Innovations in Incidence Geometry
Volume 5 (2007), Pages 13–34
ISSN 1781-6475

New multiple hyper-regulus planes

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Abstract

New classes of multiple hyper-regulus translation planes of orders q^n, for $n \geq 3$, are constructed that extend the classes of Culbert-Ebert planes of orders q^3.

MSC 2000: primary: 51E23; secondary: 51A40
Keywords: reguli, hyper-reguli, multiple hyper-regulus planes

1 Introduction

This article considers the possibility of the construction of translation planes of order q^n and kernel containing $\mathsf{GF}(q)$ by the replacement of a set of mutually disjoint hyper-reguli. A 'hyper-regulus' is a vector space partial spread $\mathcal{H}$ of degree $(q^n-1)/(q-1)$ and order q^n which has a replacement 'hyper-regulus' spread $\mathcal{H}^*$, each of whose components intersects each of the components of $\mathcal{H}$, which is to say that the intersections between the two hyper-reguli are 1-dimensional $\mathsf{GF}(q)$-subspaces. Hence, when $n=2$, a hyper-regulus is a regulus.

Consider a Desarguesian plane Σ with spread

$$x=0, y=xm; m \in \mathsf{GF}(q^n)\,.$$

Define the partial spread

$$A_\delta = \left\{ y = xm; m^{(q^n-1)/(q-1)} = \delta; m \in \mathsf{GF}(q^n)^* \right\}.$$

Then there are replacement partial spreads

$$A_\delta^i = \left\{ y = x^{q^i} m; m^{(q^n-1)/(q-1)} = \delta; m \in \mathsf{GF}(q^n)^* \right\}$$

and since for $(i,n)=1$, the intersections between the two partial spreads are 1-dimensional $\mathsf{GF}(q)$-subspaces, we obtain a pair of hyper-reguli. Such hyper-reguli are called 'André hyper-reguli'.

It was shown by Bruck [2] that every hyper-regulus in a Desarguesian plane of order q^3 is an André hyper-regulus. Furthermore, Pomareda [8] has shown that there are exactly two hyper-regulus replacement partial spreads, namely A_δ^i, for $i = 1, 2$, when $n = 3$.

The set of André nets described above in the Desarguesian affine plane

$$\Sigma \{A_\delta; \delta \in \mathsf{GF}(q)\} \text{ together with } \{x = 0, y = 0\}$$

partition the spread for Σ and form what is called a 'linear set' with carrying lines $x = 0$ and $y = 0$. On the other hand, it is possible to find sets of mutually disjoint hyper-reguli of cardinality at least two that are not linear, in the above sense. Of course, when $n > 3$, it might be the case that a hyper-regulus is not André.

In a recent article, Culbert and Ebert [3] constructed several sets of mutually disjoint hyper-reguli of order q^3. In particular, there are sets of mutually disjoint hyper-reguli of cardinalities $(q-1)/2$ and $(q-3)/2$ for q odd and $(q-2)/2$, for q even. Furthermore, no subset of cardinality at least two is linear, thus producing a great variety of new translation planes that are not André or generalized André.

When $n > 3$, Jha and Johnson have found a new type of hyper-regulus that is not André. Furthermore, for $n \geq 3$, Jha and Johnson have developed the theory of the corresponding translation planes and collineation groups in the articles ([7], [6], [4], [5]). Indeed, there are classes of sets of mutually disjoint hyper-reguli of order q^n of cardinalities $(q-1)/2$ and $(q-3)/2$ for q odd and $(q-2)/2$, for q even and $n \geq 3$.

Therefore, just considering the cardinalities of the mutually disjoint hyper-reguli, it is an open question whether when $n = 3$, the classes of Culbert and Ebert and the classes of Jha and Johnson are connected or perhaps equal. When $n = 3$, every hyper-regulus is an André hyper-regulus so the replacement hyper-reguli are then determined. It it then possible to give a description of any hyper-regulus in a Desarguesian affine plane by the use of norm functions. But, this is not the method employed by Culbert and Ebert, who employed pencils of 'Sherk surfaces' to find their sets of mutually disjoint sets of hyper-reguli. The basic idea is that a Sherk surface of cardinality $1 + q + q^2$ determines a hyper-regulus by results of Bruck [2]. Culbert and Ebert note that algebraic pencils of certain mutually disjoint Sherk surfaces given rise to certain subsets of cardinality $1 + q + q^2$ surfaces, which are forced then to define mutually disjoint hyper-reguli.

On the other hand, the method used by Jha and Johnson is to consider a Desarguesian affine plane Σ of order q^n and a function $y = \sum_{i=0}^{n-1} a_i x^{q^i}$, for $a_i \in \mathsf{GF}(q^n)$, which has the property that the image set under the kernel homology group of elements $(x, y) \longrightarrow (xd, yd)$, for $d \in \mathsf{GF}(q^n)^*$ forms a partial spread.

When this occurs, the intersection components of Σ necessarily intersect the components of the image set in 1-dimensional $\mathsf{GF}(q)$-subspaces. Since we have partial spreads of degrees $(q^n-1)/(q-1)$ and order q^n, we then obtain a hyper-regulus. Then certain arguments show that particular subsets of functions of similar type are mutually disjoint. Note that the particular hyper-reguli in the Desarguesian plane are implicitly but not explicitly defined.

So, we see that in order to determine if the Culbert and Ebert sets of mutually disjoint hyper-reguli and related to the Jha-Johnson sets, we need to connect the associated subsets of pencils of Sherk sets to the set of replacement sets of the associated hyper-reguli. When we do this for $n = 3$, we realize that we find sets of mutually disjoint sets of hyper-reguli for arbitrary n. That is, we find classes of translation planes of orders q^n, for $n \geq 3$ that contain the classes of Culbert and Ebert for $n = 3$. Furthermore, some of these classes have been previously constructed by Jha and Johnson [5], but certainly not all.

2 Jha-Johnson method

In this section, we give some background required to understand the constructions of Jha and Johnson.

Definition 2.1. Let Σ denote a Desarguesian affine plane of order q^n, for $n > 2$, coordinatized by a field F_{q^n} isomorphic to $\mathsf{GF}(q^n)$. Assume that k is an integer less than n such that $n/(n,k) > 2$. Let ω be a primitive element of $\mathsf{GF}(q^n)^*$, then for ω^i, attach an element $f(i)$ of the cyclic subgroup $C_{(q^n-1)/(q^{(n,k)}-1)}$ of $\mathsf{GF}(q^n)^*$ of order

$$(q^n-1)/(q^{(n,k)}-1),$$

and for $\omega^{-iq^{n-k}}$, attach an element $f(i)^{-q^{n-k}}$.

Hence, we have a coset representative set

$$\mathcal{C}_{\omega,f} = \{\omega f(1), \omega^2 f(2), \ldots, \omega^{(q^{(n,k)}-1)} f(q^{(n,k)}-1)\}$$

for $C_{(q^n-1)/(q^{(n,k)}-1)}$.

Assume that λ is a subset of $\{1, 2, \ldots, q^{(n,k)}-1\}$. Let $b, c \in F_{q^n}^*$ then (b, c) is said to be an '(ω, λ)-admissible pair' if and only if

$$\left(\frac{b}{c}\right)^{(q^n-1)/(q^{(n,k)}-1)} \notin (\omega^{i_j+i_z})^{(q^n-1)/(q^{(n,k)}-1)},$$

for all $i_j, i_z \in \lambda$.

The main construction theorem of Jha and Johnson is as follows:

Theorem 2.2 (Jha and Johnson [5]). *Assume that (b, c) is an '(ω, λ)-admissible pair' and let $\mathcal{C}_{\omega,f}$ be a coset representative set of $C_{(q^n-1)/(q^{(n,k)}-1)}$ as in Definition 2.1. Let*

$$\mathcal{H}^* = \left\{ \begin{array}{c} y = x^{q^k}\omega^{i_j} f(i_j) c d^{1-q^k} + x^{q^{n-k}}\omega^{-i_j q^{n-k}} f(i_j)^{-q^{n-k}} b d^{1-q^{n-k}} \\ \text{for all } d \in \mathsf{GF}(q^n)^*, \text{ for all } i_j \in \lambda \end{array} \right\}.$$

Then $\mathcal{H}^*$ is a set of $|\lambda|$ mutually disjoint hyper-reguli of order q^n and of degree $(q^n - 1)/(q^{(n,k)} - 1)$.

Since the hyper-reguli for $n > 3$ are not André, it is an open question as to how many hyper-regulus replacements there actually are — there is, of course, at least one, from the construction. For n odd, we have:

Theorem 2.3 (Jha and Johnson [5]). *If $n/(n, k)$ is odd and*

$$\mathcal{H}^*_{(\alpha,c,b)} = \left\{ \begin{array}{c} y = x^{q^k}\alpha c d^{1-q^k} + x^{q^{n-k}}\alpha^{-q^{n-k}} b d^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n) - \{0\} \end{array} \right\}$$

is a hyper-regulus which replaces the hyper-regulus $\mathcal{H}$ in the associated Desarguesian affine plane Σ then

$$\mathcal{H}^*_{(\alpha^{-1}, b^{q^k}, c^{q^{n-k}})} = \left\{ \begin{array}{c} y = x^{q^k}\alpha^{-1} b^{q^k} d^{1-q^k} + x^{q^{n-k}}\alpha^{q^{n-k}} c^{q^{n-k}} d^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n) - \{0\} \end{array} \right\}$$

is also a replacement for the hyper-regulus $\mathcal{H}$.

Furthermore, there are a variety of particular examples of such sets of mutually disjoint hyper-reguli.

Definition 2.4. Let Σ denote a Desarguesian affine plane of order q^n, for $n > 2$, coordinatized by a field F_{q^n} isomorphic to $\mathsf{GF}(q^n)$. Let $e \neq 1$ be a divisor of $q^{(n,k)} - 1$ and define define $\lambda = \{ei; i = 1, 2, \ldots, (q^{(n,k)} - 1)/e\}$.

Assume that k is an integer less than n such that $n/(n, k) > 2$. Let ω be a primitive element of $\mathsf{GF}(q^n)^*$, then for ω^i, attach an element $f(i)$ of the cyclic subgroup $C_{(q^n-1)/(q^{(n,k)}-1)}$ of $\mathsf{GF}(q^n)^*$ of order

$$(q^n - 1)/(q^{(n,k)} - 1),$$

and for $\omega^{-iq^{n-k}}$, attach an element $f(i)^{-q^{n-k}}$.

As before, we have a coset representative set

$$\mathcal{C}_{\omega,f} = \{\omega f(1), \omega^2 f(2), \ldots, \omega^{(q^{(n,k)}-1)} f(q^{(n,k)} - 1)\}$$

for $C_{(q^n-1)/(q^{(n,k)}-1)}$.

Let $b, c \in F_{q^n}^*$ then (b, c) is said to be an '(e, ω, λ)-group admissible pair' if and only if

$$\left(\frac{b}{c}\right)^{(q^n-1)/(q^{(n,k)}-1)} \notin C_{(q^{(n,k)}-1)/e}.$$

Theorem 2.5 (Jha and Johnson [5]). *Assume that (b, c) is an '(e, ω, λ)-group admissible pair' and let $\mathcal{C}_{\omega,f}$ be a coset representative set of $C_{(q^n-1)/(q^{(n,k)}-1)}$ as in Definition 2.4. Let*

$$\mathcal{H}_{e,(n,k)}^* = \left\{ \begin{array}{c} y = x^{q^k}\omega^{i_j} f(i_j) c d^{1-q^k} + x^{q^{n-k}}\omega^{-i_j q^{n-k}} f(i_j)^{-q^{n-k}} b d^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n)^*, \text{for } i_j \in \lambda \end{array} \right\}.$$

Then $\mathcal{H}_{e,(n,k)}^*$ is a set of $(q^{(n,k)} - 1)/e$ mutually disjoint hyper-reguli of order q^n and degree $(q^n - 1)/(q^{(n,k)} - 1)$.

2.1 Non-group constructions

If q is odd, we may obtain sets of $(q^{(n,k)} - 1)/2$ mutually disjoint hyper-reguli using $C_{(q^{(n,k)}-1)/2}$. However, other non-group-like sets are possible.

Theorem 2.6 (Jha and Johnson [5]). *If q is odd, let $\lambda_{odd} = \{1, 2, \ldots, (q^{(n,k)} - 3)/2\}$ and if q is even let $\lambda_{even} = \{1, 2, \ldots, (q^{(n,k)}/2 - 1\}$. For either case odd or even, let (b, c) be a (ω, λ)-admissible pair for $\lambda = \lambda_{odd}$ or λ_{even}, respectively. Let $\mathcal{C}_{\omega,f}$ be a coset representative set of $C_{(q^n-1)/(q^{(n,k)}-1)}$ as in Definition 2.1.*

(1) *If q is odd let*

$$\mathcal{H}_{(q^{(n,k)}-3)/2}^* = \left\{ \begin{array}{c} y = x^{q^k}\omega^{i_j} f(i_j) c d^{1-q^k} + x^{q^{n-k}}\omega^{-i_j q^{n-k}} f(i_j)^{-q^{n-k}} b d^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n)^*, \ i_j \in \lambda_{odd} \end{array} \right\}.$$

Then $\mathcal{H}_{(q^{(n,k)}-1)/3}^*$ is a set of $(q^{(n,k)} - 3)/2$ mutually disjoint hyper-reguli of order q^n and degree $(q^n - 1)/(q^{(n,k)} - 1)$.

(2) *If q is even let*

$$\mathcal{H}_{q^{(n,k)}/2-1}^* = \left\{ \begin{array}{c} y = x^{q^k}\omega^{i_j} f(i_j) c d^{1-q^k} + x^{q^{n-k}}\omega^{-i_j q^{n-k}} f(i_j)^{-q^{n-k}} b d^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n)^*, \text{for } i_j \in \lambda_{even} \end{array} \right\}.$$

Then $\mathcal{H}_{q^{(n,k)}/2-1}^*$ is a set of $q^{(n,k)}/2 - 1$ mutually disjoint hyper-reguli of order q^n and degree $(q^n - 1)/(q^{(n,k)} - 1)$.

Remark 2.7. If a choice of subset λ produces a set of mutually disjoint hyper-reguli, so does $\lambda + i_0$, for any fixed integer i_0, for $i_0 = 1, 2, \ldots, (q^{(n,k)} - 1)$.

Corollary 2.8 (Jha and Johnson [5]). *Assume the general conditions of the previous Theorem 2.6.*

(1) *If q is odd let $\mathcal{H}^*_{(q^{(n,k)}-3)/2,i_0}$ be the set*

$$\left\{ \begin{array}{c} y = x^{q^k}\omega^{i_j} f(i_j)cd^{1-q^k} + x^{q^{n-k}}\omega^{-i_j q^{n-k}} f(i_j)^{-q^{n-k}} bd^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n)^*, \\ \textit{for } i_j \in \lambda_{odd} + i_0 \end{array} \right\}.$$

Then $\mathcal{H}^*_{(q^{(n,k)}-3)/2,i_0}$ is a set of $(q^{(n,k)}-3)/2$ mutually disjoint hyper-reguli of order q^n and degree $(q^n-1)/(q^{(n,k)}-1)$.

(2) *If q is even let $\mathcal{H}^*_{q^{(n,k)}/2-1,i_0}$ be the set*

$$\left\{ \begin{array}{c} y = x^{q^k}\omega^{i_j} f(i_j)cd^{1-q^k} + +x^{q^{n-k}}\omega^{-i_j q^{n-k}} f(i_j)^{-q^{n-k}} bd^{1-q^{n-k}}; \\ d \in \mathsf{GF}(q^n)^*, \\ \textit{for } i_j \in \lambda_{even} + i_0 \end{array} \right\}.$$

Then $\mathcal{H}^*_{q^{(n,k)}/2-1,i_0}$ is a set of $q^{(n,k)}/2-1$ mutually disjoint hyper-reguli of order q^n and degree $(q^n-1)/(q^{(n,k)}-1)$.

3 Culbert-Ebert 'Sherk surfaces'

Part of the following section also appears in the 'Handbook of Finite Translation Planes' by N.L. Johnson, V. Jha and M. Biliotti, Taylor Books, 2007, and ultimately depends on the work of Culbert-Ebert. It is repeated here for convenience of exposition.

Culbert and Ebert [3] have constructed various sets of hyper-reguli of order q^3 and degree $1+q+q^2$. Bruck [1] shows that any hyper-regulus in a Desarguesian affine plane of degree $1+q+q^2$ and order q^3 is actually an André hyper-regulus and Pomareda [8] showed there are actually two possible replacements, namely the André replacements. Recall that in the standard setting there are $q-1$ mutually disjoint André hyper-regulus with two components $x=0, y=0$ and a corresponding affine homology group H of order $1+q+q^2$ leaving invariant each André hyper-regulus. A subset of hyper-regular is said to be 'linear' if and only if the set is a subset of an André set of $q-1$ hyper-reguli which are orbits under a group isomorphic to H.

Culbert and Ebert [3] actually found sets of mutually disjoint hyper-reguli (so each is necessarily André) with the property that no subset of at least two hyper-reguli is linear. Any such subset then can be replaced in two ways producing a translation plane which cannot be André or indeed generalized André.

The construction of the sets of mutually disjoint hyper-reguli operates on the line at infinity of a Desarguesian affine plane of order q^3. In this context, consider any hyper-regulus $\mathcal{H}$ then there is a collineation of order 3 isomorphic to the Frobenius automorphism of $\mathsf{GF}(q^3)$ over $\mathsf{GF}(q)$ which fixes $\mathcal{H}$ pointwise. The group generated by such a collineation is called the 'stability group' of the hyper-regulus. Choose two points on ℓ_∞, P and Q, and denote by $\phi(P,Q)$ the group generated by the stability groups of all hyper-reguli containing P and Q. An orbit under $\phi(P,Q)$ of a point R not equal to P or Q is called a 'cover' ('Bruck cover') and Bruck [2] shows that covers are hyper-reguli (the infinite points of hyper-reguli in the associated affine Desarguesian plane).

Let N denote the norm function from $\mathsf{GF}(q^3)$ to $\mathsf{GF}(q)$, so that $N(x) = x^{1+q+q^2}$. Then any hyper-regulus has the following form:

(i) $\left\{ x \in \mathsf{GF}(q^3);\ N(x-a) = f \right\}$, for some $a \in \mathsf{GF}(q^3)$, $f \in \mathsf{GF}(q)^*$, or

(ii) $\left\{ x \in \mathsf{GF}(q^3) \cup \{\infty\}\,;\ N(\frac{x-a}{x-b}) = f \right\}$, for some $a, b \in \mathsf{GF}(q^3)$, $f \in \mathsf{GF}(q)^*$.

The main construction device depends on results on 'Sherk surfaces':

Definition 3.1. Let N and T denote the norm and trace, respectively, of $\mathsf{GF}(q^3)$ over $\mathsf{GF}(q)$. Let $f, g \in \mathsf{GF}(q)$, $\alpha, \delta \in \mathsf{GF}(q^3)$. Then the 'Sherk surface' is defined by:

$$S(f,\alpha,\delta,g) = \left\{ z \in \mathsf{GF}(q^3) \cup \{\infty\}\,;\ fN(z) + T(\alpha^{q^2} z^{q+1}) + T(\delta z) + g\ = 0 \right\}.$$

The set of Sherk surfaces can be partitioned into four orbits under $\mathsf{P\Gamma L}(2, q^3)$. Two surfaces of the same cardinality are in the same orbit. The cardinalities of the surfaces are 1, q^2-q+1, q^2+1, q^2+q+1. The Sherk surfaces of cardinality q^2+q+1 are covers or rather hyper-reguli.

Now take a $\mathsf{GF}(q)$-linear combination of two Sherk surfaces that have no intersection. Then this linear combination will define a set of Sherk surfaces which are mutually disjoint. The subset of covers of this linear combination consists then of mutually disjoint hyper-reguli. Then by various choices of generating Sherk surfaces of the linear combination, certain subsets of covers arise. For example,

Theorem 3.2 (Part of Lemma 1 and Theorems 2 and 3, Culbert and Ebert [3]). *Assume that q is an odd prime.*

(1) *For $u \in \mathsf{GF}(q)$, let $S = S(0,1,0,u)$. Then S is a cover precisely when u is a square in $\mathsf{GF}(q)$.*

(2) *Let B denote the bitrace, $B(x) = T(x^{q+1})$. Let $S = S(0,\alpha,\delta,g)$, for $\alpha, \delta \in \mathsf{GF}(q^3)$, not both zero, $g \in \mathsf{GF}(q)$. Define*

$$\Delta = 4N(\alpha)g - B((\alpha\delta)^q + (\alpha\delta)^{q^2} - \alpha\delta).$$

Then S is a cover precisely when Δ is a non-square.

(3) *Let $S = S(1, \alpha, \delta, g)$, a Sherk surface which is not a simple point. Define $\Delta' = 4N(\delta') + (g')^2$, where $\delta' = \delta - \alpha^{q^2+q}$ and $g' = g + 2N(\alpha) - T(\alpha\delta)$. Then S is a cover precisely when Δ' is a square.*

Similarly for q even, the main results identifying covers is the following theorem.

Theorem 3.3 (Part of Lemma 4 and Theorems 5 and 6, Culbert and Ebert [3]). *Let q be even. Let T_0 denote the trace function from $\mathsf{GF}(q)$ to $\mathsf{GF}(2)$.*

(1) *Let $u \in \mathsf{GF}(q)^*$, and let $S = S(0, 1, 1, u)$. Then S is a cover precisely when $T_0(u + 1) = 0$.*

(2) *Let $\alpha, \delta \in \mathsf{GF}(q^3)$, not both zero and let $g \in \mathsf{GF}(q)$. Let $S = S(0, \alpha, \delta, g)$. Then S is a cover if $T(\alpha\delta) \neq 0$ and $T_0(c) = 0$, where $c = (gN(\alpha)+B(\alpha\sigma))/T(\alpha\delta)^2$.*

(3) *Let $S = S(1, \alpha, \delta, g)$ be a Sherk surface that is not a single point. Define $\delta' = \delta + \alpha^{q^2+q}$ and $g' = g + T(\alpha\delta)$. Then S is a cover if $g \neq T(\alpha\delta)$ and $T_0(c') = 0$ where $c' = N(\delta')/(g')^2$.*

Equipped with these theorems, it is then possible to determine the set of mutually disjoint covers within a linear combination of two appropriately selected Sherk surfaces.

The construction results of Culbert and Ebert are found in the following theorem.

Theorem 3.4 (Part of Theorems 11,12,13,14, Culbert and Ebert [3]).

(i) *Let q be an odd prime power ≥ 7. Consider the F-linear combination*

$$fS(1, 0, -1, 0) + gS(0, 0, 0, 1); f, g \in \mathsf{GF}(q).$$

Then the subset of (mutually disjoint) covers has cardinality $\frac{(q-3)}{2}$. Furthermore, this set is

$$\left\{(S(1, 0, -1, g); -4 + g^2 \text{ is a non-zero square in } \mathsf{GF}(q), g \in \mathsf{GF}(q)\right\}.$$

(ii) *Let q be an odd prime power ≥ 5 and let u be a fixed non-square. Consider the F-linear combination*

$$fS(1, 0, -u, 0) + gS(0, 0, 0, 1); f, g \in \mathsf{GF}(q).$$

Then the subset of (mutually disjoint) covers has cardinality $\frac{(q-1)}{2}$. This subset of covers is

$$\left\{S(1, 0, -u, g); -4u^3 + g^2 \text{ a non-zero square in } \mathsf{GF}(q)\right\}.$$

(iii) *Let* $q = 2^m$, *with* $m \geq 3$. *Let* $v \in \mathsf{GF}(q) - \{0, 1\}$.

$$fS(0,1,1,v) + gS(1,0,0,0); f, g \in \mathsf{GF}(q).$$

Then the subset of (mutually disjoint) covers contains a subset of cardinality $\frac{(q-2)}{2}$. *Furthermore,*

(a) *if* $T_0(v+1) = 0$ *then the subset of covers is*

$$\left\{S(1, f^{-1}, f^{-1}, vf^{-1}); vf^{-1} \neq 1\,; f \in \mathsf{GF}(q) - \{0\}; T_0(c') = 0\right\} \\ \cup \left\{S(0,1,1,v); T_0(v+1) = 0\right\},$$

(b) *if* $T_0(v+1) = 1$ *then the subset of covers is*

$$\left\{S(1, f^{-1}, f^{-1}, vf^{-1}); vf^{-1} \neq 1\,; f \in \mathsf{GF}(q) - \{0\}; T_0(c') = 0\right\},$$

where T_0 *is the trace function from* $\mathsf{GF}(q)$ *to* $\mathsf{GF}(2)$ *and*

$$c' = \frac{N(f^{-1} + f^{-q^2-q})}{vf^{-1} + T(f^{-2})},$$

for $vf^{-1} \neq 1$ *in both cases.*

Since these sets of mutually disjoint hyper-reguli arise from a linear combination of Sherk surfaces, any two hyper-reguli will then linearly generate the same set. Since a linear set of $q-1$ covers is not generated in any of these these cases, it follows that no two hyper-reguli belong to a linear set. In other words, any subset of at least two hyper-reguli from either of these three situations will produce a 'non-linear' set. Furthermore, each hyper-regulus has two independent replacements and the corresponding translation planes can never be André or generalized André.

We have seen similar types of hyper-regulus replacements of Jha and Johnson of order q^3. However, since the methods of Culbert and Ebert first find the hyper-reguli as images on the Desarguesian line, and Jha and Johnson find the replacements for the hyper-reguli and not explicitly the hyper-reguli, it is not clear how these two sets of translation planes intersect, if they do at all. We make all of the connections clear in the following sections.

4 A new Sherk pencil

In this section, it is pointed out that there is another Sherk pencil for q even, not previously determined by Culbert and Ebert. In particular, consider

$$fS(1,0,u,0) + gS(0,0,0,1); f, g \in \mathsf{GF}(q),$$

for u a fixed non-zero element of GF(q), where q is even. Clearly, $S(1,0,u,0)$ and $S(0,0,0,1)$ are disjoint sets. Within this set is the subset

$$\left\{ S(1,0,u,\frac{u^2}{d}+ud);\ \text{for } u \neq d^2 \text{ in } \mathsf{GF}(q)^* \right\}.$$

According to the criteria established for hyper-reguli 'covers' for even order, we recall part (3) of Theorem 3.3. Let $S = S(1,\alpha,\delta,g)$ be a Sherk surface that is not a single point. Define $\delta' = \delta + \alpha^{q^2+q}$ and $g' = g + T(\alpha\delta)$. Then S is a cover if $g \neq T(\alpha\delta)$ and $T_0(c') = 0$ where $c' = N(\delta')/(g')^2$. Since $\alpha = 0$ and $\delta = u = \delta' m$ we see that $g' = g = \frac{u^2}{d} + ud$ and $c' = u^3/(\frac{u^2}{d} + ud)^2$.

We claim that

$$T_0(u^3/(\frac{u^2}{d}+ud)^2) = 0$$

for all such elements d in GF$(q)^*$, such that $u \neq d^2$. We note that $u^3/(\frac{u^2}{d}+ud)^2 = d^2u/(u^2+d^4)$. Let $q = 2^r$, and recall that $T_0(t) = T_0(t^2)$. Hence,

$$T_0(u^3/(\frac{u^2}{d}+ud)^2) = T_0(du^{2^{r-1}}/(u+d^2)).$$

Now let

$$\begin{aligned} du^{2^{r-1}}/(u+d^2) &= d(u+d^2+d^2)^{2^{r-1}}/(u+d^2) \\ &= d(u+d^2)^{2^{r-1}-1} + dd^{2^{2^{r-1}}}/(u+d^2) \\ &= d(u+d^2)^{2^{r-1}-1} + d^2(u+d^2)^{-1}. \end{aligned}$$

Furthermore, notice that

$$d^2(u+d^2)^{-1} = d^2(u+d^2)^{q-2} = ((d(u+d^2))^{2^{r-1}-1})^2.$$

Also,

$$T_0(((d(u+d^2))^{2^{r-1}-1})^2) = T_0((d(u+d^2))^{2^{r-1}-1}).$$

Thus,

$$\begin{aligned} T_0(u^3/(\frac{u^2}{d}+ud)^2) &= T_0(du^{2^{r-1}}/(u+d^2)) \\ &= T_0(d(u+d^2)^{2^{r-1}-1} + d^2(u+d^2)^{-1}) \\ &= T_0(d(u+d^2)^{2^{r-1}-1}) + T_0((d(u+d^2)^{2^{r-1}-1})^2) \\ &= T_0(d(u+d^2)^{2^{r-1}-1}) + T_0(d(u+d^2)^{2^{r-1}-1}) = 0. \end{aligned}$$

Hence, we have a set of mutually disjoint hyper-reguli. Note that

$$\frac{u^2}{d} + ud = \frac{u^2}{e} + ue$$

if and only if

$$de = u, and\ d \neq e.$$

So, remove the elements $\sqrt{u}$ and 0 from $\mathsf{GF}(q)$. For the remaining $q-2$ elements, partition the elements into $(q-2)/2$ sets

$$\left\{ d, \frac{u}{d} \right\}$$

and select exactly one element from each set. We then obtain $2^{(q-2)/2}$ sets of cardinality $(q-2)/2$ producing the same set of $(q-2)/2$ mutually disjoint hyper-reguli. In the next section we consider the particular hyper-reguli that are Sherk surfaces and it is then not difficult to see that each set $\left\{ d, \frac{u}{d} \right\}$ determines the same hyper-regulus $\mathcal{H}_{\left\{ d, \frac{u}{d} \right\}}$ in a Desarguesian plane and the individual choices d or u/d correspond to the two possible replacement hyper-reguli of $\mathcal{H}_{\left\{ d, \frac{u}{d} \right\}}$.

Theorem 4.1. *The previous construction produces a set*

$$\left\{ S(1, 0, u, \frac{u^2}{d} + ud);\ \textit{for}\ u \neq d^2\ \textit{in}\ \mathsf{GF}(q)^*, q\ \textit{even} \right\}$$

of $(q-2)/2$ *mutually disjoint hyper-reguli in a Desarguesian plane. There are* $3^{(q-2)/2}$ *possible translation planes obtained from this set.*

Proof. For each of the $(q-2)/2$ hyper-reguli, there are three possible replacements (one trivial). □

5 The hyper-reguli that are Sherk surfaces

We first simply work out the Sherk surfaces

$$S(f^*, \alpha, \delta, g) = \left\{ z \in \mathsf{GF}(q^3) \cup \{\infty\}\,;\ f^* N(z) + T(\alpha^{q^2} z^{q+1}) + T(\delta z) + g\ = 0 \right\}.$$

corresponding to hyper-reguli of the form:

$$\left\{ x \in \mathsf{GF}(q^3) \cup \{(\infty)\}\,;\ N(\frac{x-a}{x-b}) = f \right\},$$

that do not contain (∞), so that $x - b \neq 0$.

$$N(\frac{x-a}{x-b}) = f$$

if and only if

$$(x-a)^{1+q+q^2} = f(x-b)^{1+q+q^2},$$

which produces the following equation:

$$(f-1)N(x)+T((a-fb)^{q^2})x^{1+q}) \\ +T((fb^{q+q^2}-a^{q+q^2})x)+N(a)-fN(b)=0\,,$$

which determines the Sherk surface $S(f-1, a-fb, fb^{q+q^2}-a^{q+q^2}, N(a)-fN(b))$. Hence, we obtain:

Lemma 5.1. *The hyper-regulus*

$$N(\frac{x-a}{x-b})=f$$

for $x \neq b$ is given by the Sherk surface

$$S(f-1, a-fb, fb^{q+q^2}-a^{q+q^2}, N(a)-fN(b)).$$

We now apply this lemma to the two Sherk pencils of Culbert and Ebert of Theorem 3.4(i), as well as to the new Sherk pencil of the previous section.

Case (i). Let q be an odd prime power ≥ 7.

Consider the F-linear combination

$$fS(1,0,-1,0)+gS(0,0,0,1); f,g \in \mathsf{GF}(q).$$

Then the subset of (mutually disjoint) covers has cardinality $\frac{(q-3)}{2}$. Furthermore, this set is

$$\left\{(S(1,0,-1,g); -4+g^2 \text{ is a non-zero square in } \mathsf{GF}(q), g \in \mathsf{GF}(q)\right\}.$$

Case (ii). Let q be an odd prime power ≥ 5.

Let u be a fixed non-square. Consider the F-linear combination

$$fS(1,0,-u,0)+gS(0,0,0,1); f,g \in \mathsf{GF}(q).$$

Then the subset of (mutually disjoint) covers has cardinality $\frac{(q-1)}{2}$. This subset of covers is

$$\left\{S(1,0,-u,g); -4u^3+g^2 \text{ a non-zero square in } \mathsf{GF}(q)\right\}.$$

Case (iii). Let q be even.

In the previous sections, we have considered the set of $(q-2)/2$ hyper-reguli

$$\left\{ S(1,0,u,\frac{u^2}{d}+ud);\text{ for } u \neq d^2 \text{ in } \mathsf{GF}(q)^*,\, q \text{ even} \right\}.$$

We now therefore see that all of these three of these sets in cases (i), (ii), (iii) have the basic Sherk surface $S(1,0,u,g)$, for u a fixed element of $\mathsf{GF}(q)^*$.

From Lemma 5.1, we match this with

$$S(f-1,(a-fb)/(f-1),(fb^{q+q^2}-a^{q+q^2})/(f-1),(N(a)-fN(b))/(f-1)).$$

Therefore, $a = fb$. Now assume that a and b are in $\mathsf{GF}(q)$. Hence, we must have

$$(fb^{q+q^2}-a^{q+q^2})/(f-1) = (fb^2-a^2)/(f-1) = (fb^2-f^2b^2)/(f-1) = -fb^2 = u.$$

So,

$$f = -u/b^2.$$

So, $a = -u/b$. So

$$(N(a)-fN(b))/(f-1)) = \frac{-u^3/b^3+(u/b^2)b^3}{-(u/b^2+1)}.$$

Note that $\frac{-u^3/b^3+(u/b^2)b^3}{-(u/b^2+1)} = \frac{-u^3+ub^4}{-b(u+b^2)} = \frac{u(b^4-u^2)}{-b(u+b^2)} = \frac{u(b^2-u)}{-b} = \frac{u^2}{b} - ub$. Thus, we have

Lemma 5.2. *The hyper-regulus*

$$N(\frac{x-(-u/b)}{x-b}) = -u/b^2$$

for $x \neq b$ is given by the Sherk surface

$$S(1,0,u,\frac{u^2}{b}-ub),$$

for $b^2 \neq u$.

Theorem 5.3. *The set of Sherk surfaces*

$$\left\{ S(1,0,u,\frac{u^2}{d}-ud);\text{ for } u \neq d^2 \text{ in } \mathsf{GF}(q)^* \right\}$$

is a set of mutually disjoint hyper-reguli.

(i) *When q is odd and u is a non-square, the set has $(q-1)/2$ hyper-reguli.*

(ii) *When q is odd and u is a square, the set has $(q-3)/2$ hyper-reguli.*

(iii) *When q is even, the set has $(q-3)/2$ hyper-reguli.*

An alternative notation for the above set is given by in terms of the norm as follows:

$$\left\{\left\{y=xm; N(\frac{x+u/d}{x-d})=-u/d^2\right\}; d^2\neq u\right\}$$

is the corresponding set of André nets.

Proof. The proof is clear from the previous sections but in terms of cardinality of the sets, note when q is odd and u is a non-square then d^2 cannot be u. Then $\frac{u^2}{d}-ud=\frac{u^2}{e}-ue$ if and only $cd=u$, for $c\neq d$. Hence, we may partition the set $\mathsf{GF}(q)^*$ into $(q-1)/2$ sets $\{d,u/d\}$. If q is odd and u is a non-zero square then we need to avoid $\pm\sqrt{u}$, and 0, leaving $q-3$ elements, which again are partitioned into sets $\{d,u/d\}$, producing $(q-3)/3$ hyper-reguli. When q is even, the arguments of the previous section are similar and we obtain a set of $(q-2)/2$ hyper-reguli. □

In the next section, we shall generalize the sets of mutually disjoint hyper-reguli of orders q^3 and degree $(q^3-1)/(q-1)$ to mutually disjoint hyper-reguli of orders q^n and degree $(q^n-1)/(q-1)$. When $n>3$, the hyper-reguli turn out not to be André hyper-reguli so our description cannot be in terms of norms as given in the last part of the previous theorem. We then turn to considering the description in terms of the replacement hyper-reguli. In order to do this, we consider certain mappings.

First a general lemma.

Lemma 5.4. *Consider an André hyper-regulus A_δ in standard form and consider the mapping $(x,y)\longrightarrow(x,y)\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, where $\Delta=ad-bc\neq 0$. Assume that $ac\neq 0$. The image of $y=xm$ for $m^{1+q+q^2}=\delta$ is $y=x\left(\frac{b+dm}{a+cm}\right)$, assuming that the image does not contain $x=0$, (i.e. the corresponding (∞)). Letting $m^*=\left(\frac{b+dm}{a+cm}\right)$, we see that*

$$\left(\frac{m^*-\frac{b}{a}}{m^*-\frac{d}{c}}\right)=\left(\frac{(d-\frac{b}{a}c)m}{b-\frac{d}{c}a}\right)=\left(\frac{\Delta m}{-\Delta}\right)\frac{c}{a}=-\frac{c}{a}m.$$

Hence, the image hyper-regulus is of the form

$$N\left(\frac{m^*-\frac{b}{a}}{m^*-\frac{d}{c}}\right)=-N(\frac{c}{a})\delta.$$

Since we are interested in hyper-reguli with form $\left\{y = xm; N(\frac{x+u/d}{x-d}) = -u/d^2\right\}$, we take $a = c = 1$, $b = -u/d$ and d in $\mathsf{GF}(q)^*$. Hence, for $\delta = -u/d^2$, we see that André net A_δ maps to $\left\{y = xm; N(\frac{x+u/d}{x-d}) = -u/d^2\right\}$. Consider a component $y = x^q z$, such that $N(z) = \delta = -u/d^2$ of one of the replacement hyper-reguli of A_δ. We work out the image under $(x, y) \longrightarrow (x, y)\begin{bmatrix}1 & -u/d\\ 1 & d\end{bmatrix}$. The image is the set of elements $(x + x^q z, -xu/d + x^q d)$ which is clearly on the set $y = x^q(zd) + x^{q^2}(-z^{1+q}d)u$. Note that $-z^{1+q}d = u/d^2 z^{-q^2} d = (zd)^{-q^2}u$.

Now from Theorem 2.3, we see that

$$\left\{y = x^q(zd) + x^{q^2}(zd)^{-q^2}u; N(zd) = -u/d^2\right\}$$

and

$$\left\{y = x^q(d^{-1}u)z + x^{q^2}((d^{-1}u)z)^{-q^2}u; N(z) = -u/(d^{-1}u)^2\right\}$$
$$= \left\{y = x^q(d^{-1}u)z + x^{q^2}(d^{-1}z); N(z) = -u/(d^{-1}u)^2\right\}$$

are replacements for each other. That is, d is the basic element in the first set then u/d is the basic element in the second set. Hence, a choice of one element of each set $\{d, u/d\}$ determines what replacement set one is considering and we may do this per element d (such that $d^2 \neq u$). Hence, we have the following theorem.

Theorem 5.5. *A set of replacement hyper-reguli for the hyper-reguli of the set*

$$\left\{\left\{y = xm; N(\frac{x+u/d}{x-d}) = -u/d^2\right\}; d^2 \neq u\right\}$$

is

$$\left\{\left\{y = x^q(zd) + x^{q^2}(zd)^{-q^2}u; N(z) = -u/d^2\right\}; d^2 \neq u\right\},$$

where one element is chosen from each set $\{d, u/d\}$ for each element d in $\mathsf{GF}(q)^*$ such that $d^2 \neq u$ for u in $\mathsf{GF}(q)^*$.

6 Generalization to q^n, n odd

We now define a set of mutually disjoint hyper-reguli of order q^n, for n odd and degree $(q^n - 1)/(q - 1)$ as follows. Consider the set

$$\left\{\left\{y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u; z^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1}\right\};\ d \in \mathsf{GF}(q)^*\right\}.$$

Actually, underlying this set is an associated Desarguesian affine plane Σ with spread as follows:

$$x = 0, y = xm; m \in \mathsf{GF}(q^n).$$

We first maintain that each set

$$\left\{ y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u; z^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1} \right\}$$

defines a replacement hyper-regulus for some hyper-regulus in the Σ, which is implicitly defined as follows: For a given set

$$y = x^q(z_0 d) + x^{q^{-1}}(z_0 d)^{-q^{-1}}u,$$

that is for a specific z_0 such that $z_0^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1}$, we ask if the intersections with the associated components of the form $y = xm$ are always 1-dimensional $\mathsf{GF}(q)$-subspaces. If this is so then applying the kernel homology mappings $(x, y) \longmapsto (xt, yt)$, for $t \in \mathsf{GF}(q^n)^*$, we obtain functions

$$y = x^q(z_0 d)t^{1-q} + x^{q^{-1}}(z_0 d)^{-q^{-1}}t^{1-q^{-1}}u.$$

Since $z = z_0 t^{1-q}$ and z_0 both have the property that

$$w^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1},$$

for $w = z$ or z_0 and since $(z_0 dt^{1-q})^{-q^{-1}} = (z_0 d)^{-q^{-1}}t^{1-q^{-1}}$, we need only check that the set of $(q^n - 1)/(q - 1)$ images under the kernel homology group of order $(q^n - 1)$ forms a partial spread and then that the full set of hyper-reguli forms a partial spread. Therefore, assume that we have an intersection between two of components of putative set of hyper-reguli:

$$y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u \text{ and } y = x^q(z^*e) + x^{q^{-1}}(z^*e)^{-q^{-1}}u$$

where

$$z^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1} \text{ and } z^{*(q^n-1)/(q-1)} = -u^{(n-1)/2}/e^{n-1}.$$

Note that if $d = e$, we are considering the same putative hyper-regulus and showing that there is no solution proves that we have a set of hyper-reguli. Then if d is not e, subject to the conditions mentioned, then proving that there is no solution shows that we have a set of mutually disjoint hyper-reguli. Therefore, assume that for some non-zero x, we have

$$x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u = x^q(z^*e) + x^{q^{-1}}(z^*e)^{-q^{-1}}u.$$

Then we must have

$$(zd - z^*e)^{(q^n-1)/(q-1)} = ((z^*e)^{-q^{-1}} - (zd)^{-q^{-1}})^{(q^n-1)/(q^n-1)}u^n.$$

Note that

$$\begin{aligned}((z^*e)^{-q^{-1}} - (zd)^{-q^{-1}})^{(q^n-1)/(q-1)} &= (zd - z^*e)^{(q^n-1)/(q-1)}/(zdz^*e)^{(q^n-1)/(q-1)} \\ &= (zd - z^*e)^{(q^n-1)/(q-1)}/(u^{(n-1)/2}du^{(n-1)/2}e) \\ &= (zd - z^*e)^{(q^n-1)/(q-1)}/(u^{n-1}de)\,.\end{aligned}$$

Assume that

$$zd = z^*e.$$

Then it follows that $u^{(n-1)/2}d = u^{(n-1)/2}e$ so that $d = e$ and hence $z = z^*$, implying that the two functions are identical. Therefore, we must have that

$$(zd - z^*e)^{(q^n-1)/(q-1)} = ((z^*e)^{-q^{-1}} - (zd)^{-q^{-1}})^{(q^n-1)/(q^n-1)}u^n$$

implies that

$$de = u.$$

Now again using Theorem 2.3, we see, considering the two elements of $\{d, u/d\}$, that each defines a replacement hyper-regulus of the other.

Now consider the situations:

(i) q odd and u non-square in $\mathsf{GF}(q)$,

(ii) q odd and u non-zero square in $\mathsf{GF}(q)$, and

(iii) q even and u non-zero in $\mathsf{GF}(q)$.

Applying the argument of Theorem 5.3, we see that we obtain $(q-1)/2$ mutually disjoint hyper-reguli in case (i), $(q-3)/2$ mutually disjoint hyper-reguli in case (ii) and $(q-2)/2$ mutually disjoint hyper-reguli in case (iii).

Hence, we obtain the following theorem:

Theorem 6.1. *Let Σ be a Desarguesian affine plane of order q^n with spread*

$$x = 0, y = xm; m \in \mathsf{GF}(q^n).$$

Choose any element u of $\mathsf{GF}(q)^*$. Let λ be any set of elements of $\mathsf{GF}(q)^*$ with the property that for d, e in λ then $de \neq u$. Then

$$\left\{\left\{y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u; z^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1}\right\};\ d \in \lambda\right\}$$

forms a set of mutually disjoint hyper-reguli.

(i) *If q is odd and u is a non-square then λ has cardinality $(q-1)/2$.*

(ii) *If q is odd and u is a non-zero square that λ has cardinality $(q-3)/2$.*

(iii) *If q is even and and u is a non-zero element of $\mathsf{GF}(q)$ then λ has cardinality $(q-2)/2$.*

(iv) *The corresponding translation plane obtained by taking components the elements of the above set and the components of the Desarguesian affine plane* Σ *that do not intersect this set will be called* $\Sigma_{\lambda,u}$.

(v) *For each set* λ, *form another set* λ^* *such that for each* $d \in \lambda$, d^* *is either* d *or* u/d. *Then* λ^* *also determines a translation plane.*
Hence, there are $2^{|\lambda|}$ *possible translation planes so constructed for each element* u.

7 The isomorphisms of the translation planes $\Sigma_{\lambda,u}$

Here we assume that we have two translation planes of order q^n, $\Sigma_{\lambda,u}$ and Σ_{λ^*,u^*} and assume these planes are isomorphic. By results of Jha and Johnson [5], we may assume that the

$$\left\{\left\{y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u; z^{(q^n-1)/(q-1)} = -u^{(n-1)/2}/d^{n-1}\right\};\ d \in \lambda\right\}$$

is mapped to

$$\left\{\left\{y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u; z^{(q^n-1)/(q-1)} = -(u^*)^{(n-1)/2}/d^{n-1}\right\};\ d \in \lambda^*\right\}$$

by a collineation of the associated Desarguesian affine plane, necessarily then of the form

$$(x, y) \longmapsto (x^\sigma, y^\sigma)\begin{bmatrix} a & b \\ c & e \end{bmatrix}.$$

Assume that

$$y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u$$

is mapped to

$$y = x^q(z^*d^*) + x^{q^{-1}}(z^*d^*)^{-q^{-1}}u^*,$$

for $d \in \lambda$ and for d^* in λ^*. The image of

$$y = x^q(zd) + x^{q^{-1}}(zd)^{-q^{-1}}u$$

has elements

$$\Big(x^\sigma a + (x^{\sigma q}(zd)^\sigma + x^{\sigma q^{-1}}(zd)^{-\sigma q^{-1}}u^\sigma)c,$$
$$x^\sigma b + (x^{\sigma q}(zd)^\sigma + x^{\sigma q^{-1}}(zd)^{-\sigma q^{-1}}u^\sigma)e\Big).$$

Assume that $n > 3$, then in order that these elements are on

$$y = x^q(z^*d^*) + x^{q^{-1}}(z^*d^*)^{-q^{-1}}u^*,$$

there cannot be an element x^{q^2} or $x^{q^{-2}}$ with non-zero coefficients. If the terms are worked out we must have

$$\begin{aligned}&(x^\sigma a + (x^{\sigma q}(zd)^\sigma + x^{\sigma q^{-1}}(zd)^{-\sigma q^{-1}}u^\sigma)c)^q(z^*d^*)\\&\quad + (x^\sigma a + (x^{\sigma q}(zd)^\sigma + x^{\sigma q^{-1}}(zd)^{-\sigma q^{-1}}u^\sigma)c)^{q^{-1}}(z^*d^*)^{-q^{-1}}u^*\\&\qquad = x^\sigma b + (x^{\sigma q}(zd)^\sigma + x^{\sigma q^{-1}}(zd)^{-\sigma q^{-1}}u^\sigma)e\,,\end{aligned}$$

for all elements $x \in \mathsf{GF}(q^n)$. Since this is a polynomial identity, we see that the coefficients on the x^{q^2} and $x^{q^{-2}}$, are both zero, since $n > 3$ and hence > 4. The coefficient on the x^{q^2}-term is $(zd)^{\sigma q}c^q(z^*d^*)$, forcing $c = 0$. But this implies in turn that $b = 0$ since this is the only coefficient on the x-term. Since all of the translation planes constructed admit the kernel homology group as a collineation group leaving invariant each hyper-regulus in question, it follows that we may assume that $a = 1$. Hence, we obtain

$$x^{\sigma q}(z^*d^*) + x^{\sigma q^{-1}}(z^*d^*)^{-q^{-1}}u^* = (x^{\sigma q}(zd)^\sigma + x^{\sigma q^{-1}}(zd)^{-\sigma q^{-1}}u^\sigma)e\,,$$

for all elements $x \in \mathsf{GF}(q^n)$. Therefore,

$$(zd)^\sigma e = z^*d^*,\ (zd)^{-\sigma q^{-1}}u^\sigma e = (z^*d^*)^{-q^{-1}}u^*.$$

So, we must have

$$((z^*d^*)e^{-1})^{-q^{-1}} = (zd)^{\sigma - q^{-1}} = (z^*d^*)^{-q^{-1}}u^*(u^\sigma e)^{-1}.$$

Hence, we see that

$$e^{q+1} = u^*(u^\sigma)^{-1}.$$

Note that since n is odd, then $e^{(q^2-1)} = 1$ and $(q^2-1, q^n-1) = q^{(2,n)}-1 = q-1$. So the order of e divides $q-1$. Then $e^{q+1} = e^{q-1+2} = e^2$ so that e^2 is in $\mathsf{GF}(q)$, implying that $\{e\alpha + \beta; \alpha, \beta \in \mathsf{GF}(q)\}$ is a subfield of order q^2 or q. Since n is odd, then the subfield is of order q, so that e is in $\mathsf{GF}(q)$. Therefore, we have

$$u^* = u^\sigma e^2 \text{ and } \lambda^* = \lambda^\sigma e.$$

So, for example, assume that q is an odd prime. Since there are $2^{|\lambda|}$ possible sets λ for the same element u, we see that $e^2 = \pm 1$, implying that there are exactly $2^{|\lambda|-1}$, mutually non-isomorphic translation planes constructed and more generally for $q = p^r$, there are at least $2^{|\lambda|-1}/r$. Thus, we have the following theorem.

Theorem 7.1. *The translation planes $\Sigma_{\lambda,u}$ and Σ_{λ^*,u^*} of order $q^n = p^{2r}$, p a prime for n odd and $n > 3$ are isomorphic if and only if*

$$u^* = u^\sigma e^2 \textit{ and } \lambda^* = \lambda^\sigma e,$$

where e is an element of $\mathsf{GF}(q)^*$. Hence, there are at least $2^{|\lambda|-1}/r$ mutually disjoint translation planes.

Remark 7.2. So for $n > 3$, when q is odd then λ has cardinality $(q-1)/2$ or $(q-3)/2$, respectively as u is a non-square or a non-zero square. Hence, there are at least $2^{(q-3)/2}/r$ or $2^{(q-5)/2}/r$ mutually non-isomorphic planes of odd order. When q is even, λ has cardinality $(q-2)/2$ and there are at least $2^{(q-2)/2}/r$ mutually non-isomorphic planes of even order. Note that since $n > 3$, there is no requirement on q other that it not be 3.

Now consider a translation plane obtained by replacement of any subset of λ. By Jha and Johnson [6], assuming that $n > 3$, we know that the full collineation group of any translation plane constructed by replacement of any non-empty subset of λ is the stablizer of this set in the associated Desarguesian affine plane. What this means is that the previous argument does not really depend on the full set λ and applies more generally to any proper subset of λ. Hence, two translation planes constructed using subsets of different cardinality are necessarily non-isomorphic and two subsets of the same cardinality are isomorphic under the same conditions as in the above theorem. Considered in this way, for each set λ, we have $3^{|\lambda|}-1$ non-Desarguesian translation planes obtained by the choice of one of three possible replacement nets for a given hyper-regulus (two proper replacements and the choice of not choosing this particular set to replace).

Theorem 7.3. *Let π be a translation plane of order q^n, $q = p^r$, for p a prime and $n > 3$, constructed from $\Sigma_{\lambda,u}$, by choosing a subset λ' and then choosing one of two possible replacements.*

(1) *Then there are at least*

$$(3^{|\lambda|}-1)/(q-1,2)r$$

mutually non-isomorphic planes.

(2) *There are at least*

$$2^{|\lambda|}(3^{|\lambda|}-1)/((q-1,2)r)^2$$

mutually non-isomorphic planes by varying the sets λ.

(3) *If q is an odd prime then there are at least*

$$2^{(q-3)/2}(3^{(q-1)/2}-1)/4$$

mutually non-isomorphic translation planes when u is a non-square and at least

$$2^{(q-5)/2}(3^{(q-3)/2}-1)/4$$

mutually non-isomorphic translation planes, when u is a non-zero square.

8 Connecting ideas

The new classes of translation planes that we have constructed may be connected to the main construction theorem of Jha and Johnson 2.5. Just from the general form of this result, it would appear that all of our translation planes may be placed within the context of this theorem. Still, however, such classes should be considered new examples of admissible pairs producing planes under the general theory.

To illustrate, consider the planes $\Sigma_{\lambda,u}$ in Theorem 6.1, when q is odd and u is a non-square. We connect this class to the class of Jha and Johnson in Theorem 2.5, where the idea is to use the cyclic group of order $(q-1)/2$, $C_{(q-1)/2}$ of $\mathsf{GF}(q)^*$ and a choice of element b such that $b^{(q^n-1)/(q-1)} \notin C_{(q-1)/2}$. To connect these two classes of planes, recall that for $\Sigma_{\lambda,u}$, there is an implicit partition of $\mathsf{GF}(q)^*$ in $(q-1)/2$ pairs $\{\beta, u/\beta\}$ to one may construct sets λ by choosing one element out of each pair of the partition. We have pointed out that this amounts to making different choices of the at least two possible replacement hyper-reguli when n is odd. Note also that if β is a square or non-square, respectively as u/β is non-square or square. Since $C_{(q-1)/2}$ contains the full set of $(q-1)/2$ non-zero squares on $\mathsf{GF}(q)$, we see that if we choose a set λ by selecting all of the squares from the sets $\{\beta, u/\beta\}$, and choose $b = u$, the plane $\Sigma_{\lambda,u}$ and the group-constructed plane of Jha-Johnson are identical.

We note that since general theory of Jha-Johnson is valid for arbitrary n, odd or even, the possible connection between admissible sets and how two such sets might be related via different choices of replacements per hyper-regulus is not considered.

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