.*

Proof. By Propositions 3.1 and 3.2, $|B| = q^{2n-1} - q^{n-1}$ and there exists a point $p \in \mathcal{Q}$ such that $|\Pi_p \cap B| = q^{2n-2} - q^{n-1}$ and $\Pi_p \cap B = \Omega \backslash \mathcal{Q}$, for a suitable hyperplane Ω of Π_p intersecting $\mathcal{Q}$ in a cone with vertex p and base a hyperbolic quadric. We divide the proof into two cases.

Case 1: $n = 2$.

The hyperplane Ω shares two distinct lines with $\mathcal{Q}$. Let L be a line in Ω on p such that $L \backslash \{p\} \subseteq B$. Consider a plane π in Π_p through L such that $\pi \cap \mathcal{Q} = \{p\}$. The q non-tangent hyperplanes Π_i, $i = 1, \ldots, q$, through π all intersect $\mathcal{Q}$ in an elliptic quadric; therefore, by Theorem 1.1 (f),

$$|B \cap \Pi_i| \geq q^2, \quad i = 1, \ldots, q. \tag{3.5}$$

Since $|\Pi_p \cap B| = q^2 - q$ and $|B| = q^3 - q$, then, counting points of B on hyperplanes through π, we obtain by (3.5) that

$$q^3 - q = |B| \geq q^2 - q + q(q^2 - q).$$

This implies that equality holds in (3.5); so, by Theorem 1.1 (f), there exists a secant plane $\Omega_i \subseteq \Pi_i$ such that $B \cap \Pi_i = \Omega_i \backslash \mathcal{Q}$, for any $i = 1, \ldots, q$. Hence, $B = (\Omega \cup \Omega_1 \cup \cdots \cup \Omega_q) \backslash \mathcal{Q}$. Now, let $L' \neq L$ be a line in Ω on p such that $L' \backslash \{p\} \subseteq B$.

By the same arguments as above, we can find q planes $\Omega'_1, \ldots, \Omega'_q$ through L' such that

$$|\Omega'_i \cap B| = q^2 \text{ and } B = (\Omega \cup \Omega'_1 \cup \cdots \cup \Omega'_q) \backslash \mathcal{Q}. \tag{3.6}$$

Observe that, since the line L is tangent to $\mathcal{Q}$ at p and since all planes Ω'_i and Ω_j, $i, j = 1, \ldots, q$, are secant planes, then $\Omega'_i \cap \Omega_j$, $i, j = 1, \ldots, q$, is either the point p or a secant line through p. Consequently,

$$|\Omega'_i \cap \Omega_j \cap B| = 0 \text{ or } q - 1. \tag{3.7}$$

Now, consider one of the planes Ω'_i, say Ω'_1. Since $\Omega'_1 \cap B = (\Omega'_1 \cap \Omega \cap B) \cup (\Omega'_1 \cap \Omega_1 \cap B) \cup \cdots \cup (\Omega'_1 \cap \Omega_q \cap B)$ and $|\Omega'_1 \cap \Omega \cap B| = q$, then (3.6) and (3.7) imply that $|\Omega'_1 \cap \Omega_j \cap B| = q - 1$ for any $j = 1, \ldots, q$.

Hence, $\Omega'_i \cap \Omega_j$ is a line, for any $i, j = 1, \ldots, q$. This implies that Ω'_i, $i = 1, \ldots, q$, is contained in the hyperplane Π joining Ω and Ω_1. Then, by (3.6), $B \subseteq \Pi$; so, $B = \Pi \setminus \mathcal{Q}$ and the statement is proved.

Case 2: $n \geq 3$.

The tangent hyperplane Π_p shares with $\mathcal{Q}$ a cone with vertex p and base a non-singular quadric of a subspace S_{2n-2}, and $\Omega \cap \mathcal{Q}$ is a cone with vertex p

and base a hyperbolic quadric of a subspace $S'_{2n-3} \subseteq S_{2n-2}$. Let S''_{2n-5} be a subspace of S'_{2n-3} such that $S''_{2n-5} \cap \mathcal{Q}$ is an elliptic quadric. Consider in S'_{2n-3} two distinct subspaces T_{2n-4} and T'_{2n-4} through S''_{2n-5} (obviously, $T_{2n-4} \cap \mathcal{Q}$ and $T'_{2n-4} \cap \mathcal{Q}$ are non-singular quadrics). Choose in S_{2n-2} a subspace U_{2n-3} such that $T_{2n-4} \subseteq U_{2n-3}$ and $U_{2n-3} \cap \mathcal{Q}$ is an elliptic quadric. Denote by W_{2n-2} the subspace joining U_{2n-3} and p. Here,

$$|W_{2n-2} \cap B| = q^{2n-3}\,. \tag{3.8}$$

The q non-tangent hyperplanes Π_i, $i = 1, \ldots, q$, through W_{2n-2} all intersect $\mathcal{Q}$ in an elliptic quadric. Therefore, by Theorem 1.1 (f),

$$|B \cap \Pi_i| \geq q^{2n-2}, i = 1, \ldots, q\,. \tag{3.9}$$

Since $|\Pi_p \cap B| = q^{2n-2} - q^{n-1}$ and $|B| = q^{2n-1} - q^{n-1}$, then, counting points of B on hyperplanes through W_{2n-2}, we obtain, by (3.8) and (3.9),

$$q^{2n-1} - q^{n-1} = |B| \geq q^{2n-2} - q^{n-1} + q(q^{2n-2} - q^{2n-3})\,.$$

This implies that equality holds in (3.9); so, by Theorem 1.1 (f), there exists a $(2n-2)$-dimensional subspace $\Omega_{(i)} \subseteq \Pi_i$ sharing a non-singular quadric with $\mathcal{Q}$, and such that $B \cap \Pi_i = \Omega_{(i)} \setminus \mathcal{Q}$, for any $i = 1, \ldots, q$. Hence, $B = (\Omega \cup \Omega_{(1)} \cup \cdots \cup \Omega_{(q)}) \setminus \mathcal{Q}$.

Now, we apply the previous arguments to T'_{2n-4}; so we find q subspaces $\Omega'_{(1)}, \ldots, \Omega'_{(q)}$ of dimension $2n-2$ such that

$$|\Omega'_{(i)} \cap B| = q^{2n-2} \text{ and } B = (\Omega \cup \Omega'_{(1)} \cup \cdots \cup \Omega'_{(q)}) \setminus \mathcal{Q}\,. \tag{3.10}$$

Let Z_{2n-4} be the subspace joining S''_{2n-5} and p. For any $\Omega_{(i)}$ and $\Omega'_{(j)}$, $\Omega_{(i)} \cap \Omega'_{(j)}$ is Z_{2n-4} or a $(2n-3)$-dimensional subspace Z_{ij} on Z_{2n-4} such that $Z_{ij} \cap \mathcal{Q}$ is an elliptic quadric with Z_{2n-4} as the tangent hyperplane in p; so,

$$|\Omega_{(i)} \cap \Omega'_{(j)} \cap B| = q^{2n-4} + q^{n-2} \text{ or } q^{2n-3} + q^{n-2}\,, \tag{3.11}$$

respectively.

Now, consider one of the spaces $\Omega'_{(i)}$, say $\Omega'_{(1)}$. Since

$$\Omega'_{(1)} \cap B = (\Omega'_{(1)} \cap \Omega \cap B) \cup (\Omega'_{(1)} \cap \Omega_{(1)} \cap B) \cup \cdots \cup (\Omega'_{(1)} \cap \Omega_{(q)} \cap B)$$

and $|\Omega'_{(1)} \cap \Omega \cap B| = |\langle p, T'_{2n-4} \rangle \cap B| = q^{2n-3}$, (3.10) and (3.11) imply that $|\Omega'_{(1)} \cap \Omega_{(j)} \cap B| = q^{2n-3} + q^{n-2}$ for any $j = 1, \ldots, q$.

Hence, $\Omega'_{(i)} \cap \Omega_{(j)}$ is a $(2n-3)$-dimensional subspace, for any $i, j = 1, \ldots, q$. This implies that $\Omega'_{(i)}$, $i = 1, \ldots, q$, is contained in the hyperplane Π joining Ω and $\Omega_{(1)}$. Then, by (3.10), $B \subseteq \Pi$; so $B = \Pi \setminus \mathcal{Q}$.

The statement is completely proved. □

4 Non-singular quadrics in PG$(2n+1, q)$

Let $\mathcal{Q}$ be a non-singular quadric of PG$(2n+1, q)$, $n \geq 2$ and $q \geq 9$. Throughout this section, B denotes a minimum size blocking set with respect to the set $\mathcal{F}$ of all external lines to $\mathcal{Q}$.

Proposition 4.1. *There exists a line tangent to $\mathcal{Q}$ and skew to B.*

Proof. Firstly, let $\mathcal{Q}$ be a hyperbolic quadric. By (e) in Section 1,

$$|B| \leq q^{2n} - q^n \,. \tag{4.1}$$

Assume that no tangent line is skew to B. Then, counting in two ways the point-line pairs (p, L), $p \in B \cap L$ and L a tangent line, gives

$$|B|(q^{2n-1} + \cdots + 1) \geq (q^{2n-1} - q^{n-1})(q^{2n} + \cdots + q^{n+1} + 2q^n + q^{n-1} + \cdots + 1),$$

a contradiction to (4.1).

The same argument applies to an elliptic quadric. So, the statement is proved. □

Applying the arguments of the proof of Proposition 3.2 to $\mathcal{Q}$ gives the following result.

Proposition 4.2. *If $\mathcal{Q}$ is a hyperbolic quadric, then*

(i) $|B| = q^{2n} - q^n$;

(ii) *if a tangent line through a point $p \in \mathcal{Q}$ skew to B exists, then $|\Pi_p \cap B| = q^{2n-1} - q^n$ and $\Pi_p \cap B = \Omega \backslash \mathcal{Q}$, for a suitable tangent hyperplane Ω of Π_p.*

If $\mathcal{Q}$ is an elliptic quadric, then

(i) $|B| = q^{2n}$;

(ii) *if a tangent line through a point $p \in \mathcal{Q}$ skew to B exists, then $|\Pi_p \cap B| = q^{2n-1}$ and $\Pi_p \cap B = \Omega \backslash \mathcal{Q}$, for a suitable non-tangent hyperplane Ω of Π_p through p.*

Theorem 4.3. *Let $\mathcal{Q}$ be a hyperbolic (elliptic) quadric in PG$(2n+1, q)$, $n \geq 2$ and $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $|B| = q^{2n} - q^n$ ($|B| = q^{2n}$) and $B = \Pi \backslash \mathcal{Q}$, for a suitable tangent (non-tangent) hyperplane Π to $\mathcal{Q}$.*

Proof. Firstly, assume that $\mathcal{Q}$ is a hyperbolic quadric. By Propositions 4.1 and 4.2, $|B| = q^{2n} - q^n$ and there exists a point $p \in \mathcal{Q}$ such that $|\Pi_p \cap B| = q^{2n-1} - q^n$ and such that $\Pi_p \cap B = \Omega \backslash \mathcal{Q}$, for a suitable tangent hyperplane Ω of Π_p. The tangent hyperplane Π_p shares with $\mathcal{Q}$ a cone with vertex p and base a hyperbolic quadric of a subspace S_{2n-1}. Set $\Omega \cap S_{2n-1} = S'_{2n-2}$ and observe that $\Omega \cap \mathcal{Q}$ is a cone with vertex a line L through p and base a hyperbolic quadric of a $(2n-3)$-dimensional subspace of S'_{2n-2}. Consider in S'_{2n-2} two distinct subspaces T_{2n-3} and T'_{2n-3} such that $T_{2n-3} \cap \mathcal{Q}$ and $T'_{2n-3} \cap \mathcal{Q}$ are non-singular hyperbolic quadrics and such that $T_{2n-3} \cap T'_{2n-3} \cap \mathcal{Q}$ is either a non-singular quadric or the empty set according as $n \geq 3$ or $n = 2$. Choose a point p' in $\mathcal{Q} \cap (S_{2n-1} \backslash S'_{2n-2})$ such that the subspace U_{2n-2} joining p' and T_{2n-3} intersects $\mathcal{Q}$ in a non-singular quadric, and denote by W_{2n-1} the subspace joining U_{2n-2} and p. Since $T_{2n-3} \cap \mathcal{Q}$ is a hyperbolic quadric,

$$|W_{2n-1} \cap B| = q^{2n-2} - q^{n-1}\,. \tag{4.2}$$

Let Π_i, $i = 1, \ldots, q$, be the non-tangent hyperplanes through W_{2n-1}. By Theorem 1.1 (d),

$$|B \cap \Pi_i| \geq q^{2n-1} - q^{n-1}, i = 1, \ldots, q\,. \tag{4.3}$$

Since $|\Pi_p \cap B| = q^{2n-1} - q^n$ and $|B| = q^{2n} - q^n$, then, counting points of B on hyperplanes through W_{2n-1}, we obtain, by (4.2) and (4.3), that

$$q^{2n} - q^n = |B| \geq q^{2n-1} - q^n + q(q^{2n-1} - q^{2n-2}).$$

This implies that equality holds in (4.3); so, by Theorem 1.1 (d), there exists a $(2n-1)$-dimensional subspace $\Omega_{(i)} \subseteq \Pi_i$ sharing with $\mathcal{Q}$ a hyperbolic quadric and such that $B \cap \Pi_i = \Omega_{(i)} \backslash \mathcal{Q}$, for any $i = 1, \ldots, q$. Hence, $B = (\Omega \cup \Omega_{(1)} \cup \cdots \cup \Omega_{(q)}) \backslash \mathcal{Q}$.

Now, we apply the previous arguments to the subspace T'_{2n-3}; so we find q subspaces $\Omega'_{(1)}, \ldots, \Omega'_{(q)}$ of dimension $2n-1$ such that

$$|\Omega'_{(i)} \cap B| = q^{2n-1} - q^{n-1} \text{ and } B = (\Omega \cup \Omega'_{(1)} \cup \cdots \cup \Omega'_{(q)}) \backslash \mathcal{Q}\,. \tag{4.4}$$

As in the proof of Theorem 3.3, we can prove that $\Omega'_{(i)} \cap \Omega_{(j)}$ is a $(2n-2)$-dimensional subspace, for any $i, j = 1, \ldots, q$. This implies that $\Omega'_{(i)}$, $i = 1, \ldots, q$, is contained in the hyperplane Π joining Ω and $\Omega_{(1)}$. Then, by (4.4), $B \subseteq \Pi$; so $B = \Pi \backslash \mathcal{Q}$.

Now, let $\mathcal{Q}$ be an elliptic quadric. The proof proceeds in the same way as above. Consider Proposition 4.2. For the hyperbolic quadric, $|B| = q^{2n} - q^n$ and for the elliptic quadric, $|B| = q^{2n}$. But we use in the arguments of this proof, a point p of $\mathcal{Q}$ for which $|\Pi_p \cap B| = q^{2n-1} - q^n$ for the hyperbolic quadric and for which $|\Pi_p \cap B| = q^{2n-1}$ for the elliptic quadric. We see that when looking at the numbers $|B|$ and $|\Pi_p \cap B|$ for the hyperbolic quadric and for the elliptic quadric, in both cases, the difference is q^n. This makes that all the arguments can be copied; everywhere the same equalities are obtained, leading to the analogous conclusions. $\square$

References

[1] **A. Aguglia** and **G. Korchmáros**, Blocking sets of external lines to a conic in PG$(2, q)$, q odd, *Combinatorica* **26** (2006), 379–394.

[2] **P. Biondi** and **P. M. Lo Re**, On blocking sets of external lines to a quadric in PG$(3, q)$, q prime, to appear in *Ars Combin.*

[3] ———, On blocking sets of external lines to a hyperbolic quadric in PG$(3, q)$, q even, submitted.

[4] **P. Biondi**, **P. M. Lo Re** and **L. Storme**, On minimum size blocking sets of external lines to a quadric in PG$(3, q)$, to appear in *Contrib. Algebra Geom.*

[5] **A. Blokhuis**, Blocking sets in Desarguesian planes, *Paul Erdös is eighty, Bolyai Society Math. Studies* **2** (1993), 1–20.

[6] **M. Giulietti**, Blocking sets of external lines to a conic in PG$(2, q)$, q even, *European J. Combin.* **28** (2007), 36–42.

[7] **J. W. P. Hirschfeld** and **J. A. Thas**, General Galois Geometries, *Oxford University Press* (1991).

[8] **K. Metsch**, Blocking sets in projective spaces and polar spaces, *J. Geom.* **76** (2003), 216–232.

[9] **T. Szőnyi**, Blocking sets in Desarguesian affine and projective spaces, *Finite Fields Appl.* **3** (1997), 88–101.

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