The general structure of the projective planes admitting PSL$(2, q)$ as a collineation group

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Abstract

Projective planes of order n admitting PSL$(2, q)$, $q > 3$, as a collineation group are investigated for $n \leq q^2$. As a consequence, affine planes of order n admitting PSL$(2, q)$, $q > 3$, as a collineation group are classified for $n < q^2$ and $(q, n) \neq (5, 16)$. Finally, a complete classification of the translation planes order n that admitting PSL$(2, q)$, $q > 3$, as a collineation group is obtained for $n \leq q^2$.

Keywords: projective plane, collineation group, orbit

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1. Introduction and main results

A classical problem in finite geometry is classifying finite projective planes Π of order n admitting a collineation group G isomorphic to PSL$(2, q)$. The first significant result related to this problem dates back to 1964 and is due to Lüneburg [21] and to Yaqub [26]. In their papers, the authors provide a characterization of the Desarguesian projective planes of order $n = q$. Some years later, Kantor [19], Hering [9], Hering and Walker [11, 12], Reifart and Stroth [25] extended the investigation to planes of more arbitrary order but with additional assumptions: G does not fix points, lines or triangles of Π and G contains involutory perspectivities. In 1989, Moorhouse obtains significant progress along these lines in two ways: he classifies projective planes of order n admitting PSL$(2, q)$ as a collineation group for $n < q$ and he investigates the structure of the planes of order q^2 for q odd. In the second case, Moorhouse shows that Π cannot be the projective extension of an affine plane admitting PSL$(2, q)$ as a collineation group, except for $q = 5$ or 9 which remain still unsolved. In particular, Moorhouse provides a new proof for q odd of the characterization, due to Foulser

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and to Johnson [6, 7], of the translation planes of order q^2 admitting $\mathsf{PSL}(2,q)$. In 1991, Dempwolff [3] obtains a complete characterization of the projective planes of order 16 admitting $\mathsf{PSL}(2,7)$ as collineation group. In that paper, Dempwolff shows that, beside the Desarguesian plane of order 16, the Lorimer-Rahilly plane of order 16, the Johnson-Walker plane of order 16, and their duals also occur. A similar result for translation planes of order 16 was obtained by Johnson [18] in 1984. In 1994, Ho [13] and Ho-Gonçalves [15] investigate the projective planes of order n admitting G isomorphic to $\mathsf{PSL}(2,q)$ for q odd, under the assumption that $G_P \neq \langle 1 \rangle$ for each point P of Π. The authors prove that G does not fix points, lines or triangles of Π. In particular, Π cannot be the projective extension of an affine plane that admits $\mathsf{PSL}(2,q)$ as a collineation group. They also obtain a characterization of the Desarguesian plane of order q under the assumption that G contains involutory homologies and that G_P has a particular order for each point P of Π. Recently, Liu and Li [20] proved that the unique projective plane Π of order n admitting $\mathsf{PSL}(2,q)$ as transitive collineation group is $\Pi \cong \mathsf{PG}(2,2)$ and the group is isomorphic to $\mathsf{PSL}(2,7)$.

This paper focuses on the main problem cited above. In particular, the projective planes of order n admitting a collineation group G isomorphic to $\mathsf{PSL}(2,q)$, $q > 3$, for $n \leq q^2$, are investigated and the following results are obtained.

Theorem 1.1. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,q)$, $q > 3$. If $n \leq q^2$, then one of the following occurs:*

(1) *$n < q$ and one of the following occurs:*

  (a) *$n = 4$, $\Pi \cong \mathsf{PG}(2,4)$ and $G \cong \mathsf{PSL}(2,5)$;*
  (b) *$n = 2$ or 4, $\Pi \cong \mathsf{PG}(2,2)$ or $\mathsf{PG}(2,4)$, respectively, and $G \cong \mathsf{PSL}(2,7)$;*
  (c) *$n = 4$, $\Pi \cong \mathsf{PG}(2,4)$ and $G \cong \mathsf{PSL}(2,9)$.*

(2) *$n = q$, $\Pi \cong \mathsf{PG}(2,q)$ and one of the following occurs:*

  (a) *G fixes a line or a point and q is even;*
  (b) *G is strongly irreducible and q is odd.*

(3) *$q < n < q^2$ and one of the following occurs:*

  (a) *G fixes a point or a line, and one of the following occurs:*
    (i) *$n = 16$ and $G \cong \mathsf{PSL}(2,5)$;*
    (ii) *$n = 16$, Π is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and $G \cong \mathsf{PSL}(2,7)$;*
  (b) *G fixes a subplane Π_0 of Π, q is odd and one of the following occurs:*

(i) $n = 16$, $\Pi_0 \cong \mathsf{PG}(2,4)$ *and* $G \cong \mathsf{PSL}(2,5)$*;*

(ii) $\Pi_0 \cong \mathsf{PG}(2,2)$ *or* $\mathsf{PG}(2,4)$*, and* $G \cong \mathsf{PSL}(2,7)$*;*

(iii) $\Pi_0 \cong \mathsf{PG}(2,4)$ *and* $G \cong \mathsf{PSL}(2,9)$.

(c) *G is strongly irreducible and q is odd;*

(4) *$n = q^2$ and one of the following occurs:*

(a) *G fixes a point or a line, and one of the following occurs:*

(i) $n = 25$ *and* $G \cong \mathsf{PSL}(2,5)$*;*

(ii) $n = 81$ *and* $G \cong \mathsf{PSL}(2,9)$*;*

(iii) $n = q^2$*, q even, and* $G \cong \mathsf{PSL}(2,q)$.

(b) *G fixes a subplane Π_0 of Π, q is odd and one of the following occurs:*

(i) $n = q^2$, $\Pi_0 \cong \mathsf{PG}(2,q)$ *and* $G \cong \mathsf{PSL}(2,q)$*;*

(ii) $n = 25$, $\Pi_0 \cong \mathsf{PG}(2,4)$ *and* $G \cong \mathsf{PSL}(2,5)$*;*

(iii) $n = 81$, $\Pi_0 \cong \mathsf{PG}(2,4)$ *and* $G \cong \mathsf{PSL}(2,9)$*;*

(iv) $n = 81$, *Π_0 is a Hughes plane of order 9 and* $G \cong \mathsf{PSL}(2,9)$*;*

(c) *G is strongly irreducible.*

The Theorem 1.1 under the additional assumptions $n \leq q$, or $n = q^2$ with q odd yields the cases (1), (2), and (4) for q odd. So, we need to prove that (3) occurs when $q < n < q^2$, and either (4a.iii) or (4c) for $n = q^2$ and q even.

Examples corresponding to case (1) or (2) really occur (see [24] and [21, 26], respectively). Examples of the case (3a.i) occur in the Dempwolff plane of order 16 (see [7]), those of type (3a.ii) really occur (see [3]). Examples of the case (3b.i) occur in the Hall plane of order 16, those corresponding to the case (3b.ii) occur in the Desarguesian plane of order 16 when $\Pi_0 \cong \mathsf{PG}(2,2)$ by [3]. Also, examples of the case (3b.iii) occur in the Desarguesian plane of order 64 or the Figueroa plane of order 64. See section 8 for a description of the latter. Furthermore, examples of the case (3c) occurs in the Desarguesian planes of prime order. In these cases $G \cong \mathsf{PSL}(2,q)$ with $q = 5, 7$ or 9 and the involutions in G are a homologies of Π. For a description of these examples see [15] and [13]. While cases (4a.i) and (4a.ii) are open, examples of the case (4a.iii) typically occurs in the Desarguesian planes, in the Hall planes and in the Ott-Schaeffer planes (see [7]). The case (4b.i) occurs in the Desarguesian or Generalized Hughes planes (see [22]). Finally the cases (4b.ii), (4b.iii), (4b.iv) and (4c) are open. Other examples are obtained in section 8.

A special case of the previous theorem is the following which focuses on the projective extensions of affine planes of order n that admit $G \cong \mathsf{PSL}(2,q)$, $q > 3$,

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as a collineation group when $n \leq q^2$. It should be stressed that it furnishes a complete classification of such affine planes, when $n < q^2$ and $(q,n) \neq (5,16)$.

Theorem 1.2. *Let Π be the projective extension of an affine plane of order n that admits a collineation group $G \cong \mathsf{PSL}(2,q)$, $q > 3$. If $n \leq q^2$, then one of the followings occurs:*

(1) $n = q$, $q = 2^h$, $h > 1$, $\Pi \cong \mathsf{PG}(2,q)$ *and* $G \cong \mathsf{PSL}(2,q)$;

(2) $n = 16$ *or* 25, *and* $G \cong \mathsf{PSL}(2,5)$;

(3) $n = 16$, Π *is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and* $G \cong \mathsf{PSL}(2,7)$;

(4) $n = 81$ *and* $G \cong \mathsf{PSL}(2,9)$;

(5) $n = q^2$, $q = 2^h$, $h > 1$, *and* $G \cong \mathsf{PSL}(2,q)$.

Finally, the previous theorem leads to a complete classification of the projective extensions of translation planes of order n that admit a collineation group $G \cong \mathsf{PSL}(2,q)$, $q > 3$, for $n \leq q^2$. In particular, it represents an extension of the Foulser-Johnson Theorems [6] and [7], when $G \cong \mathsf{PSL}(2,q)$ and q is even.

Theorem 1.3. *Let Π be the projective extension of a translation plane of order n that admits a collineation group $G \cong \mathsf{PSL}(2,q)$, $q > 3$. If $n \leq q^2$, then one of the following occurs:*

(1) $n = q$, $q = 2^h$, $h > 1$, $G \cong \mathsf{PSL}(2,q)$ and $\Pi \cong \mathsf{PG}(2,q)$;

(2) $n = 16$, $G \cong \mathsf{PSL}(2,7)$ and Π is the Lorimer-Rahilly plane or the Johnson-Walker plane;

(3) $n = q^2$, $q = 2^h$, $h > 1$, $G \cong \mathsf{PSL}(2,q)$ and Π is the Desarguesian or Hall plane of even order q^2, or the Ott-Schaeffer plane of order q^2, or the Dempwolff plane of order 16 (in this case $q = 4$).

The paper is structured as follows. In section 2, we fix notation and introduce some geometrical and group-theoretical background. In section 3, we provide a reduction for the group-structure of G_P, where P is a point of a line l of Π fixed by G. A reduction is also provided for types and numbers of G-orbits on l. The same is also made for G_m, where m is a line of $[Q]$ and Q is a point of Π fixed by G. Sections 4, 5, 6 and 7 are devoted to the proof of Theorem 1.1 for $q \equiv 1,3,5,7 \mod 8$, respectively. Finally, in section 8 the proofs of Theorems 1.1, 1.2 and 1.3 are completed and some examples are provided. In particular, in this section, the case q even is resolved.

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2. The background

In this section, we introduce the background for the problem investigated and we state the group-theoretical theorems that are used in the proof of our main result. Furthermore some useful numerical and group-theoretical results are proved.

For what concerns finite groups and in particular the group $\mathsf{PSL}(2,q)$ the reader is referred to [4] and [17]. The necessary background about finite projective planes may be found in [16].

Let $\Pi = (\mathcal{P},\mathcal{L})$ be a finite projective plane of order n. If H is a collineation group of Π and $P \in \mathcal{P}$ ($l \in \mathcal{L}$), we denote by $H(P)$ (by $H(l)$) the subgroup of H consisting of perspectivities with centre P (axis l). Also, $H(P,l) = H(P) \cap H(l)$. Furthermore, we denote by $H(P,P)$ (by $H(l,l)$) the subgroup of H consisting of elations with centre P (axis l).

Let Π be a finite projective plane of order n admitting a collineation group G isomorphic to $\mathsf{PSL}(2,q)$, and assume that $n \le q^2$. The following theorems deal with the case $n<q$, $n=q$ and $n=q^2$, respectively.

Theorem 2.1 (Moorhouse). *If Π is a projective plane of order $n<q$ admitting a collineation group G isomorphic to $\mathsf{PSL}(2,q)$, then Π is Desarguesian and $(n,q) = (2,3),(2,7),(4,5),(4,7)$ or $(4,9)$. Moreover, each of the latter cases indeed occurs.*

Proof. See [24, Theorem 1.1]. □

Theorem 2.2 (Lüneburg-Yaqub). *If Π is a projective plane of order q admitting a collineation group G isomorphic to $\mathsf{PSL}(2,q)$, then Π is Desarguesian.*

Proof. See [21] and [26]. □

Theorem 2.3 (Moorhouse). *Suppose that a projective plane Π of order q^2 admits a collineation group G isomorphic to $\mathsf{PSL}(2,q)$, where q is odd. Then one of the following must hold:*

(1) *G acts irreducibly on Π;*

(2) *$q=3$ and G fixes a triangle but no point or line of Π;*

(3) *$q=5$, $\mathrm{Fix}(G)$ consists of an antiflag (X,l) and G has point orbits of length $5,5,6$ and 10 on l;*

(4) *$q=9$ and $\mathrm{Fix}(G)$ consists of a flag.*

Proof. See [24, Theorem 1.2]. □

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Theorem 2.4 (Moorhouse). *Suppose that a projective plane Π of order q^2 admits a collineation group G isomorphic to $\mathsf{PSL}(2,q)$ (where q is odd), and that G leaves invariant a subplane Π_0 of Π. If $q \neq 5, 9$, then Π_0 is a Desarguesian Baer subplane of Π.*

Proof. See [24, Corollary 5.2(i)]. □

As a consequence of the previous theorems, it follows that we may consider projective planes Π of order n admitting a collineation group G isomorphic to $\mathsf{PSL}(2,q)$ for $q < n < q^2$ when q is odd, and for $n < q^2$ and $n \neq q$ when q is even.

Before starting our investigation, we introduce some tools that will be used throughout the paper. Let P^G be an orbit on l, let X be any subgroup of G and let α be any element of G. Set $\mathrm{Fix}_{P^G}(X) = \mathrm{Fix}(X) \cap P^G$ and $\mathrm{Fix}_{P^G}(\alpha) = \mathrm{Fix}(\alpha) \cap P^G$. If r^G is an orbit of lines of Π, set $\mathrm{Fix}_{r^G}(X) = \mathrm{Fix}(X) \cap r^G$ and $\mathrm{Fix}_{r^G}(\alpha) = \mathrm{Fix}(\alpha) \cap r^G$.

Proposition 2.5 (Moorhouse). *Let G be a collineation group of a finite projective plane Π of order n, let $P \in l$ and let H be a subgroup of G. Then*

$$|\mathrm{Fix}_{P^G}(H)| = \frac{|N_G(H)|}{|G_P|} \cdot |\{U \leq G_P : U \text{ is conjugate to } H \text{ in } G\}| \,. \tag{1}$$

Proof. See [24, relation (9)]. □

Note that (1) still works if we replace $\mathrm{Fix}_{P^G}(H)$ with $\mathrm{Fix}_{r^G}(X)$ and G_P with G_r.

Theorem 2.6 (Ho). *Let G be a collineation group of a finite projective plane Π of order n. Suppose that either n is not a square or $n = m^2$ with $m \equiv 2$ or $3 \bmod 4$. If $4 \mid |G|$, then G contains an involutory perspectivity.*

Proof. See [14, Theorem A]. □

As we will see, the following lemmas play a central role in section 4.

Lemma 2.7. *Let q be an even power of an odd prime, let x be a positive integer, let u be a positive divisor of $\frac{\sqrt{q}\pm 1}{2}$ and let $h = 2$ or 4. Then the following hold:*

(I) *The quadruple $(x, h, u, \sqrt{q}) = (1, 4, 1, 3)$ is the unique solution of the Diophantine equation*

$$x\sqrt{q} = h\frac{\sqrt{q}-1}{2u} - 1\,. \tag{2}$$

(II) *The quadruple* $(x,h,u,\sqrt{q}) = (1,h,h/2,\sqrt{q})$ *for* $\sqrt{q} \equiv 3 \bmod h$ *is the unique solution of the Diophantine equation*

$$x\sqrt{q} = h\frac{\sqrt{q}+1}{2u} - 1\,. \tag{3}$$

Proof. Consider the Diophantine equation (2). Assume that $h=2$. Then (2) becomes $x\sqrt{q} = \frac{\sqrt{q}-1}{u} - 1$. Since $u \geq 1$, then $x\sqrt{q} \leq \sqrt{q}-2$. Nevertheless, this is impossible, since $x \geq 1$. So, no solutions arise for $h=2$.

Assume that $h=4$. Then (2) becomes

$$x\sqrt{q} = 2\frac{\sqrt{q}-1}{u} - 1\,. \tag{4}$$

If $u \geq 2$, then $x\sqrt{q} \leq \sqrt{q}-2$. Thus, we again obtain a contradiction, since $x \geq 1$. Therefore, $u=1$. By substituting this value in (4), we obtain $x\sqrt{q} = 2\sqrt{q}-3$. This one has a unique solution $(x,\sqrt{q}) = (1,3)$. Hence, $(x,h,u,\sqrt{q}) = (1,4,1,3)$ is the unique solution of the Diophantine equation (2) for $h=4$. From this and bearing in mind that (2) has no solutions for $h=2$, we obtain the assertion (I).

Now, consider the Diophantine equation (3). Assume that $h=2$. Then (3) becomes

$$x\sqrt{q} = \frac{\sqrt{q}+1}{u} - 1\,. \tag{5}$$

If $u>1$, then $\frac{\sqrt{q}+1}{u} - 1 < \sqrt{q} \leq x\sqrt{q}$. Thus, (5) has no solutions in this case. So, assume that $u=1$. By substituting this value in (5), we obtain $x\sqrt{q} = \sqrt{q}$ and hence $x=1$. Therefore, we have proved that $(x,h,u,\sqrt{q}) = (1,2,1,\sqrt{q})$ is the unique solution of (3) for $h=2$.

Now, assume that $h=4$. Then (3) becomes

$$x\sqrt{q} = 2\frac{\sqrt{q}+1}{u} - 1\,. \tag{6}$$

If $u>2$, then $2\frac{\sqrt{q}+1}{u} - 1 < \sqrt{q} \leq x\sqrt{q}$. Therefore, (6) has no solutions in this case. So, there are admissible solutions for (6) only for $u \leq 2$. If $u=1$, then (6) becomes $x\sqrt{q} = 2\sqrt{q}+1$. This one has no solutions, since the first part is divisible by $\sqrt{q}$, while the second is not. Thus, $u=2$. At this point, it is a straightforward computation to see that $(x,h,u,\sqrt{q}) = (1,4,2,\sqrt{q})$ for $\sqrt{q} \equiv 3 \bmod 4$ is a solution of (6) and hence of (3). From this and bearing in mind that $(x,h,u,\sqrt{q}) = (1,2,1,\sqrt{q})$ is the unique solution of (3) for $h=2$, we obtain the assertion (II). $\square$

Lemma 2.8. *Let q be an even power of an odd prime, let x be a positive integer and let u_1 and u_2 be two positive divisors of $\frac{\sqrt{q}-1}{2}$. Furthermore, let $h=2$ or 4.*

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If $u_1 \leq u_2$*, then* $(x, h, u_1, u_2, \sqrt{q}) = (1, 2, 1, \frac{\sqrt{q}-1}{2}, \sqrt{q})$*,* $(3, 4, 1, 1, 5)$*,* $(1, 4, 3, 3, 7)$*, or* $(1, 4, 3, 5, 31)$ *are the unique solutions of the Diophantine equation*

$$x\sqrt{q} = h\frac{\sqrt{q}-1}{2u_1} + h\frac{\sqrt{q}-1}{2u_2} - 1\,. \tag{7}$$

Proof. Multiplying by $2u_1u_2$ each term of (7), we have

$$2u_1u_2x\sqrt{q} = hu_2(\sqrt{q}-1) + hu_1(\sqrt{q}-1) - 2u_1u_2\,.$$

Now, collecting the terms with respect to $\sqrt{q}$, we obtain

$$[h(u_1+u_2) - 2u_1u_2x]\sqrt{q} = h(u_1+u_2) + 2u_1u_2\,. \tag{8}$$

Since $h(u_1+u_2) + 2u_1u_2 > 0$, then

$$2u_1u_2x < h(u_1+u_2)\,. \tag{9}$$

Assume that $h = 2$. Then $u_1u_2x < u_1 + u_2$ by (9). In particular, $u_1u_2 < u_1 + u_2$, as $x \geq 1$. This, in turn, yields $u_1u_2 < 2u_2$, since $u_1 \leq u_2$ by our assumption. Thus, $u_1 < 2$. That is $u_1 = 1$. Now, by substituting $h = 2$ and $u_1 = 1$ in (9), we obtain $2u_2x < 2(1+u_2)$ and hence $x < 1 + 1/u_2$. Then $x = 1$, as $u_2 \geq 1$. By substituting the values $x = 1$, $h = 2$ and $u_1 = 1$ in (8), and then by elementary calculations of this one, we have $2\sqrt{q} = 2 + 4u_2$. Hence, $u_2 = \frac{\sqrt{q}-1}{2}$. Consequently, $(x, h, u_1, u_2, \sqrt{q}) = (1, 2, 1, \frac{\sqrt{q}-1}{2}, \sqrt{q})$ is a solution of (7).

Assume that $h = 4$. Then

$$u_1u_2x < 2(u_1+u_2) \tag{10}$$

by (9).

If $x \geq 4$, then $2u_1u_2 < u_1 + u_2 \leq 2u_2$ by (10), since $u_1 \leq u_2$ by our assumption. This yields $u_1 < 1$, which is a contradiction.

If $x = 3$, then $3u_1u_2 < 2(u_1+u_2)$ by (10). Since $u_1 \leq u_2$, we have $3u_1u_2 < 4u_2$ and hence $u_1 = 1$. Now, by substituting $(x, h, u_1) = (3, 4, 1)$ in (9), we obtain $u_2 < 2$. Actually, $u_2 = 1$. Finally, by substituting $(x, h, u_1, u_2) = (3, 4, 1, 1)$ in (8), we have $\sqrt{q} = 5$. So, $(x, h, u_1, u_2, \sqrt{q}) = (3, 4, 1, 1, 5)$ is a solution of (7).

If $x = 2$, then $u_1u_2 < u_1 + u_2$ by (10). By arguing as above, we obtain $u_1 = 1$. By substituting $(x, h, u_1) = (2, 4, 1)$ in (8), we have $4\sqrt{q} = 4 + 6u_2$ and hence $u_2 = \frac{2(\sqrt{q}-1)}{3}$. Nevertheless, this contradicts the assumption $u_2 \mid \frac{\sqrt{q}-1}{2}$.

If $x = 1$, then $u_1u_2 < 4(u_1+u_2)$ by (9). Thus, $u_1 < 8$, since $u_1 \leq u_2$. On the other hand, by substituting $h = 4$ and $x = 1$ in (7), we have

$$\frac{\sqrt{q}+1}{2} = \frac{\sqrt{q}-1}{u_1} + \frac{\sqrt{q}-1}{u_2}\,. \tag{11}$$

If $u_1 = 1$, no solutions arise, since

$$\sqrt{q} - 1 < \frac{\sqrt{q}-1}{u_1} + \frac{\sqrt{q}-1}{u_2} = \frac{\sqrt{q}+1}{2}$$

and since $\sqrt{q}$ is odd. So, $u_1 \geq 2$. Assume that $u_1 \geq 4$. Then $u_2 \geq 4$ as $u_2 \geq u_1$. Hence,

$$\frac{\sqrt{q}+1}{2} = \frac{\sqrt{q}-1}{u_1} + \frac{\sqrt{q}-1}{u_2} \leq \frac{\sqrt{q}-1}{2}\,.$$

Therefore, there are also no solutions in this case. Consequently, $u_1 = 2$ or 3. Assume that $u_1 = 2$. Then (11) becomes $\frac{\sqrt{q}+1}{2} = \frac{\sqrt{q}-1}{2} + \frac{\sqrt{q}-1}{u_2}$. This yields $u_2 = \sqrt{q} - 1$. Nevertheless, this cannot occur, since $u_2 \mid \frac{\sqrt{q}-1}{2}$ by our assumptions. Hence, $u_1 = 3$. Then $u_2 = \frac{6(\sqrt{q}-1)}{\sqrt{q}+5}$ from (11). This yields $\sqrt{q} + 5 \mid 36$, since $(\sqrt{q}+5, \sqrt{q}-1) \mid 6$. As a consequence, $\sqrt{q} = 7, 13$ or 31. Then $u_2 = 3, 4$ or 5, respectively, since $u_2 = \frac{6(\sqrt{q}-1)}{\sqrt{q}+5}$. Nevertheless, only the cases $u_2 = 3$ or 5 are admissible, since $u_2 \mid \frac{\sqrt{q}-1}{2}$ by our assumption. Actually, $(x, h, u_1, u_2, \sqrt{q}) = (1, 4, 3, 3, 7)$ or $(1, 4, 3, 5, 31)$ are solutions of (7). This completes the proof. □

Lemma 2.9. *Let q be an even power of an odd prime, let x be a positive integer and let u_1 and u_2 be two positive divisors $\frac{\sqrt{q}+1}{2}$. Furthermore, let $h = 2$ or 4. If $u_1 \leq u_2$, then $(x, h, u_1, u_2, \sqrt{q}) = (5, 4, 1, 1, 3)$, $(3, 4, 1, 3, 5)$, $(1, h, h, h, \sqrt{q})$ and $\sqrt{q} \equiv -1 \bmod 2h$, or $(1, 4, 3, 6, \sqrt{q})$ and $\sqrt{q} \equiv -1 \bmod 12$ are the unique solutions of the Diophantine equation*

$$x\sqrt{q} = h\frac{\sqrt{q}+1}{2u_1} + h\frac{\sqrt{q}+1}{2u_2} - 1\,. \tag{12}$$

Proof. Multiplying by $2u_1u_2$ each term of (12), we have

$$2u_1u_2x\sqrt{q} = hu_2(\sqrt{q}+1) + hu_1(\sqrt{q}+1) - 2u_1u_2\,.$$

Now, collecting the terms with respect to $\sqrt{q}$, we obtain

$$[2u_1u_2x - h(u_1+u_2)]\sqrt{q} = h(u_1+u_2) - 2u_1u_2\,. \tag{13}$$

We treat the cases $2u_1u_2x - h(u_1+u_2) \neq 0$ and $2u_1u_2x - h(u_1+u_2) = 0$ separately. Assume the former occurs. Then $\sqrt{q} = \frac{h(u_1+u_2)-2u_1u_2}{2u_1u_2x-h(u_1+u_2)}$. That is

$$\sqrt{q} = -\frac{h(u_1+u_2) - 2u_1u_2}{h(u_1+u_2) - 2u_1u_2x}\,. \tag{14}$$

Note that $h(u_1+u_2) < 2u_1u_2$ implies $h(u_1+u_2) < 2u_1u_2x$, since $x \geq 1$. Then, by (14), we have that $h(u_1+u_2) > 2u_1u_2$, $h(u_1+u_2) < 2u_1u_2x$ and $x \geq 2$. In particular, since $x \geq 2$ and $\sqrt{q} \geq 3$, from the first part of (13), we have

$$[2u_1u_2x - h(u_1+u_2)]\sqrt{q} \geq 3\,[4u_1u_2 - h(u_1+u_2)]\,. \tag{15}$$

Combining (13) with (15), we obtain

$$3\left[4u_1u_2 - h(u_1+u_2)\right] \le h(u_1+u_2) - 2u_1u_2\,.$$

Elementary calculations of the previous inequality yield $14u_1u_2 \le 4h(u_1+u_2)$. That is

$$7u_1u_2 \le 2h(u_1+u_2)\,. \tag{16}$$

Therefore, $7u_1u_2 \le 4hu_2$ and hence

$$1 \le u_1 \le \frac{4}{7}h\,, \tag{17}$$

since $u_1 \le u_2$.

Assume that $h=2$. Then $u_1=1$ by (17). So, $h(u_1+u_2)-2u_1u_2=2$. Then $\sqrt{q} \mid 2$ by (13). Nevertheless, this is impossible, since $\sqrt{q}$ is a power of an odd prime. Hence, (12) has no solutions for $h=2$.

Assume that $h=4$. Then either $u_1=1$ or $u_1=2$ by (17). Assume the latter occurs, then $14u_2 \le 8(2+u_2)$ by (16). As a consequence, $u_2 \le \frac{16}{6}$. On the other hand, $u_2 \ge 2$, since $u_1=2$ and $u_1 \le u_2$. So, $2 \le u_1 \le u_2 \le \frac{16}{6}$. Therefore, $u_1=u_2=2$. Then $h(u_1+u_2)-2u_1u_2=8$, since $h=4$. Thus, $\sqrt{q} \mid 8$ by (13). Again, this is impossible, since $\sqrt{q}$ is a power of an odd prime. For this reason, we have $u_1=1$. Then (13) becomes

$$\left[(2x-4)u_2-4)\right]\sqrt{q} = 4+2u_2\,. \tag{18}$$

Note that $x \ge 3$, otherwise the first part of (18) is negative while the second one is positive, as $u_2 \ge u_1 = 1$.

Assume that $x \ge 4$. Then $12(u_2-1) \le 4+2u_2$ by (18), as $\sqrt{q} \ge 3$. This yields $u_2=1$ and $4+2u_2=6$. Then $\sqrt{q}=3$ and $x=5$ again by (18), since $\sqrt{q}$ is a power of an odd prime. So, $(x,h,u_1,u_2,\sqrt{q}) = (5,4,1,1,3)$ is the unique solution of (12) for $h=4$ and $x \ge 4$.

Assume that $x=3$. Then $(u_2-2)\sqrt{q} = u_2+2$ by (18). Now, collecting with respect to u_2, we have $(\sqrt{q}-1)u_2 = 2(\sqrt{q}+1)$. That is $u_2 = 2+4/(\sqrt{q}-1)$. This Diophantine equation has solutions $(u_2,\sqrt{q}) = (4,3)$ or $(3,5)$. Actually, the former is not admissible, since $u_2 \mid \frac{\sqrt{q}+1}{2}$ by our assumption. Hence, $(x,h,u_1,u_2,\sqrt{q}) = (3,4,1,3,5)$ is the unique solution of (12) for $h=4$ and $x=3$.

Assume that $2u_1u_2x - h(u_1+u_2) = 0$. Then

$$h(u_1+u_2) - 2u_1u_2 = 0 \tag{19}$$

by (13). As a consequence, $x=1$.

If $h = 2$, then $2(u_1 + u_2) - 2u_1u_2 = 0$ by (19). So, $u_1 + u_2 = u_1u_2$. Now, it is plain to see that $u_1 = u_2 = 2$. Then $\sqrt{q} \equiv -1 \bmod 4$, since u_1 and u_2 are positive divisors of $\frac{\sqrt{q}+1}{2}$ by our assumption. Thus, by substituting $(x, h, u_1, u_2) = (1, 2, 2, 2)$ in (13), we have that $(x, h, u_1, u_2, \sqrt{q}) = (1, 2, 2, 2, \sqrt{q})$ is a solution of (12).

If $h = 4$, then $4(u_1 + u_2) - 2u_1u_2 = 0$ by (19). So, $2(u_1 + u_2) = u_1u_2$. As $u_2 \mid 2u_1$ and $u_1 \le u_2$, then either $u_2 = u_1$ or $u_2 = 2u_1$. Assume that $u_2 = u_1$. Then $u_1 = u_2 = 4$ by $2(u_1 + u_2) = u_1u_2$. As a consequence, $\sqrt{q} \equiv -1 \bmod 8$, since u_1 and u_2 are positive divisors of $\frac{\sqrt{q}+1}{2}$ by our assumption. Thus, by substituting $(x, h, u_1, u_2) = (1, 4, 4, 4)$ in (13), we have that $(x, h, u_1, u_2, \sqrt{q}) = (1, 4, 4, 4, \sqrt{q})$ is a solution of (12). Finally, assume that $u_2 = 2u_1$. Then $u_1 = 3$ and $u_2 = 6$. Now, since $u_1 = 3$ and $u_2 = 6$ are two positive divisors of $\frac{\sqrt{q}+1}{2}$, we have that $\sqrt{q} \equiv -1 \bmod 12$. Moreover, by substituting $(x, h, u_1, u_2) = (1, 4, 3, 6)$ in (13), we see that $(x, h, u_1, u_2, \sqrt{q}) = (1, 4, 3, 6, \sqrt{q})$ is a solution of (12). This completes the proof. □

In Lemmas 2.8 and 2.9 the assumption $u_1 \le u_2$ can be dropped. Indeed, if $u_1 \ge u_2$ we obtain the 'same' solutions for (7) and (12) but with the values of u_1 and u_2 exchanged.

Lemma 2.10. *Let D be a dihedral group acting on a projective plane Π. Assume that D fixes a line l and there exists a dihedral subgroup D_0 of D which fixes two distinct points on l and contains a non central involutory homology. Then one of the following occurs:*

(1) *There exists a subgroup D_1 of D, such that $D_0 \le D_1$ and $[D : D_1] \le 2$, fixing at least one point on l;*

(2) *$D_0 \cong E_4$.*

Proof. Suppose that D fixes a line l of Π and that there exists a subgroup D_0 of D which fixes two distinct points X and Y on l and which contains an involutory homology. Set $|D| = 2m$ and $|D_0| = 2m_0$, where $m, m_0 > 1$. Also, set $D_0 = \langle \alpha, \beta \rangle$, where $\alpha^{m_0} = \beta^2 = 1$ and $\alpha^\beta = \alpha^{-1}$. We may assume that β is an involutory homology, since D_0 contains a non central one by our assumption. If $a_\beta = l$ then $\beta \in N$, where N is the kernel of the action of D on l. Clearly, it holds that $N \trianglelefteq D$. Thus, $N = D$ if m is odd, and $N \cong D_m$ or $N = D$ for m even, since D is dihedral and $\beta \in N$. If we set $D_1 = D_0N$, we obtain the assertion (1).

Assume that $a_\beta \neq l$. Thus, either $C_\beta = X$ and $a_\beta \cap l = \{Y\}$, or $C_\beta = Y$ and $a_\beta \cap l = \{X\}$, since D_0 fixes two distinct points X and Y on l and since $\beta \in D_0$.

We may assume that $C_\beta = X$ and $\{Y\} = a_\beta \cap l$. Assume also that m_0 is odd. Then each involution in D_0 is a homology of center X and axis intersecting l in Y, since D_0 fixes X and Y, with $X \neq Y$, and since D_0 contains a unique conjugate class of involutions as m_0 is odd.

If two distinct involutions in D_0 have distinct axes (passing through Y), then $D_0(X, X) \neq \langle 1 \rangle$ by [16, Theorem 4.25], since each involution in D_0 is homology of center X. In particular, $D_0(X, X) \leq S(D_0)$, where $S(D_0)$ denotes the unique maximal (normal) cyclic subgroup of D_0. Therefore, $D_0(X, X) \lhd D$, since $S(D_0) \trianglelefteq S(D) \lhd D$ and D is dihedral (actually, $D_0(X, X) = D_0(X, r)$ by [16, Theorem 4.14]). Thus, D fixes X and we again obtain the assertion (1).

If all involutions in D_0 have the same axis a_β, then $D_0 = D_0(X, a_\beta)$ as m_0 is odd. In particular, $S(D_0) = S(D_0)(X, a_\beta)$, with $S(D_0) \neq \langle 1 \rangle$, as m_0 is odd and $m_0 > 1$. Thus, D fixes X, a_β and hence Y, where $\{Y\} = a_\beta \cap l$, since $S(D_0) \lhd D$. Hence, we obtain the assertion (1) also in this case.

Assume that m_0 is even and $m_0 > 2$. Thus, $\alpha^{m_0/2}$ is a homology by [19, Proposition 3.3], since $Z(D_0) = \left\langle \alpha^{m_0/2} \right\rangle$ (actually, $Z(D) = \left\langle \alpha^{m_0/2} \right\rangle$) and since β is a homology. Set $\delta = \alpha^{m_0/2}$. If $C_\delta \in l$, then D fixes C_δ, since $\langle \delta \rangle = Z(D)$, D being a dihedral group. Thus, we still obtain the assertion (1).

Now, recall that $a_\beta \neq l$ by our assumption. Set $K = \langle \delta, \beta \rangle$. Then K is a Klein group consisting of commuting involutory homologies whose vertices lie in the triangle $\{X, Y, C_\delta\}$, as D_0 fixes X and Y on l and since $a_\beta \neq l$. Let $\rho \in D_0$ and consider K^ρ. Then $K^\rho = \langle \delta, \beta^\rho \rangle$ as δ is central in D_0. Furthermore, K^ρ is still a Klein group consisting of commuting involutory homologies whose vertices lying in the triangle $\{X, Y, C_\delta\}$, since D_0 fixes X, Y and C_δ. Then $K^\rho = K$ by [19, Lemma 3.1]. Hence, $K \trianglelefteq D_0$. Thus, $D_0 \cong D_8$ as $m_0 > 2$ by our assumptions. Therefore, $\delta = \alpha^2$ and $K = \left\langle \alpha^2, \beta \right\rangle$. Since β is an involutory (X, a_β)-homology with $a_\beta = C_\delta Y$, since D_0 fixes X an Y and C_δ and since $\alpha \in D_0$, then β^α is still an involutory (X, a_β)-homology. This is a contradiction, since $\beta^\alpha = \alpha^2\beta$ and the collineation $\alpha^2\beta$ is a $(Y, C_\delta X)$ homology lying in K, as K is a Klein group consisting of commuting involutory homologies whose centres are the vertices of the triangle $\{X, Y, C_\delta\}$. Thus, $m_0 = 2$. That is, $D_0 \cong E_4$. So, we have proved the assertion (2). □

3. General reductions

In this section, we provide some reductions for the action of G on Π. In particular, when G fixes a line l (resp. a point Q) of Π, we determine the admissible stabilizer of a point (resp. line) on l (resp. on $[Q]$), the length of the corresponding G-orbit on l (resp. on $[Q]$). Finally, we provide some upper bounds

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for the number of some G-orbits of points on l (resp. G-orbits of lines on $[Q]$).

Lemma 3.1. *If $G \cong \mathsf{PSL}(2,q)$, with q odd and $q > 3$, does not fix points or lines of Π, then G is irreducible on Π. Furthermore, one of the following occurs:*

(1) *G fixes a subplane $\Pi_0 \cong \mathsf{PG}(2,m)$, where $(m,q) = (2,7),(4,7),(4,9)$;*

(2) *G is strongly irreducible on Π.*

Proof. Assume that G does not fix lines or points of Π. Then G does not fix triangles of Π, since G is non abelian simple as $q > 3$. So, G is irreducible on Π. Now, assume that G fixes a subplane Π_0 of Π of order m. Then $m < q$ by [16, Theorem 3.7], since $n < q^2$ by our assumption. Thus, $\Pi_0 \cong \mathsf{PG}(2,m)$, where $(m,q) = (2,7),(4,5),(4,7),(4,9)$, by Theorem 2.1, as $q > 3$. Since G acts irreducibly on Π, then it does the same on Π_0. Hence, the case $(m,q) = (4,5)$ is ruled out, since $\mathsf{PSL}(2,5)$ fixes always a point or a line in $\mathsf{PG}(2,4)$. So, $(m,q) = (2,7),(4,7),(4,9)$ and hence we obtain the assertion. □

Lemma 3.2. *The following holds:*

(1) *If G fixes a line l, then $G(l) = \langle 1 \rangle$ and hence G acts faithfully on l.*

(2) *If G fixes a point P, then $G(P) = \langle 1 \rangle$ and hence G acts faithfully on $[P]$.*

Proof. Assume that G fixes l. If $G(l,l) \neq \langle 1 \rangle$, then $G = G(l,l)$, since G is simple as $q > 3$. Actually, $G = G(A,l)$ for some point $A \in l$ by [16, Theorem 4.14], since G is non abelian. So, $|G| \mid n$ and hence $|G| < q^2$, as $n < q^2$, which is a contradiction. Thus, $G(l,l) = \langle 1 \rangle$. Now, assume that G contains homologies of axis l. Each involution in G of axis l must have the same center, say C, otherwise $G(l,l) \neq \langle 1 \rangle$ by [16, Theorem 4.25], as G fixes l. Therefore, G fixes C and hence $\langle 1 \rangle < G(C,l) \lhd G$. Then $G = G(C,l)$, since G is simple as $q > 3$. So, $|G| \mid n-1$ and hence $|G| < q^2$ as $n < q^2$. Hence, we arrive at a contradiction. As a consequence, $G(l) = \langle 1 \rangle$ and hence G acts faithfully on l. So, we have proved the assertion (1). Now, dualizing the previous proof, we obtain also the assertion (2). □

Lemma 3.3. *If $q > 3$ and G fixes a line l of Π, then the involutions in G are Baer collineations of Π. In particular, $\sqrt{n} \equiv 0,1 \bmod 4$.*

Proof. Let σ be any involution of G. Assume that σ is a (C_σ, a_σ)-perspectivity of G. Then $C_\sigma \in l$ and $a_\sigma \neq l$ by Lemma 3.2(1), since G fixes l. Clearly, $C_G(\sigma)$ fixes C_σ, the lines l and a_σ and hence the point $l \cap a_\sigma$ (note that the points C_σ and $l \cap a_\sigma$ might coincide or not according to whether n is even or odd, respectively). Hence, $C_G(\sigma) \leq G_{C_\sigma}$. Furthermore, $G_{C_\sigma} < G$ by Lemma 3.2(2).

Then $G_{C_\sigma} = C_G(\sigma)$, since $C_G(\sigma)$ is maximal in G, being $C_G(\sigma) \cong D_{q\pm 1}$ by [17, Hauptsatz II.8.27], according to whether $q \equiv 3 \bmod 4$ or $q \equiv 1 \bmod 4$, respectively. Then σ fixes exactly either $(q+3)/2$ points or $(q+1)/2$ points on C_σ^G by (1) of Proposition 2.5, for either $q \equiv 3 \bmod 4$ or $q \equiv 1 \bmod 4$, respectively. Thus, σ fixes at least 3 points on l in each case as $q > 3$. This is a contradiction, since σ is (C_σ, a_σ)-perspectivity of G with $C_\sigma \in l$ and $a_\sigma \neq l$. Thus, σ is a Baer collineation of Π. Then each involution of G is a Baer collineation of Π, since $G \cong \mathsf{PSL}(2,q)$ contains a unique conjugate class of involutions. This yields $\sqrt{n} \equiv 0, 1 \bmod 4$ by Theorem 2.6. □

Lemma 3.4. *Let Π be a finite projective plane of order n and let $G \cong \mathsf{PSL}(2,q)$, $q > 3$, be a collineation group of Π fixing a line l of Π. If $P \in l$, then one the following occurs:*

(1) $G_P = G$;

(2) $G_P \cong D_{q-1}$;

(3) $G_P \cong D_{q+1}$;

(4) $G_P \cong A_4$ *and* $q = 5, 7, 9, 11, 13, 17, 19$;

(5) $G_P \cong A_5$ *and* $q = 5, 9, 11, 19, 25, 29, 31, 41, 49, 59, 61, 71, 79, 81, 89, 101, 109$;

(6) $G_P \cong S_4$ *and* $q = 7, 9, 17, 23, 25, 31, 41$;

(7) $G_P \cong \mathsf{PSL}(2, \sqrt{q})$;

(8) $G_P \cong \mathsf{PGL}(2, \sqrt{q})$;

(9) $G_P \cong E_{p^{m-e}}.Z_{p^e-1}$, *where* $2e \mid m$;

(10) $G_P \cong F_q.Z_d$.

Proof. Note that $n \leq (q-1)^2$, since $n < q^2$ and n is a square by Lemma 3.3. Since $P^G \subset l$ and $n+1 \leq (q-1)^2+1$, then $\left|P^G\right| \leq (q-1)^2+1$. That is $\frac{q(q^2-1)}{2|G_P|} \leq (q-1)^2+1$. Actually, $\frac{q(q^2-1)}{2|G_P|} < (q-1)^2+1$ and hence $\frac{q(q^2-1)}{2|G_P|} \leq (q-1)^2$. Then $2\,|G_P| \geq \frac{(q-1+1)(q+1)}{(q-1)}$ and consequently $2\,|G_P| > q+1$. So,

$$|G_P| > \frac{q+1}{2}\,. \tag{20}$$

Now, filtering the list of the proper subgroups of G given in [17, Hauptsatz II.8.27], with respect to (20) and bearing in mind [24, Lemma 2.8], when $G_P \leq F_{p^m}.Z_{\frac{p^m-1}{2}}$, we obtain the assertion. □

Let $P \in l$. We say that G_P is of type (i), where $1 \leq i \leq 10$, if G_P is a group isomorphic to the i-th group of the list given in the previous lemma. Also, we say that the orbit P^G is of type (i) if G_P is of type (i). So, for example, P^G and

G_P are of type (6) if $G_P \cong S_4$. Finally, we denote by x_i, the number of G-orbits on l of type (i).

The G-orbits on l are of type (i), with $1 \le i \le 8$, i fixed, have the same length. Hence, they cover exactly $x_i \left|P^G\right|$ points on l, where P^G is of type (i). The G-orbits on l of type (9) or (10) might have different lengths depending on e and d, respectively. Nevertheless, there exists at most on G-orbit on l of type (9), as we will see in the following lemma (that is $x_9 \le 1$). So, let us focus on the G-orbits of type (10) and on the points of l that they cover. Each G-orbit of type (10) has length $\frac{q^2-1}{2d}$ which depends on the particular divisor d of $\frac{q-1}{2}$. Therefore, all the G-orbits on l of type (10) cover exactly $\sum_{j=1}^{x_{10}} \frac{q^2-1}{2d_j}$ points of l. Set $\mathcal{S} = \sum_{j=1}^{x_{10}} \frac{q^2-1}{2d_j}$. We introduce the following abbreviations for the G-orbits on l of type (10): $\mathcal{S}_1 = \sum_{j=1}^{x_{10}} \frac{q-1}{d_j}$ and $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_4$, $\mathcal{S}_{2,4}$ (sum with the same summands $\frac{q-1}{d_j}$ but over $2 \mid d_j$, $2 \nmid d_j$, $4 \mid d_j$ and $d_j \equiv 2 \bmod 4$, respectively). In particular, we have the following relations $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$, $\mathcal{S}_1 = \mathcal{S}_2 + \mathcal{S}_{2'}$ and $\mathcal{S}_2 = \mathcal{S}_4 + \mathcal{S}_{2,4}$.

When investigating the admissible orbital decomposition of l under G, the following situation might arise (as we will see, in some cases it actually does): G fixes at least a point Q on l and the admissible orbital decomposition of G on set of lines of $[Q]$ it is easier to be investigated than the admissible one l, since the first one has some influences on the second one. In order to do so, we introduce further notation as follows.

If G fixes a point Q, clearly, G acts on $[Q]$. Now, consider Π^*, the dual of Π. The group G acts on Π^* fixing the line $[Q]$. Then we may apply Lemma 3.4 to Π^*. As a result, we obtain the same list of admissible groups with G_m, where m is a point of $[Q]$. Then we may extend the notation previously introduced to the groups G_m. Hence, we say that G_m is of type $(i)^*$, where $1 \le i \le 10$, if G_m is a group isomorphic to the i-th group of the list given in Lemma 3.4. Now, going back to Π, we obtain the same list of admissible groups with G_m in the role of G_P, where m is a line of $[Q]$ and Q is a point of Π fixed by G. So, we are actually applying the dual of Lemma 3.4 referred to G-orbits of lines through a point Q fixed by G. At this point, continuing with this notation, we say that the orbit m^G is of type $(i)^*$ if the respective G_m is of type $(i)^*$. So, for example, m^G and G_m are of type $(6)^*$ if $G_m \cong S_4$. Finally, we denote by x_i^*, the number of G-orbits on $[Q]$ of type $(i)^*$. In particular, since we might have G-orbits of type $(10)^*$, it makes sense considering $\mathcal{S}^* = \sum_{j=1}^{x_{10}^*} \frac{q^2-1}{2d_j}$ and hence $\mathcal{S}_2^*$, $\mathcal{S}_{2'}^*$, $\mathcal{S}_4^*$, $\mathcal{S}_{2,4}^*$ with the same meaning of $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_4$, $\mathcal{S}_{2,4}$, respectively, but referred to lines instead of points. As a consequence, we have $\mathcal{S}^* = \frac{q+1}{2}\mathcal{S}_1^*$, $\mathcal{S}_1^* = \mathcal{S}_2^* + \mathcal{S}_{2'}^*$ and $\mathcal{S}_2^* = \mathcal{S}_4^* + \mathcal{S}_{2,4}'^*$. It should be stressed that, the notation used depends on the particular point Q fixed by G (the same could be made for l). So, it would be correct using $x_i^*(Q)$ instead of x_i^*. Nevertheless, we shall use the second notation, since it will be clear from the context which point we are focusing on.

Lemma 3.5. *If $q > 9$, then the following hold:*

(1) $x_2 \leq 1$;

(2) $x_3 \leq 1$;

(3) $x_4 \leq 1$;

(4) $x_5 \leq 3$;

(5) $x_6 \leq 1$ *for* $q \neq 17$ *and* $x_6 \leq 2$ *for* $q = 17$;

(6) $x_9 \leq 1$.

Proof. Assume that l contains x_i orbits of G of type (i). Assume also that $2 \leq i \leq 6$ with i fixed. Clearly, these G-orbits have same length. So, they cover exactly $x_i \left|P^G\right|$ points on l, where P^G is any orbit of type (i). Therefore, $x_i \left|P^G\right| \leq n+1$ and hence $x_i \frac{q(q^2-1)}{2|G_P|} \leq n+1$. Now, arguing as in Lemma 3.4, we have $x_i \frac{q(q^2-1)}{2|G_P|} \leq (q-1)^2+1$ and consequently

$$|G_P| > x_i \frac{q+1}{2}.$$

Assume that $x_i \geq 2$. Then $|G_P| > q+1$. This is a contradiction by Lemma 3.4. Thus, we have proved the assertion for $i = 2$ or 3.

Assume that $i = 4$. Hence, $G_P \cong A_4$. Then $q < 11$, as $|G_P| > q+1$. Actually, we have $q \leq 9$, which is a contradiction by our assumption. So, $x_4 \leq 1$ and we obtain the assertion also in this case.

Assume that $i = 5$. Then $G_P \cong A_5$. If $x_5 \geq 4$, then $|G_P| > 2(q+1)$. So $2(q+1) < 60$. Hence, $q < 29$. Actually, $q = 11, 19$ or 25 by Lemma 3.4. Let σ be an involution lying in G_P. By [4], there exists one conjugate class of involutions in G. If $q = 11$ or 19, then $C_G(\sigma) \cong D_{q+1}$ again by [4], since $q \equiv 3 \bmod 4$. Therefore, using (1) of Proposition 2.5, we obtain that σ fixes exactly $\frac{q+1}{4}$ points on P^G. As a consequence, σ fixes at least $q+1$ points on l, since $x_5 \geq 4$. Hence $\sqrt{n} \geq q$, since σ is a Baer collineation of Π by Lemma 3.3. So, we arrived at a contradiction, since $n < q^2$ by our assumptions. Thus, $q = 25$. In this case, $C_G(\sigma) \cong D_{q-1}$. Arguing as above, we see that $\sqrt{n} \geq q-1$. Actually, $\sqrt{n} = q-1$, since $\sqrt{n} < q$. That is $\sqrt{n} = 24$. Let T be a Klein subgroup of G such that $\sigma \in T$ and $T \leq G_P$. Then $N_G(T) \cong S_4$ by [4]. Furthermore, all Klein subgroups in $G_P \cong A_5$ are conjugate, since they are Sylow 2-subgroups of it. Then, using (1) of Proposition 2.5, we obtain that T fixes exactly 2 points on P^G. Hence, $\mathrm{Fix}(T) \cap l \subset \mathrm{Fix}(\sigma) \cap l$. Since $x_5 \geq 4$, then T induces a Baer collineation on $\mathrm{Fix}(\sigma)$. This is a contradiction, since $\sqrt{n} = 24$. Thus, $x_5 \leq 3$ and we obtain the assertion also in this case.

Assume that $i = 6$. So, $G_P \cong S_4$. Then $q + 1 < 24$ as $|G_P| > q + 1$. Therefore $x_6 = 2$ and $q = 17$ by Lemma 3.4. Hence, we have proved the assertion in this case.

Assume that $i = 9$. Let P^G be a G-orbit on l of type (9). Then $\left|P^G\right| = \frac{p^e(q^2-1)}{2(p^e-1)}$, where q is a square $p^e \mid \sqrt{q}$ and $e \geq 1$ by Lemma 3.4. Clearly. $\left|P^G\right| > \frac{q^2-1}{2}$ and hence $x_9 \leq 1$, since $n + 1 \leq (q-1)^2 + 1$. □

Clearly, we may consider the dual of Lemma 3.5. In other words, we may apply the previous lemma to $[Q]$ if Q is a point fixed by G on Π. So, we have $x_i^* \leq 1$ for $2 \leq i \leq 5$ or $i = 9$, and $x_6^* \leq 1$ for $q \neq 17$ and $x_6^* \leq 2$ for $q = 17$. We shall do the same for any lemma or proposition in the sequel whenever it is needed.

Lemma 3.6. *Let $q > 9$. If $x_2 + x_3 > 0$, then the following hold:*

(1) $x_2 + x_3 = 1$*;*

(2) $x_4 = 0$*;*

(3) $x_5 \leq 2$ *and if* $x_5 > 0$*, then* $q = 11, 19, 25, 29, 31, 41, 49$*;*

(4) $x_6 = 0$ *for* $q \neq 17$ *and* $x_6 \leq 1$ *for* $q = 17$*;*

(5) $x_9 = 0$.

Proof. Assume $x_2 + x_3 > 0$. Let P^G be on orbit on l of type (2) or (3). If P^G is of type (2), then $\left|P^G\right| = \frac{q(q-1)}{2}$, and if P^G is of type (3), then $\left|P^G\right| = \frac{q(q+1)}{2}$. Hence, $\left|P^G\right| \geq \frac{q(q-1)}{2}$ in each case. Then $\left|l - P^G\right| \leq n+1-\frac{q(q-1)}{2}$. In particular, $\left|l - P^G\right| \leq (q-1)^2+1-\frac{q(q-1)}{2}$ as $n+1 \leq (q-1)^2+1$. So $\left|l - P^G\right| \leq \frac{1}{2}(q^2-3q+4)$. Assume there are x_i orbits of G of type (i) on $l - P^G$, where $2 \leq i \leq 6$ or $i = 9$, i fixed. Let Q^G be one of these orbits. It is a plain that, $x_i \left|Q^G\right| \leq \left|l - P^G\right|$ and hence $x_i \left|Q^G\right| \leq \frac{1}{2}(q^2 - 3q + 4)$. As a consequence, $|G_Q| \geq x_i \frac{q(q^2-1)}{q^2-3q+4}$. Easy computation, similar to that used in the first part of the proof of Lemma 3.5, yield the assertion, unless $i = 5$ and $q = 11$.

Assume that $i = 5$ and $q = 11$ and assume that $x_5 \geq 3$. Now, arguing in the second part of the proof of Lemma 3.5, we have that $\sqrt{n} + 1 \geq 3^{\frac{q-1}{4}}$ and $\sqrt[4]{n}$ is an integer. Then $\sqrt[4]{n} = 3$ and hence $\sqrt{n} = 9$. So, $3\left|Q^G\right| \leq 3^4 + 1 - \frac{11(11-1)}{2}$ and hence $\left|Q^G\right| \leq 9$. Hence, we arrive at a contradiction, since $\left|Q^G\right| = \frac{q(q^2-1)}{120}$ and $q = 11$. Thus, we have proved the assertion in any case. □

Now, we recall some known facts about $G \cong \mathrm{PSL}(2, q)$ which are useful hereafter. By [4], there exists a unique conjugate class of involutions in G and there are either one or two conjugate class of Klein subgroups of G according to whether $q \equiv 3, 5 \bmod 8$ or $q \equiv 1, 7 \bmod 8$, respectively. Let σ be a representative of the involutions in G. Let T_1 and T_2 the representatives of the two conjugate classes of Klein subgroups of G. We may choose T_1 an T_2 in order to contain σ (see [4] or [24]). Clearly, T_1 and T_2 are conjugate if $q \equiv 3, 5 \bmod 8$. So, if $q \equiv 3, 5 \bmod 8$, we shall just denote by T the representative of the unique conjugate classes of Klein subgroups of G. Hence, by [4], the following admissible cases arise:

(1) $q \equiv 1 \bmod 8$. Then $C_G(\sigma) \cong D_{q-1}$ and $N_G(T_j) \cong S_4$, where $j = 1$ or 2;

(2) $q \equiv 3 \bmod 8$. Then $C_G(\sigma) \cong D_{q+1}$ and $N_G(T) \cong A_4$;

(3) $q \equiv 5 \bmod 8$. Then $C_G(\sigma) \cong D_{q-1}$ and $N_G(T) \cong A_4$;

(4) $q \equiv 7 \bmod 8$. Then $C_G(\sigma) \cong D_{q+1}$ and $N_G(T_j) \cong S_4$, where $j = 1$ or 2.

We investigate these cases separately.

4. The case $q \equiv 1 \bmod 8$

This section is devoted to the cases $q \equiv 1 \bmod 8$. By [4], there are two conjugate classes of subgroups isomorphic to A_4 (type (4)), to A_5 (type (5)), to S_4 (type (6)), to $\mathrm{PSL}(2, \sqrt{q})$ (type (7)), to $\mathrm{PGL}(2, \sqrt{q})$ (type (8)). Since there are two conjugate classes of subgroups of type (4) regarded as stabilizer of a point P on l, we may extend our preceding notation as follows: we label the subgroups G_P isomorphic to A_4 and belonging to the first conjugate class under G to be of type (4a), while those belonging to the second one to be of type (4b). Moreover, P^G is a G-orbit of type either (4a) or (4b) if the corresponding G_P is of type (4a) or (4b), respectively. We denote by x_{4a} and x_{4b} the number of G-orbits on l of type (4a) or (4b), respectively. Clearly, $x_4 = x_{4a} + x_{4b}$. Extending the previous notation, when P^G and G_P are of type (i), for $4 \le i \le 8$, we actually say that they are of type $(i$a) or $(i$b) depending on the particular conjugate class under G the group G_P lies. Hence, we write $x_i = x_{ia} + x_{ib}$ for $4 \le i \le 8$.

The usual argument involving Proposition 2.5 yields the following table containing all the informations we need about the admissible stabilizers in G of any point P of l. It should also be stressed that the G-orbits of type (7), (8) or (9) might occur only when q is a square.

For $\pm$ and $\mp$ read the *upper* sign if $\sqrt{q} \equiv 1 \bmod 4$ and the *lower* sign if $\sqrt{q} \equiv 3 \bmod 4$ (for q square). This convention is followed throughout this section.

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Table I

type	G_P	$[G:G_P]$	$\lvert\mathrm{Fix}_{PG}(\sigma)\rvert$	$\lvert\mathrm{Fix}_{PG}(T_1)\rvert$	$\lvert\mathrm{Fix}_{PG}(T_2)\rvert$
1	G	1	1	1	1
2	D_{q-1}	$\frac{q(q+1)}{2}$	$\frac{q+1}{2}$	3	3
3	D_{q+1}	$\frac{q(q-1)}{2}$	$\frac{q-1}{2}$	0	0
4a	A_4	$\frac{q(q^2-1)}{24}$	$\frac{q-1}{4}$	2	0
4b	A_4	$\frac{q(q^2-1)}{24}$	$\frac{q-1}{4}$	0	2
5a	A_5	$\frac{q(q^2-1)}{120}$	$\frac{q-1}{4}$	2	0
5b	A_5	$\frac{q(q^2-1)}{120}$	$\frac{q-1}{4}$	0	2
6a	S_4	$\frac{q(q^2-1)}{48}$	$\frac{3(q-1)}{8}$	$4\,,\ q\equiv_{16}1$ $1\,,\ q\equiv_{16}9$	$4\,,\ q\equiv_{16}1$ $3\,,\ q\equiv_{16}9$
6b	S_4	$\frac{q(q^2-1)}{48}$	$\frac{3(q-1)}{8}$	$4\,,\ q\equiv_{16}1$ $3\,,\ q\equiv_{16}9$	$4\,,\ q\equiv_{16}1$ $1\,,\ q\equiv_{16}9$
7a	$\mathsf{PSL}(2,\sqrt{q})$	$\sqrt{q}(q+1)$	$\sqrt{q}\pm 1$	1 ± 1	1 ∓ 1
7b	$\mathsf{PSL}(2,\sqrt{q})$	$\sqrt{q}(q+1)$	$\sqrt{q}\pm 1$	1 ∓ 1	1 ± 1
8a	$\mathsf{PGL}(2,\sqrt{q})$	$\frac{\sqrt{q}(q+1)}{2}$	$\sqrt{q}$	2 ± 1	2 ∓ 1
8b	$\mathsf{PGL}(2,\sqrt{q})$	$\frac{\sqrt{q}(q+1)}{2}$	$\sqrt{q}$	2 ∓ 1	2 ± 1
9	$E_{p^{m-e}}.Z_{p^e-1}$	$\frac{p^e(q^2-1)}{2(p^e-1)}$	$\frac{q-1}{p^e-1}$	0	0
10	$F_q.Z_d$	$\frac{q^2-1}{2d}$	$\frac{q-1}{d}\,,\quad 2\mid d$ $0\,,\quad 2\nmid d$	0	0

Recall that the Sylow p-subgroups of G are elementary abelian. Furthermore, by [4], there are two conjugate classes of p-elements. Let ρ_1 and ρ_2 be the representatives of these two classes lying in a Sylow p-subgroup S of G which is normalized by σ. Since σ acts as the inversion on S, then σ normalizes $\langle\rho_1\rangle$ and $\langle\rho_2\rangle$ and hence $\langle\rho_1,\sigma\rangle \cong \langle\rho_2,\sigma\rangle \cong D_{2p}$. Again by [4], there is a unique conjugate class of elements of order for 4 in G. Let γ be a representative of this class such that $\gamma^2=\sigma$. By using (1) of Proposition 2.5, we obtain the following table.

The sign $\pm$ has the same meaning as above. In particular, the non negative integers k_1 and k_2 are such that $k_1+k_2=\frac{q-p^e}{p^e-1}$, where $2e \mid m$ (see [24], Table IV* and related remarks).

It should be pointed out that Tables I and II, with types and entries in differ-

Table II

(We use the abbreviation F for Fix_{PG} in the top line of this table.)

Type	$\lvert\mathrm{F}(\rho_1)\rvert$	$\lvert\mathrm{F}(\rho_2)\rvert$	$\lvert\mathrm{F}(\rho_1,\sigma)\rvert$	$\lvert\mathrm{F}(\rho_2,\sigma)\rvert$	$\lvert\mathrm{F}(\gamma)\rvert$
1	1	1	1	1	1
2	0	0	0	0	1
3	0	0	0	0	0
4a	0	0	0	0	0
4b	0	0	0	0	0
5a	0	0	0	0	0
5b	0	0	0	0	0
6a	0	0	0	0	$\frac{q-1}{8}$
6b	0	0	0	0	$\frac{q-1}{8}$
7a	$2\sqrt{q}$	0	1 ± 1	0	$\sqrt{q} \pm 1\,,\quad q \equiv_{16} 1$ $0\,,\quad q \equiv_{16} 9$
7b	0	$2\sqrt{q}$	0	1 ± 1	$\sqrt{q} \pm 1\,,\quad q \equiv_{16} 1$ $0\,,\quad q \equiv_{16} 9$
8a	$\sqrt{q}$	0	1	0	$\frac{\sqrt{q}\pm 1}{2}$
8b	0	$\sqrt{q}$	0	1	$\frac{\sqrt{q}\pm 1}{2}$
9	$k_1 p^e$	$k_2 p^e$	k_1	k_2	$\frac{q-1}{p^e-1},\quad p^e \equiv_4 1$ $0\,,\quad p^e \equiv_4 3$
10	$\frac{q-1}{2d}$	$\frac{q-1}{2d}$	$\frac{q-1}{2d},\quad 2 \mid d$ $0\,,\quad 2 \nmid d$	$\frac{q-1}{2d},\quad 2 \mid d$ $0\,,\quad 2 \nmid d$	$\frac{q-1}{d},\quad 4 \mid d$ $0\,,\quad 4 \nmid d$

ent order, can be extracted from Tables III* and IV* of [24], respectively.

Now, if G acts on $[Q]$, where Q is any point of Π, then $[Q]$ consists of G-orbits of lines of type $(i)^*$ for $1 \le i \le 10$, following the notation introduced in section 3. As G contains two conjugate classes of subgroups isomorphic to A_4 (type (4)*), to A_5 (type (5)*), to S_4 (type (6)*), to $\mathsf{PSL}(2,\sqrt{q})$ (type (7)*), to $\mathsf{PGL}(2,\sqrt{q})$ (type (8)*), the distinction made for G-orbits of points of Π inside a fixed type (i) in subtypes $(i\text{a})$ and $(i\text{a})$ can be extended in G-orbits of lines of Π in the following sense. Let m be any line of $[Q]$ and assume that a subgroup G_m of G is isomorphic to A_4. We say that G_m is either of type (4a)* or of type (4b)* depending on which of the two conjugate classes of subgroups isomorphic to A_4 the group G_m lies. So, we denote by x^*_{4a} and x^*_{4b} the number of G-orbits on $[Q]$ of type (4a)* and (4b)*, respectively. Clearly $x^*_4 = x^*_{4a} + x^*_{4b}$. Extending the previous notation, when m^G and G_m are of type $(i)^*$, for $4 \leq i \leq 8$, we actually say that they are of type $(i\text{a})^*$ or $(i\text{b})^*$ depending on the particular conjugate class under G the group G_m lies. Hence, we write $x^*_i = x^*_{ia} + x^*_{ib}$ for $4 \leq i \leq 8$.

It is a plain that, at this point, we may use Tables I and II referred to G-orbits of lines of Π. So in this case, the first column containing types (i) is replaced by types $(i)^*$ and G_P is replaced by G_m. So, when we use Tables I and II referred to G-orbits of lines of Π through some point fixed by G, we actually use *the duals of Tables I and II*, respectively.

The strategy of the proof in this section is the following. Assuming that G fixes a line l of Π, we show that each T_j induces either a Baer collineation or a perspectivity of axis distinct from l on $\mathrm{Fix}(\sigma)$ (Lemma 4.2). We use this fact to show that γ, where $\gamma^2 = \sigma$, induces either the identity or a Baer collineation on $\mathrm{Fix}(\sigma)$ (Lemma 4.4). Then, using Tables I and II, we show that, if the first case occurs, the group T_j induces a homology on $\mathrm{Fix}(\sigma)$ (Lemma 4.10). Nevertheless, this is impossible (Lemma 4.11). Thus γ induces a Baer collineation on $\mathrm{Fix}(\sigma)$. Again, Table I and II imply that each T_j induces a Baer collineation on $\mathrm{Fix}(\sigma)$ and on $\mathrm{Fix}(\gamma)$ by Propositions 4.12 and 4.18, respectively. Thus, G fixes necessarily a subplane of Π of order $\sqrt[8]{n}$ pointwise (Lemma 4.19), which is a contradiction (Proposition 4.21).

Recall that σ is a Baer collineation of Π by Lemma 3.3. Set $C = C_G(\sigma)$. Then C acts on $\mathrm{Fix}(\sigma)$ with kernel K. Hence, let $\bar{C} = C/K$. Clearly, $\langle\sigma\rangle \trianglelefteq K \trianglelefteq C$. Furthermore, either $K \trianglelefteq Z_{\frac{q-1}{2}}$ or $K \cong D_{\frac{q-1}{2}}$ or $K = C$, since $C \cong D_{q-1}$ and $q \equiv 1 \bmod 8$. Now, we need to investigate the admissible structure of K in order to show that T_j cannot induce on $\mathrm{Fix}(\sigma)$ either the identity or a perspectivity of axis $\mathrm{Fix}(\sigma) \cap l$ for each $j = 1, 2$.

Lemma 4.1. *If* $\mathrm{Fix}(T_j) \cap l = \mathrm{Fix}(\sigma) \cap l$ *for some* $j = 1$ *or* 2*, then either* $K \cong D_{\frac{q-1}{2}}$ *or* $K = C$.

Proof. Assume that $\mathrm{Fix}(T_1) \cap l = \mathrm{Fix}(\sigma) \cap l$ and that $K \trianglelefteq Z_{\frac{q-1}{2}}$. Then $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ by Table I, since $q > 9$. Set $l_0 = \mathrm{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \mathrm{Fix}(G) \cap l$.

Assume that $\bar{C} = \bar{C}(l_0, l_0)$. Then T_1 induces a perspectivity $\bar{\beta}_1$ of center $C_{\bar{\beta}_1}$ and axis l_0 on $\mathrm{Fix}(\sigma)$. Suppose that $\bar{\beta}_1$ is an elation. Hence, $C_{\bar{\beta}_1} \in l$. Thus, G fixes $C_{\bar{\beta}_1}$. So, $\mathrm{Fix}(G) \cap [C_{\bar{\beta}_1}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$, by dual of Table I, since $\mathrm{Fix}(T_1) \cap [C_{\bar{\beta}_1}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$. Therefore, $\bar{C} = \bar{C}(C_{\bar{\beta}_1}, l_0)$. Let $X \in l_0 - \{C_{\bar{\beta}_1}\}$.

For each line $t \in [X] \cap \mathrm{Fix}(\sigma)$, we have that $\sigma \in G_t$ but G_t does not contain Klein groups. Clearly, $\mathrm{Fix}(G_t) \subseteq \mathrm{Fix}(\sigma)$. Actually, $\mathrm{Fix}(G_t) = \mathrm{Fix}(\sigma)$, since $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$, since $\mathrm{Fix}(G) \cap [C_{\bar{\beta}_1}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$, and since $t \in \mathrm{Fix}(G_t) \cap \mathrm{Fix}(\sigma)$ and $t \notin [C_{\bar{\beta}_1}]$. So, $\mathrm{Fix}(G_t)$ is a Baer subplane of Π. Assume that $p \mid |G_t|$ and let S_0 be a Sylow p-subgroup of G_t. Then $\mathrm{Fix}(S_0) = \mathrm{Fix}(G_t)$, since $\mathrm{Fix}(G_t)$ is a Baer subplane of Π. Furthermore, either q is a square and $|S_0| = p^e$ with $p^e \mid \sqrt{q}$, or $|S_0| = q$ by dual of Table II. Assume the latter occurs. Then $q \mid n - \sqrt{n}$, since S_0 must be semiregular on $l - \mathrm{Fix}(S_0)$, as $\mathrm{Fix}(S_0) = \mathrm{Fix}(\sigma)$ and $\mathrm{Fix}(\sigma)$ is a Baer subplane of Π. This yields that either $q \mid \sqrt{n} - 1$ or $q \mid \sqrt{n}$, since q is a prime power. This gives a contradiction, since $\sqrt{n} < q$ by our assumption. Thus, q is a square and $|S_0| = p^e$ with $p^e \mid \sqrt{q}$. In particular, G_t is of type (9)*. Moreover, $\mathrm{Fix}(S_0) \cap [X] = \mathrm{Fix}(\sigma) \cap [X]$, since $\mathrm{Fix}(S_0) = \mathrm{Fix}(\sigma)$. This yields $\mathrm{Fix}_{t^G}(S_0) = \mathrm{Fix}_{t^G}(\sigma)$ and hence $|\mathrm{Fix}_{t^G}(S_0)| = |\mathrm{Fix}_{t^G}(\sigma)|$, since G fixes X. Then $k_1 p^e = \frac{q-1}{p^e-1}$ by duals of Tables I and II, which is a contradiction. As a consequence, $(p, |G_t|) = 1$. Therefore, $G_t \cong D_{q+1}$ by dual of Table I, since $\sigma \in G_t$ but G_t does not contain Klein groups. Then $q + 1 \mid n - \sqrt{n}$, since G_t must be semiregular on $l - \mathrm{Fix}(G_t)$, as $\mathrm{Fix}(G_t) = \mathrm{Fix}(\sigma)$ and $\mathrm{Fix}(\sigma)$ is a Baer subplane of Π. Furthermore, $K \leq G_t$. Thus $K = \langle\sigma\rangle$, since $\langle\sigma\rangle \trianglelefteq K \leq Z_{\frac{q-1}{2}}$ and since $G_t \cong D_{q+1}$. So, $\bar{C} \cong D_{\frac{q-1}{2}}$ and hence $\frac{q-1}{2} \mid \sqrt{n}$, since $\bar{C} = \bar{C}(C_{\bar{\beta}_1}, l_0)$. Actually, either $\sqrt{n} = \frac{q-1}{2}$ or $\sqrt{n} = q - 1$, since $\sqrt{n} < q$ by our assumptions. On the other hand, $t^G \subset [X] - \{l\}$, as G fixes X. Then $n \geq \frac{q(q-1)}{2}$, since $|t^G| = \frac{q(q-1)}{2}$ as $G_t \cong D_{q+1}$. Since $n \geq \frac{q(q-1)}{2}$ and since $\frac{q(q-1)}{2} > \left(\frac{q-1}{2}\right)^2$, the case $\sqrt{n} = \frac{q-1}{2}$ cannot occur. Hence, $\sqrt{n} = q - 1$. Then $q + 1 \mid (q-1)(q-2)$, since $q+1 \mid n - \sqrt{n}$, being G_t semiregular on $l - \mathrm{Fix}(G_t)$. Since $(q+1, q-1) = 2$ and $(q+1, q-2) \mid 3$, then $q + 1 \mid 6$. This gives a contradiction, since $q > 9$ by our assumptions. Thus, $\bar{C}(l_0, l_0) < \bar{C}$.

Assume that $\bar{C}(l_0, l_0) \neq \langle 1\rangle$. Note that $\bar{C}(Y, l_0) \neq \langle 1\rangle$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$, since $\bar{C}(l_0, l_0) < \bar{C}$ and $\bar{C} = \bar{C}(l_0)$. In particular, $\bar{C}(l_0, l_0) \leq Z_{\frac{q-1}{k}}$, since $\bar{C}(l_0, l_0) < \bar{C}$ and $\bar{C} \cong D_{\frac{q-1}{k}}$. Actually, $\bar{C}(l_0, l_0) = \bar{C}(V, l_0) \cong Z_{\frac{q-1}{k}}$ and $\bar{C}(Y, l_0) \cong Z_2$ by [16, Theorems 4.14 and 4.25], since $\bar{C}(l_0, l_0) \cong D_{\frac{q-1}{k}}$, $q \equiv 1 \bmod 8$ and k is even. Let $u \in [V] \cap \mathrm{Fix}(\sigma) - \{l, VY\}$, then u is fixed by K and by $\bar{C}(V, l_0)$. Therefore, $Z_{\frac{q-1}{2}} \leq G_u$, where $Z_{\frac{q-1}{2}} \lhd C_G(\sigma)$. Since G fixes $l \cap \mathrm{Fix}(\sigma)$, since $q > 9$ and by dual of Lemma 3.4, we have that either $G_u \cong F_q.Z_{\frac{q-1}{2}}$ or $G_u = C_G(\sigma)$ or $G_u = G$. The two latter cases cannot occur, since $C_u \cong Z_{\frac{q-1}{2}}$. So, $G_u \cong F_q.Z_{\frac{q-1}{2}}$ for each $u \in [V] \cap \mathrm{Fix}(\sigma) - \{l, VY\}$. Note also that $x_1^* = 1$, since G fixes only the line l through V, and $x_2^* \geq 1$ since $G_{VY} = C$. Actually, $x_2^* = 1$ by dual of Lemma 3.5(2). Moreover, $|\mathrm{Fix}_{u^G}(\sigma)| = 2$

by dual of Table II. Then

$$\sqrt{n}+1 = 1+\frac{q+1}{2}+\mathcal{S}_2 \tag{21}$$

by dual of Table I, since $x_1^* = x_2^* = 1$ and since $G_u \cong F_q.Z_{\frac{q-1}{2}}$ for each $u \in [V] \cap \mathrm{Fix}(\sigma) - \{l, VY\}$. Let W be the Sylow p-subgroup of G normalized by σ. Then, by (21), W fixes exactly $1+\frac{1}{2}\mathcal{S}_2$ lines through V, namely l and the lines lying in the G-orbits corresponding to stabilizer isomorphic to $F_q.Z_{\frac{q-1}{2}}$. Furthermore, if $R \in l - \mathrm{Fix}(G)$, then G_R must have odd order, since $|\mathrm{Fix}(G) \cap l| = \sqrt{n}+1$ and since the involutions in G are Baer involutions of Π by Lemma 3.3. Then G_R must be of type (10) by Lemma 3.4. Henceforth, $W \leq G_L$ for some point $L \in R^G$. Consequently, W fixes at least $\sqrt{n}+2$ points on l and at least $1+\frac{1}{2}\mathcal{S}_2$ lines through V. Thus, the p-elements in G cannot be planar. So, if $Z \in \mathrm{Fix}(G) \cap l$, $Z \neq V$, for each line $r \in [Z] \cap \mathrm{Fix}(\sigma) - \{l, ZV\}$, the group G_r contains σ but does not contain Klein groups and $(p, |G_t|) = 1$. This implies that $G_r \cong D_{q+1}$ by dual of Table I. Now, as $K \leq G_r$ and $\langle\sigma\rangle \trianglelefteq K \leq Z_{\frac{q-1}{2}}$, then $K = \langle\sigma\rangle$. As a consequence, $\bar{C}(V, l_0) \cong Z_{\frac{q-1}{2}}$, being $k = |K|$. In particular, $\bar{C}(V, l_0)$ has even order. Nevertheless, this is a contradiction, since $\bar{C}(Y, l_0) \cong Z_2$ and $Y \in \mathrm{Fix}(\sigma) - l_0$. So, $\bar{C}(l_0, l_0) = \langle 1\rangle$.

Assume that $\bar{C} = \bar{C}(Z, l_0)$ for some $Z \in \mathrm{Fix}(\sigma) - l_0$. Let $Q \in l_0$ and $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, YQ\}$. Then $\sigma \in G_m$ but G_m does not contain Klein groups. Therefore, by dual of Table I, we have that $G_m \cong D_{q+1}$ or $G_m \cong E_{p^{m-e}}.Z_{p^e-1}$ or $G_m \cong F_q.Z_d$, since G fixes Q. Thus, $x_i^* > 0$ for either $i = 3$ or 9 or 10, since G acts on $[Q]$. The cases $i = 3$ or 9 cannot occur by dual of Lemma 3.6(1) and (5), since $x_2^* > 0$, as $G_{ZQ} = C \cong D_{q-1}$. As a consequence, $G_m \cong F_q.Z_d$. Let S be Sylow p-subgroup of G which is normalized by σ. Then $|\mathrm{Fix}_{m^G}(S)| \geq 1$ for each $Q \in l_0$ and $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, ZQ\}$. Assume that $\left|\mathrm{Fix}_{m_1^G}(S)\right| \geq 2$ for some line $m_1 \in [Q_1] \cap \mathrm{Fix}(\sigma) - \{l, ZQ_1\}$ and for some point $Q_1 \in l_0$. Then S is planar, since $|\mathrm{Fix}_{m^G}(S)| \geq 1$ for each other $Q \in l_0$ and each $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, ZQ\}$ and since $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$. Then S fixes a Baer subplane of Π, since $\mathrm{Fix}(S) \cap l = \mathrm{Fix}(\sigma) \cap l$ and $|\mathrm{Fix}(\sigma) \cap l| = \sqrt{n}+1$. Now, arguing as above with S in the role of S_0, we obtain a contradiction. Thus, $|\mathrm{Fix}_{m^G}(S)| = 1$ for each $Q \in l_0$ and $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, ZQ\}$. Nevertheless, we still have a contradiction if $|\mathrm{Fix}(S) \cap [Q_2]| \geq 2$ for some point $Q_2 \in l_0$. So, $|\mathrm{Fix}(S) \cap [Q]| = 1$ for each point $Q \in l_0$. Consequently, $G_{f_Q} \cong F_q.Z_{\frac{q-1}{2}}$, where $\{f_Q\} = \mathrm{Fix}(S) \cap [Q]$ by the dual of Table II. Therefore, σ fixes exactly two lines in f_Q^G for each $Q \in l_0$ by Table I. Actually, σ fixes exactly two lines in $[Q] - \{l, ZQ\}$ for each $Q \in l_0$, since $|\mathrm{Fix}_{m^G}(S)| \geq 1$ for $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, ZQ\}$, while $|\mathrm{Fix}(S) \cap [Q]| = 1$. So, $\sqrt{n}-1 = 2$. That is $\sqrt{n} = 9$. On the other hand, by dual of Table II, we have $\sqrt{n}+1 \geq 1+\frac{q+1}{2}+2$, since $x_1^* = 1$ as G fixes only the l through Z, since $x_2^* \geq 1$

as $G_{ZQ} = C$ and since $x^*_{10} \geq 1$ as $G_{f_Q} \cong F_q.Z_{\frac{q-1}{2}}$. Then $\sqrt{n} \geq 11$ since $q \geq 17$ being $q \equiv 1 \bmod 8$ and $q > 9$. Hence, we arrive at a contradiction, since it was proved above that $\sqrt{n} = 9$. □

Lemma 4.2. *It holds that* $\mathrm{Fix}(T_j) \cap l \subset \mathrm{Fix}(\sigma) \cap l$ *for each* $j = 1, 2$.

Proof. Assume that $K = C$. Let P be any point of l_0 and let r be any line of $[P] - \{l\}$. Then $C \leq G_r$. Since $q > 9$, then C is maximal in G and hence either $G_r = C$ or $G_r = G$. Assume that $G_r = C$. Again by the maximality of C in G, the line r is the unique one in r^G fixed by C. Furthermore, $x^*_2 = 1$ dual of Lemma 3.5. Therefore, r is the unique line in $[P]$ fixed by C. So, the remaining lines are fixed by G. Now, by repeating the previous argument for each point U of $\mathrm{Fix}(\sigma) \cap l$, we see that C fixes exactly one line of $[U] \cap \mathrm{Fix}(\sigma)$ and the remaining ones are fixed by G. If $\sqrt{n} > 2$, then G is planar. Thus, $\mathrm{Fix}(G) = \mathrm{Fix}(\sigma)$, since $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ and $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(\sigma)$. Then G fixes r, since $r \in \mathrm{Fix}(\sigma)$. This is a contradiction, since $G_r = C$ by our assumptions. So, $\sqrt{n} = 2$ and $n = 4$, which is a contradiction, since $q < n < q^2$ and $q > 9$. As a consequence, $G_r = G$. Now, by repeating the previous argument for each point of $\mathrm{Fix}(\sigma) \cap l$, we again obtain $\mathrm{Fix}(G) = \mathrm{Fix}(\sigma)$. Thus, G fixes a Baer subplane of Π. Then G is semiregular on $l - \mathrm{Fix}(G)$ and hence $|G| \mid n - \sqrt{n}$. Hence, we arrive at a contradiction, since $n < q^2$.

Finally, assume that $K \cong D_{\frac{q-1}{2}}$. We may also assume that $T_1 \leq K$ and $C \cong D_{q-1}$. Then $\mathrm{Fix}(T_1) \cap [B] = \mathrm{Fix}(\sigma) \cap [B]$ for each point $B \in l_0$. As $l_0 = \mathrm{Fix}(G) \cap l$, then $\mathrm{Fix}(G) \cap [B] = \mathrm{Fix}(\sigma) \cap [B]$ for each point $B \in l_0$ by dual of Table I. So, $\mathrm{Fix}(G) = \mathrm{Fix}(\sigma)$ and we have a contradiction as above. Thus, $\mathrm{Fix}(T_1) \cap l \subset \mathrm{Fix}(\sigma) \cap l$.

Now repeating the above arguments with T_2 in the role of T_1, we obtain $\mathrm{Fix}(T_2) \cap l \subset \mathrm{Fix}(\sigma) \cap l$. □

Lemma 4.3. *If* $|\mathrm{Fix}(\gamma) \cap l| \leq 2$*, then the following hold:*

(1) $|\mathrm{Fix}(\gamma) \cap l| = x_1 + x_2 = 1$ *or* 2*;*

(2) $x_6 = 0$*;*

(3) $x_7 > 0$*, if* q *is a square and* $q \equiv 9 \bmod 16$*;*

(4) $x_8 = 0$*;*

(5) $x_9 > 0$*, if* q *is a square and* $p^e \equiv 3 \bmod 4$*, where* $p^e \mid \sqrt{q}$*;*

(6) $\mathcal{S}_4 = 0$*;*

(7) T_j *induces a Baer involution on* $\mathrm{Fix}(\sigma)$ *for each* $j = 1, 2$.

Proof. Assume that $|\mathrm{Fix}(\gamma) \cap l| \leq 2$. Then γ induces an involutory perspectivity $\bar{\gamma}$ on $\mathrm{Fix}(\sigma)$ and hence $|\mathrm{Fix}(\gamma) \cap l| = 1$ or 2. Clearly, $C_{\bar{\gamma}} \in l \cap \mathrm{Fix}(\sigma)$ and

$a_{\bar{\gamma}} \neq l \cap \mathrm{Fix}(\sigma)$. Set $\{X\} = a_{\bar{\gamma}} \cap l$. The points $C_{\bar{\gamma}}$ and X might coincide or not according to whether $\bar{\gamma}$ is either an elation or a homology of $\mathrm{Fix}(\sigma)$, respectively. Let $\bar{\beta}_j$ be the involution induced on $\mathrm{Fix}(\sigma)$ by the Klein subgroup T_j containing σ (and hence lying in C), $j = 1, 2$. As $\bar{\gamma}$ is central in $\bar{C}$, then $\bar{C}$ fixes $C_{\bar{\gamma}}$, $a_{\bar{\gamma}}$ and so X. Thus, C does it. Therefore, $C \leq G_{C_{\bar{\gamma}}}$ and $C \leq G_X$. Then, by Table II and since $q > 9$, we have that

(1) $|\mathrm{Fix}(\gamma) \cap l| = x_1 + x_2 = 1$ or 2;

(2) $x_6 = 0$;

(3) $x_7 > 0$ if q is a square and $q \equiv 9 \mod 16$;

(4) $x_8 = 0$;

(5) $x_9 > 0$ if q is a square and $p^e \equiv 3 \mod 4$, where $p^e \mid \sqrt{q}$;

(6) $\mathcal{S}_4 = 0$.

It remains to prove the assertion (7). If $C < G_{C_{\bar{\gamma}}}$ and $C < G_X$. Then $G_{C_{\bar{\gamma}}} = G_X = G$, since C is maximal in G as $q > 9$. As a consequence, $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}(G) \cap l$. Assume that $\bar{\beta}_1$ is an involutory $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-perspectivity. Then $C_{\bar{\beta}_1} \in l$ and $a_{\bar{\beta}_1} \neq l$ by Lemma 4.2. So, $C_{\bar{\beta}_1} \in \{C_{\bar{\gamma}}, X\}$, since $G_{C_{\bar{\gamma}}} = G_X = G$. Therefore, G fixes $C_{\bar{\beta}_1}$. Hence, we arrive at a contradiction by dual of Lemma 4.2, since $\mathrm{Fix}(T_1) \cap [C_{\bar{\beta}_1}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$. Thus, $\bar{\beta}_1$ is a Baer involution of $\mathrm{Fix}(\sigma)$. The previous argument with T_2 in the role of T_1, yields that $\bar{\beta}_2$ is also a Baer involution of $\mathrm{Fix}(\sigma)$.

Assume there exists $Q \in \{C_{\bar{\gamma}}, X\}$ such that $G_Q = C$. Then $\bar{\beta}_j$ is a Baer involution of $\mathrm{Fix}(\sigma)$ for each $j = 1, 2$, since $\left|\mathrm{Fix}_{Q^G}(T_j)\right| = 3$ by Table I for each $j = 1, 2$. Therefore, $\bar{\beta}_j$ is a Baer involution of $\mathrm{Fix}(\sigma)$ for each $j = 1$ or 2 in any case. This completes the proof. □

Lemma 4.4. *It holds that* $|\mathrm{Fix}(\gamma) \cap l| \geq 3$.

Proof. Suppose that $|\mathrm{Fix}(\gamma) \cap l| \leq 2$. Then either $|\mathrm{Fix}(\gamma) \cap l| = 1$ or $|\mathrm{Fix}(\gamma) \cap l| = 2$, as G fixes l and γ induces and involution $\bar{\gamma}$ on $\mathrm{Fix}(\sigma)$.

Assume that $|\mathrm{Fix}(\gamma) \cap l| = 1$. Then $\bar{\gamma}$ is an involutory $(C_{\bar{\gamma}}, a_{\bar{\gamma}})$-elation of $\mathrm{Fix}(\sigma)$ with $C_{\bar{\gamma}} \in l \cap \mathrm{Fix}(\sigma)$ and $a_{\bar{\gamma}} \neq l \cap \mathrm{Fix}(\sigma)$. Thus, $x_1 + x_2 = 1$ by Lemma 4.3(1). Moreover, by Table II in conjunction with Lemma 4.3(2)–(5), we have $|\mathrm{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_{4a} + 2x_{5a} + 2x_{7a}$ and $|\mathrm{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_{4b} + 2x_{5b} + 2x_{7b}$ for $\sqrt{q} \equiv 1 \mod 4$, and $|\mathrm{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_{4b} + 2x_{5b} + 2x_{7b}$ and $|\mathrm{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_{4a} + 2x_{5a} + 2x_{7a}$ for $\sqrt{q} \equiv 3 \mod 4$. Then

$$\sqrt[4]{n} + 1 = x_1 + 3x_2 + 2x_{4a} + 2x_{5a} + 2x_{7a} \tag{22}$$

$$\sqrt[4]{n} + 1 = x_1 + 3x_2 + 2x_{4b} + 2x_{5b} + 2x_{7b}\,, \tag{23}$$

in each case, being $|\mathrm{Fix}(T_j)\cap l| = \sqrt[4]{n}+1$ for each $j=1,2$ by Lemma 4.3(7). Now, summing up (22) and (23), we have

$$\sqrt[4]{n}+1 = x_1+3x_2+x_4+x_5+x_7\,.$$

Bearing in mind that $x_1+x_2=1$, we actually obtain

$$\sqrt[4]{n} = 2x_2+x_4+x_5+x_7\,. \tag{24}$$

Assume that $x_4>0$. Then $q=17$ and $x_5=0$ by Lemma 3.4, since $q\equiv 1 \bmod 8$. Furthermore, $x_4=1$ and $x_2=0$ by Lemma 3.5(3) and Lemma 3.6(3), respectively. Then $x_1=1$, as $x_1+x_2=1$ by the above argument. So, $x_7=\sqrt[4]{n}-1$ by (24). On the other hand, $\sqrt{n}+1\geq 1+x_7(\sqrt{q}\pm 1)$ by Table I. By substituting $x_7=\sqrt[4]{n}-1$ in the previous inequality and by elementary calculations of the inequality, we have $\sqrt[4]{n}+1\geq\sqrt{q}\pm 1$. Then $\sqrt[4]{n}=\sqrt{q}-1$ and $\sqrt{q}\equiv 3 \bmod 4$, since $\sqrt[4]{n}<\sqrt{q}$ by our assumptions. Nevertheless, this contradicts [16, Theorem 13.18], since $\bar{\gamma}$ acts non trivially on the plane $\mathrm{Fix}(T_j)$ and since $q>9$. So, $x_4=0$.

Assume that $x_5>0$. If $x_2=1$, then $x_5\leq 2$ and $q=25$, 41 or 49 by Lemma 3.6(4), since $q\equiv 1 \bmod 8$. Then $n+1\geq\frac{q(q+1)}{2}+\frac{q(q^2-1)}{120}$, with $n<q^2$, and n a fourth power. This is impossible, since $q=25$, 41 or 49. Then $x_2=0$ and hence $x_1=1$, since $x_1+x_2=1$. Then $x_7\geq\sqrt[4]{n}-3$ by (24), since $x_5\leq 3$ by Lemma 3.5(4). If $\sqrt[4]{n}>3$, then $x_7>0$. This implies that q is a square and $q\equiv 9 \bmod 16$ by Lemma 4.3(3). As a consequence, $q=25$. Then $\sqrt[4]{n}=4$, since $q<n<q^2$, n is a fourth power and $\sqrt[4]{n}>3$. As $n+1\geq 1+x_5\frac{q(q^2-1)}{120}$ by Table I, where $n=4^4$ and $q=25$, then $x_5=1$ and hence $x_7=3$. Therefore, by Table I, $n+1\geq x_1+x_5\frac{q(q^2-1)}{120}+x_7\frac{\sqrt{q}(q+1)}{2}$, where $x_1=x_5=1$ and $x_7=3$ and $q=25$. That is $n\geq 325$. Nevertheless, this contradicts the fact that $n=4^4$. Then $\sqrt[4]{n}\leq 3$ and hence $n\leq 3^4$. Nevertheless, $n\geq 130$, being $n\geq\frac{q(q^2-1)}{120}$ with $q\geq 25$ by Lemma 3.4. So, we again obtain a contradiction. Thus, $x_5=0$.

Since $x_4=x_5=0$, then $\sqrt[4]{n}=2x_2+x_7$ by (24). If $x_2=0$, then $x_7=\sqrt[4]{n}$ and hence $\sqrt{n}+1\geq 1+\sqrt[4]{n}(\sqrt{q}\pm 1)$ by Table I. Consequently, $\sqrt[4]{n}\geq\sqrt{q}\pm 1$. Actually, $\sqrt[4]{n}=\sqrt{q}-1$, since $\sqrt[4]{n}<\sqrt{q}$ by our assumptions. At this point the above argument rules out this case. Then $x_2=1$ and hence $x_7=\sqrt[4]{n}-2$. If $x_7>0$, then $\sqrt[4]{n}>2$ and hence

$$\sqrt{n}+1\geq\frac{q+1}{2}+(\sqrt[4]{n}-2)(\sqrt{q}\pm 1) \tag{25}$$

by Table I. Note that $\sqrt{n}+1<q+1$. So $\sqrt{n}+1>\frac{\sqrt{n}+1}{2}+(\sqrt[4]{n}-2)(\sqrt{q}\pm 1)$ by (25). Collecting with respect to $\sqrt{n}+1$, we have $\sqrt{n}+1>2(\sqrt[4]{n}-2)(\sqrt{q}\pm 1)$. Since $\sqrt[4]{n}>2$, then $\frac{\sqrt{n}}{2(\sqrt[4]{n}-2)}<\sqrt[4]{n}$ and therefore $\sqrt[4]{n}>(\sqrt{q}\pm 1)$. In particular,

$\sqrt[4]{n} > \sqrt{q} - 1$ in each case. On the other hand, $\sqrt[4]{n} < \sqrt{q}$, since $n < q^2$ by our assumption. So, $\sqrt{q} - 1 < \sqrt[4]{n} < \sqrt{q}$, where $\sqrt{q}$ is integer by Lemma 4.3(3), being $x_7 > 0$. Clearly, this is a contradiction. Thus, $x_7 = 0$. Then $\sqrt[4]{n} = 2$, since $x_7 = \sqrt[4]{n} - 2$. So, $n = 16$. Nevertheless, $n + 1 \geq \frac{q(q+1)}{2}$, as $x_2 = 1$ and $q > 9$, which is still a contradiction.

Assume that $|\mathrm{Fix}(\gamma) \cap l| = 2$. Then $\bar{\gamma}$ is an involutory $(C_{\bar{\gamma}}, a_{\bar{\gamma}})$-homology of $\mathrm{Fix}(\sigma)$ with $C_{\bar{\gamma}} \in l \cap \mathrm{Fix}(\sigma)$ and $a_{\bar{\gamma}} \neq l \cap \mathrm{Fix}(\sigma)$. Then $x_1 + x_2 = 2$ by Lemma 4.3(1). Recall that $\{X\} = a_{\bar{\gamma}} \cap l$ (clearly $C_{\bar{\gamma}} \neq X$) and each T_j induces a Baer involution on $\mathrm{Fix}(\sigma)$ by Lemma 4.3(7). Then either $x_1 = x_2 = 1$ or $x_1 = 2$ and $x_2 = 0$, since $x_2 \leq 1$ by Lemma 3.5(1).

Assume that $x_1 = x_2 = 1$. Arguing as above, we have $\sqrt[4]{n} + 1 = x_1 + 3x_2 + x_4 + x_5 + x_7$ by Table I in conjunction with Lemma 4.3(2)–(7). Actually, $x_4 = 0$ by Lemma 3.5(3), since $x_2 = 1$. Therefore,

$$\sqrt[4]{n} = 3 + x_5 + x_7\,, \tag{26}$$

as $x_1 = x_2 = 1$. If $x_5 > 0$, then $x_5 \leq 2$ and $q = 25, 41, 49, 81$ or 89 by Lemma 3.6(4), since $q \equiv 1 \bmod 8$. Then $n + 1 \geq \frac{q(q+1)}{2} + \frac{q(q^2-1)}{120}$, with $n < q^2$ and n a fourth power, which is a contradiction as above. Thus, $x_5 = 0$. So, $x_7 = \sqrt[4]{n} - 3$ by (26). On the other hand, $\sqrt{n} + 1 \geq 1 + \frac{q+1}{2} + x_7(\sqrt{q} \pm 1)$ by Table I, since $x_1 = x_2 = 1$. Then

$$\sqrt{n} + 1 \geq 1 + \frac{q+1}{2} + (\sqrt[4]{n} - 3)(\sqrt{q} \pm 1)\,, \tag{27}$$

since $x_7 = \sqrt[4]{n} - 3$. If $\sqrt[4]{n} > 3$, then $x_7 > 0$. Hence, q is a square and $q \equiv 9 \bmod 16$ by Lemma 4.3(3). Thus the cases $q = 41, 49, 81$ or 89 are ruled out. As a consequence, $q = 25$. This yields $\sqrt[4]{n} < 5$, since $n < q^2$ by our assumptions. Then $\sqrt[4]{n} = 4$, since $\sqrt[4]{n} > 3$. This is a contradiction, since $\bar{\gamma}$ is an involutory homology of $\mathrm{Fix}(\sigma)$. Therefore $\sqrt[4]{n} = 3$ and hence $n = 3^4$. Then $\frac{q(q+1)}{2} \leq 82$, since $\frac{q(q+1)}{2} \leq n + 1$ by (27), being $x_2 = 1$. This is still a contradiction, since $q = 25, 41, 49, 81$ or 89.

Assume that $x_1 = 2$ and $x_2 = 0$. Recall that $\bar{\beta}_j$ is a Baer involution of $\mathrm{Fix}(\sigma)$. Hence $|\mathrm{Fix}(T_j) \cap l| = \sqrt[4]{n} + 1$ for $j = 1, 2$ by Lemma 4.3(7). Therefore,

$$\sqrt[4]{n} = 1 + x_4 + x_5 + x_7\,, \tag{28}$$

arguing as above, as $x_1 = 2$ and $x_2 = 0$.

Assume that $x_4 > 0$. Then $x_4 = 1$ by Lemma 3.5(3). Then $q = 17$ by Lemma 3.4, since $q \equiv 1 \bmod 8$. Moreover, $x_5 = 0$ again by Lemma 3.4, and $x_7 = 0$ since q is a non square. So, $\sqrt[4]{n} = 2$ by (28). That is $n = 16$. Nevertheless, this contradicts the fact that $q < n$ by our assumptions. So, $x_4 = 0$.

Assume that $x_5 > 0$, then $q = 25, 41, 49, 81$ or 89 by Lemma 3.4, since $q \equiv 1 \bmod 8$. Furthermore, $x_5 \leq 3$ by Lemma 3.5(4). Thus, $x_7 \geq \sqrt[4]{n} - 4$ by (28). If $\sqrt[4]{n} > 4$, then $x_7 > 0$ and hence q is a square and $q \equiv 9 \bmod 16$. Therefore, only the case $q = 25$ is admissible. Nevertheless, $\sqrt[4]{n} < 5$, since $q < n < q^2$ by our assumptions. This is a contradiction, since $\sqrt[4]{n} > 4$. As consequence, $\sqrt[4]{n} = 4$ and $x_7 = 0$. Then $\sqrt{n} = 16$, and we again obtain a contradiction, since $\bar{\gamma}$ is an involutory homology of $\mathrm{Fix}(\sigma)$. So, $x_5 = 0$.

Since $x_4 = x_5 = 0$, then $x_7 = \sqrt[4]{n} - 1$ by (28). Now, bearing in mind that $x_1 = 2$, $x_2 = x_4 = x_5 = 0$ and $x_7 = \sqrt[4]{n} - 1$, we have $\sqrt{n} \geq 1 + (\sqrt[4]{n} - 1)(\sqrt{q} \pm 1)$ by Table I. Therefore, $\sqrt[4]{n} + 1 \geq (\sqrt{q} \pm 1)$. Then $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt[4]{n} = \sqrt{q} - 2$ or $\sqrt{q} - 1$, since $\sqrt[4]{n} < \sqrt{q}$ by our assumptions. Actually, only the case $\sqrt[4]{n} = \sqrt{q} - 2$ is admissible, since $\sqrt[4]{n}$ is odd, as $\bar{\gamma}$ is an involutory homology of $\mathrm{Fix}(\sigma)$. Then $x_7 = \sqrt{q} - 3$ by (28). Hence $\sqrt[4]{n} = \sqrt{q} - 2$. Now, by substituting these values in $\sqrt{n} \geq 1 + (\sqrt[4]{n} - 1)(\sqrt{q} - 1)$ (obtained by Table I), we actually obtain an equality. Thus, there are exactly two points on l fixed by G ($x_1 = 2$) and the stabilizer in G of any of the remaining ones on $l \cap \mathrm{Fix}(\sigma)$ is isomorphic to $\mathsf{PSL}(2, \sqrt{q})$. Then $\mathcal{S}_2 = x_3 = x_9 = 0$ by Table I. Therefore, $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_{2'}$, being $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$ and $\mathcal{S}_1 = \mathcal{S}_2 + \mathcal{S}_{2'}$. By this and by Table I, we have

$$n + 1 = 2 + \frac{\sqrt{q}(\sqrt{q} - 3)}{2}(q + 1) + \frac{q + 1}{2}\mathcal{S}_{2'}\,,$$

since $x_1 = 2$, $x_2 = 0$ by our assumption, since $x_3 = x_4 = x_5 = x_9 = 0$ and $x_7 = \sqrt{q} - 3$ by the above argument, and since $x_6 = x_8 = 0$ by Lemma 4.3(2) and (4). Since $q \equiv 1 \bmod 8$, then $\mathcal{S}_{2'}$ is even (see its definition) and hence $q+1 \mid n-1$. That is $q+1 \mid (\sqrt{q}-2)^4 - 1$, since $\sqrt[4]{n} = \sqrt{q} - 2$. Easy computations yield $q + 1 \mid 40\sqrt{q} - 8$. As $q + 1 \neq 40\sqrt{q} - 8$, then $q + 1 \leq 80\sqrt{q} - 16$ and so $\sqrt{q} \leq 79$. Actually, since $\sqrt{q} \leq 41$, since $(\sqrt{q})^2 + 1 \leq 40\sqrt{q} - 8$. Now, it is straightforward computation to show that there are no $\sqrt{q}$, such that $\sqrt{q} \leq 41$ and $(\sqrt{q})^2 + 1 \mid 40\sqrt{q} - 8$. Thus, we have proved the assertion. $\square$

Let $C = C_G(\sigma)$ and let K and K^* be the kernels of the action of C on $\mathrm{Fix}(\sigma)$ and on $\mathrm{Fix}(\sigma) \cap l$, respectively. Clearly $\langle\sigma\rangle \trianglelefteq K \trianglelefteq K^* \trianglelefteq C$. Moreover, either $K^* \trianglelefteq Z_{\frac{q-1}{2}}$ or $K^* \cong D_{\frac{q-1}{2}}$ or $K^* = C$, since $q \equiv 1 \bmod 8$. Actually, the cases $K^* \cong D_{\frac{q-1}{2}}$ or $K^* = C$ are ruled out by Lemma 4.2. Then $\langle\sigma\rangle \trianglelefteq K \trianglelefteq K^* \trianglelefteq Z_{\frac{q-1}{2}}$. Let $\gamma \in C$ such that $\gamma^2 = \sigma$. The previous lemma shows that either $\gamma \in K^*$ or γ induces a Baer involution on $\mathrm{Fix}(\sigma)$. Now, we investigate these two configurations separately.

4.1. The collineation $\gamma \in K^*$

Lemma 4.5. *If* $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}(\sigma) \cap l$, *then the following hold:*

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(1) $\mathcal{S}_{2,4} = 0$, $\mathcal{S}_2 = \mathcal{S}_4$ *and hence* $\mathcal{S}_1 = \mathcal{S}_{2'} + \mathcal{S}_4$*;*

(2) $x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0$*;*

(3) *If* $x_7 > 0$ *then* q *is a square and* $q \equiv 1 \bmod 16$*;*

(4) *If* $x_9 > 0$ *then* q *is a square and* $p^e \equiv 1 \bmod 4$, *where* $p^e \mid \sqrt{q}$*;*

(5) *We may assume that* $|\mathrm{Fix}(T_1) \cap l| = x_1 + 2x_{7a}$ *and* $|\mathrm{Fix}(T_2) \cap l| = x_1 + 2x_{7b}$.

Proof. Assume that $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}(\sigma) \cap l$. Note that $|\mathrm{Fix}(\gamma) \cap l| = \sum_h \left|\mathrm{Fix}_{P_h^G}(\gamma)\right|$ and $|\mathrm{Fix}(\sigma) \cap l| = \sum_h \left|\mathrm{Fix}_{P_h^G}(\sigma)\right|$. Then $\left|\mathrm{Fix}_{P_h^G}(\gamma)\right| = \left|\mathrm{Fix}_{P_h^G}(\gamma)\right|$ for each admissible P_h^G on l, since $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}(\sigma) \cap l$ and $\left|\mathrm{Fix}_{P_h^G}(\gamma)\right| \leq \left|\mathrm{Fix}_{P_h^G}(\gamma)\right|$. Thus, the assertions (1)–(4) follow by a direct inspection of the Tables I and II.

It remains to show the assertion (5). In order to do so, note that $|\mathrm{Fix}(T_1) \cap l| = x_1 + 2x_{7a}$ and $|\mathrm{Fix}(T_2) \cap l| = x_1 + 2x_{7b}$ for $\sqrt{q} \equiv 1 \bmod 4$, while $|\mathrm{Fix}(T_1) \cap l| = x_1 + 2x_{7b}$ and $|\mathrm{Fix}(T_2) \cap l| = x_1 + 2x_{7a}$ by Table I, since $x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0$ by (2). Therefore, we have proved the assertion (5). □

Lemma 4.6. *If* $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}(\sigma) \cap l$ *then one of the following occurs:*

(1) *The group* T_j *induces a homology on* $\mathrm{Fix}(\sigma)$ *for either* $j = 1$ *or* $j = 2$, *and the following occur:*

  (a) $x_1 = 0$*;*

  (b) $x_{7a}, x_{7b} > 0$. *In particular, either* $x_{7a} = 1$ *or* $x_{7b} = 1$.

(2) *The group* T_j *induces a Baer involution on* $\mathrm{Fix}(\sigma)$ *for each* $j = 1, 2$, *and the following occur:*

  (a) $\sqrt[4]{n} + 1 = x_1 + x_7$*;*

  (b) $x_1 \geq 3$*;*

  (c) *The collineation* γ *induces the identity on* $\mathrm{Fix}(\sigma)$*;*

  (d) *The group* G *fixes a subplane of* Π *of order* $x_1 - 1$.

Proof. Let $\bar{\beta}_j$ be the involution induced on $\mathrm{Fix}(\sigma)$ by a Klein subgroup T_j containing σ (and hence lying in C), $j = 1, 2$. Assume that $\bar{\beta}_1$ is a $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-perspectivity. Then $C_{\bar{\beta}_1} \in \mathrm{Fix}(\sigma) \cap l$ and $a_{\bar{\beta}_1} \neq l$ by Lemma 4.2. Set $\{X\} = a_{\bar{\beta}_1} \cap l$. If $\bar{\beta}_1$ is an elation $\mathrm{Fix}(\sigma)$, then $C_{\bar{\beta}_1} = X$ and hence $|\mathrm{Fix}(T_1) \cap l| = 1$. Then $x_1 = 1$ and $x_{7a} = 0$, since $|\mathrm{Fix}(T_1) \cap l| = x_1 + 2x_{7a}$ by Lemma 4.5(5). So, G fixes $C_{\bar{\beta}_1}$, which is a contradiction by dual of Lemma 4.2, since $\mathrm{Fix}(T_1) \cap [C_{\bar{\beta}_1}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$. Thus, $\bar{\beta}_1$ is a $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-homology of $\mathrm{Fix}(\sigma)$. Then $C_{\bar{\beta}_1} \neq X$ and hence $|\mathrm{Fix}(T_1) \cap l| = 2$. Therefore, either $x_1 = 2$ and $x_{7a} = 0$ or $x_1 = 0$

and $x_{7a} = 1$, since $|\mathrm{Fix}(T_1) \cap l| = x_1 + 2x_{7a}$ by Lemma 4.5(5). Assume the former occurs. Then G fixes $C_{\bar{\beta}_1}$ and X, which is a contradiction by the same argument as above. Consequently, $x_1 = 0$ and $x_{7a} = 1$. Moreover, $x_{7b} > 0$, since $|\mathrm{Fix}(T_1) \cap l| = x_1 + 2x_{7b}$, being $x_1 = 0$. The previous argument still works with T_2 in the role of T_1. Hence, we obtain the assertion (1a) and (1b).

Assume that T_j induces a Baer involution on $\mathrm{Fix}(\sigma)$ for each $j = 1, 2$. Then $|\mathrm{Fix}(T_j) \cap l| = \sqrt[4]{n} + 1$ for each $j = 1, 2$. Then $\sqrt[4]{n} + 1 = x_1 + 2x_{7a}$ or $\sqrt[4]{n} + 1 = x_1 + 2x_{7b}$ by Lemma 4.5(5). As a consequence,

$$\sqrt[4]{n} + 1 = x_1 + x_7\,, \tag{29}$$

since $x_7 = x_{7a} + x_{7b}$. Thus, we have proved the assertion (2a).

Now, note that $\sqrt{n} + 1 \geq x_1 + x_7(\sqrt{q} \pm 1)$ by Table I. By composing this one with (29), we obtain

$$\sqrt{n} + 1 \geq x_1 + (\sqrt[4]{n} + 1 - x_1)(\sqrt{q} \pm 1)\,. \tag{30}$$

Assume that $x_7 > 0$. Thus $q \equiv 1 \bmod 16$ by Lemma 4.5(3). If $x_1 = 0$, then $x_7 = \sqrt[4]{n} + 1$ by (29) and hence $\sqrt{n} + 1 \geq (\sqrt[4]{n} + 1)(\sqrt{q} \pm 1)$ by (30). That is $\sqrt[4]{n} - 1 \geq (\sqrt{q} \pm 1) \geq \sqrt{q} - 1$. This yields $\sqrt[4]{n} \geq \sqrt{q}$, which is a contradiction, since $n < q^2$. Then $x_1 \geq 1$. If $x_1 = 1$, then $x_7 = \sqrt[4]{n}$ by (29). Furthermore, $\sqrt{n} + 1 \geq 1 + \sqrt[4]{n}(\sqrt{q} \pm 1)$ by (30). That is $\sqrt[4]{n} \geq (\sqrt{q} \pm 1)$. As $\sqrt[4]{n} < \sqrt{q}$ by our assumption, then $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt[4]{n} = \sqrt{q} - 1$. By substituting the determined values of x_1, x_7 and of $\sqrt[4]{n}$ in (30), we see that this one is satisfied as an equality. So, $x_9 = 0$ and $\mathcal{S}_4 = 0$ by Table I. Therefore, $n + 1 = 1 + \mathcal{S} + (\sqrt{q} - 3)(q + 1)$. As $q + 1 \mid \mathcal{S}$, being $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$ and being $\mathcal{S}_1$ even by its definition, then $q + 1 \mid n$. Consequently, $q + 1 \mid (\sqrt{q} - 1)^4$. Elementary calculations of the previous relation yield $q + 1 \mid 4$. Hence, we arrive at a contradiction, since q is a square as $x_7 > 0$. Therefore, $x_1 \geq 2$. Assume that $x_1 = 2$. Then $x_7 = \sqrt[4]{n} - 1$ by (29). Furthermore, $\sqrt{n} + 1 \geq 2 + (\sqrt[4]{n} - 1)(\sqrt{q} \pm 1)$ by (30). This yields $\sqrt[4]{n} + 1 \geq \sqrt{q} \pm 1$. As $\sqrt[4]{n} < \sqrt{q}$ by our assumption, then $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt[4]{n} \geq \sqrt{q} - 2$. Then either $\sqrt[4]{n} = \sqrt{q} - 1$ or $\sqrt[4]{n} = \sqrt{q} - 2$, again since $\sqrt[4]{n} < \sqrt{q}$. Actually, the case $\sqrt[4]{n} = \sqrt{q} - 1$ is ruled out by the above argument. So, $\sqrt[4]{n} = \sqrt{q} - 2$. This forces $\sqrt{n} + 1 \geq 2 + (\sqrt[4]{n} - 1)(\sqrt{q} \pm 1)$ to be an equality. As a consequence, $x_9 = 0$ and $\mathcal{S}_4 = 0$ by Table II. Thus $n + 1 = 2 + \mathcal{S}_1 + (\sqrt{q} - 3)(q + 1)$ by Table I. As $q + 1 \mid \mathcal{S}_1$, then $q + 1 \mid n - 1$. Hence $q + 1 \mid (\sqrt{q} - 2)^4 - 1$, since $\sqrt[4]{n} = \sqrt{q} - 2$. Easy computations yield a contradiction. Therefore, $x_1 \geq 3$ for $x_7 > 0$. Actually, $x_1 \geq 3$ also for $x_7 = 0$ by (29), since $\sqrt[4]{n} \geq 2$. Thus $x_1 \geq 3$ in each case, which is the assertion (2b).

Now, G and hence γ acts on $[X]$ for each point X of the x_1 ones fixed by G on l. Then γ fixes at least 3 lines of $[X]$ for each point X of the x_1 ones fixed by

G on l by dual of Lemma 4.4. So, γ induces the identity on $\mathrm{Fix}(\sigma)$, since $x_1 \geq 3$ and since $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}(\sigma) \cap l$. Thus, we have proved the assertion (2c).

Now, we may apply the dual of the above argument to $[X]$ in the role of l for each point X of the x_1 ones fixed by G on l. This yields that G fixes at least 3 lines through each of these x_1 points on l. Then G is planar, which is the assertion (2d). □

Proposition 4.7. *The group T_j induces a homology on $\mathrm{Fix}(\sigma)$ for either $j = 1$ or $j = 2$, and the following hold:*

(1) $x_1 = 0$;

(2) $x_{7a}, x_{7b} > 0$. *In particular, either $x_{7a} = 1$ or $x_{7b} = 1$.*

Proof. Assume that T_j induces a Baer involution on $\mathrm{Fix}(\sigma)$ for each $j = 1, 2$. Then $x_1 \geq 3$ and γ induces the identity on $\mathrm{Fix}(\sigma)$ by Lemma 4.6(2b) and (2c), respectively. Then, by Table I and by Lemma 4.6(2b), we have the following system of Diophantine equations:

$$\sqrt[4]{n} + 1 = x_1 + x_7 \tag{31}$$

$$\sqrt{n} + 1 = x_1 + x_7(\sqrt{q} \pm 1) + x_9 \frac{q-1}{p^e - 1} + \mathcal{S}_4 \tag{32}$$

$$n + 1 = x_1 + x_7\sqrt{q}(q+1) + x_9 \frac{p^e q^2 - 1}{2(p^e - 1)} + \frac{q+1}{2}\mathcal{S}_1 . \tag{33}$$

Suppose that $x_7 = 0$. Then $\sqrt[4]{n} + 1 = x_1$ by (31) and hence $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$ by Lemma 4.6(2d). Thus ,$\sqrt{n}+1 = \mathcal{S}_4 + x_9 \frac{q-1}{p^e-1} + \sqrt[4]{n}$ by (32). As a consequence, $\mathcal{S}_4 + x_9 > 0$. Let ρ_t, where $t = 1$ or 2, be the representatives the two conjugates of p-elements in G. Then ρ_t is planar for each $t = 1, 2$ since G is planar. In particular, ρ_t fixes a Baer subplane of Π by [16, Theorem 3.7], since $\mathrm{Fix}(G) \subset \mathrm{Fix}(\rho_t)$ and $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$. Furthermore, $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(\langle \rho_t, \sigma \rangle) \subseteq \mathrm{Fix}(\rho_t)$. Then either $\mathrm{Fix}(\langle \rho_t, \sigma \rangle) = \mathrm{Fix}(G)$ or $\mathrm{Fix}(\langle \rho_t, \sigma \rangle) = \mathrm{Fix}(\rho_t)$ again by [16, Theorem 3.7], since $\mathrm{Fix}(G)$ is a Baer subplane of $\mathrm{Fix}(\rho_t)$. Actually, $\mathrm{Fix}(\langle \rho_t, \sigma \rangle) = \mathrm{Fix}(\rho_t)$, since $\mathcal{S}_4 + x_9 > 0$. This yields $x_9 = 0$ and $\mathcal{S}_1 = \mathcal{S}_4$ again by Table II. In particular, $\mathcal{S}_1 > 0$, since $\mathcal{S}_4 + x_9 > 0$ and $x_9 = 0$. Moreover, $\mathrm{Fix}(\rho_t) \subseteq \mathrm{Fix}(\sigma)$. Actually, $\mathrm{Fix}(\rho_t) = \mathrm{Fix}(\sigma)$, since $\mathrm{Fix}(\rho_t)$ and $\mathrm{Fix}(\sigma)$ are Baer subplanes of Π. Then $\frac{1}{2}\mathcal{S}_1 = \mathcal{S}_4$, since $o(\mathrm{Fix}(\rho_t)) = \frac{1}{2}\mathcal{S}_1 + \sqrt[4]{n}$ and $o(\mathrm{Fix}(\sigma)) = \mathcal{S}_4 + \sqrt[4]{n}$ by Tables II and I, respectively. Hence, we arrive at a contradiction, since $\mathcal{S}_1 = \mathcal{S}_4$ and $\mathcal{S}_1 > 0$. Thus, $x_7 > 0$.

Let us focus on the group $\langle \rho_t, \sigma \rangle$, $t = 1, 2$. Then $\langle \rho_t, \sigma \rangle$ is planar for each $t = 1, 2$, since $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(\langle \rho_t, \sigma \rangle$. In particular, $o(\mathrm{Fix}(\langle \rho_t, \sigma \rangle)) \leq \sqrt[4]{n} + 1$ by [16, Theorem 3.7], since $o(\mathrm{Fix}(\langle \rho_t, \sigma \rangle))$ is a proper subplane of $\mathrm{Fix}(\sigma)$ as $x_7 > 0$, and since $\mathrm{Fix}(\sigma)$ is a Baer subplanes of Π. Moreover, by Table II,

$o(\mathrm{Fix}(\langle\rho_t,\sigma\rangle))+1 = x_1 + x_7\varepsilon + x_9 k_t + \frac{1}{2}\mathcal{S}_4$, where $\varepsilon = 2$ or 0 according to whether $\sqrt{q} \equiv 1 \bmod 4$ or $\sqrt{q} \equiv 3 \bmod 4$, respectively. So,

$$x_1 + x_7\varepsilon + x_9 k_t + \frac{1}{2}\mathcal{S}_4 \leq x_1 + x_7, \text{ for each } t = 1, 2\,, \tag{34}$$

since $o(\mathrm{Fix}(\langle\rho_t,\sigma\rangle)) \leq \sqrt[4]{n}+1$ and since $\sqrt[4]{n}+1 = x_1+x_7$ by (31). It follows from (34) that $\varepsilon = 0$ and hence $\sqrt{q} \equiv 3 \bmod 4$, as $\varepsilon = 2$ or 0 according to whether $\sqrt{q} \equiv 1 \bmod 4$ or $\sqrt{q} \equiv 3 \bmod 4$. Then summing up the two inequalities in (34) (one for $t = 1$ and the other for $t = 2$) and then subtracting $2x_1$ to the sum, we obtain

$$\mathcal{S}_4 + x_9(k_1 + k_2) \leq 2x_7\,. \tag{35}$$

Assume that $\mathcal{S}_4 + x_9 > 0$. If $x_9 = 0$, then $\mathcal{S}_4 > 0$. Moreover, $\mathcal{S}_4 \leq 2x_7$ by (35). Since $\mathrm{Fix}(G)$ is a proper subplane of $\mathrm{Fix}(T_j)$, then $(x_1-1)^2 \leq (x_1-1)+x_7$ by [16, Theorem 3.7], and hence $x_1 - 1 \leq x_7$. Then $x_1 - 1 + \mathcal{S}_4 \leq 3x_7$, since $\mathcal{S}_4 \leq 2x_7$. Now, note that $\sqrt{n}+1 = x_1 + x_7(\sqrt{q}-1) + \mathcal{S}_4$ by (32), being $x_9 = 0$ and $\sqrt{q} \equiv 3 \bmod 4$. This produces $\sqrt{n} \leq 3x_7 + x_7(\sqrt{q}-1)$ as $\mathcal{S}_4 \leq 2x_7$. Hence, $x_7 \geq \frac{\sqrt{n}}{\sqrt{q}+2}$. On the other hand, $n+1 \geq x_7\sqrt{q}(q+1)+1$ by (33), since $x_1 \geq 1$ (actually, $x_1 \geq 3$). Now, by substituting $x_7 \geq \frac{\sqrt{n}}{\sqrt{q}+2}$ in the last inequality, we obtain $n \geq \frac{\sqrt{n}}{\sqrt{q}+2}\sqrt{q}(q+1)$. Since $\frac{q+1}{\sqrt{q}+2} = \sqrt{q} - 2 + \frac{5}{\sqrt{q}+2}$, we actually obtain $\sqrt{n} > (\sqrt{q}-1)^2$ and hence $\sqrt{n} \geq (\sqrt{q})^2$, since $\sqrt{n}$ is a square. This is impossible, since $n < q^2$ by our assumptions. Therefore, $x_9 > 0$. Actually, $x_9 = 1$ by Lemma 3.5(6). Then $2x_7 \geq k_1 + k_2$ by (35). Hence, $x_7 \geq \frac{q-p^e}{2p^e(p^e-1)}$, since $k_1 + k_2 = \frac{q-p^e}{p^e(p^e-1)}$. Recall that $q = p^{2we}$, $w \geq 1$. Hence $x_7 \geq \frac{p^{2we}-p^e}{2p^e(p^e-1)}$. Now, by substituting these value in $\sqrt{n} \geq x_7(\sqrt{q}-1) + \frac{q-1}{p^e-1}$ which is obtained by (32), as $x_1 \geq 1$ and $x_9 = 1$ and $\sqrt{q} \equiv 3 \bmod 4$, we have

$$\sqrt{n} \geq \frac{p^{(2w-1)e}-1}{2(p^e-1)}(p^{we}-1) + \frac{p^{2we}-1}{p^e-1}\,.$$

Furthermore, since $q = p^{2we}$ and $\sqrt{q} \equiv 3 \bmod 4$, then w is odd. Assume that $w \geq 3$. Then $\frac{p^{we}-1}{p^e-1} \geq p^{2e} + p^e + 1$ and hence $\frac{p^{we}-1}{p^e-1} > 2p^e$. Then $\sqrt{n} > \frac{p^{2we}-1}{p^e-1} + (p^{(2w-1)e}-1)p^e$ and hence $\sqrt{n} > p^{2we}$. Thus $\sqrt{n} > q$, which is contradiction, since $\sqrt{n} < q$ by our assumptions. Then $w < 3$ and hence $w = 1$, since w is odd. So, $p^e = \sqrt{q}$ and hence $p^e \equiv 3 \bmod 4$, since $\sqrt{q} \equiv 3 \bmod 4$. This is a contradiction, by Lemma 4.5(4).

Finally, assume that $\mathcal{S}_4 = x_9 = 0$. Then $n+1 \geq x_1 + x_7\sqrt{q}(q+1)$ by (33). If we subtract (31) from (32), and then (31) from $n+1 \geq x_1 + x_7\sqrt{q}(q+1)$, by bearing in mind that $\sqrt{q} \equiv 3 \bmod 4$, we obtain

$$\sqrt{n} - \sqrt[4]{n} = x_7(\sqrt{q}-2) \tag{36}$$

$$n - \sqrt[4]{n} \geq x_7\left[\sqrt{q}(q+1) - 1\right]\,. \tag{37}$$

Now, combining (36) and (37), and bearing in mind that $x_7 > 0$, we obtain

$$\frac{n-\sqrt[4]{n}}{\sqrt{n}-\sqrt[4]{n}} \geq \frac{\sqrt{q}(q+1)-1}{\sqrt{q}-2}\,. \tag{38}$$

Since $n - \sqrt[4]{n} = (\sqrt{n}-\sqrt[4]{n})(\sqrt{n}+\sqrt[4]{n}+1)$ and since $\frac{\sqrt{q}(q+1)-1}{\sqrt{q}-2} > q$, then $\sqrt{n}+\sqrt[4]{n}+1 > q$ by (38). Then $(\sqrt[4]{n}+1)^2 > q$ and hence $\sqrt[4]{n} > \sqrt{q}-1$, as q is a square. On the other hand, $\sqrt[4]{n} < \sqrt{q}$ by our assumptions. So, $\sqrt{q}-1 < \sqrt[4]{n} < \sqrt{q}$, with $\sqrt[4]{n}$ and $\sqrt{q}$ integers. This is clearly a contradiction. At this point, the assertion easily follows by Lemma 4.6. □

Lemma 4.8. *The following hold:*

(1) $x_9 = 0$;

(2) $\mathcal{S}_4 > 0$;

(3) *Let* $h = 2$ *or* 4. *Then* $\frac{\sqrt{q}\mp 1}{h} \mid |K|$ *for* $\sqrt{q} \equiv \pm 1 \bmod 8$, *respectively.*

Proof. The group T_j induces a homology on $\mathrm{Fix}(\sigma)$ for at least one $j = 1$ or 2 by Proposition 4.7. Furthermore, $x_1 = 0$, $x_{7a}, x_{7b} > 0$ and either $x_{7a} = 1$ or $x_{7b} = 1$. We may assume that T_1 does and that $x_{7a} = 1$. Let $\bar{\beta}_1$ is a $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-homology induced by T_1 on $\mathrm{Fix}(\sigma)$. Set $\{X\} = a_{\bar{\beta}_1} \cap l$. Then, by Table I and by Lemma 4.5, we have

$$\sqrt{n}+1 = \mathcal{S}_4 + x_7(\sqrt{q}\pm 1) + x_9\frac{q-1}{p^e-1} \tag{39}$$

$$n+1 = \frac{q+1}{2}\mathcal{S}_1 + x_7\sqrt{q}(q+1) + x_9\frac{p^e q^2-1}{2(p^e-1)}\,. \tag{40}$$

Since $\mathcal{S}_1 \geq \mathcal{S}_4$, we compose (39) and (40) obtaining

$$n+1 \geq \frac{q+1}{2}(\sqrt{n}+1) + x_7\frac{q+1}{2}(\sqrt{q}\mp 1) + x_9\frac{q^2-1}{2}\,. \tag{41}$$

In particular, $n+1 \geq \frac{q+1}{2}(\sqrt{n}+1) + x_9\frac{q^2-1}{2}$. Since $q+1 > \sqrt{n}+1$, we have $n+1 > (\frac{q+1}{2} + x_9\frac{q-1}{2})(\sqrt{n}+1)$. Since $\sqrt{n}+1$ does not divide $n+1$ or n, then $n-1 \geq (\frac{q+1}{2} + x_9\frac{q-1}{2})(\sqrt{n}+1)$ being the second part an integer. Hence, dividing each term by $\sqrt{n}+1$, we obtain $\sqrt{n} \geq 1 + \frac{q+1}{2} + x_9\frac{q-1}{2}$. If $x_9 \geq 1$, then $\sqrt{n} \geq 1+q$, which is a contradiction. Therefore, $x_9 = 0$ and we have proved the assertion (1).

Assume that $\mathcal{S}_4 = 0$. Then $x_7 = \frac{\sqrt{n}+1}{\sqrt{q}\pm 1}$ by (39) and hence $n+1 = \frac{q+1}{2}\mathcal{S}_1 + \frac{\sqrt{n}+1}{\sqrt{q}\pm 1}\sqrt{q}(q+1)$ by (40), as $x_9 = 0$. Note that $\sqrt{q}\pm 1 \mid \sqrt{n}+1$, since x_7 is an integer, otherwise we would have a contradiction. In particular, $n+1 \geq \frac{\sqrt{n}+1}{\sqrt{q}\pm 1}\sqrt{q}(q+1)$. As $\sqrt{n} \geq 2$, we have $\sqrt{n}+1 \nmid n+1$. Furthermore, $\sqrt{n}+1 \nmid n$.

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Hence, $n-1 \geq \frac{\sqrt{n}+1}{\sqrt{q}\pm 1}\sqrt{q}(q+1)$ since the second part is an integer. So, $\sqrt{n} \geq 1+\frac{\sqrt{q}(q+1)}{\sqrt{q}\pm 1}$. If $\sqrt{q} \equiv 3 \bmod 4$, then $\sqrt{n} > q+1$, which is a contradiction by our assumptions. As a consequence, $\sqrt{q} \equiv 1 \bmod 4$. Then $\sqrt{n} \geq 1+\frac{\sqrt{q}(q+1)}{\sqrt{q}+1}$ and hence $\sqrt{n} > (\sqrt{q}-1)^2$.

If $x_7 > 2$, then T_2 must induce a Baer collineation $\mathrm{Fix}(\sigma)$ and consequently $\sqrt[4]{n}$ must be an integer. Since $n < q^2$, then $\sqrt[4]{n} < \sqrt{q}$. Actually, $\sqrt[4]{n} \leq \sqrt{q}-1$, since $\sqrt[4]{n}$ is an integer. Therefore, $\sqrt{n} \leq (\sqrt{q}-1)^2$ by squaring. This is a contradiction, since $\sqrt{n} > (\sqrt{q}-1)^2$ by the above argument.

If $x_7 = 2$. Then $\sqrt{n} = 2(\sqrt{q}+1)-1$ by (39), since $x_9 = 0$, and since $\mathcal{S}_4 = 0$ by our assumption. Hence $\sqrt{n} = 2\sqrt{q}+1$. On the other hand, $n \geq 2\sqrt{q}(q+1)-1$ by (40). By composing these inequalities, we have $(2\sqrt{q}+1)^2 \geq 2\sqrt{q}(q+1)-1$. Easy computations yield a contradiction, since $q > 9$. Thus, $\mathcal{S}_4 > 0$ which is the assertion (2).

As $x_{7a}, x_{7b} > 0$, it follows that q is a square and $q \equiv 1 \bmod 16$ by Lemma 4.5(3). So, either $\sqrt{q} \equiv 1 \bmod 8$ and $\sqrt{q} \equiv 7 \bmod 8$. Since T_1 fixes exactly two points on l, by Table I, these ones must lie in either a G-orbit on l of type (7a) or in a G-orbit on l of type (7b) according to whether$\sqrt{q} \equiv 1 \bmod 8$ or $\sqrt{q} \equiv 7 \bmod 8$, respectively.

Assume that $\sqrt{q} \equiv 1 \bmod 8$. Then $C_{\bar{\beta}_1}, X \in C_{\bar{\beta}_1}^G$, where $G_{C_{\bar{\beta}_1}} \cong \mathsf{PSL}(2,\sqrt{q})$, since T_1 fixes exactly two points in G-orbit of type (7a). In particular, $G_{C_{\bar{\beta}_1}} = G_X$, since $G_{C_{\bar{\beta}_1}} \lhd \mathsf{PGL}(2,\sqrt{q}) < \mathsf{PSL}(2,q)$. Recall that $C = C_G(\sigma)$ and that $\bar{C} = C/K$, where K is the kernel of C on $\mathrm{Fix}(\sigma)$. Clearly, $C \cong D_{q-1}$ and $\bar{\beta}_1 \in \bar{C}$. Note that $C_X = C \cap G_X = C_{G_X}(G)$ and hence $C_X \cong D_{\sqrt{q}-1}$, since $G_X \cong \mathsf{PSL}(2,\sqrt{q})$. In particular, $C_X = C_{X,C_{\bar{\beta}_1}}$. Set $C_0 = C_{X,C_{\bar{\beta}_1}}$. Clearly, $K \trianglelefteq C_0$ and $\bar{\beta}_1 \in \bar{C}_0$, where $\bar{C}_0 = C_0/K$. Then X and $C_{\bar{\beta}_1}$ are the unique points on $\mathrm{Fix}(\sigma) \cap l$ fixed by $\bar{C}_0$, as $\bar{\beta}_1 \in \bar{C}_0$. Set $h = \left|\bar{C}_0\right|$. Then h is even, as $\bar{\beta}_1 \in \bar{C}_0$. If $\left|\bar{C}_0\right| > 4$, then $\bar{C}_0$ is dihedral and therefore exists a point $Y \in l$ such that $\bar{C}_Y = \bar{C}$ by Lemma 2.10. That is $C \leq G_Y$. Then $G_Y = C$, where $C = C_G(\sigma)$, since C is maximal in G as $q > 9$, and since $x_1 = 0$ by Proposition 4.7(1). Nevertheless, this is a contradiction, since $x_2 = 0$ by Lemma 4.5(2). Thus, $h = \left|\bar{C}_0\right| \leq 4$. Actually, either $h = 2$ or 4, since h is even. On the other hand, $|C_0| = h\,|K|$. Hence, $\frac{\sqrt{q}-1}{h} \mid |K|$, where $h = 2, 4$, since $C_0 \cong D_{\sqrt{q}-1}$ as $C_0 = C_{G_X}(G)$ and $G_X \cong \mathsf{PSL}(2,\sqrt{q})$.

Now, repeating the previous argument for $\sqrt{q} \equiv 7 \bmod 8$, we find that $C_0 \cong D_{\sqrt{q}+1}$ and hence $\frac{\sqrt{q}+1}{h} \mid |K|$, where $h = 2, 4$. So, we have proved the assertion (3). $\square$

Lemma 4.9. *If the p-elements are not planar, then $\mathcal{S}_1 = \mathcal{S}_4 = 2(\sqrt{q}+1)$.*

Proof. Let ρ_t, $t = 1, 2$, be the representatives the two conjugate classes of p-elements in G. Since $x_{7a}, x_{7b} > 0$, the collineation ρ_t fixes at least $2\sqrt{q}$ points on l for each $t = 1, 2$ by Table II. Then ρ_t must fixes at least $2\sqrt{q}$ lines on Π by [16, Theorem 13.3]. Since ρ_t cannot be planar, all these lines must concur to a unique point X_t of Π. It is a plain that X_1 and X_2 might coincide. Let S be the Sylow p-subgroup of G containing ρ_t for each $t = 1, 2$. Clearly, ρ_t fixes X_t^S and at least $2\sqrt{q}$ lines through each point of X_t^S, since S is abelian. Then $\left|X_t^S\right| = 1$, since ρ_t cannot be planar on Π by our assumption. Thus S fixes X_t for each $t = 1, 2$. Assume that $X_t \in \Pi - l$ for at least one $t = 1$ or 2. As $\mathcal{S}_4 > 0$ by Lemma 4.8(2), then there exists a point Y on l fixed by S. Hence S acts on $X_tY - \{X_t, Y\}$. Assume that $S_R \neq \langle 1 \rangle$ for some point $R \in X_tY - \{X_t, Y\}$. Let $\psi \in S_R$, $\psi \neq 1$. Clearly, ψ fixes Y and R on $\Pi - l$. Furthermore, ψ fixes at least $2\sqrt{q}$ points on l. Indeed, ψ conjugate either to ρ_1 or ρ_2 and each of these collineations fixes at least $2\sqrt{q}$ points on l as $x_{7a}, x_{7b} > 0$. So, ψ is planar. This is impossible by our assumption. Thus, S is semiregular on $X_tY - \{X_t, Y\}$. Hence, $q \mid n - 1$. That is $n = aq + 1$ for some positive integer a. On the other hand,

$$n + 1 = \frac{q+1}{2}\mathcal{S}_4 + x_7\sqrt{q}(q + 1) \tag{42}$$

by (40) of Lemma 4.8, since $x_9 = 0$ by Lemma 4.8(1). Since $\mathcal{S}_1$ is even, then $q + 1 \mid n + 1$ by (42). Then $q + 1 \mid a - 1$, since $n = aq + 1$. If $a = 1$, it follows that $n = q + 1$. As $q \equiv 1 \bmod 8$, we have that $n \equiv 2 \bmod 4$. Hence, we arrive at a contradiction by [16, Theorem 13.18], since $q > 3$. Thus, $a > 1$. Hence, $a = \theta(q + 1) + 1$, with $\theta \geq 1$. Therefore, $n = \theta q(q + 1) + q + 1$. This yields $n > q^2$, as $\theta \geq 1$, which is a contradiction. So, $X_t \in l$ for each $t = 1, 2$. Then $G_{X_t} \cong F_q.Z_{d_t}$ by Table I, since S fixes X_t, $|S| = q$, and since $x_1 = 0$ by Proposition 4.7(1). As a consequence, $x_{10} > 0$.

Note that σ normalizes S and it acts as the inversion on S. Thus, σ normalizes $\langle \rho_t \rangle$ for each $t = 1, 2$. If d_t is odd, then σ moves X_t. Then ρ_t fixes at least $2\sqrt{q}$ lines through X_t and at least other $2\sqrt{q}$ ones though $X_t\sigma$. As a consequence, ρ_t is planar on Π. Nevertheless, this contradicts our assumptions. So, d_t must be even. This implies $\mathcal{S}_{2'} = 0$. Therefore, $\mathcal{S}_1 = \mathcal{S}_4$ by Lemma 4.5(1). In particular, by (42), we have $n + 1 = \frac{q+1}{2}\mathcal{S}_4 + x_7\sqrt{q}(q + 1)$.

Let r_t be a line of $[X_t] - \{l\}$ fixed by ρ_t. Clearly each line of r_t^G intersect l in a (unique) point of X_t^G, where $G_{X_t} \cong F_q.Z_{d_t}$ and d_t is even. In particular each element in ρ_t^G fixes at least one line of r_t^G. Since the p-elements in G cannot be planar, then for each element τ in ρ_t^G actually there exists a unique point Q_τ in X_t^G such that each line of Π fixed by τ lies in $[Q_\tau]$. As a consequence, each p-element in G fixes a subset of a pencil of lines concurrent to a point lying either in X_1^G or in X_2^G. If $x_{10} > 2$, there exists a G-orbit of type (10), say Q^G, such that S is semiregular on $[Q] - \{l\}$. Thus $q \mid n$ and hence $n = bq$ where

$b \geq 1$. Then $q+1 \mid b-1$, since $q+1 \mid n+1$ arguing as above. As $n > q$ by our assumptions, then $b > 1$ and hence $b = f(q+1)+1$. Therefore, $n = fq(q+1)+q$, which is a contradiction, since $n < q^2$ by our assumption. Thus, $0 < x_{10} \leq 2$. That is $x_{10} = 1$ or 2.

Assume that $x_{10} = 1$. Then $X_1 = X_2$ and $d_1 = d_2$, since ρ_1 and ρ_2 lies in the same Sylow p-subgroup S of G. Set $X = X_1 = X_2$ and $d = d_1 = d_2$. Then $\mathcal{S}_4 = \frac{q-1}{d}$. Now, recall that $\frac{\sqrt{q}\mp 1}{h} \mid |K|$, $h = 2, 4$, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, by Lemma 4.8(3). Thus, $\frac{\sqrt{q}\mp 1}{h} \mid |C_X|$, $h = 2, 4$, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, since $K \leq C_X$. This fact, in conjunction with the fact that $G_X \cong F_q.Z_d$, yields $C_X \cong Z_d$, where $d = \frac{\sqrt{q}\mp 1}{h} u$, $h = 2, 4$, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively. Here u is a positive divisor of d. So, $\mathcal{S}_4 = h\frac{\sqrt{q}\pm 1}{u}$ for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, since $\mathcal{S}_4 = \frac{q-1}{d}$. Now, let P be a point of l such that $G_P \cong \mathrm{PSL}(2, \sqrt{q})$ and let $S_P = S \cap G_P$. Then S_P must be semiregular on $[P]$, since $P \notin X^G$, since the lines fixed by any non trivial element in S lie in X^G and since S does not contain planar elements. Hence, $\sqrt{q} \mid n$ as $|S_P| = \sqrt{q}$. Then $\sqrt{q} \mid \frac{1}{2}\mathcal{S}_4 - 1$ by (42). That is $\sqrt{q} \mid h\frac{\sqrt{q}\pm 1}{2u} - 1$ for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively. So, there exists a positive integer x such that $x\sqrt{q} = h\frac{\sqrt{q}\pm 1}{2u} - 1$ for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively. Since $\sqrt{q} > 3$ by our assumption, there are no admissible solutions of the Diophantine equation $x\sqrt{q} = h\frac{\sqrt{q}-1}{2u} - 1$ by Lemma 2.7(1). Hence $x\sqrt{q} = h\frac{\sqrt{q}+1}{2u} - 1$ and $\sqrt{q} \equiv 1 \mod 8$. Then $(x, h, u, \sqrt{q}) = (1, 2, 1, \sqrt{q})$ by Lemma 2.7(2). Therefore, $\mathcal{S}_4 = 2(\sqrt{q}+1)$. Since $\mathcal{S}_1 = \mathcal{S}_4$, we have the assertion.

Assume that $x_{10} = 2$. Then $\mathcal{S}_4 = \frac{q-1}{d_1} + \frac{q-1}{d_2}$ and $X_1 \neq X_2$ for $x_{10} = 2$, since $\mathcal{S}_1 = \mathcal{S}_4$. Now, recall that $\frac{\sqrt{q}\mp 1}{h} \mid |K|$, $h = 2, 4$, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, by Lemma 4.8(3). Thus, $\frac{\sqrt{q}\mp 1}{h} \mid |C_{X_t}|$, $h = 2, 4$, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, since $K \leq C_{X_t}$ for each $t = 1, 2$. On the other hand, $C_{X_t} \cong Z_{d_t}$, since $G_{X_t} \cong F_q.Z_{d_t}$ for each $t = 1, 2$. So, $d_t = \frac{\sqrt{q}\mp 1}{h} u_t$, $h = 2, 4$, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively. Here, u_t is a positive divisor of d_t. Then $\mathcal{S}_4 = h\frac{\sqrt{q}\mp 1}{u_1} + h\frac{\sqrt{q}\mp 1}{u_2}$, where $h = 2$ or 4, for $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, since $\mathcal{S}_4 = \frac{q-1}{d_1} + \frac{q-1}{d_2}$. Arguing as above, we have $\sqrt{q} \mid \frac{1}{2}\mathcal{S}_4 - 1$ by (42). Thus, $\sqrt{q} \mid h\frac{\sqrt{q}\mp 1}{2u_1} + h\frac{\sqrt{q}\mp 1}{2u_2} - 1$, where $h = 2$ or 4 or $\sqrt{q} \equiv \pm 1 \mod 8$, respectively, since $\mathcal{S}_4 = h\frac{\sqrt{q}\mp 1}{u_1} + h\frac{\sqrt{q}\mp 1}{u_2}$. Then there exists a positive integer x such that $x\sqrt{q} = h\frac{\sqrt{q}\mp 1}{2u_1} + h\frac{\sqrt{q}\mp 1}{2u_2} - 1$, where $h = 2$ or 4 or $\sqrt{q} \equiv \pm 1 \mod 8$, respectively (in particular, $\mathcal{S}_4 = 2(x\sqrt{q}+1)$). Then $x = 1$ in any case by Lemma 2.8 and 2.9, since $\sqrt{q} \equiv \pm 1 \mod 8$. Therefore, $\mathcal{S}_4 = 2(\sqrt{q}+1)$. Since $\mathcal{S}_1 = \mathcal{S}_4$, we have the assertion. □

Lemma 4.10. *The group T_j induces a homology on* $\mathrm{Fix}(\sigma)$ *for each $j = 1$ or 2.*

Proof. The group T_j induces a homology on $\mathrm{Fix}(\sigma)$ for either $j = 1$ or $j = 2$

by Proposition 4.7. Assume that T_1 does it. We may also assume that $x_{7a} = 1$. Assume also that T_2 induces a Baer collineation on $\mathrm{Fix}(\sigma)$. Then by Table I in conjunction with Lemmas 4.5, 4.8 and Proposition 4.7, we have the following system of Diophantine equations:

$$\sqrt[4]{n} + 1 = 2x_{7b} \tag{43}$$

$$\sqrt{n} + 1 = \mathcal{S}_4 + x_7(\sqrt{q} \pm 1) \tag{44}$$

$$n + 1 = \frac{q+1}{2}\mathcal{S}_1 + x_7\sqrt{q}(q+1)\,. \tag{45}$$

By substituting $x_7 = 1 + x_{7b}$ in (43), we have that $x_7 = \frac{\sqrt[4]{n}+3}{2}$. Now, by substituting this value of x_7 in and (44), we obtain $\sqrt{n} + 1 = \mathcal{S}_4 + (\sqrt[4]{n} + 3)\frac{\sqrt{q}\pm 1}{2}$. That is

$$\sqrt{n} - 9 = (\mathcal{S}_4 - 10) + (\sqrt[4]{n} + 3)\frac{\sqrt{q} \pm 1}{2}\,. \tag{46}$$

Assume that $\mathcal{S}_4 = 10$. Then $\sqrt[4]{n} = 3 + \frac{\sqrt{q}\pm 1}{2}$ by (46). As $x_7 > 0$, being $x_7 = 1 + x_{7b}$, then $q \equiv 1 \bmod 16$ by Lemma 4.5(3). This yields $\sqrt{q} \equiv 1, 7 \bmod 8$. So, $\sqrt{n} \equiv 3 \bmod 4$, which is a contradiction by Lemma 3.3. Hence, $\mathcal{S}_4 \neq 10$. Nevertheless, $\sqrt[4]{n} + 3 \mid \mathcal{S}_4 - 10$ again by (46).

Assume that $\mathcal{S}_4 < 10$, hen $\sqrt[4]{n} + 3 \mid 10 - \mathcal{S}_4$. As $\sqrt[4]{n} \geq 2$, then $\sqrt[4]{n} + 3 \geq 5$ and therefore $10 - \mathcal{S}_4 \geq 5$. That is $\mathcal{S}_4 \leq 5$. Then $\mathcal{S}_4 = 2$ or 4, since $\mathcal{S}_4$ is even by its definition and since $\mathcal{S}_4 > 0$ by Lemma 4.8(2). Assume that $\mathcal{S}_4 = 4$. It follows that $\sqrt{n} = 3 + x_7(\sqrt{q} \pm 1)$. As $x_7 > 0$, then $\sqrt{q} \equiv \pm 1 \bmod 8$ by the above argument. Then $\sqrt{n} \equiv 3 \bmod 4$, as $\sqrt{n} = 3 + x_7(\sqrt{q} \pm 1)$. Nevertheless, this contradicts Lemma 3.3. Therefore, $\mathcal{S}_4 = 2$. Hence, $\sqrt[4]{n}+3 \mid 8$, as $\sqrt[4]{n}+3 \mid \mathcal{S}_4 - 10$. Then $\sqrt[4]{n} + 3 = 8$, since $\sqrt[4]{n} \geq 2$. As a consequence, $\sqrt[4]{n} = 5$. This yields $x_7 = 4$, as $x_7 = \frac{\sqrt[4]{n}+3}{2}$. Thus, $4(\sqrt{q} \pm 1) = 24$ by (44), as $\mathcal{S}_4 = 2$. So, $\sqrt{q} \pm 1 = 6$. On the other hand, $\sqrt{q} \pm 1 \equiv 0 \bmod 8$ by the previous argument, as $x_7 > 0$. Nevertheless, this contradicts $\sqrt{q} \pm 1 = 6$.

Assume that $\mathcal{S}_4 > 10$. Then $\mathcal{S}_4 = \theta(\sqrt[4]{n} + 3) + 10$ with $\theta \geq 1$. Assume θ is odd. Then $\theta(\sqrt[4]{n} + 3) = \mathcal{S}_4 - 10$ and hence $\sqrt[4]{n} - 3 = \theta + \frac{\sqrt{q}\pm 1}{2}$. Note that $\sqrt[4]{n} = (3 + \theta) + \frac{\sqrt{q}\pm 1}{2}$ is even, as θ is odd. Thus $\sqrt{n}$ is even, which is a contradiction, since T_1 induces a homology on $\mathrm{Fix}(\sigma)$. Then θ is even and hence $\theta \geq 2$, as $\theta \geq 1$. Since $\sqrt[4]{n} + 1 = 2x_{7b}$, then $\mathcal{S}_4 > 4x_{7b}$.

Let ρ_t, $t = 1$ or 2, be the representative of the two conjugate classes p-elements in G. Suppose that ρ_t is planar for either $t = 1$ or $t = 2$. Then $o(\mathrm{Fix}(\rho_t)) + 1 = \frac{1}{2}\mathcal{S}_1 + x_{7v}2\sqrt{q}$ by Table II, where $v = a$ for $t = 1$ and $v = b$ for $t = 2$, since $x_1 = 0$ by Proposition 4.7, since $x_8 = 0$ by Lemma 4.5(2) and since $x_9 = 0$ by Lemma 4.8(1). Clearly, σ acts on $\mathrm{Fix}(\rho_t)$, since σ inverts ρ_t. Furthermore, it follows from Table II that $|\mathrm{Fix}(\langle \rho_t, \sigma\rangle) \cap l| = \frac{1}{2}\mathcal{S}_4 + x_{7v}\varepsilon$,

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where ε is either 2 or 0 according to whether $\sqrt{q} \equiv 1 \bmod 4$ or $\sqrt{q} \equiv 3 \bmod 4$, respectively. Since $\mathcal{S}_4 > 10$, then $|\mathrm{Fix}(\langle \rho_t, \sigma \rangle) \cap l| > 3$. On the other hand, $|\mathrm{Fix}(\langle \rho_t, \sigma \rangle) \cap l| < \frac{1}{2}\mathcal{S}_1 + x_{7v} 2\sqrt{q}$. So, $\langle \rho_t, \sigma \rangle$ induces a Baer collineation on $\mathrm{Fix}(\rho_t)$. Therefore, $\langle \rho_t, \sigma \rangle$ is planar. In particular, $\mathrm{Fix}(\langle \rho_t, \sigma \rangle)$ is a subplane of $\mathrm{Fix}(\sigma)$ of order $\frac{1}{2}\mathcal{S}_4 + x_{7v}\varepsilon - 1$, where $v = a$ for $t = 1$ and $v = b$ for $t = 2$. Then $\frac{1}{2}\mathcal{S}_4 + x_{7v}\varepsilon \le \sqrt[4]{n} + 1$ by [16, Theorem 3.7], since $\mathrm{Fix}(\sigma)$ is a Baer subplane of Π. This yields $\frac{1}{2}\mathcal{S}_4 + x_{7v}\varepsilon \le 2x_{7b}$ by (43). In particular, $\frac{1}{2}\mathcal{S}_4 \le 2x_{7b}$ and hence $\mathcal{S}_4 \le 4x_{7b}$. Hence, we arrive at a contradiction, since $\mathcal{S}_4 > 4x_{7b}$ by the above argument. Thus, G cannot contain p-planar elements. Then $\mathcal{S}_1 = \mathcal{S}_4 = 2(\sqrt{q}+1)$ by Lemma 4.9. This yields $x_7 = \frac{\sqrt{n}-1}{\sqrt{q}-1} - 2$ by (44). By substituting these values of $\mathcal{S}_4$ and x_7 in (45), we have

$$n + 1 = (q + 1)(\sqrt{q} + 1) + \left(\frac{\sqrt{n} - 1}{\sqrt{q} - 1} - 2 \right) \sqrt{q}(q + 1)\,.$$

By elementary calculations of this one, we have

$$n - q = \frac{\sqrt{n} - \sqrt{q}}{\sqrt{q} - 1} \sqrt{q}(q + 1)\,.$$

Thus, $\sqrt{n} + \sqrt{q} \ge q + 1$ and $\sqrt{n} \ge q - \sqrt{q} + 1$. On the other hand, $\sqrt{n} \le (\sqrt{q} - 1)^2$ since $n < q^2$, q is a square and n is a fourth power, since T_2 induces a Baer collineation on $\mathrm{Fix}(\sigma)$ by our assumption. So, we obtain a contradiction, since $q - \sqrt{q} + 1 > (\sqrt{q} - 1)^2$. Hence, T_j induces a homology on $\mathrm{Fix}(\sigma)$ for each $j = 1$ or 2. □

Proposition 4.11. *The collineation γ induces a Baer collineation on $\mathrm{Fix}(\sigma)$ and hence $K \trianglelefteq Z_{\frac{q-1}{4}}$.*

Proof. The group T_j induces a homology on $\mathrm{Fix}(\sigma)$ for each $j = 1$ or 2 by Lemma 4.10. Then $x_{7a} = x_{7b} = 1$ by Table I, since $x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0$ by Lemma 4.5(2), since $x_1 = 0$ by Proposition 4.7(1), and since $x_9 = 0$ by Lemma 4.8(1). Therefore, $x_7 = 2$. Then we obtain the following system of Diophantine equations:

$$\sqrt{n} + 1 = \mathcal{S}_4 + 2(\sqrt{q} \pm 1) \qquad (47)$$

$$n + 1 = \frac{q + 1}{2}\mathcal{S}_1 + 2\sqrt{q}(q + 1)\,. \qquad (48)$$

Since $\mathcal{S}_1 \ge \mathcal{S}_4$, then $n + 1 \ge \frac{q+1}{2}\mathcal{S}_4 + 2\sqrt{q}(q + 1)$ by (48). Now, composing this inequality with (47), we obtain

$$n + 1 \ge \frac{q + 1}{2}(\sqrt{n} + 1) + (q + 1)(\sqrt{q} \mp 1) \qquad (49)$$

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and hence $n+1 > \left[\frac{q+1}{2}+\sqrt{q}\mp 1\right](\sqrt{n}+1)$, since $\sqrt{n}+1 < q+1$. As $\sqrt{n}\geq 2$, we $\sqrt{n}+1 \nmid n+1$. Furthermore, $\sqrt{n}+1 \nmid n$. So, $n-1 \geq \left[\frac{q+1}{2}+\sqrt{q}\mp 1\right](\sqrt{n}+1)$. Dividing each term by $\sqrt{n}+1$ in the previous inequality, we obtain $\sqrt{n}-1 > \frac{q+1}{2}+\sqrt{q}\pm 1$. This implies $\sqrt{n}-1 > \frac{q-1}{2}+\sqrt{q}\pm 1$ and therefore

$$\sqrt{n}+1 > \left(\sqrt{q}\pm 1\right)\left(\frac{\sqrt{q}\mp 1}{2}+1\right). \tag{50}$$

Let ρ_t, $t=1$ or 2, be the representatives of the two conjugate classes p-elements in G. Suppose that ρ_t is planar. Then $o(\mathrm{Fix}(\rho_t))+1 = \frac{1}{2}\mathcal{S}_1+2\sqrt{q}$ by Table II, since $x_{7a}=x_{7b}=1$. Clearly, σ acts on $\mathrm{Fix}(\rho_t)$, since σ inverts ρ_t. Again by Table II, we have $|\mathrm{Fix}(\langle\rho_t,\sigma\rangle)\cap l| = \frac{1}{2}\mathcal{S}_4+\varepsilon$, where ε is either 2 or 0 according to whether $\sqrt{q}\equiv 1 \bmod 4$ or $\sqrt{q}\equiv 3 \bmod 4$, respectively.

Assume that $\frac{\sqrt{q}\mp 1}{2}\geq 5$. Then $\sqrt{n}+1 > 6\left(\sqrt{q}\pm 1\right)$ and hence $\mathcal{S}_4 > 4\left(\sqrt{q}\pm 1\right)$ by (47). In particular, $\mathcal{S}_4 > 8$. Then $|\mathrm{Fix}(\langle\rho_t,\sigma\rangle)\cap l| > 3$. On the other hand, $|\mathrm{Fix}(\langle\rho_t,\sigma\rangle)\cap l| < \frac{1}{2}\mathcal{S}_1+2\sqrt{q}$. Hence, $\langle\rho_t,\sigma\rangle$ induces a Baer collineation on $\mathrm{Fix}(\rho_t)$. Then $\left(\frac{1}{2}\mathcal{S}_4+\varepsilon-1\right)^2 \leq \frac{1}{2}\mathcal{S}_1+2\sqrt{q}$ by [16, Theorem 3.7]. Note that $(\frac{1}{2}\mathcal{S}_4+\varepsilon-1)^2 > \mathcal{S}_4$, as $\mathcal{S}_4 > 8$. So, $\mathcal{S}_4 < \frac{1}{2}\mathcal{S}_4+2\sqrt{q}-1$. Hence, $\mathcal{S}_4 < 4\sqrt{q}-2$. On the other hand, we proved $\mathcal{S}_4 > 4\left(\sqrt{q}\pm 1\right)$. Combining these two inequalities involving $\mathcal{S}_4$, we obtain $\sqrt{q}\equiv 3 \bmod 4$ and $4\sqrt{q}-2 > \mathcal{S}_4 > 4\left(\sqrt{q}-1\right)$. Therefore, $\mathcal{S}_4 = 4\sqrt{q}-3$, which is a contradiction, since $\mathcal{S}_4$ must be even.

Assume that $\frac{\sqrt{q}\mp 1}{2}\leq 4$. Recall that the *upper* sign if $\sqrt{q}\equiv 1 \bmod 4$ and the *lower* sign if $\sqrt{q}\equiv 3 \bmod 4$ This yields $q=25$ or 49, since q is odd and $q>9$. Actually, only the case $q=49$ is admissible, since $q\equiv 1 \bmod 16$ by Lemma 4.5(3), being $x_7>0$. Now, by substituting $q=49$ in (49), we have $\sqrt{n}\geq 35$. Hence $35\leq\sqrt{n}<49$, since $\sqrt{n}<\sqrt{q}$ by our assumptions. Furthermore, $\frac{q+1}{2}\mid n+1$ by (48). That is $25\mid n+1$, since $q=49$. Now, filtering the list $35\leq\sqrt{n}<49$ with respect to the conditions $25\mid n+1$, and $\sqrt{n}$ odd, as the T_j induces a homology on $\mathrm{Fix}(\sigma)$ for each $j=1,2$, we obtain $\sqrt{n}=43$. Nevertheless, this contradicts Lemma 3.3. As a consequence, the p-elements in G cannot be planar. Then $\mathcal{S}_4 = 2(\sqrt{q}+1)$ by Lemma 4.9. This is still a contradiction, since $\mathcal{S}_4 > 4\left(\sqrt{q}\pm 1\right)$ by the above argument, being $q>9$. □

4.2. The collineation γ induces a Baer involution on $\mathrm{Fix}(\sigma)$

Proposition 4.12. *The group T_j induces a Baer collineation on $\mathrm{Fix}(\sigma)$ for each $j=1,2$.*

Proof. Recall that $C=C_G(\sigma)$ and let K and K^* be the kernels of the action of C on $\mathrm{Fix}(\sigma)$ and on $\mathrm{Fix}(\sigma)\cap l$, respectively. In particular, $K\trianglelefteq Z_{\frac{q-1}{4}}$, since

γ induces a Baer involution $\bar{\gamma}$ on $\mathrm{Fix}(\sigma)$ by Proposition 4.11. Let $\bar{\beta}_j$ be the involution induced on $\mathrm{Fix}(\sigma)$ by a Klein subgroup T_j containing σ (and hence lying in C) for $j = 1, 2$.

Suppose that $K \cong Z_{\frac{q-1}{4}}$. Assume also that $\bar{\beta}_j$ is an involutory $(C_{\bar{\beta}_j}, a_{\bar{\beta}_j})$-perspectivity. Then $C_{\bar{\beta}_j} \in l \cap \mathrm{Fix}(\sigma)$ and $a_{\bar{\beta}_j} \neq l$ by Lemma 4.2. Thus, $K \leq G_{C_{\bar{\beta}_j}}$. This implies $C \leq G_{C_{\bar{\beta}_j}}$, since the collineation $\bar{\gamma}$ fixes $C_{\bar{\beta}_j}$ as $\bar{\gamma}$ centralizes $\bar{\beta}_j$ and since $K \cong Z_{\frac{q-1}{4}}$. Note that $N_G(T_j) \cap C \cong D_8$, where $N_G(T_j) \cong S_4$. Then $N_G(T_j) \leq G_{C_{\bar{\beta}_j}}$, since $|\mathrm{Fix}(T_j) \cap l| = 1$ or 2, since $C_{\bar{\beta}_j} \in \mathrm{Fix}(T_j) \cap l$ and since $C \leq G_{C_{\bar{\beta}_j}}$. So, G fixes $G_{C_{\bar{\beta}_j}}$, since $\langle C, N_G(T_j) \rangle \leq G_{C_{\bar{\beta}_j}}$ and $G = \langle C, N_G(T_j) \rangle$. Hence, we arrive at a contradiction by dual of Lemma 4.2, since $\mathrm{Fix}(T_j) \cap \left[G_{C_{\bar{\beta}_j}} \right] = \mathrm{Fix}(\sigma) \cap \left[G_{C_{\bar{\beta}_j}} \right]$.

Suppose that $K < Z_{\frac{q-1}{4}}$. Then $C/K \cong D_{2m}$ with $m \equiv 0 \bmod 4$, as $q \equiv 1 \bmod 8$. If n is odd, then each involution induced on $\mathrm{Fix}(\sigma)$ by a Klein subgroup T_j containing σ is a Baer involution by [19, Proposition 3.3], since $\bar{\gamma}$ is a Baer involution of $\mathrm{Fix}(\sigma)$. Thus, we have proved the assertion for n odd.

Assume that n is even. Assume also that $\bar{\beta}_1$ is an involutory $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-elation of $\mathrm{Fix}(\sigma)$. As $\mathrm{Fix}(T_1) \cap l = \left\{ C_{\bar{\beta}_1} \right\}$, then $N_G(T_1) \leq G_{C_{\bar{\beta}_1}}$, where $N_G(T_1) \cong S_4$ as $q \equiv 1 \bmod 8$. Clearly, $G_{C_{\bar{\beta}_1}} < G$, otherwise, we would have a contradiction by the above argument. Then either $G_{C_{\bar{\beta}_1}} \cong S_4$ and $q \equiv 9 \bmod 16$ or $G_{C_{\bar{\beta}_1}} \cong \mathsf{PGL}(2, \sqrt{q})$ by Table I, since $|\mathrm{Fix}(T_1) \cap l| = 1$. If $G_{C_{\bar{\beta}_1}} \cong S_4$, then $q = 25$ or 41 by Lemma 3.4, since $q \equiv 9 \bmod 16$. So, $\left| C_{\bar{\beta}_1}^G \right| = \frac{q(q^2-1)}{48}$. Then $\frac{q(q^2-1)}{48} \leq n+1 < q^2+1$, since $C_{\bar{\beta}}^G \subseteq l$. Furthermore, n is a fourth power by Proposition 4.11, and n is even. This is a contradiction, since $q = 25$ or 41. Thus, $G_{C_{\bar{\beta}}} \cong \mathsf{PGL}(2, \sqrt{q})$. Now, since $|\mathrm{Fix}(\gamma) \cap l| = \sqrt[4]{n} + 1$ and $|\mathrm{Fix}(T_1) \cap l| = 1$, then

$$\sqrt[4]{n} + 1 = x_8 \frac{1}{2}(\sqrt{q} \pm 1) + \mathcal{S}_4\,, \tag{51}$$

where $x_{8a} = 1$. If $x_8 \geq 2$, then $\sqrt[4]{n} + 1 \geq \sqrt{q} \pm 1$. Then $\sqrt{q} \equiv 3 \bmod 4$ and hence $\sqrt[4]{n} \geq \sqrt{q} - 2$, since $\sqrt[4]{n} < \sqrt{q}$. Actually, $\sqrt[4]{n} = \sqrt{q} - 1$, since $\sqrt[4]{n}$ is even as $\bar{\beta}_1$ is an involutory $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-elation of $\mathrm{Fix}(\sigma)$. Therefore, $\sqrt[4]{n} \equiv 2 \bmod 4$ as $\sqrt{q} \equiv 3 \bmod 4$. Then $\sqrt[4]{n} = 2$ by [16, Theorem 13.18], since $\bar{\beta}_1$ acts non trivially on $\mathrm{Fix}(\bar{\gamma})$. As a consequence, $\sqrt{q} = 3$. Nevertheless, this is a contradiction, since $q > 9$ by our assumptions. Then $x_8 = x_{8a} = 1$ and hence $|\mathrm{Fix}(T_2) \cap l| = 3$ by Table I. Thus, T_2 induces a Baer collineation on $\mathrm{Fix}(\sigma)$. Therefore, $\sqrt[4]{n} + 1 = 3$. Now, by substituting $x_8 = 1$ and $\sqrt[4]{n} = 2$ in (51), we obtain $\frac{1}{2}(\sqrt{q} \pm 1) + \mathcal{S}_4 = 3$. As a consequence, $\mathcal{S}_4 = 0$ or $\mathcal{S}_4 = 2$, since $\mathcal{S}_4$ is even. If the latter occurs, then $\sqrt{q} \pm 1 = 2$. Nevertheless, we again obtain a contradiction, since $q > 9$. So, $\mathcal{S}_4 = 0$ and $\sqrt{q} \pm 1 = 6$. Consequently, $\sqrt{q} = 5$ or 7 and $n = 2^4$, which is a contradiction, since $q < n$ by our assumptions. Hence, we have proved the assertion when the order n of Π is even. This completes the proof. $\square$

Lemma 4.13. *The following occur:*

(1) $x_4 = x_5 = x_6 = 0$;

(2) *if* $x_7 > 0$, *then* $q \equiv 9 \bmod 16$;

(3) $x_8 \leq 1$;

(4) *if* $x_9 > 0$, *then* $p^e \equiv 3 \bmod 4$.

Proof. Recall that γ induces a Baer collineation of $\mathrm{Fix}(\sigma)$ by Proposition 4.11. Therefore, n is a fourth power. Clearly, $\sqrt[4]{n} \geq 2$.

(1) Note that $q \geq 17$, since $q \equiv 1 \bmod 8$ and $q > 9$. Then $\sqrt[4]{n} > 2$, since $q < n$ by our assumption. Assume that $\sqrt[4]{n} = 3$ or 7. Thus, $\mathrm{Fix}(\sigma)$ has order 9 or 49, respectively. Furthermore, the group induced by $C_G(\sigma)$ on $\mathrm{Fix}(\sigma)$ has order divisible by 4 and each its involution is Baer collineation of $\mathrm{Fix}(\sigma)$ by Propositions 4.11 and 4.12. Nevertheless, this is a contradiction by Theorem 2.6, since $\sqrt[4]{n} \equiv 3 \bmod 4$. Thus $\sqrt[4]{n} \geq 4$ and $\sqrt[4]{n} \neq 7$. Moreover, the case $\sqrt[4]{n} \neq 6$ by [16, Theorem 3.6]. Hence, $\sqrt[4]{n} \geq 4$ and $\sqrt[4]{n} \neq 6, 7$. Then $q > 17$, since $q < n < q^2$. Therefore, $x_4 = 0$ by Lemma 3.4, since $q \equiv 1 \bmod 8$.
Assume that $x_6 > 0$. Then $q = 25$ or 41 by Lemma 3.4(4), since $q \equiv 1 \bmod 8$ and $q \neq 17$. Then $\sqrt[4]{n} = 4$, since $q < n < q^2$, since n is a fourth power with $\sqrt[4]{n} \geq 4$ and $\sqrt[4]{n} \neq 6$. On the other hand, $n + 1 \geq \frac{q(q^2-1)}{48}$ by Table I, as $x_6 > 0$. Thus, either $n \geq 325$ or $n \geq 1435$ according to whether $q = 25$ or 41, respectively. This is impossible, since $n = 4^4$. So, $x_6 = 0$.
Assume that $x_5 > 0$. Then $n+1 \geq \frac{q(q^2-1)}{120}$ by Table I. Hence, $\frac{q(q^2-1)}{120} x_5 - 1 \leq n < q^2$. Furthermore, $q = 25, 41, 49, 81$ or 89 by Lemma 3.4(5), since $q \equiv 1 \bmod 8$. In addition, n is a fourth power with $\sqrt[4]{n} \geq 4$ and $\sqrt[4]{n} \neq 6, 7$ by the above argument. Thus, $(q, n) = (25, 4^4)$ or $(41, 5^4)$ or $(89, 9^4)$. Moreover, $x_5 = 1$ in each of these cases.
Assume that $(q, n) = (41, 5^4)$. Let $S \cong Z_{41}$ which is normalized by σ. Since $n = 5^4$, then $n + 1 \equiv 11 \bmod 41$ and $n^2 \equiv 10 \bmod 41$. Hence, S is planar. In particular, $o(\mathrm{Fix}(S)) = 10 + \theta 41$, where $\theta \geq 0$. Actually, $\theta = 0$ by [16, Theorem 3.7], since $n = 5^4$. Therefore, $o(\mathrm{Fix}(S)) = 10$. Since σ normalizes S, it acts on $\mathrm{Fix}(S)$. Note that σ must act trivially on $\mathrm{Fix}(S)$, otherwise we would have a contradiction by [16, Theorem 13.18]. Thus, $\mathrm{Fix}(S) \subset \mathrm{Fix}(\sigma)$. So, we arrive at a contradiction by [16, Theorem 3.7], since $o(\mathrm{Fix}(S)) = 10$, while $o(\mathrm{Fix}(\sigma)) = 25$.
Assume that $(q, n) = (89, 9^4)$. Let $U \leq G$ such that $U \cong Z_{89}$. Since $n = 9^4$, then $n + 1 \equiv 65 \bmod 89$ and $n^2 \equiv 64 \bmod 89$. Hence, U is planar. In

particular, $o(\mathrm{Fix}(U)) = 64 + \lambda 89$, where $\lambda \geq 0$. Actually, $\lambda = 0$ by [16, Theorem 3.7], since $n = 9^4$. Thus, $o(\mathrm{Fix}(U)) = 64$. Let $V \leq N_G(U)$ such that $V \cong Z_{11}$. Clearly, V acts on $\mathrm{Fix}(U)$. Since $65 \equiv 10 \bmod 11$ and $64^2 \equiv 4 \bmod 11$, then V fixes a subplane of $\mathrm{Fix}(U)$ of order 9 at least. Then $\mathrm{Fix}(U) \subseteq \mathrm{Fix}(V)$, otherwise we would have a contradiction by [16, Theorem 3.7], since $o(\mathrm{Fix}(U)) = 64$. If $\mathrm{Fix}(U) \subset \mathrm{Fix}(V)$, we obtain a contradiction by [16, Theorem 3.7], since $o(\mathrm{Fix}(U)) = 64$, while $o(\mathrm{Fix}(V)) \leq 81$ as $n = 9^4$. Then $\mathrm{Fix}(U) = \mathrm{Fix}(V)$ and hence $o(\mathrm{Fix}(V)) = 64$. Clearly, $V \cong Z_{11}$ must be semiregular on $l - \mathrm{Fix}(V)$. So, $11 \mid |l - \mathrm{Fix}(V)|$. This is a contradiction, since $|l - \mathrm{Fix}(V)| = 6497$, as $n = 9^4$.

Assume that $(q, n) = (25, 4^4)$ and $x_5 = 1$. Let us focus on the action of the involution σ of Π. Clearly, σ fixes exactly 17 points on l, since it induces a Baer collineation on Π and $n = 4^4$. Let X^G the orbit of type (5). Then $|\mathrm{Fix}_{X^G}(\sigma)| = 6$ by Table I. Hence, σ fixes exactly 11 points on $l - X^G$. If Y^G is a on orbit on l of type (3), then $Y^G \subseteq l - X^G$. Furthermore, $|\mathrm{Fix}_{Y^G}(\sigma)| = 12$ again by Table I. Nevertheless, this is a contradiction. Thus, $x_3 = 0$. Then each admissible non trivial G-orbit on l has length divisible by 13. Indeed, one can compute each length orbit on l using Table I for $q = 25$. Therefore, $13 \mid |l - (l \cap \mathrm{Fix}(G))|$. That is $13 \mid n - x_1$, since $|l \cap \mathrm{Fix}(G)| = x_1$. Then $x_1 \geq 10$, since $n = 256$ and $257 \equiv 10 \bmod 13$. So, γ, where $\gamma^2 = \sigma$, fixes at least 10 points on l. This contradicts the facts that γ fixes exactly 5 points on l by Lemma 4.11, being $n = 4^4$.

(2) Assume that $x_7 > 0$ and $q \equiv 1 \bmod 16$. Then, by Table II, the collineation γ fixes at least $x_7(\sqrt{q} \pm 1)$ points on $l \cap \mathrm{Fix}(\sigma)$ according to whether $\sqrt{q} \equiv 1 \bmod 4$ or $\sqrt{q} \equiv 3 \bmod 4$, respectively. Then $\sqrt[4]{n} + 1 \geq x_7(\sqrt{q} \pm 1)$. On the other hand, $\sqrt[4]{n} < \sqrt{q}$ by our assumption. Hence, $\sqrt[4]{n} + 1 \leq \sqrt{q}$. By composing, we have $x_7(\sqrt{q} \pm 1) \leq \sqrt{q}$. Actually, $x_7(\sqrt{q} \pm 1) \leq \sqrt{q} - 1$. So, $x_7 = 1$ and $\sqrt{q} \equiv 3 \bmod 4$. Therefore, $\sqrt[4]{n} = \sqrt{q} - 2$. Let P^G the G-orbit of type (7). Note that $\mathrm{Fix}(\gamma) \cap l = \mathrm{Fix}_{P^G}(\gamma)$, since $\sqrt[4]{n} = \sqrt{q} - 2$. Hence, $x_1 = x_2 = x_8 = 0$ by Table I, being $x_4 = x_5 = x_6 = 0$ by part (1). This yields $\sqrt[4]{n} + 1 = 2x_7$ again by Table I, where $x_7 = 1$, since T_1 induces a Baer collineation on $\mathrm{Fix}(\sigma)$. Nevertheless, we again obtain a contradiction, since $\sqrt[4]{n} \geq 2$. Thus, we have proved the assertion (1).

(3) Assume that $x_8 \geq 2$. Then, by Table II, the collineation γ fixes at least $\sqrt{q} \pm 1$ points on $\mathrm{Fix}(\sigma) \cap l$ according to whether $\sqrt{q} \equiv 1 \bmod 4$ or $\sqrt{q} \equiv 3 \bmod 4$, respectively. Then $\sqrt[4]{n} + 1 \geq \sqrt{q} \pm 1$. If $\sqrt{q} \equiv 1 \bmod 4$, then $\sqrt[4]{n} + 1 \geq \sqrt{q} + 1$ and hence $\sqrt[4]{n} \geq \sqrt{q}$. Nevertheless, this contradicts our assumption. Hence, $\sqrt{q} \equiv 3 \bmod 4$, then $\sqrt[4]{n} \geq \sqrt{q} - 2$. Then either $\sqrt[4]{n} = \sqrt{q} - 1$ or $\sqrt[4]{n} = \sqrt{q} - 2$, since $\sqrt[4]{n} < \sqrt{q}$ and q is a square.

Assume that $\sqrt[4]{n} = \sqrt{q} - 1$. Then $\sqrt[4]{n} \equiv 2 \bmod 4$ as $\sqrt{q} \equiv 3 \bmod 4$. Let

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$C = C_G(\sigma)$ and recall that $K \trianglelefteq Z_{\frac{q-1}{4}}$, where K is the kernel of the action of C on $\mathrm{Fix}(\sigma)$. Thus, $4 \mid |\bar{C}|$, where $\bar{C} = C/K$. Note also that each involution in $\bar{C}$ is a Baer collineation of $\mathrm{Fix}(\sigma)$. Indeed, each involution in $\bar{C}$ is induced either by γ or by the T_j for each $j = 1, 2$, and all these ones are Baer collineations of $\mathrm{Fix}(\sigma)$ by Propositions 4.11 and 4.12, respectively. Nevertheless, this is impossible by Theorem 2.6.
Assume that $\sqrt[4]{n} = \sqrt{q} - 2$. Let P_1^G an P_2^G be two distinct orbits on l both of type (9). Since $\sqrt[4]{n} + 1 = \sqrt{q} - 1$, since $x_9 = 2$ and since $\left|\mathrm{Fix}_{P_1^G}(\gamma)\right| = \left|\mathrm{Fix}_{P_2^G}(\gamma)\right| = \frac{\sqrt{q}-1}{2}$, then $\mathrm{Fix}_{P_1^G}(\gamma) \cup \mathrm{Fix}_{P_2^G}(\gamma) = \mathrm{Fix}(\gamma) \cap l$. Thus, $x_1 = x_2 = 0$ by Table II. Note that $|\mathrm{Fix}(T_1) \cap l| = \sqrt[4]{n} + 1$. Also, $|\mathrm{Fix}(T_1) \cap l| = 2x_{7a} + 3x_{8a} + x_{8b}$ by Table I, since $x_1 = x_2 = 0$ by the previous argument, since $x_4 = x_5 = x_6 = 0$ by part (1), and being $\sqrt{q} \equiv 3 \bmod 4$. Hence $\sqrt[4]{n} + 1 = 2x_{7a} + 3x_{8a} + x_{8b}$. Arguing as above with T_2 in the role of T_1, we obtain $\sqrt[4]{n} + 1 = 2x_{7b} + x_{8a} + 3x_{8b}$ (see Table I). Summing up these two equations and bearing in mind that $x_7 = x_{7a} + x_{7b}$ and $x_8 = x_{8a} + x_{8b}$, we have $2(\sqrt[4]{n} + 1) = 2x_7 + 4x_8$. Hence, $\sqrt[4]{n} + 1 = x_7 + 2x_8$. As $x_8 = 2$, then $x_7 = \sqrt[4]{n} - 5$. That is $x_7 = \sqrt{q} - 7$, as $\sqrt[4]{n} = \sqrt{q} - 2$. On the other hand, we have $\sqrt{q}(q+1)x_7 + x_8\frac{\sqrt{q}(q+1)}{2} \leq n + 1$ again by Table I. That is $\sqrt{q}(q+1)(\sqrt{q} - 7) + \sqrt{q}(q+1) \leq (\sqrt{q} - 2)^4 + 1$, since $x_7 = \sqrt{q} - 7$, $x_8 = 2$ and $\sqrt[4]{n} = \sqrt{q} - 2$. Easy computations yield a contradiction, since $q > 9$. Thus, we have proved the assertion (3).

(4) Suppose that $p^e \equiv 1 \bmod 4$ and $x_9 > 0$, then γ fixes $\frac{q-1}{p^e-1}$ points on $\mathrm{Fix}(\sigma) \cap l$, where $q = p^{2ew}$, $w \geq 1$, by Table II and following remark. Then $\sqrt[4]{n} + 1 \geq \frac{p^{2we}-1}{p^e-1}$ and hence $\sqrt[4]{n} + 1 \geq p^{we} + 1$, as $\frac{p^{2we}-1}{p^e-1} \geq p^{we} + 1$. That is $\sqrt[4]{n} + 1 \geq \sqrt{q} + 1$. So, $n \geq q^2$. Therefore, we arrive at a contradiction by our assumption. Then the assertion (4) follows by Table I. □

Lemma 4.14. *It holds that* $\sqrt[4]{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8$.

Proof. Assume that $\sqrt{q} \equiv 1 \bmod 4$. Then $|\mathrm{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_{7a} + 3x_{8a} + x_{8b}$ by Table I, since $x_4 = x_5 = x_6 = 0$ by Lemma 4.13(1). Then $\sqrt[4]{n} + 1 = x_1 + 3x_2 + 2x_{7a} + 3x_{8a} + x_{8b}$, since T_1 induces a Baer involution on $\mathrm{Fix}(\sigma)$ by Proposition 4.12. Arguing as above with T_2 in the role of T_1, we have $|\mathrm{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_{7b} + x_{8a} + 3x_{8b}$ and hence $\sqrt[4]{n} + 1 = x_1 + 3x_2 + 2x_{7b} + x_{8a} + 3x_{8b}$. Summing up, the two relations involving $\sqrt[4]{n} + 1$, we obtain $\sqrt[4]{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8$, as $x_7 = x_{7a} + x_{7b}$ and $x_8 = x_{8a} + x_{8b}$.

Assume that $\sqrt{q} \equiv 3 \bmod 4$. Then, arguing as above, we obtain $|\mathrm{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_{7b} + x_{8a} + 3x_b$ and $|\mathrm{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_{7a} + 3x_{8a} + x_{8b}$. Thus, the role of T_1 and T_2, in term of fixed points, are exchanged. This yields $\sqrt[4]{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8$ as above. □

Let $H_j = \langle T_j, \gamma \rangle$ for each $j = 1, 2$. Clearly, $H_j \cong D_8$, since $\gamma^2 = \sigma$, $T_j \cong E_4$, $\sigma \in T_j$ and $H_j \leq C_G(\sigma)$. By [4], two cases arise:

(1) $q \equiv 1 \bmod 16$. In this case, H_1 and H_2 are the representative of the two distinct conjugate classes under G. Moreover, $N_G(H_j) \cong D_{16}$ for each $j = 1, 2$;

(2) $q \equiv 9 \bmod 16$. In this case, the dihedral subgroups of order 8 are Sylow 2-subgroups of G and hence they are conjugate. In particular, $H_1 = H_2$. Set $H = H_1$, then $N_G(H) = H \cong D_8$.

Lemma 4.15. *One of the following occurs:*

(1) $q \equiv 1 \bmod 16$ *and one of the following occurs:*

  (a) H_j *induces the identity on* $\mathrm{Fix}(\gamma)$ *for each* $j = 1, 2$*,* G *fixes a subplane of order* $\sqrt[4]{n}$ *and* $\mathcal{S}_4 = x_2 = x_7 = x_8 = 0$*;*

  (b) H_j *induces a perspectivity of axis* $\mathrm{Fix}(\gamma) \cap l$ *on* $\mathrm{Fix}(\gamma)$ *for each* $j = 1, 2$*. Furthermore,* $x_1 = \sqrt[4]{n} + 1$ *and* $x_2 = x_7 = x_8 = 0$*;*

  (c) H_j *induces a perspectivity on* $\mathrm{Fix}(\gamma)$ *of axis distinct from* $\mathrm{Fix}(\gamma) \cap l$ *for each* $j = 1, 2$*. In particular,* $x_1 + x_2 + x_8 = 1, 2$*;*

  (d) H_j *induces a Baer involution on* $\mathrm{Fix}(\gamma)$ *for each* $j = 1, 2$ *and hence* $x_1 + x_2 + x_8 = \sqrt[8]{n} + 1$.

(2) $q \equiv 9 \bmod 16$ *and one of the following occurs:*

  (a) H *induces the identity on* $\mathrm{Fix}(\gamma)$*,* G *fixes a subplane of order* $\sqrt[4]{n}$ *and* $\mathcal{S}_4 = x_2 = x_7 = x_8 = 0$*;*

  (b) H *induces a perspectivity on* $\mathrm{Fix}(\gamma)$ *of axis* $\mathrm{Fix}(\gamma) \cap l$*. Furthermore,* $x_1 = \sqrt[4]{n} + 1$ *and* $x_2 = x_7 = x_8 = 0$*;*

  (c) H *induces a perspectivity on* $\mathrm{Fix}(\gamma)$ *of axis distinct from* $\mathrm{Fix}(\gamma) \cap l$ *and* $x_1 + x_2 + x_8 = 1, 2$*;*

  (d) H *induces a Baer involution on* $\mathrm{Fix}(\gamma)$ *and hence* $x_1 + x_2 + x_8 = \sqrt[8]{n} + 1$.

Proof. Assume that $q \equiv 1 \bmod 16$. In this case, H_1 and H_2 are the representatives of the two distinct conjugate classes under G of dihedral subgroups of order 8. Moreover, $N_G(H_j) \cong D_{16}$ for each $j = 1, 2$. In particular, the unique G-orbits on l on which H fixes points are those of type (1),(2),(8) by Tables I and II, since $x_6 = x_7 = 0$ by Lemma 4.13(1) and (2) as $q \equiv 1 \bmod 16$. Clearly, $\left|\mathrm{Fix}_{Q^G}(H_j)\right| = 1$ if Q^G is of type (1). Since $H_j \leq C_G(\sigma)$ for each $j = 1, 2$, and since two subgroups of G isomorphic to D_8 are conjugate in G if they are conjugate $C_G(\sigma)$ by [4, §246], then, by Proposition 2.5, $\left|\mathrm{Fix}_{Q^G}(H_j)\right| = 1$ if Q^G is of type (2). Also, $\left|\mathrm{Fix}_{Q^G}(H_j)\right| = 1$ if Q^G is of type (8) by Proposition 2.5. Indeed, in this case, $G_Q \cong \mathsf{PGL}(2, \sqrt{q})$ and hence $\left|H_j^{G_Q}\right| = |G_Q|/16$ for each $j = 1, 2$ again by [4, §246]. Thus, $|\mathrm{Fix}(H_j) \cap l| = x_1 + x_2 + x_8$ for each $j = 1, 2$. Assume that $\mathrm{Fix}(H_j) \cap l = \mathrm{Fix}(\gamma) \cap l$ for each $j = 1, 2$. Then $x_1 + x_2 + x_8 = \sqrt[4]{n} + 1$ since γ induces a Baer collineation on $\mathrm{Fix}(\sigma)$ by Proposition 4.11. On the other hand, $\sqrt[4]{n} + 1 = x_1 + 3x_2 + 2x_8$ by Lemma 4.14 (note that $x_7 = 0$ by Lemma 4.13(2) as $q \equiv 1 \mod 16$). Hence $x_1 + x_2 + x_8 = x_1 + 3x_2 + 2x_8$. This yields $x_2 = x_8 = 0$ and $\sqrt[4]{n} + 1 = x_1$. Then we obtain the assertion (1a) or (1b) according to whether H_j induces the identity or a perspectivity of axis $\mathrm{Fix}(\gamma) \cap l$ on $\mathrm{Fix}(\gamma)$, respectively. At this point, the assertions (1b)–(1c) easily follow.

Assume that $q \equiv 9 \mod 16$. Then H is a Sylow 2-subgroup of G and $N_G(H) = H$. In particular, the unique G-orbits on l on which H fixes points are those of type (1),(2),(8) by Table I and II. Indeed, $x_6 = 0$ by Lemma 4.13(1). Also, $x_7 = 0$. Namely, if P^G is of type (7), we have $G_P \cong \mathsf{PSL}(2, \sqrt{q})$, where $\sqrt{q} \equiv 3, 5 \mod 8$, as $q \equiv 9 \mod 16$. Hence, $8 \nmid |G_P|$. Now, by Proposition 2.5, we obtain that H fixes 1 point for each G-orbit on l of type (1),(2) or (8), since H is a Sylow 2-subgroup of G and $N_G(H) = H$. Therefore, $|\mathrm{Fix}(H) \cap l| = x_1 + x_2 + x_8$. Assume that $\mathrm{Fix}(H_j) \cap l = \mathrm{Fix}(\gamma) \cap l$. Then $x_1 + x_2 + x_8 = \sqrt[4]{n} + 1$, since γ induces a Baer collineation on $\mathrm{Fix}(\sigma)$ by Proposition 4.11. At this point, the same argument as $q \equiv 1 \mod 16$ can be applied to obtain the assertions (2a)–(2d). $\square$

Lemma 4.16. *If H_j induces a perspectivity on $\mathrm{Fix}(\gamma)$ of axis distinct from $\mathrm{Fix}(\gamma) \cap l$ for each $j = 1, 2$, then $x_2 = x_8 = 0$.*

Proof. Assume that H_j induces a perspectivity on $\mathrm{Fix}(\gamma)$ of axis distinct from $\mathrm{Fix}(\gamma) \cap l$ for each $j = 1, 2$. We treat the cases $q \equiv 1 \mod 16$ and $q \equiv 9 \mod 16$ at the same time, bearing in mind that $H_j = H$ when the latter occurs. Hence, $x_1 + x_2 + x_8 = 1$ or 2 by Lemma 4.15.

$x_2 = 0$. Assume that $x_2 > 0$. Then $x_2 = 1$ by Lemma 3.5(1). If $x_1 = x_8 = 0$, then

$$\sqrt[4]{n} = 2 + x_7 \tag{52}$$

$$\sqrt{n} + 1 \geq \frac{q(q+1)}{2} + (\sqrt{q} \pm 1)x_7 \tag{53}$$

by Lemma 4.14 and Table I, respectively. So, $x_7 = \sqrt[4]{n} - 2$ by (52). Now, by substituting this value in (53) and then elementary computations of this one, we obtain

$$(\sqrt{q} \pm 1)(\sqrt[4]{n} - 2) + \frac{(q-9)}{2} \leq \sqrt{n} - 4\,. \tag{54}$$

As $q > 9$, then $(\sqrt{q} \pm 1)(\sqrt[4]{n} - 2) < \sqrt{n} - 4$. As $x_2 = 1$ and $x_1 = x_8 = 0$, then H must induce an elation on $\mathrm{Fix}(\gamma)$. Thus $\sqrt[4]{n}$ must be even. So, the case $\sqrt[4]{n} = \sqrt{q} - 2$ is ruled out as q is odd. Therefore, $\sqrt[4]{n} = \sqrt{q} - 1$. Nevertheless, this case cannot occur by [16, Theorem 13.18], since $\sqrt[4]{n} \equiv 2 \bmod 4$ and $\sqrt[4]{n} > 2$, as $\sqrt{q} \equiv 3 \bmod 4$ with $\sqrt{q} > 3$, and since H induces a non trivial involutory collineation on $\mathrm{Fix}(\gamma)$. Then either $x_1 = 1$ and $x_8 = 0$ or $x_1 = 0$ and $x_8 = 1$, since $x_1 + x_2 + x_8 \leq 2$ and $x_2 = 1$.
If $x_1 = 1$ and $x_8 = 0$, then

$$\sqrt[4]{n} = 3 + x_7 \tag{55}$$

$$\sqrt{n} + 1 \geq 1 + \frac{q(q+1)}{2} + (\sqrt{q} \pm 1)x_7 \tag{56}$$

by Lemma 4.14 and Table I, respectively. If $x_7 = 0$, then $\sqrt[4]{n} = 3$ by (55). Consequently, $\frac{q(q+1)}{2} \leq 9$ by (56). A contradiction, since $q > 9$. Thus $x_7 > 0$ and hence q is a square. In particular, $x_7 = \sqrt[4]{n} - 3$ by (55). Then

$$(\sqrt{q} \pm 1)(\sqrt[4]{n} - 3) + \frac{(q-17)}{2} \leq \sqrt{n} - 9\,, \tag{57}$$

combining $x_7 = \sqrt[4]{n} - 3$ with (56). As q is an odd square number and $q > 9$, then $q \geq 25$ and hence $\frac{(q-17)}{2} > 0$. This yields $(\sqrt{q} \pm 1)(\sqrt[4]{n} - 3) < \sqrt{n} - 9$ by (57). If $\sqrt{q} \equiv 1 \bmod 4$, then $\sqrt[4]{n} > \sqrt{q} - 2$. Then $\sqrt[4]{n} = \sqrt{q} - 1$, as $\sqrt[4]{n} < \sqrt{q}$. If $\sqrt{q} \equiv 3 \bmod 4$, then $\sqrt[4]{n} > \sqrt{q} - 3$ and hence either $\sqrt[4]{n} = \sqrt{q} - 1$ or $\sqrt[4]{n} = \sqrt{q} - 2$. As $x_1 = x_2 = 1$ and $x_8 = 0$, then H must induces a homology on $\mathrm{Fix}(\gamma)$. Thus, $\sqrt[4]{n}$ must be odd. Then only $\sqrt[4]{n} = \sqrt{q} - 2$ is really admissible as q is odd. Now, by substituting this value in (57) and bearing in mind that $\sqrt{q} \equiv 3 \bmod 4$, we obtain a contradiction, since $q > 9$.
If $x_1 = 0$ and $x_8 = 1$, then

$$\sqrt[4]{n} = 4 + x_7 \tag{58}$$

$$\sqrt{n} + 1 \geq \frac{q(q+1)}{2} + (\sqrt{q} \pm 1)x_7 + \sqrt{q} \tag{59}$$

by Lemma 4.14 and Table I, respectively. Then $x_7 = \sqrt[4]{n} - 4$. If $\sqrt[4]{n} = 4$, then $x_7 = 0$. Now, by substituting theses values in (59), we have $\frac{q(q+1)}{2} + \sqrt{q} \leq 17$. Nevertheless, this yields contradiction, since $q \geq 25$ as q is an odd square number and $q > 9$. Then $\sqrt[4]{n} > 4$ and hence $x_7 > 0$. Note also that $q \neq 25$, since $q < n < q^2$ with $q = 25$, and since n a fourth power with $\sqrt[4]{n} > 4$. So, $q \geq 49$. Indeed, q is an odd square number and $q > 9$. Now, by substituting $x_7 = \sqrt[4]{n} - 4$ in (59), we obtain

$$\frac{(q+1)}{2} + (\sqrt{q} \pm 1)(\sqrt[4]{n} - 4) + (\sqrt{q} - 1) \leq \sqrt{n} \tag{60}$$

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and hence

$$(\sqrt{q}-1)(\sqrt[4]{n}-4)+\frac{(q-31)}{2}\leq\sqrt{n}-16\,. \tag{61}$$

As $q\geq 49$, then $\frac{(q-31)}{2}>0$. Moreover, by (61), we have $(\sqrt{q}-1)(\sqrt[4]{n}-4)<\sqrt{n}-16$ and hence $\sqrt[4]{n}+4>\sqrt{q}-1$. That is $\sqrt[4]{n}>\sqrt{q}-5$. Then $\sqrt[4]{n}=\sqrt{q}-\theta$, where $1\leq\theta\leq 4$, as $\sqrt[4]{n}<\sqrt{q}$. As $x_2=x_8=1$ and $x_1=0$, it follows that H must induce a homology on $\mathrm{Fix}(\gamma)$. Thus, $\sqrt[4]{n}$ must be odd. Therefore, we actually have either $\sqrt[4]{n}=\sqrt{q}-2$ or $\sqrt[4]{n}=\sqrt{q}-4$, as q is odd. Nevertheless, these cases cannot occur. Indeed, if we substitute each of them in (60), we obtain a contradiction.

$x_8=0$. Assume that $x_8>0$. Then $x_8=1$ by Lemma 4.13(3). The previous point implies $x_2=0$. Thus, either $x_1=0$ or $x_1=1$ as $x_1+x_2+x_8\leq 2$ and $x_8=1$. Assume that $x_1=0$. Then

$$\sqrt[4]{n}=1+x_7 \tag{62}$$
$$\sqrt{n}+1\geq(\sqrt{q}\pm 1)x_7+\sqrt{q} \tag{63}$$

by Lemma 4.14 and Table I, respectively. It follows that $x_7=\sqrt[4]{n}-1$ and hence $(\sqrt{q}\pm 1)(\sqrt[4]{n}-1)+\sqrt{q}-2\leq\sqrt{n}-1$ by (62) and (63). This yields $\sqrt{n}+1>\sqrt{q}\pm 1$. So, $\sqrt{n}>\sqrt{q}\pm 1-1$. Then $\sqrt{q}\equiv 3 \bmod 4$ and $\sqrt{n}>\sqrt{q}-2$, as $\sqrt{n}<\sqrt{q}$. That is $\sqrt[4]{n}=\sqrt{q}-1$. Therefore $\sqrt[4]{n}\equiv 2 \bmod 4$ and $\sqrt[4]{n}>2$, as $\sqrt{q}\equiv 3 \bmod 4$ and $\sqrt{q}>3$. Nevertheless, this is a contradiction by [16, Theorem 13.18], since H induces a non trivial involutory collineation on $\mathrm{Fix}(\gamma)$.
If $x_1=1$, then

$$\sqrt[4]{n}=2+x_7 \tag{64}$$
$$\sqrt{n}+1\geq\sqrt{q}+1+(\sqrt{q}\pm 1)x_7 \tag{65}$$

by Lemma 4.14 and Table I, respectively. If $\sqrt[4]{n}=2$, then $x_7=0$. By substituting these vales in (65), we obtain $\sqrt{q}\leq 4$. Then $\sqrt{q}=3$, since $\sqrt{q}$ is odd. Hence, we arrive at a contradiction, since $q>9$ by our assumptions. Then $\sqrt[4]{n}>2$ and hence $x_7=\sqrt[4]{n}-2$ by (64). Again combining the previous equation with (65), we have

$$(\sqrt{q}\pm 1)(\sqrt[4]{n}-2)+\sqrt{q}-4\leq\sqrt{n}-4\,. \tag{66}$$

This yields $\sqrt{n}+2>\sqrt{q}\pm 1$. So, $\sqrt{n}>\sqrt{q}\pm 1-2$. Then $\sqrt{q}\equiv 3 \bmod 4$ and $\sqrt{n}>\sqrt{q}-3$ as $\sqrt{n}<\sqrt{q}$. Consequently, either $\sqrt[4]{n}=\sqrt{q}-1$ or $\sqrt[4]{n}=\sqrt{q}-2$. As $x_1=x_8=1$ and $x_2=0$, then H must induce an involutory homology on $\mathrm{Fix}(\gamma)$. Thus, $\sqrt[4]{n}$ must be odd. Therefore, the case $\sqrt[4]{n}=\sqrt{q}-1$ is ruled out, as q is odd. Hence, $\sqrt[4]{n}=\sqrt{q}-2$. Now, by substituting this value in (66) and bearing in mind that $\sqrt{q} \equiv 3 \bmod 4$, we obtain an equality. Then $\mathcal{S}_2 = \mathcal{S}_4 = 0$, since $\sqrt{n}+1 \geq (\sqrt{q}\pm 1)x_7 + \sqrt{q} + 1 + \mathcal{S}_2$ by Table I. It follows that $|\mathrm{Fix}(\gamma) \cap l| = 1 + \frac{\sqrt{q}-1}{2}$ by Table II, since $x_1 = x_8 = 1$ and $\mathcal{S}_4 = 0$ by the previous argument, since $q \equiv 9 \bmod 16$ for $x_7 > 0$ by Lemma 4.13(2), and since $p^e \equiv 3 \bmod 4$ for $x_9 > 0$ by Lemma 4.13(4) (note that the collineation γ does not fix points on the G-orbits on l of type (7) for $q \equiv 9 \bmod 16$ or (9) for $p^e \equiv 3 \bmod 4$ by Table II). That is $\sqrt[4]{n} = \frac{\sqrt{q}-1}{2}$, as γ induces a Baer collineation on $\mathrm{Fix}(\sigma)$. This is a contradiction, since $\sqrt[4]{n} = \sqrt{q} - 2$. □

Lemma 4.17. *The group H_j induces either the identity or a Baer involution on* $\mathrm{Fix}(\gamma)$ *for each* $j = 1, 2$.

Proof. Assume that H_j induces a perspectivity on $\mathrm{Fix}(\gamma)$. We treat the cases $q \equiv 1 \bmod 16$ and $q \equiv 9 \bmod 16$ at the same time, bearing in mind that $H_j = H$ when the latter occurs. If the axes of he perspectivities induced on $\mathrm{Fix}(\gamma)$ by H_j are distinct from $\mathrm{Fix}(\gamma) \cap l$ for each $j = 1, 2$, then

$$\sqrt[4]{n} + 1 = x_7 + x_1 \tag{67}$$
$$\sqrt{n} + 1 \geq (\sqrt{q} \pm 1)x_7 + x_1 \tag{68}$$

by Lemma 4.14 and Table I, respectively, since $x_2 = x_8 = 0$ by Lemma 4.16. In particular, either $x_1 = 1$ or $x_1 = 2$, since $x_1 + x_2 + x_8 = 1$ or 2 by Lemma 4.15 and being $x_2 = x_8 = 0$.

Assume that $x_1 = 1$. Then $x_7 = \sqrt[4]{n}$ and hence $(\sqrt{q} \pm 1)\sqrt[4]{n} + 1 \leq \sqrt{n} + 1$ by (67) and (68). By calculations of the previous inequality, we have $\sqrt[4]{n} \geq \sqrt{q} \pm 1$. It follows that $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt[4]{n} = \sqrt{q} - 1$, as $\sqrt[4]{n} < \sqrt{q}$. Then $q \geq 49$ and hence $\sqrt[4]{n} \geq 6$, as $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt{q} > 3$. Moreover, $\sqrt[4]{n} \equiv 2 \bmod 4$ and $\sqrt[4]{n} > 2$, as $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt{q} > 3$, respectively. Nevertheless, this contradicts [16], Theorem 13.18, since H_j induces a non trivial involutory collineation on $\mathrm{Fix}(\gamma)$.

Assume that $x_1 = 2$. Then $x_7 = \sqrt[4]{n} - 1$ by (67). Now, by substituting this value in (68), we obtain $(\sqrt{q} \pm 1)(\sqrt[4]{n} - 1) + 2 \leq \sqrt{n} + 1$ and so

$$(\sqrt{q} \pm 1)(\sqrt[4]{n} - 1) \leq \sqrt{n} - 1\,. \tag{69}$$

This yields $\sqrt[4]{n} + 1 \geq \sqrt{q} \pm 1$ and hence $\sqrt[4]{n} \geq \sqrt{q} \pm 1 - 1$. Then $\sqrt{q} \equiv 3 \bmod 4$ and $\sqrt[4]{n} \geq \sqrt{q} - 2$, as $\sqrt[4]{n} < \sqrt{q}$. Therefore, either $\sqrt[4]{n} = \sqrt{q} - 1$ or $\sqrt[4]{n} = \sqrt{q} - 2$. As $x_1 = 2$ and $x_2 = x_8 = 0$, then H_j must induces a homology on $\mathrm{Fix}(\gamma)$. Thus, $\sqrt[4]{n}$ must be odd. Then we actually have $\sqrt[4]{n} = \sqrt{q} - 2$, as q is odd. Now, by substituting this value in (69) and bearing in mind that $\sqrt{q} \equiv 3 \bmod 4$, we have $(\sqrt{q} - 1)(\sqrt{q} - 2) + 2 \leq (\sqrt{q} - 2)^2 + 1$. It is a straightforward computation to see that the previous inequality is impossible.

Assume that H_j induces a perspectivity on $\mathrm{Fix}(\gamma)$ of axis $\mathrm{Fix}(\gamma) \cap l$. Then $x_1 = \sqrt[4]{n} + 1$ again by Lemma 4.15. Thus $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\gamma) \cap l$. Now, dualizing the above argument, we obtain that $|\mathrm{Fix}(H_j) \cap [P]| \geq 3$ for each point $P \in \mathrm{Fix}(\gamma) \cap l$ and for each $j = 1, 2$. Nevertheless, this is impossible, since the H_j induces a perspectivity on $\mathrm{Fix}(\gamma)$ of axis $\mathrm{Fix}(\gamma) \cap l$. At this point, the assertion follows by Lemma 4.15. □

Proposition 4.18. *The group H_j induces a Baer involution on $\mathrm{Fix}(\gamma)$ for each $j = 1, 2$.*

Proof. The group H_j induces either the identity or a Baer involution on $\mathrm{Fix}(\gamma)$ for each $j = 1, 2$ by Lemma 4.17. Assume the former occurs. Then G fixes a subplane of order $\sqrt[4]{n}$ and $\mathcal{S}_4 = x_2 = x_7 = x_8 = 0$ by Lemma 4.15. That is $\mathrm{Fix}(G) = \mathrm{Fix}(\gamma)$. Then

$$\sqrt{n} = \sqrt[4]{n} + \frac{q-1}{2} x_3 + \frac{q-1}{p^e - 1} x_9 + \mathcal{S}_{2,4} \tag{70}$$

$$n = \sqrt[4]{n} + \frac{q(q-1)}{2} x_3 + \frac{p^e(q^2-1)}{2(p^e-1)} x_9 + \frac{q+1}{2} \mathcal{S}_1 \tag{71}$$

by Table I. Assume that $x_3 > 0$. Then $x_3 = 1$ by Lemma 3.5(2). Hence, let $P \in l$ such that $G_P \cong D_{q+1}$. Then G_P fixes exactly one point P^G, since it is maximal in G. Thus, $\mathrm{Fix}(G_P) \cap l = \{P\} \cup \mathrm{Fix}(G_P) \cap l$. Furthermore, G_P is planar, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(G_P)$ and G is planar. Actually $\mathrm{Fix}(G) \subset \mathrm{Fix}(G_P) \subset \mathrm{Fix}(\sigma)$. So, we arrive at a contradiction by [16, Theorem 3.7], since $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$ and $o(\mathrm{Fix}(\sigma)) = \sqrt{n}$. Then $x_3 = 0$ and hence $x_9 + \mathcal{S}_{2,4} > 0$ by (70).

Let ρ_t, where $t = 1, 2$ be the representatives of the two conjugate classes of p-elements in G. Since $x_9 + \mathcal{S}_{2,4} > 0$, then $\mathrm{Fix}(G) \subset \mathrm{Fix}(\langle \rho_t \rangle)$ for each $t = 1, 2$. It follows that the group $\langle \rho_t \rangle$ fixes a Baer subplane of Π for each $t = 1, 2$ by [16, Theorem 3.7], since $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$. Clearly, σ inverts ρ_t for each $t = 1, 2$. Furthermore, $\mathrm{Fix}(G) \subset \mathrm{Fix}(\langle \rho_t, \sigma \rangle) \subset \mathrm{Fix}(\langle \rho_t \rangle)$, since $x_9 + \mathcal{S}_{2,4} > 0$ (see Table II). This still contradicts [16, Theorem 3.7], since $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$ and $o(\mathrm{Fix}(\rho_t)) = \sqrt{n}$ for each $t = 1, 2$. Thus, the group H_j induces a Baer involution on $\mathrm{Fix}(\gamma)$ for each $j = 1, 2$. □

Lemma 4.19. *The group G fixes a subplane of Π of order $\sqrt[8]{n}$ pointwise.*

Proof. By Proposition 4.18 and by Lemma 4.15, we have $\sqrt[8]{n} + 1 = x_1 + x_2 + x_8$. Recall that $x_2 \leq 1$ by Lemma 3.5(1) and $x_8 \leq 1$ by Lemma 4.13(3). Since $\sqrt[8]{n} \geq 2$, then $x_1 \geq 1$. Assume that $x_1 \leq 2$. Hence either $x_1 = 1$ or $x_1 = 2$. Then we have the following admissible triples $(x_1, x_2, x_8) = (1, 1, 1)$, $(2, 0, 1)$,

$(2,1,0)$, $(2,1,1)$, since $\sqrt[8]{n} \geq 2$. Furthermore, $\sqrt[4]{n}+1 = x_1+3x_2+x_7+2x_8$ by Lemma 4.14. Thus,

$$(x_1+x_2+x_8-1)^2 = x_1+3x_2+x_7+2x_8-1\,. \tag{72}$$

By substituting the values found of (x_1,x_2,x_8) in (72), we see that $(x_1,x_2,x_8) = (1,1,1)$ is ruled out.

If $(x_1,x_2,x_8) = (2,0,1)$, then $\sqrt[8]{n} = 2$ and $x_7 = 1$. So, $\sqrt{n}+1 \geq 2+\frac{\sqrt{q}\pm 1}{2}+\sqrt{q}$ by Table I. It follows that $\frac{\sqrt{q}\pm 1}{2}+\sqrt{q} \leq 15$ as $\sqrt{n} = 16$. This yields $3\sqrt{q} \leq 31$. That is $\sqrt{q} = 5$, 7 or 9, as $\sqrt{q} > 3$. Therefore, $q = 5^2$, 7^2 or 9^2. Nevertheless, only the case $q = 5^2$ is admissible, since it must be $q \equiv 9 \bmod 16$ by Lemma 4.13(2), being $x_7 = 1$. If $x_3 > 0$, then $n+1 \geq \frac{q(q-1)}{2}$ by Table I. Nevertheless, this is impossible, since $n = 2^8$, while $q = 5^2$. Then the length of any admissible non trivial G-orbit on l is divisible by $\frac{q+1}{2}$ by Table I, since $x_4 = x_5 = x_6 = 0$ by Lemma 4.13(1). Thus, $\frac{q+1}{2}$ must divide $|l - \mathrm{Fix}(G)|$. That is $\frac{q+1}{2} \mid n+1-x_1$, being $|l-\mathrm{Fix}(G)| = n+1-x_1$. Hence, we arrive at a contradiction, since $\frac{q+1}{2} = 13$, as $q = 5^2$, while $n+1-x_1 = 2^8-1$, as $n = 2^8$ and $x_1 = 2$.

If $(x_1,x_2,x_8) = (2,1,0)$, then $\sqrt[8]{n} = 2$ and $x_7 = 1$. As a consequence, $\frac{q+1}{2}+\sqrt{q} \leq 15$. So, we again obtain a contradiction, since $q \equiv 9 \bmod 16$ and $q > 9$.

Finally, assume that $(x_1,x_2,x_8) = (2,1,1)$. Then $\sqrt[8]{n} = 3$ and $x_7 = 3$. Then $\sqrt{n}+1 \geq 2+\frac{q+1}{2}+3\frac{\sqrt{q}\pm 1}{2}+\sqrt{q}$, which is a contradiction. Then $x_1 \geq 3$ and hence G fixes at least 3 points on l. Let P_1, P_2, P_3 three distinct points on l which are fixed by G. Now, repeating all the arguments with $[P_i]$ in the role of l, for each $i = 1,2,3$, we see that G fixes at least three lines at least 3 lines, including l, on $[P_i]$ for each $i = 1,2,3$. Thus, G is planar on Π. In particular, $\mathrm{Fix}(G)$ is a subplane of $\mathrm{Fix}(H)$ of order $x_1 - 1$.

Assume that $\mathrm{Fix}(G) \subset \mathrm{Fix}(H)$. This yields $x_2+x_8 > 0$, since $\sqrt[8]{n}+1 = x_1+x_2+x_8$. Then, by [16, Theorem 3.7], either

$$(x_1-1)^2 = x_1+x_2+x_8-1, \text{ or} \tag{73}$$

$$(x_1-1)^2+(x_1-1) \leq x_1+x_2+x_8-1\,, \tag{74}$$

since $o(\mathrm{Fix}(G)) = x_1-1$ and $o(\mathrm{Fix}(H)) = x_1+x_2+x_8-1$. If $x_2 = 1$, then either $(x_1-1)^2 = x_1+x_8$ or $(x_1-1)^2+(x_1-1) \leq x_1+x_8$. Note that $x_8 \leq 1$ by Lemma 4.13(2). Assume that $x_8 = 0$. Then either $(x_1-1)^2 = x_1$ or $(x_1-1)^2 \leq 1$. This yields a contradiction in any case, as $x_1 \geq 3$. Then $x_8 = 1$. Thus, either $(x_1-1)^2 = x_1+1$ or $(x_1-1)^2 \leq 2$. Actually, only the former occurs and hence $x_1 = 3$. Therefore, $o(\mathrm{Fix}(H)) = 4$. Then $\sqrt[8]{n} = 4$ and hence $n = 4^8$. In particular, $x_3 = 0$ by Lemma 3.6(1), as $x_2 = 1$. Thus the length of any admissible non trivial G-orbit on l is divisible by $\frac{q+1}{2}$ (see Table I). Therefore,

$\frac{q+1}{2} \mid n+1-x_1$, as $|l - \mathrm{Fix}(G)| = n - x_1$. That is $\frac{q+1}{2} \mid 4^8 - 2$, as $n = 4^8$ and $x_1 = 3$. Now, it is a plain that $4^8 - 2$ has no divisors of the form $\frac{q+1}{2}$ with q an even power of an odd prime. Hence, $x_2 = 0$. Then $x_8 = 1$, as $x_8 \leq 1$ and $x_2 + x_8 > 0$. Now, by substituting the couple $(x_2, x_8) = (0, 1)$ in (73) and (74), we have either $(x_1 - 1)^2 = x_1$ or $(x_1 - 1)^2 + (x_1 - 1) \leq x_1$. While the first equation has no solutions, the second one yields $x_1 \leq 2$. Actually, $x_1 = 2$, as $x_1 \geq 2$, being $\sqrt[8]{n} = x_1$. Therefore, $n = 2^8$. Now, recall that $x_3 \leq 1$ by Lemma 3.5(2). If $x_3 = 0$, then $\frac{q+1}{2} \mid n+1-x_1$ arguing as above. Then $\frac{q+1}{2} \mid 2^8 - 1$, as $n = 2^8$ and $x_1 = 2$. Easy computations show that $q = 13^2$, since q is an even power of a prime and $q \geq 11$. Recall that $x_8 = 1$. Since γ fixes a subplane of Π of order $\sqrt[4]{n}$, since $13^2 \equiv 9 \bmod 16$ and by Table I, we see that $\sqrt[4]{n} + 1 \geq 7x_8 = 7$. Nevertheless, this is a contradiction, since $\sqrt[4]{n} = 4$. So, $x_3 = 1$. Then the length of any admissible non trivial G-orbit on l is divisible by $\frac{q+1}{2}$, unless this one is of type (3) by Table I, since $x_4 = x_5 = x_6 = 0$ by Lemma 4.13(1). Thus, $\frac{q+1}{2} \mid n+1-x_1 - \frac{q(q-1)}{2}$. This yields $\frac{q+1}{2} \mid n+1-x_1$ and hence $\frac{q+1}{2} \mid 2^8 - 2$. Since $q \equiv 1 \bmod 8$, then $\frac{q+1}{2}$ is odd. Consequently, $\frac{q+1}{2} \mid 2^7 - 1$. Actually, $\frac{q+1}{2} = 2^7 - 1$, since $2^7 - 1$ is prime and $\frac{q+1}{2} > 1$. So, $q = 253$. So, we arrive at a contradiction, since q must be a square as $x_8 = 1$. Thus, $\mathrm{Fix}(G) = \mathrm{Fix}(H)$. Therefore, we have proved the assertion. □

Lemma 4.20. *If $q > 9$, then the group G does not fix lines of Π.*

Proof. Assume that G fixes a subplane of Π of order $\sqrt[8]{n}$ pointwise. Assume that $x_i > 0$ for either $i = 2$ or 3. Then $x_i = 1$ for either $i = 2$ or 3 by Lemma 3.6(1). Hence, let $P \in l$ such that $G_P \cong D_{q\pm 1}$. Then G_P fixes exactly one point P^G, since it is maximal in G. Thus, $\mathrm{Fix}(G_P) \cap l = \{P\} \cup \mathrm{Fix}(G_P) \cap l$. Furthermore, G_P is planar, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(G_P)$ and G is planar. In particular, $o(\mathrm{Fix}(G_P)) = \sqrt[8]{n}+1$. Moreover, $\sqrt[4]{n} \leq \sqrt[8]{n}+1$ by [16, Theorem 3.7], since $\mathrm{Fix}(G) \subset \mathrm{Fix}(G_P)$. Nevertheless, this is a contradiction. Therefore, $x_2 = x_3 = 0$. In addition, $x_4 = x_5 = x_6 = 0$ by Lemma 4.13(1). So, we have the following system of Diophantine equations:

$$\sqrt[4]{n} = \sqrt[8]{n} + x_7 + 2x_8 \tag{75}$$

$$\sqrt[4]{n} = \sqrt[8]{n} + \frac{\sqrt{q} \pm 1}{2} x_8 + \mathcal{S}_4 \,. \tag{76}$$

By subtracting (76) from (75), we obtain

$$x_7 = \left[\frac{\sqrt{q} \pm 1}{2} - 2 \right] x_8 + \mathcal{S}_4 \,. \tag{77}$$

Let ρ_t be the representative of the two conjugate classes of p-elements in G for $t = 1, 2$. We may assume that ρ_t, for each $t = 1, 2$, lie in the Sylow

p-subgroup S of G normalized by σ. Then ρ_t is planar, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(\rho_t)$ for each $t = 1, 2$. In particular, by Table II,

$$o(\mathrm{Fix}(\rho_t)) \geq \sqrt[8]{n} + x_7 2\sqrt{q} + x_8\sqrt{q} + \frac{1}{2}\mathcal{S}_1 \,. \tag{78}$$

Assume that $x_8 > 0$. So q is a square. Actually, $x_8 = 1$ by Lemma 4.13(3). Then $o(\mathrm{Fix}(\rho_t)) > \sqrt[8]{n}$ and hence $o(\mathrm{Fix}(\rho_t)) \geq \sqrt[4]{n}$ by [16, Theorem 3.7], since $o(\mathrm{Fix}(G)) = \sqrt[8]{n}$ and $\mathrm{Fix}(G) \subset \mathrm{Fix}(\rho_t)$. If $o(\mathrm{Fix}(\rho)) = \sqrt[4]{n}$, it follows that $\sqrt[4]{n} > \sqrt{q}$, as $o(\mathrm{Fix}(\rho_t)) \geq \sqrt[8]{n} + x_7 2\sqrt{q} + x_8\sqrt{q}$ with $\sqrt[8]{n} \geq 2$ and $x_8 = 1$. Nevertheless, this contradicts the assumption $n < q^2$. Thus, $o(\mathrm{Fix}(\rho_t)) > \sqrt[4]{n}$. Nevertheless, $o(\mathrm{Fix}(\rho_t)) \leq \sqrt{n}$ by [16, Theorem 3.7]. Note that $x_7 \geq \frac{\sqrt{q}\pm 1}{2} - 2$ by (77), since $x_8 = 1$. By substituting $x_7 \geq \frac{\sqrt{q}\pm 1}{2} - 2$ in (78), we obtain

$$o(\mathrm{Fix}(\rho_t)) \geq \sqrt[8]{n} + 2\sqrt{q}(\frac{\sqrt{q}\pm 1}{2} - 2) + \sqrt{q} + \frac{1}{2}\mathcal{S}_1 \,. \tag{79}$$

Then $2 + 2\sqrt{q}(\frac{\sqrt{q}\pm 1}{2} - 2) + \sqrt{q} + \frac{1}{2}\mathcal{S}_1 \leq \sqrt{n}$, since $o(\mathrm{Fix}(\rho_t)) \leq \sqrt{n}$ and $\sqrt[8]{n} \geq 2$. By elementary calculations of the previous inequality, we obtain $\sqrt{n} \geq q \pm \sqrt{q} - 3\sqrt{q} + 2 + \frac{1}{2}\mathcal{S}_1$. Assume that $\sqrt{q} \equiv 3 \bmod 4$. Hence, $\sqrt{n} \geq q - 4\sqrt{q} + 2 + \frac{1}{2}\mathcal{S}_1$. That is, $\sqrt{n} > (\sqrt{q} - 2)^2$. So, $\sqrt[4]{n} > \sqrt{q} - 2$. Then $\sqrt[4]{n} = \sqrt{q} - 1$, since $\sqrt[4]{n} < \sqrt{q}$, as $n < q^2$ by our assumption. Note that $\sqrt[4]{n} \equiv 2 \bmod 4$ and $\sqrt[4]{n} > 2$, since $\sqrt{q} \equiv 3 \bmod 4$ and $q > 9$. Nevertheless, this yields a contradiction by [16, Theorem 13.18], since H_j acts non trivially on $\mathrm{Fix}(\gamma)$ by Proposition 4.18 and since $o(\mathrm{Fix}(\gamma)) = \sqrt[4]{n}$. Hence, $\sqrt{q} \equiv 1 \bmod 4$. Then $\sqrt{n} \geq (\sqrt{q} - 1)^2 + \frac{1}{2}\mathcal{S}_1$ by (79) as $\sqrt[8]{n} \geq 2$. Actually, $\sqrt{n} = (\sqrt{q} - 1)^2$ and $\mathcal{S}_1 = 0$, since $\sqrt[4]{n} < \sqrt{q}$ being $n < q^2$ by our assumption, and being n a fourth power and q as square. That is $\sqrt[4]{n} = \sqrt{q} - 1$. Now note that $\mathcal{S}_4 = 0$, since $\mathcal{S}_1 \geq \mathcal{S}_4 \geq 0$ and since $\mathcal{S}_1 = 0$. Then $x_7 = \frac{\sqrt{q}-3}{2}$ by (77), since $\sqrt{q} \equiv 1 \bmod 4$ and $x_8 = 1$. Now, by substituting $x_7 = \frac{\sqrt{q}-3}{2}$, $x_8 = 1$ and $\sqrt[4]{n} = \sqrt{q} - 1$ in (75), we obtain $\sqrt{q} - 1 = \sqrt[8]{n} + \frac{\sqrt{q}-3}{2} + 2$. By elementary calculations of the previous equality, we have $\sqrt[8]{n} = \frac{\sqrt{q}-3}{2}$. Then $\left(\frac{\sqrt{q}-3}{2}\right)^2 = \sqrt{q} - 1$, since $\sqrt[4]{n} = \sqrt{q} - 1$, which is a contradiction. Thus, $x_8 = 0$. Then $x_7 = \mathcal{S}_4$ by (77). If $\mathcal{S}_4 = 0$, then $x_7 = 0$ and we have a contradiction by (75), since also $x_8 = 0$. So $\mathcal{S}_4 > 0$. In particular, $\sqrt[4]{n} = \sqrt[8]{n} + \mathcal{S}_4$ by (76).

Finally, let us consider the subgroup W of G, where $W = S\langle\gamma\rangle$ and S is the Sylow p-subgroup of G normalized by σ and hence by γ. Then W fixes at least a point Q on l since $\mathcal{S}_4 > 0$. Hence, let Q^G be an orbit of type (10). Clearly, $G_Q \cong F_q.Z_d$, where $d \equiv 0 \bmod 4$. In particular, $\left|\mathrm{Fix}_{Q^G}(W)\right| = \frac{q-1}{2d}$ by (1) of Proposition 2.5. Thus, the number of points coming out from G-orbits on l of type (10) which are fixed by W are exactly $\sum_{d_j \equiv 0 \bmod 4} \frac{q-1}{2d_j}$. These turn out to be $\frac{1}{2}\mathcal{S}_4$ as $\mathcal{S}_4 = \sum_{d_j \equiv 0 \bmod 4} \frac{q-1}{d_j}$. Then $|\mathrm{Fix}(W) \cap l| = \sqrt[8]{n} + 1 + \frac{1}{2}\mathcal{S}_4$, since

$|\mathrm{Fix}(G) \cap l| = \sqrt[8]{n}+1$ by Lemma 4.19. Furthermore, W is planar, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(W)$. On the other hand, $\mathrm{Fix}(W) \subset \mathrm{Fix}(\gamma)$, since $o(\mathrm{Fix}(W)) = \sqrt[8]{n} + \frac{1}{2}\mathcal{S}_4$ and $o(\mathrm{Fix}(\gamma)) = \sqrt[8]{n} + \mathcal{S}_4$ being $\mathcal{S}_4 > 0$. Therefore, $\mathrm{Fix}(G) \subset \mathrm{Fix}(W) \subset \mathrm{Fix}(\gamma)$, where $o(\mathrm{Fix}(G)) = \sqrt[8]{n}$ and $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$. Nevertheless, this contradicts [16, Theorem 3.7]. Thus, G does not fix lines of Π. □

Proposition 4.21. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,q)$. If $q < n < q^2$ and $q \equiv 1 \bmod 8$, then G cannot fix lines of Π.*

Proof. Assume that G fixes a line l of Π. Then $q \leq 9$ by Lemma 4.20. Actually, $q = 9$, since $q \equiv 1 \bmod 8$. Then $n = 16, 25, 36, 49, 64$, since $q < n < q^2$, with n a square by Lemma 3.3. The case $n = 36$ and $n = 49$ are ruled out by Lemma 3.3. Thus, $n = 16, 25$ or 64.

Assume that $n = 16$. Let P^G be a non trivial orbit on l. Then $\left|P^G\right| \leq 17$. Then G_P is isomorphic either to $Z_9.Z_4$ or to S_4 or to A_5. If $G_P \cong Z_9.Z_4$, then $\left|P^G\right| = 10$. In particular, G acts 2-transitively on P^G, which contradicts [23, Theorem 1], since $n = 16$. If $G_P \cong A_5$, then $\left|P^G\right| = 15$ and hence $\left|l - P^G\right| = 11$. Let $Q \in l - P^G$. Then $\left|Q^G\right| \leq 11$, since $\left|l - P^G\right| = 11$. Clearly, $G_Q \ncong Z_9.Z_4$ by the previous argument. Furthermore, $G_Q \ncong S_4$, otherwise $\left|Q^G\right| = 15$. Thus, $G_Q \cong A_5$. Therefore, $\left|l - (P^G \cup Q^G)\right| = 5$. Then G fixes $l - (P^G \cup Q^G)$ pointwise, since the minimal primitive permutation representation degree of $G \cong \mathsf{PSL}(2,9)$ is 6. So, any involution in G fixes at least 8 points on l. Hence, we arrive at a contradiction, since each involution in G is a Baer collineation of Π by Lemma 3.3 and since $n = 16$. Thus, $G_P \cong S_4$. Then $\left|P^G\right| = 15$ and hence $\left|l - P^G\right| = 2$. So, G fixes $l - P^G$ pointwise. Set $\{X, Y\} = l - P^G$. It follows that $G_r \cong S_4$ for some line $r \in [X]$ and $G_u \cong S_4$ for some line $u \in [Y]$ by dual of the above argument, since G acts on $[X]$ and on $[Y]$ fixing two lines through each of them (clearly, l is one of them). Therefore, G fixes a triangle $\Delta = \{X, Y, Z\}$. Let ρ_1 and ρ_2 are the representatives of the 3-elements in G. We may assume that the lie in the Sylow 3-subgroup of G normalized by σ. As a consequence, σ inverts each of them. Since $G_P \cong S_4$, then one of them fixes exactly 3 points on P^G, 3 on XZ and 3 on YZ by Table IV* of [24], since $q = 9$. We may assume that ρ_1 does it. Hence, ρ_1 fixes a Baer subplane of Π, since ρ_1 fixes exactly 3 points on P^G and the points X and Y. So, $o(\mathrm{Fix}(\rho_1)) = 4$. The involution σ acts on $\mathrm{Fix}(\rho_1)$, since it inverts ρ_1. Note that $\langle \rho_1, \sigma \rangle$ does not fix point on P^G by Table IV* of [24], since $G_P \cong S_4$ and $q = 9$. Therefore, σ fixes exactly 2 points on $\mathrm{Fix}(\rho_1) \cap l$, namely X and Y. So, σ induces a homology on $\mathrm{Fix}(\rho_1)$. Nevertheless, this is impossible, since $o(\mathrm{Fix}(\rho_1)) = 4$.

Assume that $n = 25$ or 64. Assume also that $\mathrm{Fix}(T_j) \cap l = \mathrm{Fix}(\sigma) \cap l$ for some $j = 1$ or 2. Then $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ by Table III* of [24], since

$q = 9$. Therefore, for each point $A \in l - \mathrm{Fix}(G)$, the group G_A has odd order. Then $G_A \cong E_9$ by Table III* of [24], since $G \cong \mathsf{PSL}(2,9)$. Hence $\left|A^G\right| = 40$ for each point $A \in l - \mathrm{Fix}(G)$. Then $40 \mid |l - \mathrm{Fix}(G)|$. That is $40 \mid n - \sqrt{n}$, since $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ and $|\mathrm{Fix}(\sigma) \cap l| = \sqrt{n} + 1$. This is a contradiction, since $n = 25$ or 64. Thus, $|\mathrm{Fix}(T_j) \cap l| = 2$ or 1 for each $j = 1, 2$, according to whether $n = 25$ or 64, respectively. Therefore, T_j induces a non trivial perspectivity $\bar{\beta}_j$ on $\mathrm{Fix}(\sigma)$ for each $j = 1, 2$. Clearly, T_1 and T_2 are subgroups of $C_G(\sigma) \cong D_8$. Furthermore, $C_G(\sigma)$ acts on $\mathrm{Fix}(\sigma)$ inducing a subgroup $\bar{C}$ isomorphic either to E_4 or to Z_2. In each case $\bar{\beta}_1 \in \bar{C}$ and $\bar{\beta}_1 \neq 1$, since $\bar{\beta}_1$ is a non trivial perspectivity of $\mathrm{Fix}(\sigma)$. Then $\bar{C}$ fixes $C_{\bar{\beta}_1}$, since $\bar{\beta}_1$ is central in $\bar{C}$. So, $C_G(\sigma)$ fixes $C_{\bar{\beta}_1}$. That is $C_G(\sigma) \leq G_{C_{\bar{\beta}_1}}$. Let $U \leq N_G(T_1)$ such that $U \cong A_4$. Then U fixes $\mathrm{Fix}(T_1) \cap l$ pointwise, since $T_1 \lhd U$, $U \cong A_4$ and $|\mathrm{Fix}(T_1) \cap l| = 1$ or 2. Then $U \leq G_{C_{\bar{\beta}_1}}$ and therefore $\langle C_G(\sigma), U\rangle \leq G_{C_{\bar{\beta}_1}}$. Note that $\langle C_G(\sigma), U\rangle \cong S_4$, since $\langle C_G(\sigma), U\rangle \leq N_G(T_1)$ as $G \cong \mathsf{PSL}(2,9)$. So, either $G_{C_{\bar{\beta}_1}} \cong S_4$ or $G_{C_{\bar{\beta}_1}} = G$ by Table III* of [24] as $q = 9$. Actually, the case $G_{C_{\bar{\beta}_1}} \cong S_4$ cannot occur, since $\left|\mathrm{Fix}_{C_{\bar{\beta}_1}^G}(T_j)\right| = 3$ with $C_{\bar{\beta}_1}^G \subset l$, while we proved that $|\mathrm{Fix}(T_j) \cap l| = 2$ or 1 for each $j = 1, 2$. As a consequence, $G_{C_{\bar{\beta}_1}} = G$. This implies that G acts on $\left[C_{\bar{\beta}_1}\right]$ and $\mathrm{Fix}(T_1) \cap \left[C_{\bar{\beta}_1}\right] = \mathrm{Fix}(\sigma) \cap \left[C_{\bar{\beta}_1}\right]$, which is a contradiction by dual of the above argument. Thus, we have proved the assertion. $\square$

Theorem 4.22. *Let Π be a projective plane of order n admitting $G \cong \mathsf{PSL}(2,q)$ as a collineation group. If $n \leq q^2$, $q \equiv 1 \bmod 8$, then one of the following occurs:*

(1) $n < q$, $\Pi \cong \mathsf{PG}(2,4)$ *and* $G \cong \mathsf{PSL}(2,9)$;

(2) $n = q$, $\Pi \cong \mathsf{PG}(2,q)$ *and G is strongly irreducible on* Π;

(3) $q < n < q^2$, *one of the following occurs:*

 (a) *G is strongly irreducible on* Π;

 (b) $G \cong \mathsf{PSL}(2,9)$ *fixes a proper subplane* $\Pi_0 \cong \mathsf{PG}(2,4)$ *of* Π;

(4) $n = q^2$ *and one of the following occurs:*

 (a) *G is strongly irreducible on* Π;

 (b) $n = 81$ *and* $G \cong \mathsf{PSL}(2,9)$ *fixes a point and line of* Π;

 (c) *G fixes a subplane Π_0 of Π. Furthermore, either* $\Pi_0 \cong \mathsf{PG}(2,q)$ *is a Baer subplane of Π, or $n = 81$, Π_0 is the Hughes plane of order 9 and* $G \cong \mathsf{PSL}(2,9)$.

Proof. If $n < q$ or $n = q$, the assertions (1) and (2) easily follow by Theorems 2.1 and 2.2, respectively. If $q < n < q^2$, the group G does not fix lines or points of Π by Proposition 4.21 and its dual. At this point, the assertion (3a) and (3b) easily follow by Lemma 3.1, since $q \equiv 1 \bmod 8$. The assertions (4a) and (4b) follow by Theorem 2.3. Finally, the assertion (4c) follows by Theorem 2.4. □

Clearly, Theorem 1.1 easily follows from Theorem 4.22 when $q \equiv 1 \bmod 8$.

5. The case $q \equiv 3 \bmod 8$

In this section, we deal with the case $q \equiv 3 \bmod 8$. Recall that there exists a unique conjugate class of in involutions and one of Klein subgroups of G. Let σ be an involution of G and let T be a representative of this class containing σ. As pointed out at the end of section 3, we have $C_G(\sigma) \cong D_{q+1}$ and $N_G(T) \cong A_4$.

We filter the list given in Lemma 3.4 with respect to the condition $q \equiv 3 \bmod 8$. For each of the resulting groups, we find its corresponding index in G. Thus, we determine the length of the orbit P^G, with P a point of l, when G_P is isomorphic to one of these groups. Next, for each of these groups, using (1) of Proposition 2.5, we obtain the number of points fixed by σ, and by T, in the orbit P^G. All these informations are displayed in the following table.

Table III

Type	G_P	$[G : G_P]$	$\lvert\mathrm{Fix}_{P^G}(\sigma)\rvert$	$\lvert\mathrm{Fix}_{P^G}(T)\rvert$
1	G	1	1	1
2	D_{q-1}	$\frac{q(q+1)}{2}$	$\frac{q+1}{2}$	0
3	D_{q+1}	$\frac{q(q-1)}{2}$	$\frac{q+3}{2}$	3
4	A_4	$\frac{q(q^2-1)}{24}$	$\frac{q+1}{4}$	1
5	A_5	$\frac{q(q^2-1)}{120}$	$\frac{q+1}{4}$	1
10	$F_q.Z_d$	$\frac{q^2-1}{2d}$	0	0

Recall that the G-orbits of type (10) on l cover exactly $\mathcal{S}$ points of l, where $\mathcal{S} = \sum_{j=1}^{x_{10}} \frac{q^2-1}{2d_j}$. Recall also that $\mathcal{S}_1 = \sum_{j=1}^{x_{10}} \frac{q-1}{d_j}$ and $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_4$, $\mathcal{S}_{2,4}$ (sum with the same summands $\frac{q-1}{d_j}$ but over $2 \mid d_j$, $2 \nmid d_j$, $4 \mid d_j$ and $d_j \equiv 2 \bmod 4$, respectively). Note that $\mathcal{S}_2 = \mathcal{S}_{2,4} = \mathcal{S}_4 = 0$, since $q \equiv 3 \bmod 8$. Hence, $\mathcal{S}_1 = \mathcal{S}_{2'}$ and $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_{2'}$.

Finally, if G fixes a point Q and acts on $[Q]$, we may focus on the G-orbits of lines in $[Q]$. So, following the notation introduced in section 4, we obtain a

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table, namely *the dual of Table III*, where type $(i)^*$ replaces (i), the group G_m replaces G_P and the orbit m^G replaces P^G. Here m is any line of $[Q]$. Remind that, we denote by x_i^*, the number of G-orbits on $[Q]$ of type $(i)^*$. As mentioned in section 4, we write x_i^* instead of $x_i^*(Q)$, even if the second notation would be correct. Nevertheless we use the first one, since it will be clear from the context which point we are focusing on. In particular, since we might have G-orbits of type $(10)^*$, it makes sense considering $\mathcal{S}^* = \sum_{j=1}^{x_{10}^*} \frac{q^2-1}{2d_j}$ and hence $\mathcal{S}_2^*$, $\mathcal{S}_{2'}^*$, $\mathcal{S}_4^*$, $\mathcal{S}_{2,4}^*$ with the same meaning of $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_4$, $\mathcal{S}_{2,4}$, respectively, but referred to lines instead of points. Clearly, $\mathcal{S}_2^* = \mathcal{S}_{2,4}^* = \mathcal{S}_4^* = 0$, since $q \equiv 3 \bmod 8$.

Note that σ is a Baer collineation of Π by Lemma 3.3. Set $C = C_G(\sigma)$. Then C acts on $\mathrm{Fix}(\sigma)$ with kernel K. Hence, let $\bar{C} = C/K$. Clearly, $\langle\sigma\rangle \trianglelefteq K \trianglelefteq C$. Furthermore, either $K \trianglelefteq Z_{\frac{q+1}{2}}$ or $K = C$, since $C \cong D_{q+1}$ and $q \equiv 3 \bmod 8$. We need to investigate the admissible structure of K in order to show that T induces a Baer collineation on $\mathrm{Fix}(\sigma)$.

Lemma 5.1. *If* $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$*, then* $K = C$*.*

Proof. Assume that $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$ and that $K \trianglelefteq Z_{\frac{q+1}{2}}$. Then $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ by Table III, since $q > 9$. Set $l_0 = \mathrm{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \mathrm{Fix}(G) \cap l$. In particular, $\bar{C} \cong D_{\frac{q+1}{k}}$, where $k = |K|$, since $C \cong D_{q+1}$. On the other hand, $\bar{C}$ is the semidirect product of $\bar{C}(l_0, l_0)$ with $\bar{C}(Y, l_0)$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$ by [16, Theorem 4.25].

Assume that $\bar{C}(l_0, l_0) \neq \langle 1 \rangle$. If $\bar{C}(l_0, l_0) = \bar{C}$, then $\bar{C} = \bar{C}(V, l_0)$, for some point $V \in l_0$ by [16, Theorem 4.14], since $\bar{C} \cong D_{\frac{q+1}{k}}$ and $q \equiv 3 \bmod 8$. Hence for each point $X \in l_0 - \{V\}$ and for each line $t \in [X] \cap \mathrm{Fix}(\sigma)$, we have that $\sigma \in G_t$ but G_t does not contain Klein groups. Then, by dual of Table III, we have that $G_t \cong D_{q-1}$, since G fixes X. Moreover, $K \leq G_t$. Thus, $K = \langle\sigma\rangle$, since $\langle\sigma\rangle \trianglelefteq K \leq Z_{\frac{q+1}{2}}$. Therefore, $\bar{C} \cong D_{\frac{q+1}{2}}$ and hence $\frac{q+1}{2} \mid \sqrt{n}$ as $\bar{C} = \bar{C}(V, l_0)$. Actually, $\sqrt{n} = \frac{q+1}{2}$, since $\sqrt{n} < q$ by our assumptions. Then $\sqrt{n} \equiv 2 \bmod 4$, since $q \equiv 3 \bmod 8$. Hence, we arrive at a contradiction by Lemma 3.3. So, $\bar{C}(l_0, l_0) < \bar{C}$. Then $\bar{C}(l_0, l_0) \leq Z_{\frac{q+1}{k}}$, since $\bar{C}(l_0, l_0) \triangleleft \bar{C}$, $\bar{C} \cong D_{\frac{q+1}{k}}$ and $q \equiv 3 \bmod 8$. Actually, $\bar{C}(l_0, l_0) = \bar{C}(V, l_0) \cong Z_{\frac{q+1}{k}}$ and $\bar{C}(Y, l_0) \cong Z_2$ by [16, Theorems 4.14 and 4.25]. Let $u \in [V] \cap \mathrm{Fix}(\sigma) - \{l, VY\}$. Then u is fixed by K and by $\bar{C}(V, l_0)$. Therefore, $Z_{\frac{q+1}{2}} \leq G_u < C_G(\sigma)$. Thus either $G_u = C_G(\sigma)$ or $G_u = G$ by dual of Lemma 3.4, since G fixes l_0 and $q > 9$. We again obtain a contradiction, since $G_u < C_G(\sigma)$. Hence, $\bar{C}(l_0, l_0) = \langle 1 \rangle$.

Assume that $\bar{C} = \bar{C}(Y, l_0)$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$. Let $Q \in l_0$ and let $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, YQ\}$. Then $\sigma \in G_m$ but G_m does not contain Klein groups. Then $G_m \cong D_{q-1}$ by dual of Table III. Thus, $x_2^* \geq 1$. Furthermore,

$x_3^* \geq 1$, since $G_{YQ} = C$ and $C \cong D_{q+1}$ as $q \equiv 3 \bmod 8$. So, $x_2^* + x_3^* \geq 2$. This is a contradiction by dual of Lemma 3.6(1), as $q > 9$. $\square$

Lemma 5.2. $\mathrm{Fix}(T) \cap l \subset \mathrm{Fix}(\sigma) \cap l$.

Proof. Assume that $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$. Then $K = C$ by Lemma 5.1. As a consequence $\mathrm{Fix}(T) = \mathrm{Fix}(\sigma)$. Let P be any point of $\mathrm{Fix}(\sigma) \cap l$ and let r be any line of $[P] - \{l\}$. Then $C \leq G_r$. Since $q > 9$, then C is maximal in G and hence either $G_r = C$ or $G_r = G$. If the former occurs, then $|\mathrm{Fix}_{r^G}(T)| = 3$ and $|\mathrm{Fix}_{r^G}(\sigma)| = \frac{q+3}{2}$ by dual of table III. Hence $|\mathrm{Fix}_{r^G}(\sigma)| > |\mathrm{Fix}_{r^G}(T)|$ as $q > 9$. A contradiction, since $\mathrm{Fix}(T) = \mathrm{Fix}(\sigma)$. Hence $G_r = G$ for any point P of $\mathrm{Fix}(\sigma) \cap l$ and for any line r of $[P] - \{l\}$. Thus $\mathrm{Fix}(G) = \mathrm{Fix}(\sigma)$, since $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ and $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(\sigma)$. So, G fixes a Baer subplane of Π. Then G is semiregular on $l - \mathrm{Fix}(G)$ and hence $|G| \mid n - \sqrt{n}$, which is a contradiction. Thus, we have proved the assertion. $\square$

The previous lemma rules the possibility for T to induce either the identity or a perspectivity of axis $\mathrm{Fix}(\sigma) \cap l$ on $\mathrm{Fix}(\sigma)$. Hence, T induces either a perspectivity of axis distinct from $\mathrm{Fix}(\sigma) \cap l$ or a Baer involution on $\mathrm{Fix}(\sigma)$. The following lemma shows that only the second case is admissible.

Lemma 5.3. *The group T induces a Baer collineation on* $\mathrm{Fix}(\sigma)$.

Proof. The group T induces an involution $\bar{\beta}$ on $\mathrm{Fix}(\sigma)$ by Lemma 5.2. Assume that $\bar{\beta}$ is an involutory $(C_{\bar{\beta}}, a_{\bar{\beta}})$-perspectivity on $\mathrm{Fix}(\sigma)$. Then $C_{\bar{\beta}} \in l$ and $a_{\bar{\beta}} \neq l$ again by Lemma 5.2, since G fixes l. Then $|\mathrm{Fix}(T) \cap l| = 1$ or 2, where $|\mathrm{Fix}(T) \cap l| = x_1 + 3x_3 + x_4 + x_5$ by table III. Clearly, $x_3 = 0$. Furthermore, by table III, we have the following system of Diophantine equations:

$$\sqrt{n} + 1 = x_1 + \frac{q+1}{2} x_2 + \frac{q+1}{4} x_4 + \frac{q+1}{4} x_5 \tag{80}$$

$$n + 1 = x_1 + \frac{q(q+1)}{2} x_2 + \frac{q(q^2-1)}{24} x_4 + \frac{q(q^2-1)}{120} x_5 + \mathcal{S}\,. \tag{81}$$

Suppose that $\bar{\beta}$ is an involutory $(C_{\bar{\beta}}, a_{\bar{\beta}})$-elation of $\mathrm{Fix}(\sigma)$. Then $\mathrm{Fix}(T) \cap l = \{C_{\bar{\beta}}\}$, since $C_{\bar{\beta}} \in l$ and $a_{\bar{\beta}} \neq l$ by the above argument. Thus, $x_1 + x_4 + x_5 = 1$, since $|\mathrm{Fix}(T) \cap l| = x_1 + x_4 + x_5$. Clearly, G cannot fix $C_{\bar{\beta}}$, otherwise we have a contradiction by dual of Lemma 5.2, since $\mathrm{Fix}(T) \cap [C_{\bar{\beta}}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}}]$. Consequently $x_1 = 0$ and $x_4 + x_5 = 1$, since $x_1 + x_4 + x_5 = 1$. Then either $x_4 = 1$ and $x_1 = x_5 = 0$, or $x_5 = 1$ and $x_1 = x_4 = 0$.

Assume that $x_4 = 1$ and $x_1 = x_5 = 0$. Then $x_2 = 0$ by Lemma 3.6(3) being $q \neq 17$. Moreover, $\sqrt{n} + 1 = \frac{q+1}{4}$ by (80). Hence, $\sqrt{n} = \frac{q-3}{4}$. Since

$\sqrt{n} > \sqrt{q}$ by our assumptions, then $\frac{q-3}{4} > \sqrt{q}$. This yields $q > 21$. So we obtain a contradiction, since $q = 11$ or 19 by Lemma 3.4, as $x_4 = 1$ and $q \equiv 3 \bmod 8$.

Assume that $x_5 = 1$ and $x_1 = x_4 = 0$. Then $q = 11, 19, 59$ by Lemma 3.4, since $q \equiv 3 \bmod 8$. If $x_2 = 1$, then $\sqrt{n} + 1 = \frac{q+1}{4} + \frac{q+1}{2}$ (80). Therefore, $\sqrt{n} = \frac{3q-1}{4}$. Actually, $q \neq 59$ by Lemma 3.6(6). If $q = 19$, then $\sqrt{n} = 14$. Nevertheless, this case cannot occur by Lemma 3.3. Hence, $q = 11$ and $\sqrt{n} = 8$. This contradicts the fact that $n \geq 65$, since $n + 1 \geq \frac{q(q+1)}{2}$ by (81), as $x_2 = 1$. Thus, $x_2 = 0$. Then $\sqrt{n} = \frac{q-3}{4}$, by (80), as $x_1 = x_2 = x_4 = 0$. This is impossible for $q = 11$ or 19 by the above argument. As a consequence, $q = 59$ and $\sqrt{n} = 14$. Nevertheless, this case cannot occur by Lemma 3.3.

Suppose that $\bar{\beta}$ is an involutory $(C_{\bar{\beta}}, a_{\bar{\beta}})$-homology of $\mathrm{Fix}(\sigma)$. Again, $C_{\bar{\beta}} \in l$ and $a_{\bar{\beta}} \neq l$ by the above argument. Hence, $|\mathrm{Fix}(T) \cap l| = 2$. Then $x_1 + x_4 + x_5 = 2$, since $|\mathrm{Fix}(T) \cap l| = x_1 + 3x_3 + x_4 + x_5$ and $x_2 = 0$. It follows that, $x_1 \leq 1$. Therefore, $x_4 + x_5 \geq 1$, since G cannot fix $C_{\bar{\beta}}$ and since $\mathrm{Fix}(G) \subset \mathrm{Fix}(T)$. Thus, either $x_1 = x_4 = 1$ and $x_5 = 0$, or $x_1 = x_5 = 1$ and $x_4 = 0$, or $x_1 = 0$ and $x_4 + x_5 = 2$.

Assume that $x_1 = x_4 = 1$ and $x_5 = 0$. Then $x_2 = 0$ by Lemma 3.6(3). So, $\sqrt{n} = \frac{q+1}{4}$ by (80). Furthermore, $q = 11$ or 19 by Lemma 3.4(4), since $q \equiv 3 \bmod 8$. We obtain a contradiction as above, since $\frac{q+1}{4} > \sqrt{q}$ being $\frac{q+1}{4} > \frac{q-3}{4}$.

Assume that $x_1 = x_5 = 1$ and $x_4 = 0$. Then $q = 11, 19, 59$ by Lemma 3.4(5), since $q \equiv 3 \bmod 8$. If $x_2 = 1$, then $\sqrt{n} = \frac{q+1}{4} + \frac{q+1}{2}$ by (80) and hence $\sqrt{n} = \frac{3q+3}{4}$. Furthermore, $q \neq 59$ by Lemma 3.6(6). If $q = 11$, then $\sqrt{n} = 9$. In addition, $\mathcal{S} = 4$ by (81), since $x_1 = x_2 = x_5 = 1$. This is impossible, since $\frac{q+1}{2} = 6$ must divide $\mathcal{S}$ by the definition of this one. Hence, $q = 19$ and $\sqrt{n} = 15$, which is a contradiction by Lemma 3.3. Thus, $x_2 = 0$ and $\sqrt{n} = \frac{q+1}{4}$. If $q = 59$, then $\sqrt{n} = 15$ and we have a contradiction by the previous argument. Consequently, $q = 11$ or 19. Moreover, $\sqrt{n} = \frac{q+1}{4}$ by (80), since $x_1 = x_2 = x_4 = 0$ and $x_5 = 1$. Nevertheless this cannot occur by the above argument, since $q = 11$ or 19.

Finally, assume that $x_1 = 0$ and $x_4 + x_5 = 2$. Then $\sqrt{n} + 1 = \frac{q+1}{2} x_2 + \frac{q+1}{2}$ by (80). If $x_2 \geq 1$, then $\sqrt{n} \geq q$. Nevertheless, this cannot occur by our assumption. So, $x_2 = 0$ and hence $\sqrt{n} = \frac{q+1}{2} - 1$. That is $\sqrt{n} = \frac{q-1}{2}$. If $x_4 > 0$, then $\left(\frac{q-1}{2}\right)^2 + 1 \geq \frac{q(q^2-1)}{24}$ by (81), where $q = 11$ or 19 by Lemma 3.4(4), as $q \equiv 3 \bmod 8$. Easy computations yield a contradiction. Therefore, $x_4 = 0$ and $x_5 = 2$, since $x_4 + x_5 = 2$. Then $n + 1 = \frac{q(q^2-1)}{60} + \mathcal{S}$ by (81), where $n = \left(\frac{q-1}{2}\right)^2$. It follows that, $\mathcal{S} = \left(\frac{q-1}{2}\right)^2 + 1 - \frac{q(q^2-1)}{60}$. In particular, $q = 11, 19, 59$ by Lemma 3.4(5), since $q \equiv 3 \bmod 8$. Easy computation yield $\mathcal{S} = 4$ or -32 or -2580. So the cases $q = 19$ or 59 are ruled out, since $\mathcal{S} \geq 0$ by the definition of this one. Hence $q = 11$ and $\mathcal{S} = 4$. Nevertheless, this case cannot occur, since $\frac{q+1}{2} = 6$ must divide $\mathcal{S}$ again by the definition of this one. Thus, T induces a

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Baer collineation on $\mathrm{Fix}(\sigma)$. $\square$

Lemma 5.4. *For each point $P \in l$, the group G_P cannot be isomorphic either to A_4 or to A_5.*

Proof. The group T induces a Baer collineation on $\mathrm{Fix}(\sigma)$ by Lemma 5.3. So, $|\mathrm{Fix}(T) \cap l| = \sqrt[4]{n}+1$. By Table III, we have the following system of Diophantine equations:

$$\sqrt[4]{n}+1 = x_1+3x_3+x_4+x_5 \tag{82}$$

$$\sqrt{n}+1 = x_1+\frac{q+1}{2}x_2+\frac{q+3}{2}x_3+\frac{q+1}{4}x_4+\frac{q+1}{4}x_5 \tag{83}$$

$$n+1 = x_1+\frac{q(q+1)}{2}x_2+\frac{q(q-1)}{2}x_3+\frac{q(q^2-1)}{24}x_4+\frac{q(q^2-1)}{120}x_5+\mathcal{S}\,. \tag{84}$$

Assume that $x_4 > 0$. Then $x_4 = 1$ by Lemma 3.5(3). Consequently $x_2 = x_3 = 0$ by Lemma 3.6(3). Furthermore, $q = 11$ or 19 by Lemma 3.4, since $q \equiv 3 \bmod 8$. If $q = 11$, then either $\sqrt[4]{n} = 2$ or 3, since $n < q^2$ by our assumption. On the other hand, $n + 1 \geq \frac{q(q^2-1)}{24}$, since $x_4 = 1$. Thus the case $\sqrt[4]{n} = 2$ cannot occur. Hence, $\sqrt[4]{n} = 3$. Then $x_1+x_5 = 3$ and $x_1+3x_5 = 7$ by (82) and (83), since $x_2 = x_3 = 0$ and $x_4 = 1$. Thus, $x_1 = 1$ and $x_5 = 2$. So, $\mathcal{S} = 59$ by (84), which is a contradiction, since $\frac{q+1}{2} = 6$ must divide $\mathcal{S}$ by the definition of this one. As a consequence $q = 19$ and hence $\sqrt[4]{n} = 3$ or 4, since $q < n < q^2$. Nevertheless, this contradicts the fact that $n + 1 \geq \frac{q(q^2-1)}{24}$, being $x_4 = 1$. Therefore $x_4 = 0$.

Assume that $x_5 > 0$. Then $q = 11, 19$ or 59 by Lemma 3.4, since $q \equiv 3 \bmod 8$. If $x_3 > 0$, then $x_3 = 1$ by Lemma 3.5(2). Furthermore, $x_2 = 0$ an $q \neq 59$ by Lemma 3.6(2) and (3). Thus, $x_1 = 0$ and $x_5 = 1$ by (82), since $x_4 = 0$ and $x_5 > 0$. Now, by substituting $x_1 = x_2 = x_4 = 0$ and $x_3 = x_5 = 1$ in (83), we obtain $\sqrt{n} = \frac{3(q+1)}{4}$. Then $\sqrt{n} = 9$ for $q = 11$ and $\sqrt{n} = 15$ for $q = 19$. The latter is ruled out by Lemma 3.3. Hence, $\sqrt{n} = 9$ and $q = 11$, which is a contradiction, since $n + 1 \geq \frac{q(q-1)}{2}$ as $x_3 = 1$. So, $x_3 = 0$.

Now, assume that $x_2 > 0$. Then $x_2 = 1$ by Lemma 3.5(1). Then $q \neq 59$ by Lemma 3.6(3). Then $\sqrt{n} + 1 \geq \frac{q+1}{2} + \frac{q+1}{4}$ by (83), as $x_2, x_5 > 0$. Therefore $\frac{3q-1}{4} \leq \sqrt{n} < \sqrt{q}$. Then $\sqrt{n} = 9$ for $q = 11$ and $\sqrt{n} = 16$ for $q = 19$. Assume the former occurs. Then $x_1 + x_5 = 4$ and $x_1 + 3x_5 = 4$ by (82) and (83), respectively, since $x_2 = x_4 = 0$. Consequently, $x_5 = 0$. Hence, we arrive at a contradiction by our assumptions. Thus, $\sqrt{n} = 16$ for $q = 19$. Then $x_1 + x_5 = 5$ and $x_1 + 5x_5 = 7$ by (82) and (83), respectively, since $x_2 = x_4 = 0$. Since x_1 and x_5 must be integers, the previous equation have no solutions. As a consequence, $x_2 = x_3 = 0$.

Now, subtracting (82) from (83), we obtain $\sqrt{n} - \sqrt[4]{n} = \frac{q-3}{4}x_5$, since $x_2 = x_3 = x_4 = 0$. Easy computations for $q = 11, 19$ or 59, being $0 < x_5 \leq 3$

by Lemma 3.5(4), show that the admissible solutions for $\sqrt{n}-\sqrt[4]{n}=\frac{q-3}{4}x_5$ and $x_1=\sqrt[4]{n}+1-x_5$ are $(q,\sqrt[4]{n},x_1,x_5)=(11,2,2,1),(11,3,1,3),(19,4,2,3)$ and $(59,7,5,3)$. Now, by substituting these values in (84) and bearing in mind that $x_2=x_3=x_4=0$, we obtain $\mathcal{S}=4,48,84$ or -2736, respectively. The case $(q,\sqrt[4]{n},x_1,x_5)=(59,7,5,3)$ cannot occur, since it must be $\mathcal{S}\geq 0$. Furthermore, $\frac{q+1}{2}$ must divide $\mathcal{S}$ by the definition of this one. Then also the cases $(q,\sqrt[4]{n},x_1,x_5)=(11,2,2,1)$ and $(19,4,2,3)$ cannot occur. Thus, $(q,\sqrt[4]{n},x_1,x_5)=(11,3,1,3)$ and $\mathcal{S}=48$. Let Y_h^G, $h=1,2,3$, the three distinct orbits of type (5) on l. Then $\left|Y_h^G\right|=11$ for each $h=1,2,3$ and hence $\left|l-\cup_{h=1}^{3}Y_h^G\right|=48$. As $\mathcal{S}=48$, then $x_{10}>0$. Hence, let $X\in l$ such that $G_X\leq Z_{11}.Z_5$. Then $G_X\cong Z_{11}.Z_5$, since $\left|X^G\right|\leq 48$ as $X^G\subset l-\cup_{h=1}^{3}Y_h^G$. Thus, each orbit of type (10) has length 12. Therefore, $x_{10}=3$, since $\mathcal{S}=48$. In particular, G acts 2-transitively on each of the orbits of type (10). Let A be a subgroup of G such that $A\cong Z_{11}$. Then A fixes exactly 1 point in each of the three G-orbits of type (10), since G acts 2-transitively on each of them. Hence, A fixes exactly 4 points on l by Table III, since $x_1=1$ and $x_5=x_{10}=3$. This is impossible, since A must fix at least 5 points on l and since $n+1\equiv 5 \bmod 11$, being $n=3^4$. So, $x_4=x_5=0$ and we have proved the assertion. □

Proposition 5.5. *Let Π be a projective plane of order n admitting a collineation group $G\cong \mathsf{PSL}(2,q)$, $q>3$. If $q<n<q^2$ and $q\equiv 3 \bmod 8$, then G does not fix lines of Π.*

Proof. Assume that G fixes a line l of Π. Note that $q>9$, since $q\equiv 3 \bmod 8$ and $q>3$. Now, $|\mathrm{Fix}(T)\cap l|=\sqrt[4]{n}+1$ by Lemma 5.3. Furthermore, $x_4=x_5=0$ by Lemma 5.4. Then, by table III, we have

$$\sqrt[4]{n}+1=x_1+3x_3 \tag{85}$$

$$\sqrt{n}+1=x_1+\frac{q+1}{2}x_2+\frac{q+3}{2}x_3 \tag{86}$$

$$n+1=x_1+\frac{q(q+1)}{2}x_2+\frac{q(q-1)}{2}x_3+\mathcal{S}\,. \tag{87}$$

Assume that $x_3>0$. Then $x_3=1$ and $x_2=0$ by Lemma 3.6(1). Thus, $\sqrt[4]{n}=x_1+2$ and $\sqrt{n}=x_1+\frac{q+3}{2}$ by (85) and (86), respectively. By composing these equations, we have $(x_1+2)^2=x_1+\frac{q+1}{2}$ and hence $x_1^2+x_1-\frac{q-7}{2}=0$. If $x_1\leq 2$, it is easily seen that, $(n,x_1,q)=(3^4,1,11)$ or $(4^4,2,19)$. Let B^G be the G-orbit on l of type (3). Then $l-(\mathrm{Fix}(G)\cup B^G)\neq\emptyset$. Moreover, it consists of G-orbits of type (10). Then $q+1\mid n+1-x_1-\left|B^G\right|$, since $\left|l-(\mathrm{Fix}(G)\cup B^G)\right|=n+1-x_1-\left|B^G\right|$ and since each G-orbit of type (10) has length divisible by $q+1$. Hence, we arrive at a contradiction in any case, since $\left|B^G\right|=55$ for $(n,x_1,q)=(3^4,1,11)$ and $\left|B^G\right|=171$ for $(n,x_1,q)=(4^4,2,19)$. Thus, $x_1\geq 3$

for $x_3 > 0$. Actually, $x_1 \geq 3$ also for $x_3 = 0$, since $\sqrt[4]{n} \geq 2$. So, $x_1 \geq 3$ in any case. Thus, G fixes at least 3 points on l.

Let Q be any of the points fixed by G on l. Clearly, $|\mathrm{Fix}(T) \cap [Q]| = \sqrt[4]{n} + 1$ by Lemma 5.3. Applying the dual of Lemma 5.4, we obtain that G_r cannot be isomorphic either to A_4 or to A_5 for each $r \in [Q] - \{l\}$. Therefore, $x_4^* = x_5^* = 0$. Consequently, we obtain the same system of Diophantine equations as (85), (86) and (87) but referred to $[Q]$ and hence with the x_i^* in the role of the x_i. At this point, the above argument yields that G fixes at least 3 lines (including l) through any point Q of $\mathrm{Fix}(G) \cap l$. Thus, G fixes a subplane of Π pointwise, as $|\mathrm{Fix}(G) \cap l| \geq 3$. In particular, $o(\mathrm{Fix}(G)) = x_1 - 1$.

Assume that $\mathrm{Fix}(G) \subset \mathrm{Fix}(T)$. Then either $\sqrt[4]{n} = (x_1 - 1)^2$ or $\sqrt[4]{n} \geq (x_1 - 1)^2 + (x_1 - 1)$ by [16, Theorem 3.7], since T induces a Baer collineation on $\mathrm{Fix}(\sigma)$. Furthermore, there must be a G-orbit of type (3) on l. So $\sqrt[4]{n} = x_1 + 2$ by (85). It follows that, either $x_1 + 2 = (x_1 - 1)^2$ or $x_1 + 2 \geq (x_1 - 1)^2 + (x_1 - 1)$. Easy computations show that, no one of them occurs, since $x_1 \geq 3$. Hence, $\mathrm{Fix}(G) = \mathrm{Fix}(T)$. Thus, G fixes a subplane of Π of order $\sqrt[4]{n}$. This forces $x_3 = 0$ which yields $\sqrt[4]{n} + 1 = x_1$ in (85). Then $x_2 > 0$ by (86). Actually, $x_2 = 1$ by Lemma 3.5(1). So, $\sqrt{n} - \sqrt[4]{n} = \frac{q+1}{2}$ by (86). If $\mathcal{S} = 0$, then $n - \sqrt[4]{n} = \frac{q(q+1)}{2}$ by (87). Note that $n - \sqrt[4]{n} = (\sqrt{n} - \sqrt[4]{n})(\sqrt{n} + \sqrt[4]{n} + 1)$. As $n - \sqrt[4]{n} = \frac{q(q+1)}{2}$ and $\sqrt{n} - \sqrt[4]{n} = \frac{q+1}{2}$, then $\frac{q(q+1)}{2} = \frac{q+1}{2}(\sqrt{n} + \sqrt[4]{n} + 1)$. By elementary calculations of the previous equality, we obtain $\sqrt{n} + \sqrt[4]{n} = q - 1$. Thus, $\sqrt[4]{n} \mid q - 1$. On the other hand, $\sqrt[4]{n} \mid q + 1$, since $\sqrt{n} - \sqrt[4]{n} = \frac{q+1}{2}$. So, $\sqrt[4]{n} = 2$. Then $q = 3$, since $\sqrt{n} - \sqrt[4]{n} = \frac{q+1}{2}$, which is a contradiction by our assumptions. Therefore, $\mathcal{S} > 0$. Then a Sylow p-subgroup S of G fixes at least one point on $l - \mathrm{Fix}(G)$. Consequently, $\mathrm{Fix}(S)$ is a Baer subplane of Π by [16, Theorem 3.7], since $\mathrm{Fix}(G) \subset \mathrm{Fix}(S)$ and since $\mathrm{Fix}(G)$ fixes a subplane of Π of order $\sqrt[4]{n}$. It follows that, S is semiregular on $l - \mathrm{Fix}(S)$ and $q \mid n - \sqrt{n}$, since $|S| = q$. This yields that, either $q \mid \sqrt{n} - 1$ or $q \mid \sqrt{n}$, as q is a prime power. Hence, $\sqrt{n} \geq q$ in any case, which is a contradiction by our assumptions. As a consequence, G does not fix lines of Π. □

Theorem 5.6. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2, q)$, with $q \equiv 3 \bmod 8$ and $q > 3$. If $n \leq q^2$, then one of the following occurs:*

(1) *$n = q$, $\Pi \cong \mathsf{PG}(2, q)$ and G is strongly irreducible on Π;*

(2) *$q < n < q^2$ and G is strongly irreducible on Π;*

(3) *$n = q^2$ and one of the following occurs:*

   (a) *G is strongly irreducible on Π;*

(b) *G fixes a Baer subplane $\Pi_0 \cong \mathsf{PG}(2,q)$ of Π.*

Proof. No cases arise for $n < q$ by Theorem 2.1, as $q > 3$. If $n = q$, the assertions (1) easily follows by Theorem 2.2. If $q < n < q^2$, the group G does not fix lines or points of Π by Proposition 5.5 and its dual. Now, the assertion (2) follows in this case by Lemma 3.1, since $q \equiv 3 \bmod 8$ and $q > 3$. When $n = q^2$, the assertions (3a) and (3b) follow by Theorem 2.3 and Corollary 2.4, respectively. $\square$

Finally, when $q \equiv 3 \bmod 8$, Theorem 1.1 easily follows from Theorem 5.6.

6. The case $q \equiv 5 \bmod 8$

Recall that σ and T are the representatives of the unique conjugate class of involutions and Klein subgroups of G, respectively. Recall also that T is chosen such that $\sigma \in T$. Furthermore, $C_G(\sigma) \cong D_{q-1}$ and $N_G(T) \cong A_4$. We filter the list given in Lemma 3.4 with respect to the condition $q \equiv 5 \bmod 8$. Now, arguing as in the beginning of the previous section, we obtain the following table.

Table IV

Type	G_P	$[G : G_P]$	$\lvert \mathrm{Fix}_{PG}(\sigma)\rvert$	$\lvert \mathrm{Fix}_{PG}(T)\rvert$
1	G	1	1	1
2	D_{q-1}	$\frac{q(q+1)}{2}$	$\frac{q+1}{2}$	3
3	D_{q+1}	$\frac{q(q-1)}{2}$	$\frac{q-1}{2}$	0
4	A_4	$\frac{q(q^2-1)}{24}$	$\frac{q-1}{4}$	1
5	A_5	$\frac{q(q^2-1)}{120}$	$\frac{q-1}{4}$	1
10	$F_q.Z_d$	$\frac{q^2-1}{2d}$	$\frac{q-1}{d}$ $2 \mid d$ 0 $2 \nmid d$	0

By section 3, the G-orbits of type (10) on l cover exactly $\mathcal{S}$ points of l, where $\mathcal{S} = \sum_{j=1}^{x_{10}} \frac{q^2-1}{2d_j}$. Moreover, $\mathcal{S}_1 = \sum_{j=1}^{x_{10}} \frac{q-1}{d_j}$ and $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_4$, $\mathcal{S}_{2,4}$ (sum with the same summands $\frac{q-1}{d_j}$ but over $2 \mid d_j$, $2 \nmid d_j$, $4 \mid d_j$ and $d_j \equiv 2 \bmod 4$, respectively). In particular, $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$. Note also that, $\mathcal{S}_4 = 0$, since $q \equiv 5 \bmod 8$.

If G fixes a point Q and acts on $[Q]$, we may focus on the G-orbits of lines in $[Q]$. So, following the notation introduced in section 4, we obtain a table,

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namely *the dual of Table IV*, where type $(i)^*$ replaces (i), the group G_m replaces G_P and m^G replaces P^G. Here, m is any line of $[Q]$. Recall that, we denote by x_i^* the number of G-orbits on $[Q]$ of type $(i)^*$. As mentioned in section 4, we write x_i^* instead of $x_i^*(Q)$, even if the second notation would be correct. It will be clear from the context which point we are focusing on. In particular, since we might have G-orbits of type $(10)^*$, it makes sense considering $\mathcal{S}^* = \sum_{j=1}^{x_{10}^*} \frac{q^2-1}{2d_j}$ and hence $\mathcal{S}_2^*$, $\mathcal{S}_{2'}^*$, $\mathcal{S}_4^*$, $\mathcal{S}_{2,4}^*$ with the same meaning of $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_4$, $\mathcal{S}_{2,4}$, respectively, but referred to lines instead of points. Clearly, $\mathcal{S}_4^* = 0$, since $q \equiv 5 \bmod 8$.

The collineation σ is a Baer collineation of Π by Lemma 3.3. Set $C = C_G(\sigma)$. Then C acts on $\mathrm{Fix}(\sigma)$ with kernel K. Hence, let $\bar{C} = C/K$. Clearly, $\langle\sigma\rangle \trianglelefteq K \trianglelefteq C$. Furthermore, either $K \trianglelefteq Z_{\frac{q-1}{2}}$ or $K = C$, since $C \cong D_{q-1}$ and $q \equiv 5 \bmod 8$. As we will see, we need to investigate the structure of K in order to show that T induces a Baer collineation on $\mathrm{Fix}(\sigma)$.

Lemma 6.1. *If* $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$, *then* $K = C$.

Proof. Assume that $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$ and that $K \trianglelefteq Z_{\frac{q-1}{2}}$. Then $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ by table IV, since $q > 9$. Set $l_0 = \mathrm{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \mathrm{Fix}(G) \cap l$. In particular, $\bar{C} \cong D_{\frac{q-1}{k}}$, where $k = |K|$, since $K \trianglelefteq Z_{\frac{q-1}{2}}$ and $C \cong D_{q-1}$. On the other hand, $\bar{C}$ is the semidirect product of $\bar{C}(l_0, l_0)$ with $\bar{C}(Y, l_0)$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$ by [16, Theorem 4.25].

Assume that $\bar{C}(l_0, l_0) \neq \langle 1\rangle$. Assume that also that $\bar{C}(l_0, l_0) = \bar{C}$. Then $\bar{C} = \bar{C}(V, l_0)$, for some point $V \in l_0$ by [16, Theorem 4.14], since $\bar{C} \cong D_{\frac{q-1}{k}}$, $\langle\sigma\rangle \trianglelefteq K \trianglelefteq Z_{\frac{q-1}{2}}$ and $q \equiv 5 \bmod 8$. Hence, for each point $X \in l_0 - \{V\}$ and for each line $t \in [X] \cap \mathrm{Fix}(\sigma)$, we have that $\sigma \in G_t$ but G_t does not contain Klein groups. Then, by dual of table IV, we have that either $G_t \cong D_{q+1}$ or $G_t \cong F_q.Z_d$ with d even, since G fixes X. Clearly, $K \leq G_t$ and $\langle\sigma\rangle \trianglelefteq K \leq Z_{\frac{q+1}{2}}$. Thus, $K = \langle\sigma\rangle$, since $2 \mid |G_t|$ but $4 \nmid |G_t|$ as $q \equiv 5 \bmod 8$. Therefore, $\bar{C} \cong D_{\frac{q-1}{2}}$ and $\frac{q-1}{2} \mid \sqrt{n}$. Actually, either $\sqrt{n} = \frac{q-1}{2}$ or $\sqrt{n} = q-1$, since $\sqrt{n} < q$ by our assumptions. If $\sqrt{n} = \frac{q-1}{2}$, then $\sqrt{n} \equiv 2 \bmod 4$ as $q \equiv 5 \bmod 8$. This is a contradiction by Lemma 3.3. So $\sqrt{n} = q-1$. Note that, either $G_t \cong D_{q+1}$ or $G_t \cong F_q.Z_d$, with $d = d(t)$ even, for each line $t \in [X] \cap \mathrm{Fix}(\sigma)$ such that $t \neq l$. Moreover, $\mathrm{Fix}(T) \cap [V] = \mathrm{Fix}(\sigma) \cap [V]$, since $\bar{C} = \bar{C}(V, l_0)$. Then $\mathrm{Fix}(G) \cap [V] = \mathrm{Fix}(\sigma) \cap [V]$ by dual of table IV, since $q > 9$. Thus either $|G_r|$ is odd, or $2 \mid |G_r|$ but $4 \nmid |G_r|$ for each $r \in [X] - \mathrm{Fix}(\sigma)$. Consequently, either $G_t \cong D_{q+1}$ or $G_t \cong F_q.Z_d$, with $d = d(r)$, for each $r \in [X] - \mathrm{Fix}(\sigma)$ by dual of table IV, since G fixes X. In this case $d = d(r)$ might be also odd. Therefore, $[X]$ consists of G-orbits of type $(1)^*$, $(3)^*$ or $(10)^*$. Then, again by dual of table IV, we have

$$n = \frac{q(q-1)}{2} x_3^* + \frac{q+1}{2} \mathcal{S}_1^*, \tag{88}$$

since $x_1^* = 1$ (G fixes l) and since $\mathcal{S}^* = \frac{q+1}{2}\mathcal{S}_1^*$. Actually,

$$(q-1)^2 = \frac{q(q-1)}{2}x_3^* + \frac{q+1}{2}\mathcal{S}_1^* \,, \tag{89}$$

since Π has order $(q-1)^2$. Hence $\frac{q+1}{2} \mid (q-1)^2 - \frac{q(q-1)}{2}x_3^*$, where $x_3^* \leq 1$ by dual of Lemma 3.5(2). If $x_3^* = 0$, by elementary calculations of the last divisibility relation, we obtain $\frac{q+1}{2} \mid 4$. So, we arrive at a contradiction, since $q \equiv 5 \bmod 8$. Thus, $x_3^* = 1$. So, $q+1 \mid q^2 - 3q + 1$ by $\frac{q+1}{2} \mid (q-1)^2 - \frac{q(q-1)}{2}$. This is impossible, since $q+1$ is even while $q^2 - 3q + 1$ is odd. Then $\bar{C}(l_0, l_0) < \bar{C}$ and $\bar{C}(Y, l_0) \neq \langle 1 \rangle$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$. It follows that, $\bar{C}(l_0, l_0) \leq Z_{\frac{q-1}{k}}$, since $\bar{C} \cong D_{\frac{q-1}{k}}$. Actually, $\bar{C}(l_0, l_0) = \bar{C}(V, l_0) \cong Z_{\frac{q-1}{k}}$ and $\bar{C}(Y, l_0) \cong Z_2$ by [16, Theorems 4.14 and 4.25]. Let $R \in l_0 - \{V\}$ and set $f = RY$. Clearly, $\bar{C}(Y, l_0)$ fixes f. Then $D_{2k} \leq G_f$, where k is an even divisor of $\frac{q-1}{2}$, since $\langle \sigma \rangle \leq K \leq G_f$. If $k = 2$, then $\bar{C}(V, l_0) \cong Z_{\frac{q-1}{2}}$ and $\frac{q-1}{2} \mid \sqrt{n}$, arguing as above. So, $\sqrt{n}$ is even, as $q \equiv 5 \bmod 8$, which is a contradiction, since $\bar{C}(Y, l_0)$ consists of an involutory homology. Therefore, $k > 2$. Hence, $4 \mid 2k$ with $k > 2$. As a consequence, $C \leq G_f$ by dual of Table IV. Then $\bar{C}$ fixes f. Hence, we obtain a contradiction, since $\bar{C}(l_0, l_0) = \bar{C}(V, l_0) \neq \langle 1 \rangle$, while $f = RY$ with $R \in l_0 - \{V\}$. Thus, $\bar{C}(l_0, l_0) = \langle 1 \rangle$.

Assume that $\bar{C} = \bar{C}(Y, l_0)$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$. Let $Q \in l_0$ and $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, QY\}$. Then $\sigma \in G_m$ but G_m does not contain Klein subgroups of G. By dual of table IV, either $G_m \cong D_{q+1}$ or $G_m \cong F_q.Z_d$. So,

$$\sqrt{n} = \frac{q+1}{2}x_2^* + \frac{q-1}{2}x_3^* + \mathcal{S}_2^* \,. \tag{90}$$

Note that, $x_2^* > 0$, as $G_{QY} = C$. Then $x_2^* = 1$ by dual of Lemma 3.5(2). Hence, $x_3^* = 0$ by dual of Lemma 3.6(1). Furthermore, T fixes exactly 3 points on QY^G. Thus, $\sqrt{n} = 2$, since T must induce either a perspectivity or the identity on $\mathrm{Fix}(\sigma)$ as $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$. On the other hand, $\sqrt{n} \geq \frac{q+1}{2}$ by (90) as $x_2^* = 1$. This yields $\sqrt{n} \geq 5$, being $q > 9$, which is a contradiction, since we proved that $\sqrt{n} = 2$. □

Lemma 6.2. $\mathrm{Fix}(T) \cap l \subset \mathrm{Fix}(\sigma) \cap l$.

Proof. Assume that $\mathrm{Fix}(T) \cap l = \mathrm{Fix}(\sigma) \cap l$. Then $K = C$ by Lemma 6.1. Thus, $\mathrm{Fix}(T) = \mathrm{Fix}(\sigma)$. Let P be any point of $\mathrm{Fix}(\sigma) \cap l$ and let r be any line of $[P] - \{l\}$. Then $C \leq G_r$. Since $q > 9$, then C is maximal in G and hence either $G_r = C$ or $G_r = G$. If the former occurs, then $|\mathrm{Fix}_{r^G}(T_1)| = 3$ and $|\mathrm{Fix}_{r^G}(\sigma)| = \frac{q+1}{2}$ by dual of Table V. Hence, $|\mathrm{Fix}_{r^G}(\sigma)| > |\mathrm{Fix}_{r^G}(T_1)|$ as $q > 9$. This is a contradiction, since $\mathrm{Fix}(T_1) = \mathrm{Fix}(\sigma)$. So, $G_r = G$ for any point

P of $\mathrm{Fix}(\sigma)\cap l$ and for any line r of $[P]-\{l\}$. Thus $\mathrm{Fix}(G)=\mathrm{Fix}(\sigma)$, since $\mathrm{Fix}(G)\cap l=\mathrm{Fix}(\sigma)\cap l$ and $\mathrm{Fix}(G)\subseteq\mathrm{Fix}(\sigma)$. Therefore, G fixes a Baer subplane of Π. Then G is semiregular on $l-\mathrm{Fix}(G)$ and hence $|G|\mid n-\sqrt{n}$, which is a contradiction. So, we have proved the assertion. □

Lemma 6.3. *The group T induces a Baer collineation on* $\mathrm{Fix}(\sigma)$.

Proof. The group T induces a non trivial involution $\bar\beta$ on $\mathrm{Fix}(\sigma)$ by Lemma 6.2. Assume that $\bar\beta$ is an involutory $(C_{\bar\beta},a_{\bar\beta})$-perspectivity on $\mathrm{Fix}(\sigma)$. Then $a_{\bar\beta}\neq l$ again by Lemma 6.2. Thus, $|\mathrm{Fix}(T)\cap l|=1$ or 2. Therefore, $x_1+3x_2+x_4+x_5=1$ or 2, since $|\mathrm{Fix}(T)\cap l|=x_1+3x_2+x_4+x_5$ by table IV. Clearly $x_2=0$. Hence, $x_1+x_4+x_5=1$ or 2. Furthermore, by table IV, we have the following system of Diophantine equations:

$$\sqrt{n}+1=x_1+\frac{q-1}{2}x_3+\frac{q-1}{4}x_4+\frac{q-1}{4}x_5+\mathcal{S}_2 \tag{91}$$

$$n+1=x_1+\frac{q(q-1)}{2}x_3+\frac{q(q^2-1)}{24}x_4+\frac{q(q^2-1)}{120}x_5+\mathcal{S}\,. \tag{92}$$

Assume $|\mathrm{Fix}(T)\cap l|=1$. Then $\bar\beta$ is an involutory $(C_{\bar\beta},a_{\bar\beta})$-elation of $\mathrm{Fix}(\sigma)$ with $C_{\bar\beta}\in l$ and $a_{\bar\beta}\neq l$. So $\mathrm{Fix}(T)\cap l=\left\{C_{\bar\beta}\right\}$ and $x_1+x_4+x_5=1$. Clearly, G cannot fix $C_{\bar\beta}$, otherwise we obtain a contradiction by dual of Lemma 6.2, since $\mathrm{Fix}(T)\cap\left[C_{\bar\beta}\right]=\mathrm{Fix}(\sigma)\cap\left[C_{\bar\beta}\right]$. Consequently, $x_1=0$ and $x_4+x_5=1$.

Assume that $x_4=1$ and $x_5=0$. Then $x_3=0$ by Lemma 3.6(3) and $q=13$ by Lemma 3.4. So, either $\sqrt{n}=10$ or 12, since $\frac{q(q+1)}{2}\leq n<q^2+1$ with n an even square number. The former is ruled out by Lemma 3.3. Hence $\sqrt{n}=12$. Since $x_1=x_3=x_5=0$, since $\frac{q+1}{2}\mid\mathcal{S}$ and since $\frac{q+1}{2}$ divides the G-orbit of type (4) as $q=13$, then $\frac{q+1}{2}\mid n+1$ by (92). This cannot occur, since $\frac{q+1}{2}=7$ while $\sqrt{n}=12$.

Assume that $x_4=0$ and $x_5=1$. Then $q=29,61,101,109$ by Lemma 3.4, since $q\equiv 5 \bmod 8$. If $x_3=1$, then $q=29$ by Lemma 3.6(4). Let Q^G be an orbit of type (3). Clearly, $\left|Q^G\right|=\frac{q(q-1)}{2}$. Now, let R^G be an orbit of type (5), then $\left|R^G\right|=\frac{q(q^2-1)}{120}$. Since $Q^G\cup R^G\subseteq l$, it follows that, $n+1\geq\frac{q(q-1)}{2}+\frac{q(q^2-1)}{120}$. Then $n\geq 608$, since $\frac{q(q-1)}{2}+\frac{q(q^2-1)}{120}=609$, being $q=29$. So, $24<\sqrt{n}<29$, since $n<q^2$ and $q=29$. Actually, $\sqrt{n}=26$ cannot occur by Lemma 3.3. Therefore, $\sqrt{n}=28$, since $\sqrt{n}$ must be even. Thus, $\mathcal{S}=176$, since $n+1=608+\mathcal{S}$ by (92), since $x_1=x_4=0$, $x_3=x_5=1$ and $q=29$. Then $\mathcal{S}=176$, as $\sqrt{n}=28$. Hence $\frac{q+1}{2}\mid 176$, as $\frac{q+1}{2}\mid\mathcal{S}$, which is a contradiction, since $q=29$. Consequently, $x_3=0$. Then $n+1=\frac{q(q^2-1)}{120}+\mathcal{S}$ by (92), since $x_1=x_3=x_4=0$ and $x_5=1$. If $\mathcal{S}=0$, then $n=\frac{q(q^2-1)}{120}-1$ with $q=29,61,101,109$. We again obtain a contradiction, since n must be a square. Thus, $\mathcal{S}>0$. Since $\frac{q+1}{2}\mid\mathcal{S}$, then

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$\frac{q+1}{2} \mid n+1-\frac{q(q^2-1)}{120}$, since $n+1=\frac{q(q^2-1)}{120}+\mathcal{S}$. Moreover, $q=29,61,101,109$, and $\sqrt{\frac{q(q^2-1)}{120}}-1<n<q^2$, with n an even square number. Easy computations show that, only the case $\sqrt{n}=98$ and $q=101$ is admissible. Nevertheless, it cannot occur by Lemma 3.3.

Assume that $|\mathrm{Fix}(T)\cap l|=2$. Then $\bar{\beta}$ is an involutory $(C_{\bar{\beta}},a_{\bar{\beta}})$-homology of $\mathrm{Fix}(\sigma)$ with $C_{\bar{\beta}}\in l$ and $a_{\bar{\beta}}\neq l$. Furthermore, $x_1+x_4+x_5=2$, since $|\mathrm{Fix}(T)\cap l|=x_1+3x_2+x_4+x_5$ with $x_2=0$.

Assume that $x_4>0$. Then $x_4=1$ by Lemma 3.5(3). Then $x_2=x_3=0$ by Lemma 3.6(3). Moreover, $q=13$ by Lemma 3.4, since $q\equiv 5 \bmod 8$. So, $\sqrt[4]{n}=2$ or 3, since $q<n<q^2$ by our assumption. On the other hand, $n+1\geq\frac{q(q^2-1)}{24}$, with $q=13$, since $x_4=1$. Hence, we arrive at a contradiction. Thus, $x_4=0$ and either $x_1=x_5=1$, or $x_1=0$ and $x_5=2$, since $x_1\leq 1$ and $x_1+x_4+x_5=2$. In order to make easier the analysis of these two cases, we are going to show that $x_3=0$.

Assume that $x_3>0$. Then $x_3=1$ by Lemma 3.5(3). So, $q=29$ by Lemma 3.6(4), since $x_5\geq 1$. Therefore, $24<\sqrt{n}<29$, arguing as above. Then $\sqrt{n}=25$ or 27, since $\sqrt{n}$ is odd. Actually, the case $\sqrt{n}=27$ cannot occur by Lemma 3.3, as $\sqrt{n}\equiv 3 \bmod 4$. Thus, $\sqrt{n}=25$. Let X_1^G and X_2^G be the orbits on l of type (3) and (5), respectively, as $x_3=1$ and $x_5\geq 1$. Then $\left|X_1^G\right|=406$ and $\left|X_2^G\right|=203$ as $q=29$. Since $\sqrt{n}=25$, we have $\left|l-X_1^G-X_2^G\right|=17$. As the minimal primitive permutation representation of $G\cong \mathsf{PSL}(2,29)$ is 30, the group G fixes $l-X_1^G-X_2^G$ pointwise. As a consequence, $x_1=17$ and $x_5=1$, since $\left|l-X_1^G-X_2^G\right|=17$. This is impossible, since we saw $x_1\leq 1$. So, $x_3=0$.

Now, assume that $x_1=x_5=1$. Thus, $n=\frac{q(q^2-1)}{120}+\mathcal{S}$ by (92), as $x_3=x_4=0$. Furthermore, $q=29,61,101,109$ by Lemma 3.4, since $q\equiv 5 \bmod 8$. Clearly, $n>\frac{q(q^2-1)}{120}$, since n is a square by Lemma 3.3, while $\frac{q(q^2-1)}{120}$ is not for these numerical values of q. Therefore, $\mathcal{S}>0$. Since $\frac{q+1}{2}\mid\mathcal{S}$, then $\frac{q+1}{2}\mid n-\frac{q(q^2-1)}{120}$, where $q=29,61,101,109$, and $\sqrt{\frac{q(q^2-1)}{120}}<\sqrt{n}<q$ with $\sqrt{n}$ odd and hence $\sqrt{n}\equiv 1 \bmod 4$ by Lemma 3.3. Easy computations show that no cases arise.

Assume that $x_1=0$ and $x_5=2$. Therefore, $n+1\geq\frac{q(q^2-1)}{60}$ by (92), as $x_3=x_4=0$. Furthermore, $q=29,61,101,109$ by Lemma 3.4, since $q\equiv 5 \bmod 8$. Actually, the cases $q=61,101,109$ cannot occur, since they do not satisfy $\frac{q(q^2-1)}{60}-1\leq n<q^2$. Thus, $q=29$ and hence $405\leq n<29^2$. Actually, either $n=21^2$ or 25^2, since $\sqrt{n}\equiv 1 \bmod 4$ by Lemma 3.3. Then $\mathcal{S}=36$ or 220 by (92), respectively, since $x_1=x_3=x_4=0$, $x_5=2$ and $q=29$. This leads to a contradiction, since $\frac{q+1}{2}=15$ must divide $\mathcal{S}$ by the definition of this one. Hence, we have proved the assertion. $\square$

Lemma 6.4. *For each point $P \in l$, the group G_P cannot be isomorphic either to A_4 or to A_5.*

Proof. The group T induces a Baer collineation on $\mathrm{Fix}(\sigma)$ by Lemma 6.3. Thus, $|\mathrm{Fix}(T) \cap l| = \sqrt[4]{n} + 1$. Then, by Table IV, we have the following system of Diophantine equations:

$$\sqrt[4]{n}+1 = x_1+3x_2+x_4+x_5 \tag{93}$$

$$\sqrt{n}+1 = x_1+\frac{q+1}{2}x_2+\frac{q-1}{2}x_3+\frac{q-1}{4}x_4+\frac{q-1}{4}x_5+\mathcal{S}_2 \tag{94}$$

$$n+1 = x_1+\frac{q(q+1)}{2}x_2+\frac{q(q-1)}{2}x_3+\frac{q(q^2-1)}{24}x_4+\frac{q(q^2-1)}{120}x_5+\mathcal{S}\,. \tag{95}$$

Assume that $x_4 > 0$. Then $x_4 = 1$ by Lemma 3.5(3). So, $x_2 = x_3 = 0$ by Lemma 3.6(3). Furthermore, $q = 13$ by Lemma 3.4, since $q \equiv 5 \bmod 8$. Thus, $\sqrt[4]{n} = 2$ or 3, since $q < n < q^2$ by our assumption. Hence, we obtain a contradiction, since $n+1 \geq \frac{q(q^2-1)}{24}$ for $q = 13$, being $x_4 = 1$. Therefore, $x_4 = 0$.

Assume that $x_5 > 0$. Then $q = 29, 61, 101, 109$ by Lemma 3.4, since $q \equiv 5 \bmod 8$. If $x_2 + x_3 > 0$, then $x_2 + x_3 = 1$, $x_5 \leq 2$ and $q = 29$ by Lemma 3.6(2) and (4). Moreover, $\sqrt[4]{n} = 3, 4$ or 5, since $q < n < q^2$ and since n is a fourth power by Lemma 6.3. Let Q^G be an orbit of type either (2) or (3), as $x_2+x_3 = 1$. Then $\left|Q^G\right| = \frac{q(q\pm1)}{2}$, respectively. Now, let R^G be an orbit of type (5) as $x_5 > 0$, then $\left|R^G\right| \geq \frac{q(q^2-1)}{120}$. Since $Q^G \cup R^G \subseteq l$, then $n+1 \geq \frac{q(q\pm1)}{2} + \frac{q(q^2-1)}{120}$. In particular, $n+1 = 638$ or 609, since $\frac{q(q\pm1)}{2} + \frac{q(q^2-1)}{120} = 638$ or 609 according to whether Q^G is of type (2) or (3), respectively. While the cases $\sqrt[4]{n} = 3, 4$ or 5 cannot occur when Q^G is of type (2), only $\sqrt[4]{n} = 5$ is admissible when Q^G is of type (3). In this case, since $\left|l - (Q^G \cup R^G)\right| = 17$ and since the minimal primitive permutation representation of degree of $\mathrm{PSL}(2, 29)$ is 30, the group G fixes $l-(Q^G \cup R^G)$ pointwise. Hence, $x_1 = 17$ as $\left|l - (Q^G \cup R^G)\right| = 17$ when Q^G is of type (3). So $\sqrt[4]{n} \geq 16$ as $\sqrt[4]{n} + 1 = x_1 + 3x_2 + x_5$, which is a contradiction, being $\sqrt[4]{n} = 5$ by the above argument. Thus, $x_2 = x_3 = 0$.

Since $\frac{q(q^2-1)}{120} \leq n+1 < q^2+1$ and since n is a fourth power, it is a straightforward calculation to show that, $(q, n) = (29, 4^4), (29, 5^4)$, or $(61, 7^4)$, or $(101, 10^4)$. Assume that $q \neq 29$. Let u be an odd prime divisor of $q+1$. In particular, $u = 31$ when $q = 61$ and $u = 17$ when $q = 101$. Note that, $u \mid \mathcal{S}$, since $\frac{q+1}{2} \mid \mathcal{S}$. Furthermore, $u \mid \frac{q(q^2-1)}{120}$. Then $u \mid n+1-x_1$. Indeed, we have $n+1-x_1 = \frac{q(q^2-1)}{120}x_5 + \mathcal{S}$ by (95), since $x_2 = x_3 = x_4 = 0$. Hence, $n+1 \equiv x_1 \bmod u$. This yields $x_1 \equiv 15 \bmod 31$ for $q = 61$ and $x_1 \equiv 5 \bmod 31$ for $q = 101$. Since $0 < x_1 < \sqrt[4]{n}+1$ and $\sqrt[4]{n}+1 < u$ in each case, then $x_1 = 15$ for $q = 61$ and $x_1 = 5$ for $q = 101$. This is a contradiction, since $x_1 + x_5 = \sqrt[4]{n}+1$

with $\sqrt[4]{n} \le 10$. Thus, $(q, n) = (29, 4^4), (29, 5^4)$. Then $n + 1 = x_1 + 203x_5 + \mathcal{S}$ by (95), since $x_2 = x_3 = x_4 = 0$. Assume that $\mathcal{S} = 0$. Then $n + 1 = x_1 + 203x_5$. Since $\sqrt[4]{n} + 1 = x_1 + x_5$ by (93), then $n + 1 = \sqrt[4]{n} + 1 + 202x_5$. Therefore, $202 \mid n - \sqrt[4]{n}$, as $x_5 > 0$, which is a contradiction, since $\sqrt[4]{n} = 4$ or 5. Hence, $\mathcal{S} > 0$. Actually, $\mathcal{S} = n - \sqrt[4]{n} - 202x_5$. If $\sqrt[4]{n} = 4$, then $x_5 = 1$ and $\mathcal{S} = 50$. If $\sqrt[4]{n} = 5$, then $x_5 \le 3$. Furthermore, $\mathcal{S} = 418, 216$, or 19, for $x_5 = 1, 2$, or 3, respectively. On the other hand, $15 \mid \mathcal{S}$, since $\frac{q+1}{2} = 15$ and since $\frac{q+1}{2} \mid \mathcal{S}$ by the definition of $\mathcal{S}$. in each case. So, we obtain a contradiction in any case. □

Proposition 6.5. *Let Π be a projective plane of order n that admits a collineation group $G \cong \mathrm{PSL}(2, q)$ fixing a line l. If $q < n < q^2$ and $q \equiv 5 \bmod 8$, then Π has order 16 and $G \cong \mathrm{PSL}(2, 5)$.*

Proof. Suppose that G fixes a line l of Π. Assume that $q = 5$. Then $5 < n < 5^2$ by our assumptions. Actually, $n = 16$, since n must be a square and $\sqrt{n} \equiv 0, 1 \bmod 4$ by Lemma 3.3. Thus, we have proved the assertion (1).

Assume that $q > 5$. Actually, $q > 9$, since . Recall that, $|\mathrm{Fix}(T) \cap l| = \sqrt[4]{n} + 1$ by Lemma 6.3, and that for each point $P \in l$, the group G_P cannot be isomorphic either to A_4 or to A_5 by Lemma 6.4. So, $x_4 = x_5 = 0$. Hence, by table IV, we have

$$\sqrt[4]{n} + 1 = x_1 + 3x_2 \tag{96}$$

$$\sqrt{n} + 1 = x_1 + \frac{q+1}{2}x_2 + \frac{q-1}{2}x_3 + \mathcal{S}_2 \tag{97}$$

$$n + 1 = x_1 + \frac{q(q+1)}{2}x_2 + \frac{q(q-1)}{2}x_3 + \mathcal{S}\,. \tag{98}$$

Assume that $x_2 > 0$. Then $x_2 = 1$ by Lemma 3.5(1) and therefore $x_3 = 0$ by Lemma 3.6(2). It follows that, $\sqrt[4]{n} = x_1 + 2$ and $\sqrt{n} = x_1 + \frac{q+1}{2} + \mathcal{S}_2$ by (96) and (97), respectively. By elementary calculations of the previous equations, we obtain $(x_1 + 2)^2 + 1 = x_1 + \frac{q+1}{2} + \mathcal{S}_2$. So, $x_1^2 + 3x_1 = \frac{q-9}{2} + \mathcal{S}_2$. If $x_1 = 0$, then $q = 9$, since $\mathcal{S}_2 \ge 0$. This is a contradiction, since $q \equiv 5 \bmod 8$. If $x_1 = 1$, then $\sqrt[4]{n} = 3$ and hence $n = 81$. Moreover, $n+1 \ge x_1 + \frac{q(q+1)}{2}$ by (98), being $x_2 = 1$. This is a contradiction, since $\frac{q(q+1)}{2} \ge 91$ as $q \ge 13$, while $n = 81$. If $x_1 = 2$, then $\sqrt[4]{n} = 4$ and therefore $q > 13$, since $q < n < q^2$ by our assumptions. Thus, $q \ge 29$, since $q \equiv 5 \bmod 8$. Again, $n + 1 \ge x_1 + \frac{q(q+1)}{2}$, with $\frac{q(q+1)}{2} \ge 435$, being $q \ge 29$. This is impossible, since $n = 4^4$. Thus, $x_1 \ge 3$ for $x_2 > 0$. Note that, $x_1 \ge 3$ also for $x_2 = 0$, since $\sqrt[4]{n} + 1 \ge 3$. So, $x_1 \ge 3$ in any case. Thus, G fixes always at least 3 points on l.

Let P be any of these points and let r be any line of $[P] - \{l\}$. Applying the dual of Lemma 6.4, we obtain that, G_r cannot be isomorphic either to A_4 or to A_5 for each line $r \in [P] - \{l\}$. Hence, $x_4^* = x_5^* = 0$. By dual of Table IV, we obtain the same system of Diophantine equations as (96), (97) and (98) but referred to $[P]$ and with the x_i^* in the role of x_i. Now, we may repeat the above argument showing that, G fixes at least 3 lines (including l) through any point P of $\mathrm{Fix}(G) \cap l$. Thus G fixes a subplane of Π pointwise, as $|\mathrm{Fix}(G) \cap l| \geq 3$. In particular, $o(\mathrm{Fix}(G)) = x_1 - 1$.

Assume that $\mathrm{Fix}(G) \subset \mathrm{Fix}(T)$. Then either $\sqrt[4]{n} = (x_1 - 1)^2$ or $\sqrt[4]{n} \geq (x_1 - 1)^2 + (x_1 - 1)$ by [16, Theorem 3.7], since T induces a Baer collineation on $\mathrm{Fix}(\sigma)$. Furthermore, there must be a G-orbit on l of type (2) by (96). So, $\sqrt[4]{n} + 1 = x_1 + 2$. Hence, either $x_1 + 2 = (x_1 - 1)^2$ or $x_1 + 2 \geq (x_1 - 1)^2 + (x_1 - 1)$. Easy computations show that no one of them occurs, since $x_1 \geq 3$. Consequently, $\mathrm{Fix}(G) = \mathrm{Fix}(T)$. This yields $x_2 = 0$ and $\sqrt[4]{n} + 1 = x_1$.

Assume that $\mathcal{S} = 0$. Then $x_3 > 0$ by (97), as $x_2 = 0$. Actually, $x_3 = 1$ by Lemma 3.5(2). So (96), (97) and (98), respectively, become

$$\sqrt[4]{n} + 1 = x_1 \tag{99}$$
$$\sqrt{n} + 1 = x_1 + \frac{q-1}{2} \tag{100}$$
$$n + 1 = x_1 + \frac{q(q-1)}{2}\,. \tag{101}$$

Then $\sqrt{n} - \sqrt[4]{n} = \frac{q-1}{2}$ combining (99) with (100), and $n - \sqrt{n} = \frac{(q-1)^2}{2}$ combining (100) with (101). Finally, combining these ones, we have $n + \sqrt[4]{n} = q - 1$. Then $n + \sqrt[4]{n} = 2(\sqrt{n} - \sqrt[4]{n})$, as $\sqrt{n} - \sqrt[4]{n} = \frac{q-1}{2}$. Now, dividing by $\sqrt[4]{n}$, we obtain $(\sqrt[4]{n})^3 - 2\sqrt[4]{n} + 3 = 0$ which has no integer solutions. Therefore, $\mathcal{S} > 0$.

Let S be a Sylow p-subgroup of G normalized by σ and let $X \in l$ such that $S \leq G_X$ (such a point does exist, as $\mathcal{S} > 0$). Then either $G_X \cong F_q.Z_{d_X}$, with $d_X \mid \frac{q-1}{2}$, or $G_X = G$ by Table V. Then S fixes a Baer subplane of Π, since $\mathrm{Fix}(G) \subset \mathrm{Fix}(S)$, since $o(\mathrm{Fix}(G)) = \sqrt[4]{n}$ and since $\mathcal{S} > 0$. Recall that $\mathcal{S}_1 = \sum_{j=1}^{x_{10}} \frac{q-1}{d_j}$ and that $\mathcal{S}_2$ and $\mathcal{S}_{2'}$ are the sum with the same summands $\frac{q-1}{d_j}$ but over $2 \mid d_j$ and $2 \nmid d_j$, respectively. Note that, $d_X = d_h$ for some $1 \leq h \leq x_{10}$. Then $|\mathrm{Fix}_{X^G}(S)| = \frac{q-1}{2d_h}$ by Proposition 2.5, since $N_G(S) = S.Z_{\frac{q-1}{2}}$. Thus, the number of points coming out from G-orbits on l of type (10) which fixed by S are exactly $\sum_{j=1}^{x_{10}} \frac{q-1}{2d_j}$. These turn out to be $\frac{1}{2}\mathcal{S}_1$ as $\mathcal{S}_1 = \sum_{j=1}^{x_{10}} \frac{q-1}{d_j}$. Therefore, $o(\mathrm{Fix}(S)) + 1 = x_1 + \frac{1}{2}\mathcal{S}_1$. So, $\sqrt{n} + 1 = x_1 + \frac{1}{2}\mathcal{S}_1$, since $\mathrm{Fix}(S)$ is a Baer subplane of Π. Then $x_1 + \frac{1}{2}\mathcal{S}_1 = x_1 + \frac{q-1}{2}x_3 + \mathcal{S}_2$, since $\sqrt{n} + 1 = x_1 + \frac{q-1}{2}x_3 + \mathcal{S}_2$ by (97). As a consequence

$$\mathcal{S}_1 = (q-1)x_3 + 2\mathcal{S}_2\,. \tag{102}$$

Assume that $x_3 > 0$. Then $x_3 = 1$ by Lemma 3.5(2). Then $\mathcal{S}_1 \geq q - 1$ and hence $\mathcal{S} \geq \frac{q^2-1}{2}$, since $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$. Now, by substituting $\mathcal{S} \geq \frac{q^2-1}{2}$ in (98) and bearing in mind $x_3 = 1$, we obtain $n+1 \geq \frac{q(q-1)}{2} + \frac{q^2-1}{2}$. On the other hand, $n \leq (q-1)^2$ since $n < q^2$ and n is a square. Then $(q-1)^2 + 1 \geq \frac{q(q-1)}{2} + \frac{q^2-1}{2}$, which is a contradiction.

Assume that $x_3 = 0$. Then $\mathcal{S}_1 = 2\mathcal{S}_2$ by (102). Note that, $\mathcal{S}_1 > 0$, as $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$ and $\mathcal{S} > 0$. As a consequence, $\mathcal{S}_{2'} > 0$ being $\mathcal{S}_1 = 2\mathcal{S}_2$ and $\mathcal{S}_1 = \mathcal{S}_2 + \mathcal{S}_{2'}$. Now, we focus on the points on l fixed by $S\langle\sigma\rangle$. If $S\langle\sigma\rangle$ fixes a point Q on l, then G_Q is either of type (1) or of type (10). So, $S\langle\sigma\rangle$ fixes at least x_1 points on l. Furthermore, if Q^G is of type (10), then $\left|\mathrm{Fix}_{Q^G}(S\langle\sigma\rangle)\right| = \frac{q-1}{2d_j}$ for d_j even and 0 for d_j odd by Proposition 2.5. Therefore, the number of points coming out from G-orbits on l of type (10) which fixed by $S\langle\sigma\rangle$ are exactly $\sum_{d_j \equiv_2 0} \frac{q-1}{2d_j}$. These turn out to be $\frac{1}{2}\mathcal{S}_2$ as $\mathcal{S}_2 = \sum_{d_j \equiv_2 0} \frac{q-1}{d_j}$. It follows that, $S\langle\sigma\rangle$ fixes exactly $x_1 + \frac{1}{2}\mathcal{S}_2$ on l. Hence, σ fixes exactly $x_1 + \frac{1}{2}\mathcal{S}_2$ points on $\mathrm{Fix}(S) \cap l$. Then σ induces a Baer collineation on $\mathrm{Fix}(S)$, since $x_1 + \frac{1}{2}\mathcal{S}_2 \geq 3$, since $o(\mathrm{Fix}(S)) + 1 = x_1 + \frac{1}{2}\mathcal{S}_1$ with $\mathcal{S}_1 > \mathcal{S}_{2'} > 0$. Consequently, $\sqrt[4]{n} + 1 = x_1 + \frac{1}{2}\mathcal{S}_2$, since $\mathrm{Fix}(S)$ is a Baer subplane of Π. On the other hand, $\sqrt[4]{n} + 1 = x_1$ by (99). Hence, $x_1 + \frac{1}{2}\mathcal{S}_2 = x_1$. This yields $\mathcal{S}_2 = 0$. Thus, $\mathcal{S} = 0$, since $\mathcal{S}_1 = 2\mathcal{S}_2$ and $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_1$. This is a contradiction, since $\mathcal{S} > 0$. So, G does not fix lines of Π. □

Corollary 6.6. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,5)$. If $n < 25$ and G fixes a subplane Π_0 of Π, then $\Pi_0 \cong \mathsf{PG}(2,4)$ and $n = 16$.*

Proof. Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,5)$. Assume that $n < 25$ and that G fixes a subplane Π_0 of Π of order m. Clearly, $m < 5$ by [16, Theorem 3.7]. Then $\Pi_0 \cong \mathsf{PG}(2,4)$ by Theorem 2.1. In particular, G fixes a secant l of Π_0 which is the kernel of the line oval of Π_0 left invariant by G itself. Then $n = 16$ by Proposition 6.5. □

Theorem 6.7. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,q)$ with $q \equiv 5 \bmod 8$. If $n \leq q^2$, then one of the following occurs:*

(1) *$n < q$, $\Pi \cong \mathsf{PG}(2,4)$ and $G \cong \mathsf{PSL}(2,5)$;*

(2) *$n = q$, $\Pi \cong \mathsf{PG}(2,q)$ and G is strongly irreducible on Π;*

(3) *$q < n < q^2$ and one of the following occurs:*

   (a) *G is strongly irreducible on Π;*

   (b) *$n = 16$ and $G \cong \mathsf{PSL}(2,5)$ fixes a point, or a line of Π or subplane $\Pi_0 \cong \mathsf{PG}(2,4)$;*

(4) *$n = q^2$ and one of the following occurs:*

(a) *G is strongly irreducible on Π;*

(b) *G fixes a subplane Π_0 of Π. In particular, if $q \neq 5$, then $\Pi_0 \cong \mathrm{PG}(2, q)$ is a Baer subplane of Π.*

Proof. If $n \leq q$, the assertions (1) (2) easily follows by Theorems 2.1 and 2.2, respectively. If $q < n < q^2$, then either the assertion (3b) or group G does not fix lines or points of Π by Proposition 6.5 and its dual. If the latter occurs, the assertion (3a) easily follows by Lemma 3.1, since $q \equiv 5 \bmod 8$ and by Corollary 6.6. Finally, if $n = q^2$, the assertions (4a) and (4b) and follow by Theorems 2.3 and 2.4, respectively. □

At this point, Theorem 1.1 easily follows, when $q \equiv 5 \bmod 8$, from Theorem 6.7.

7. The case $q \equiv 7 \bmod 8$

Assume that $q \equiv 7 \bmod 8$. Recall that σ is a representative of the unique conjugate class of involution in G, and that T_1 and T_2 are the representatives of the two conjugate classes of Klein subgroups of G. In particular, T_1 and T_2 are chosen in order to contain σ. Furthermore, $C_G(\sigma) \cong D_{q+1}$ and $N_G(T_j) \cong S_4$ for each $j = 1$ or 2.

We filter the list given in Lemma 3.4 with respect to the condition $q \equiv 7 \bmod 8$. Then, for each point $P \in l$, either $G_P = G$ (type (1)), or $G_P \cong D_{q-1}$ (type (2)), or $G_P \cong D_{q+1}$ (type (3)), $G_P \cong A_5$ (type (5)), or $G_P \cong S_4$ (type (6)) or $G_P \cong F_q.Z_d$, where $d \mid \frac{q-1}{2}$ and d odd (type (10)). Note that there are two conjugate classes of subgroups isomorphic to A_5 and two ones of subgroups isomorphic to S_4 by [4]. So, following the notation introduced in section 4, there are admissible subgroups of type (5a) and (5b), and admissible ones of type (6a) and (6b). Hence, $x_i = x_{ia} + x_{ib}$ for $i = 5$ or 6. The usual argument, involving Proposition 2.5, yields the table on the next page containing all the required informations about the admissible G_P.

The numbers $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_{2'}$, $\mathcal{S}_{2,4}$ and $\mathcal{S}_4$ have the usual meaning. In particular, $\mathcal{S}_2 = \mathcal{S}_{2,4} = \mathcal{S}_4 = 0$, since $q \equiv 7 \bmod 8$. Consequently, $\mathcal{S}_1 = \mathcal{S}_{2'}$ and $\mathcal{S} = \frac{q+1}{2}\mathcal{S}_{2'}$.

As in the preceding sections, we may consider the *dual of table V*, that is the table referred to the G-orbits of lines through some point Q of Π fixed by G. In particular, we might have G-orbits of lines of type $(i\mathrm{a})^*$ and $(i\mathrm{b})^*$ for $i = 5$ or 6, and it makes sense considering $\mathcal{S}^*$, $\mathcal{S}_1^*$, $\mathcal{S}_2^*$, $\mathcal{S}_{2'}^*$, $\mathcal{S}_{2,4}^*$ and $\mathcal{S}_4^*$. Similarly to

Table V

Type	G_P	$[G : G_P]$	$\lvert\mathrm{Fix}_{PG}(\sigma)\rvert$	$\lvert\mathrm{Fix}_{PG}(T_1)\rvert$	$\lvert\mathrm{Fix}_{PG}(T_2)\rvert$
1	G	1	1	1	1
2	D_{q-1}	$\frac{q(q+1)}{2}$	$\frac{q+1}{2}$	0	0
3	D_{q+1}	$\frac{q(q-1)}{2}$	$\frac{q+3}{2}$	3	3
5a	A_5	$\frac{q(q^2-1)}{120}$	$\frac{q+1}{4}$	2	0
5b	A_5	$\frac{q(q^2-1)}{120}$	$\frac{q+1}{4}$	0	2
6a	S_4	$\frac{q(q^2-1)}{48}$	$\frac{3(q+1)}{8}$	$4,\quad q \equiv_{16} 15$ $1,\quad q \equiv_{16} 7$	$4,\quad q \equiv_{16} 15$ $3,\quad q \equiv_{16} 7$
6b	S_4	$\frac{q(q^2-1)}{48}$	$\frac{3(q+1)}{8}$	$4,\quad q \equiv_{16} 15$ $3,\quad q \equiv_{16} 7$	$4,\quad q \equiv_{16} 15$ $1,\quad q \equiv_{16} 7$
10	$F_q.Z_d$	$\frac{q^2-1}{2d}$	0	0	0

above, we have $\mathcal{S}_2^* = \mathcal{S}_{2,4}^* = \mathcal{S}_4^* = 0$, since $q \equiv 7 \bmod 8$, and hence $\mathcal{S}_1^* = \mathcal{S}_{2'}^*$ and $\mathcal{S}^* = \frac{q+1}{2}\mathcal{S}_{2'}^*$.

Note that σ is a Baer collineation of Π by Lemma 3.3. Set $C = C_G(\sigma)$. Then C acts on $\mathrm{Fix}(\sigma)$ with kernel K. Hence, let $\bar{C} = C/K$. Clearly, $\langle\sigma\rangle \trianglelefteq K \trianglelefteq C$. Furthermore, either $K \trianglelefteq Z_{\frac{q+1}{2}}$ or $K \cong D_{\frac{q+1}{2}}$ or $K = C$, since $C \cong D_{q+1}$ and $q \equiv 7 \bmod 8$. As we will see, we need to investigate the admissible structure of K in order to show that T_j induces a Baer collineation on $\mathrm{Fix}(\sigma)$ for each $j = 1$ or 2.

Lemma 7.1. *If* $\mathrm{Fix}(T_j) \cap l = \mathrm{Fix}(\sigma) \cap l$ *for some* $j = 1$ *or* 2, *then either* $K \cong D_{\frac{q+1}{2}}$ *or* $K = C$.

Proof. Assume that $\mathrm{Fix}(T_1) \cap l = \mathrm{Fix}(\sigma) \cap l$ and that $K \trianglelefteq Z_{\frac{q+1}{2}}$. Then $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ by table V, since $q > 9$. Set $l_0 = \mathrm{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \mathrm{Fix}(G) \cap l$. In particular, $\bar{C} \cong D_{\frac{q+1}{k}}$, where $k = |K|$, k is even and $k \mid \frac{q+1}{2}$. Then the group $\bar{C}$ is the semidirect product of $\bar{C}(l_0, l_0)$ with $\bar{C}(Y, l_0)$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$ by [16, Theorem 4.25].

Assume that $\bar{C}(l_0, l_0) \neq \langle 1\rangle$. If $\bar{C}(l_0, l_0) = \bar{C}$, then either $\bar{C} \cong E_4$ and $K \cong Z_{\frac{q+1}{4}}$ or $\bar{C} = \bar{C}(V, l_0)$ for some point $V \in l_0$ by [16, Theorem 4.14], since $C \cong D_{q+1}$ and $q \equiv 7 \bmod 8$. Suppose the former occurs. Let R_i, $i = 1, 2$ or 3, be the (unique) points on l_0, such that $\bar{C}(R_i, l_0) \neq \langle 1\rangle$. Actually, $\bar{C}(R_i, l_0) \cong Z_2$ for each $i = 1, 2, 3$. So, there are at least two points among the R_i, $i = 1, 2$ or 3, say R_2 and R_3, such that $C_h \cong D_{\frac{q+1}{2}}$ for each line $h \in [R_i] - \{l\}$,

$i = 2, 3$, since $K \cong Z_{\frac{q+1}{4}}$ and $C \cong D_{q+1}$. As a consequence, $D_{\frac{q+1}{2}} \leq G_h$ for each $h \in ([R_2] \cup [R_3]) - \{l\}$. Now, since G fixes the R_i, $i = 2, 3$, we may filter the groups listed in the dual of Lemma 3.4 with respect to the condition $D_{\frac{q+1}{2}} \leq G_h$. Easy computation show that either $G_h = C$ or $G_h = G$, which is a contradiction, since $C_h \cong D_{\frac{q+1}{2}}$. Therefore, $\bar{C} = \bar{C}(V, l_0)$ for some point $V \in l_0$ by [16, Theorem 4.14]. Thus, for each point $X \in l_0 - \{V\}$ and for each line $t \in [X] \cap \mathrm{Fix}(\sigma)$, we have $\sigma \in G_t$ but G_t does not contain Klein groups. Then $G_t \cong D_{q-1}$ by dual of table V, since G fixes X. Assume there exists $u \in [X] \cap \mathrm{Fix}(\sigma)$ such that $G_u \cong D_{q-1}$. Clearly, $K \leq G_u$. Then $K = \langle\sigma\rangle$, since $\langle\sigma\rangle \trianglelefteq K \leq Z_{\frac{q+1}{2}}$. so, $\bar{C} \cong D_{\frac{q+1}{2}}$ and hence $\frac{q+1}{2} \mid \sqrt{n}$. Actually, $\sqrt{n} = \frac{q+1}{2}$, since $\sqrt{n} < q$ by our assumptions. On the other hand, $u^G \subset [X] - \{l\}$ as G fixes X. Then $n \geq \frac{q(q+1)}{2}$ since $|u^G| = \frac{q(q+1)}{2}$ as $G_u \cong D_{q-1}$. Then $\left(\frac{q+1}{2}\right)^2 \geq \frac{q(q+1)}{2}$, since $\sqrt{n} = \frac{q+1}{2}$. This contradicts the fact that $q > 9$. Thus, $\bar{C}(l_0, l_0) < \bar{C}$. Then $\bar{C}(l_0, l_0) \leq Z_{\frac{q+1}{k}}$, since $\bar{C} \cong D_{\frac{q+1}{k}}$. Actually, $\bar{C}(l_0, l_0) = \bar{C}(V, l_0) \cong Z_{\frac{q+1}{k}}$ and $\bar{C}(Y, l_0) \cong Z_2$ by [16, Theorems 4.14 and 4.25]. Let $s \in [V] - \{l, VY\}$, then s is fixed by K and by $\bar{C}(V, l_0)$. Therefore, $G_s \cap C \cong Z_{\frac{q+1}{2}}$. It follows that $Z_{\frac{q+1}{2}} \leq G_s$. Then either $G_s = C_G(\sigma)$ or $G_s = G$ by dual of Lemma 3.4, since G fixes l_0, since $q > 9$. This is a contradiction, since $G_s \cap C \cong Z_{\frac{q+1}{2}}$. Hence, $\bar{C}(l_0, l_0) = \langle 1 \rangle$.

Assume that $\bar{C} = \bar{C}(Y, l_0)$ for some point $Y \in \mathrm{Fix}(\sigma) - l_0$. Let $Q \in \mathrm{Fix}(\sigma) \cap l$ and let $m \in [Q] \cap \mathrm{Fix}(\sigma) - \{l, YQ\}$. Then $\sigma \in G_m$ but G_m does not contain Klein groups. So, $G_m \cong D_{q-1}$ by dual of table V, since G fixes Q. Therefore, $x_2^* \geq 1$. Furthermore, $x_3^* \geq 1$, since $G_{YQ} = C$. Thus, $x_2^* + x_3^* \geq 2$, which is a contradiction by dual of Lemma 3.6(1), being $q > 9$. □

Lemma 7.2. *It holds that* $\mathrm{Fix}(T_j) \cap l \subset \mathrm{Fix}(\sigma) \cap l$ *for each* $j = 1, 2$.

Proof. Assume that $\mathrm{Fix}(T_1) \cap l = \mathrm{Fix}(\sigma) \cap l$. Then either $K \cong D_{\frac{q+1}{2}}$ or $K = C$ by Lemma 7.1.

Assume that $K = C$. Then $\mathrm{Fix}(T_1) = \mathrm{Fix}(\sigma)$. Let P be any point of $\mathrm{Fix}(\sigma) \cap l$ and let r be any line of $[P] - \{l\}$. So, $C \leq G_r$. Since $q > 9$, then C is maximal in G and hence either $G_r = C$ or $G_r = G$. If the former occurs, then $|\mathrm{Fix}_{r^G}(T_1)| = 3$ and $|\mathrm{Fix}_{r^G}(\sigma)| = \frac{q+3}{2}$ by dual of table V. Therefore, $|\mathrm{Fix}_{r^G}(\sigma)| > |\mathrm{Fix}_{r^G}(T_1)|$ as $q > 9$. This is impossible, since $\mathrm{Fix}(T_1) = \mathrm{Fix}(\sigma)$. Thus, $G_r = G$ for any point P of $\mathrm{Fix}(\sigma) \cap l$ and for any line r of $[P] - \{l\}$. Consequently, $\mathrm{Fix}(G) = \mathrm{Fix}(\sigma)$, since $\mathrm{Fix}(G) \cap l = \mathrm{Fix}(\sigma) \cap l$ and $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(\sigma)$. Thus, G fixes a Baer subplane of Π. Then G is semiregular on $l - \mathrm{Fix}(G)$ and hence $|G| \mid n - \sqrt{n}$, which is impossible.

Assume that $K \cong D_{\frac{q+1}{2}}$. Then $D_{\frac{q+1}{2}} \leq G_f$ for each line f of $\mathrm{Fix}(\sigma) - \{l\}$. Therefore, either $G_f = C$ or $G_f = G$ by dual of Lemma 3.4, being $q \equiv 7 \bmod 8$

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and $q > 9$. Now, the above argument yields $\mathrm{Fix}(G) = \mathrm{Fix}(\sigma)$ and we again obtain a contradiction. Thus, $\mathrm{Fix}(T_1) \cap l \subset \mathrm{Fix}(\sigma) \cap l$.

Now, repeating the above argument with T_2 in the role of T_1, we obtain $\mathrm{Fix}(T_2) \cap l \subset \mathrm{Fix}(\sigma) \cap l$. Hence, we have proved the assertion. □

Lemma 7.3. *The group T_j induces a Baer collineation on* $\mathrm{Fix}(\sigma)$ *for each* $j = 1, 2$.

Proof. The group T_j induces an involution $\bar{\beta}_j$ on $\mathrm{Fix}(\sigma)$ for each $j = 1, 2$ by Lemma 7.2. Assume that $\bar{\beta}_1$ is a $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-elation of $\mathrm{Fix}(\sigma)$. Then $C_{\bar{\beta}_1} \in l$ and $a_{\bar{\beta}_1} \neq l$ again by Lemma 7.2. Hence $\mathrm{Fix}(T_1) \cap l = \{C_{\bar{\beta}_1}\}$. Thus $N_G(T_1) \leq G_{C_{\bar{\beta}_1}}$, where $N_G(T_1) \cong S_4$. Then either $G_{C_{\bar{\beta}_1}} = N_G(T_1)$ or $G_{C_{\bar{\beta}_1}} = G$ by table V. Actually $G_{C_{\bar{\beta}_1}} = G$ cannot occur, otherwise we have a contradiction by dual of Lemma 6.2, since $\mathrm{Fix}(T_1) \cap [C_{\bar{\beta}_1}] = \mathrm{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$. Hence $G_{C_{\bar{\beta}_1}} = N_G(T_1)$ and hence $x_6 > 0$. Actually $x_{6a} = 1$ and $q \equiv 7 \bmod 16$ by table V. Moreover, $x_1 = x_3 = x_{5a} = x_{6b} = 0$ again by table V. Also, $q = 23$ and $x_5 = 0$ by Lemma 3.4(6), and $x_2 = 0$ by Lemma 3.6(4). Therefore $\mathrm{Fix}(\sigma) \cap l = \mathrm{Fix}(\sigma) \cap C^G_{\bar{\beta}_1}$ and hence $\sqrt{n} + 1 = \frac{3(q+1)}{8}$ again by table V, being $x_{6a} = 1$. That is $\sqrt{n} = 8$, as $q = 23$. Thus also T_2 must induce an elation on $\mathrm{Fix}(\sigma)$. Nevertheless T_2 fixes exactly 3 points on $C^G_{\bar{\beta}_1}$ by table V, since $G_{C_{\bar{\beta}_1}} = S_4$ and $q \equiv 7 \bmod 16$. Then T_2 fixes exactly 3 points on $\mathrm{Fix}(\sigma) \cap l$ as $\mathrm{Fix}(\sigma) \cap l = \mathrm{Fix}(\sigma) \cap C^G_{\bar{\beta}_1}$. This is a contradiction, since T_2 induces an elation on $\mathrm{Fix}(\sigma)$ and $\sqrt{n} = 8$.

Assume that $\bar{\beta}_1$ is a $(C_{\bar{\beta}_1}, a_{\bar{\beta}_1})$-homology of $\mathrm{Fix}(\sigma)$. Again $C_{\bar{\beta}_1} \in l$ and $a_{\bar{\beta}_1} \neq l$ by Lemma 7.2. Set $\{X\} = a_{\bar{\beta}_1} \cap l$. Hence $\mathrm{Fix}(T_1) \cap l = \{C_{\bar{\beta}_1}, X\}$. Let $\bar{\gamma}$ be the collineation induced by γ on $\mathrm{Fix}(\sigma)$, where $\gamma \in G$ and $\gamma^2 = \sigma$ (clearly such a element does exists in G, since $q \equiv 7 \bmod 8$). Then either $\bar{\gamma} = 1$ or $\bar{\gamma}$ is a Baer involution or a involutory perspectivity. Nevertheless $\bar{\gamma}$ centralizes $\bar{\beta}_1$ in each cases. Then $\bar{\gamma}$ fixes $C_{\bar{\beta}_1}$, $a_{\bar{\beta}_1}$ and hence X. Thus $N_G(T_1) \leq G_{C_{\bar{\beta}_1}}$ and $N_G(T_1) \leq G_X$. Similar argument to that used above yields $G_{C_{\bar{\beta}_1}} < G$ and hence $G_{C_{\bar{\beta}_1}} = N_G(T_1)$ by table V, since $N_G(T_1) \cong S_4$. Thus $x_{6a} > 0$. Then $x_6 = x_{6a} = 1$ by Lemma 3.5(5) as $q \equiv 7 \bmod 16$. Hence $G_X = G$. Therefore $x_1 = 1$, since $\mathrm{Fix}(G) \cap l \subset \mathrm{Fix}(T_1) \cap l$. Furthermore, $q = 23$ and $x_5 = 0$ by Lemma 3.4(6), and $x_2 = 0$ by Lemma 3.6(4). Finally, $\sqrt{n} + 1 = \frac{3(q+1)}{8} x_{6a} + x_1$ by table V, where $x_1 = x_{6a} = 1$. That is $\sqrt{n} = \frac{3(q+1)}{8}$. Then $n = 81$ as $q = 23$. On the other hand $n + 1 \geq \frac{q(q^2-1)}{48} + 1$ again by table V. Hence, $\left[\frac{3(q+1)}{8}\right]^2 \geq \frac{q(q^2-1)}{48}$, which is a contradiction, since $q = 23$. Thus, T_1 induces a Baer involution on $\mathrm{Fix}(\sigma)$.

Arguing as above, with T_2 in the role of T_1, we have that T_2 induces a Baer involution on $\mathrm{Fix}(\sigma)$. Hence, we have proved the assertion. □

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Lemma 7.4. *For each point $P \in l$ the group G_P cannot be isomorphic either to S_4 or to A_5.*

Proof. Assume that $x_6 > 0$. Then $x_6 = 1$ by Lemma 3.5(5), being $q \equiv 7 \bmod 8$. We may assume that $x_6 = x_{6a} = 1$ without of loss of generality (see Table V). Let $Q \in l$ such that $G_Q \cong S_4$. Then $\left|Q^G\right| = \frac{q(q^2-1)}{48}$ and hence $n \geq \frac{q(q^2-1)}{48} - 1$, as $Q^G \subset l$. Also, $q = 23$ or 31 by Lemma 3.4. Easy computations show that $\sqrt[4]{n} = 4$ for $q = 23$ and $\sqrt[4]{n} = 5$ for $q = 31$, since $\frac{q(q^2-1)}{48} - 1 \leq n < q^2$ with n a fourth power by Lemma 7.3. In both cases $n + 1 - \frac{q(q^2-1)}{48} < q + 1$. It follows that $\left|l - Q^G\right| < q + 1$, since $\left|l - Q^G\right| = n + 1 - \frac{q(q^2-1)}{48}$. Then G fixes $l - Q^G$ pointwise, since the minimal primitive permutation representation of G is $q+1$, being $q = 23$ or 31. That is $x_1 = \left|l - Q^G\right|$. If $q = 23$, then $o(\mathrm{Fix}(T_1)) = x_1$ and $o(\mathrm{Fix}(T_2)) = x_1 + 3$ by Table V, since $x_6 = 1$ and $q \equiv 7 \bmod 16$. Nevertheless, $o(\mathrm{Fix}(T_1)) = o(\mathrm{Fix}(T_2))$ by Lemma 7.3. Hence, we arrive at a contradiction. As a consequence, $q = 31$. Thus, $x_1 = 6$. Therefore, $\sqrt[4]{n} + 1 \geq 10$ as $\sqrt[4]{n} + 1 \geq x_1 + 4x_{6a}$. This is impossible, since $\sqrt[4]{n} = 5$. So, $x_6 = 0$.

Assume that $x_5 > 0$. Since T_1 and T_2 fix Baer subplanes of $\mathrm{Fix}(\sigma)$, we have $\sqrt[4]{n}+1 = x_1+3x_3+2x_{5a}$ and $\sqrt[4]{n}+1 = x_1+3x_3+2x_{5b}$ by Table V, since $x_6 = 0$. Then $\sqrt[4]{n} + 1 = x_1 + 3x_3 + x_5$ summing up these two equations and by bearing in mind that $x_5 = x_{5a} + x_{5b}$. Hence, by Table V, we have

$$\sqrt[4]{n} + 1 = x_1 + 3x_3 + x_5 \tag{103}$$

$$\sqrt{n} + 1 = x_1 + \frac{q+1}{2}x_2 + \frac{q+3}{2}x_3 + \frac{q+1}{4}x_5 \tag{104}$$

$$n + 1 = x_1 + \frac{q(q+1)}{2}x_2 + \frac{q(q-1)}{2}x_3 + \frac{q(q^2-1)}{120}x_5 + \mathcal{S}\,. \tag{105}$$

Note that $q = 31, 71$ or 79 by Lemma 3.4. Assume that $x_2 + x_3 > 0$. Then $x_2 + x_3 = 1$, $0 < x_5 \leq 2$ and $q = 31$ by Lemma 3.6(2) and (4). Therefore, $\sqrt[4]{n} = 3, 4$ or 5, since $q < n < q^2$ being n a fourth power by Lemma 7.3. On the other hand, $n + 1 \geq \frac{q(q-1)}{2} + \frac{q(q^2-1)}{120}$ by (105), since $x_2 + x_3 = 1$. That is $n + 1 \geq 713$, as $q \geq 31$. Nevertheless, this is a contradiction, since $n \leq 5^4$. Thus, $x_2 = x_3 = 0$. Then $\sqrt[4]{n} + 1 = x_1 + x_5$ and $\sqrt{n} + 1 = x_1 + \frac{q+1}{4}x_5$ by (103) and (104). By elementary calculations of the previous equations, we obtain $\sqrt{n} - \sqrt[4]{n} - \frac{q-3}{4}x_5 = 0$, where $x_5 \leq 3$ by Lemma 3.5(4), and where $q = 31, 71$ or 79. It is a straightforward computation to see that, no integer solutions arise. Thus, $x_5 = x_6 = 0$ and we obtain the assertion. □

Proposition 7.5. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2, q)$ fixing a line l. If $q < n < q^2$ and $q \equiv 7 \bmod 8$ then Π is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $G \cong \mathsf{PSL}(2, 7)$.*

Proof. Suppose that G fixes a line l of Π. Assume that $q > 7$. Hence, $q > 9$ as $q \equiv 7 \bmod 8$. Recall that $|\mathrm{Fix}(T_j) \cap l| = \sqrt[4]{n} + 1$ by Lemma 7.3, and that for each point $P \in l$, the group G_P cannot be isomorphic either to A_4 or to A_5 by Lemma 7.4. Then, by table V, we have the following system of Diophantine equations:

$$\sqrt[4]{n} + 1 = x_1 + 3x_3 \tag{106}$$

$$\sqrt{n} + 1 = x_1 + \frac{q+1}{2}x_2 + \frac{q+3}{2}x_3 \tag{107}$$

$$n + 1 = x_1 + \frac{q(q+1)}{2}x_2 + \frac{q(q-1)}{2}x_3 + \mathcal{S}\,. \tag{108}$$

If $x_3 > 0$, then $x_3 = 1$ by Lemma 3.5(2). Furthermore, $x_2 = 0$ by Lemma 3.6(2). Then $\sqrt[4]{n} = x_1 + 2$ and $\sqrt{n} = x_1 + \frac{q+3}{2}$. By composing these equations, we have $(x_1 + 2)^2 = x_1 + \frac{q+1}{2}$ and hence $x_1^2 + x_1 - \frac{q-7}{2} = 0$. If $x_1 \leq 2$, it is easily seen that $(n, x_1, q) = (2^4, 0, 7)$ as $q \equiv 7 \bmod 8$. Nevertheless, $n + 1 \geq 21$ by (108) as $x_3 = 1$. So, $x_1 \geq 3$ for $x_3 > 0$. Actually, $x_1 \geq 3$ also for $x_3 = 0$ by (106), since it must be $\sqrt[4]{n} \geq 2$. Consequently, $x_1 \geq 3$ in each case. Thus, G fixes always at least 3 points on l.

Let P be any of these points. Applying the dual of Lemma 7.4, we obtain that the group G_r cannot be isomorphic either to A_4 or to A_5 for each $r \in [P] - \{l\}$. Thus, $x_4^* = x_5^* = 0$. By dual of table VI, we obtain the same system of Diophantine equations as (106), (107) and (108) but referred to $[P]$ and with the x_i^* in the role of x_i. At this point, we may repeat the above argument showing that G fixes at least 3 lines (including l) through any point P of $\mathrm{Fix}(G) \cap l$. So, G fixes a subplane of Π pointwise, as $|\mathrm{Fix}(G) \cap l| \geq 3$. In particular, $o(\mathrm{Fix}(G)) = x_1 - 1$. Now, we may use the same argument of Theorem 5.5, with (106), (107) and (108) in the role of (85), (86) and (87), respectively, in order to obtain that G fixes a subplane of Π of order $\sqrt[4]{n}$. Hence, we have a contradiction.

Assume that $q \leq 7$. Actually, $q = 7$, since $q \equiv 7 \bmod 8$. Then either $n = 16$ or 25, since $q < n < q^2$ and since $\sqrt{n} \equiv 0, 1 \bmod 4$ by Lemma 3.3. Assume that $q = 25$. Let φ be any element in G of order 7. Then φ fixes at least 5 points on l and 2 on $\Pi - l$, as $n + 1 \equiv 5 \bmod 7$ and $n^2 \equiv 2 \bmod 7$. Thus, $o(\mathrm{Fix}(\varphi)) = 4 + 7\theta$, where $\theta \geq 0$. Actually, $\theta = 0$ by [16, Theorem 3.7], since $n = 25$. So, $o(\mathrm{Fix}(\varphi)) = 4$. Note that $N_G(\langle\varphi\rangle) = \langle\varphi, \psi\rangle$, where $o(\psi) = 3$ and ψ normalizes $\langle\varphi\rangle$. Also, $N_G(\langle\varphi\rangle)$ is the unique maximal subgroup of G containing φ. Therefore, for each point $Q \in \mathrm{Fix}(\varphi) \cap l$, either $G_Q = \langle\varphi\rangle$ or $G_Q = \langle\varphi, \psi\rangle$ or $G_Q = G$. Assume that $G_B = \langle\varphi\rangle$ for some $B \in \mathrm{Fix}(\varphi) \cap l$. Then $\left|B^G\right| = 24$. Thus, l consists of B^G and of 2 points fixed by G as $n + 1 = 26$. Consequently, any involution fixes exactly 2 points on l, namely those fixed by G, since $\left|B^G\right| = 24$ and $|G| = 168$. Hence, the involutions are homologies of Π, which is a contradiction by Lemma 3.3. It follows that either $G_Q = \langle\varphi, \psi\rangle$ or

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$G_Q = G$ for or each $Q \in \mathrm{Fix}(\varphi)\cap l$. Nevertheless, $\mathrm{Fix}(\varphi)\cap l = \mathrm{Fix}(\langle\varphi,\psi\rangle)\cap l$. So, $|\mathrm{Fix}(N_G(\langle\varphi,\psi\rangle)\cap l| = 5$. Assume that $|\mathrm{Fix}(\langle\varphi,\psi\rangle)\cap l - \mathrm{Fix}(G)\cap l| \geq 3$. Since this group is maximal, then there are at least 3 orbits of length 8. Therefore, l consists of three G-orbits each of length 8 and of 2 points fixed by G. Thus, any involution fixes exactly 2 points on l and we again obtain a contradiction by Lemma 3.3. It follows that $|\mathrm{Fix}(N_G(\langle\varphi,\psi\rangle))\cap l - \mathrm{Fix}(G)\cap l| \leq 2$ and hence $|\mathrm{Fix}(G)\cap l| \geq 3$. Now, we may repeat the above argument with $[X]$ in the role of l for each point $X \in \mathrm{Fix}(G)\cap l$. This yields $|\mathrm{Fix}(G)\cap [X]| \geq 3$ for each $X \in \mathrm{Fix}(G)\cap l$. Then G is planar, since $|\mathrm{Fix}(G)\cap l| \geq 3$. Therefore, $o(\mathrm{Fix}(G)) \geq 2$.

Now, let β be any involution of G. Then $o(\mathrm{Fix}(\beta)) = 5$ by Lemma 3.3 as $n = 25$. Note that $\mathrm{Fix}(G) \subsetneq \mathrm{Fix}(\beta)$, since φ and β fix exactly 4 and 6 points on l, respectively. So, we have a contradiction by [16, Theorem 3.7], since $o(\mathrm{Fix}(\beta)) = 5$ while $2 \leq o(\mathrm{Fix}(G)) < 5$. Thus, $n = 16$. Then either Π is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or one of their duals by [3]. Hence, we have proved the assertion. □

Theorem 7.6. *Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,q)$, with $q \equiv 7 \bmod 8$. If $n \leq q^2$, then one of the following occurs:*

(1) *$n < q$, $\Pi \cong \mathsf{PG}(2,2)$ or $\mathsf{PG}(2,4)$ and $G \cong \mathsf{PSL}(2,7)$;*

(2) *$n = q$ and $\Pi \cong \mathsf{PG}(2,q)$;*

(3) *$q < n < q^2$ and one of the following occurs:*

 (a) *G is strongly irreducible on Π;*

 (b) *$n = 16$, Π is the Lorimer-Rahilly plane or the Johnson-Walker plane, or one of their duals, and $G \cong \mathsf{PSL}(2,7)$;*

 (c) *$G \cong \mathsf{PSL}(2,7)$ fixes a subplane of Π isomorphic either to $\mathsf{PG}(2,2)$ or to $\mathsf{PG}(2,4)$;*

(4) *$n = q^2$ and one of the following occurs:*

 (a) *G is strongly irreducible on Π;*

 (b) *G fixes a Desarguesian Baer subplane Π_0 of Π.*

Proof. If $n \leq q$, the assertions (1) and (2) easily follow by Theorems 2.1 and 2.2, respectively. If $q < n < q^2$, then either the assertion (3b) or the group G fixes lines or points of Π by Proposition 5.5 and its dual. If the latter occurs, the assertions (3a) and (3c) easily follow by Lemma 3.1, since $q \equiv 7 \bmod 8$. Finally, the assertions (4a) and (4b) follow by Theorems 2.3 and 2.4, respectively. □

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Now, Theorem 1.1, when $q \equiv 7 \bmod 8$, easily follows from Theorem 7.6.

8. Concluding proofs and other examples

Proof of Theorem 1.1. Let Π be a projective plane of order n admitting a collineation group $G \cong \mathsf{PSL}(2,q)$, $q > 3$. Assume that $n \leq q^2$. If q is odd, the assertion of Theorem 1.1 easily follows by Theorems 4.22, 5.6, 6.7 and 7.6 for $q \equiv 1,3,5,7 \bmod 8$, respectively. It remains to investigate the case q even in order to complete the proof of the theorem. Hence, assume that $G \cong \mathsf{PSL}(2,q)$, with $q = 2^h$, $h > 1$. Since $\mathsf{PSL}(2,4) \cong \mathsf{PSL}(2,5)$ and since we have already dealt with this case in Theorem 6.7, we may assume that $q > 4$.

(I) If $n < q^2$ the involutions in G are perspectivities of Π.

Assume that $n < q^2$. Assume also that the involutions in G are Baer collineations of Π. Let H be an elementary abelian subgroup of G of order q. Then H fixes a point X of Π, since n^2+n+1 is odd. Furthermore, each non trivial element in H fixes exactly $\sqrt{n}+1$ lines through X, since $H - \{1\}$ consists of involutions. Then $q \mid (q-1)(\sqrt{n}+1)+(n+1)$ by [16, Result 1.14]. Hence, $q \mid \sqrt{n}(\sqrt{n}-1)$. Thus, either $q \mid \sqrt{n}-1$ or $q \mid \sqrt{n}$, since $q = 2^h$, $h > 1$. So, $\sqrt{n} \geq q$ and therefore $n \geq q^2$ in any case. This is a contradiction, since $n < q^2$ by our assumption. Thus, the involutions in G are perspectivities of Π, since $G \cong \mathsf{PSL}(2,q)$ contains a unique conjugate class of involutions by [4].

(II) If $n < q^2$ and $n \neq q$, then G does not fix lines of Π.

Assume that $n < q^2$, $n \neq q$. Assume also that G fixes a line l of Π. Let H be as above.

Suppose that n is even. Then $H = H(C,C)$ for some point $C \in l$ by (I), since H is an elementary abelian 2-group fixing l and since $H(l) = \langle 1 \rangle$ by Lemma 3.2(1). So, $H \leq G_C$. Furthermore, $G_C < G$ by Lemma 3.2(2). Then $G_C \leq H.Z_d$, where $d \mid q-1$ by [17, Hauptsatz II.8.27]. Note that H fixes exactly $\frac{q-1}{d}$ points in C^G by (1) of Proposition 2.5. Nevertheless, H fixes exactly 1 point on l. Then $\frac{q-1}{d} = 1$ and hence $G_C \cong H.Z_{q-1}$. In particular, $\left|C^G\right| = q+1$. Thus, $n \geq q$. Actually, $n > q$, since $n \neq q$ by our assumptions. In addition, since H is a Sylow 2-subgroup of G, then each Sylow 2-subgroup of G fixes exactly 1 point on l which lies in C^G. Therefore, G_X has odd order for each point $X \in l - C^G$. Such points do exist as $n > q$. Moreover, $\left|X^G\right| < q^2 - q$, since $X^G \subseteq l - C^G$, and since $\left|l - C^G\right| < q^2 - q$ as $n < q^2$. This yields $|G_X| > q+1$ with $|G_X|$ odd.

Hence, we arrive at a contradiction by a direct inspection of the list given in [17, Hauptsatz II.8.27].

Suppose that n is odd. Then H consists of homologies of Π by (I), since H is an elementary abelian 2-group. In particular, $H = H(C, a)$, where $C \in l$, $a \neq l$ by [19, Lemma (3.1)], since $H(l) = \langle 1 \rangle$ by Lemma 3.2(1). Set $\{Q\} = a \cap l$. Clearly, $Q \neq C$. Arguing as above, we have $G_C \leq H.Z_d$, where $d \mid q-1$. Consequently, H fixes exactly $\frac{q-1}{d}$ points in C^G. Nevertheless, H fixes exactly 2 points on l. Then $\frac{q-1}{d} = 1$, as q is even. Therefore, $\left|C^G\right| = q+1$. Thus, $n \geq q$, since $C^G \subseteq l$. Actually, $n > q$, since $n \neq q$ by our assumptions. In particular, $Q \notin C^G$. The above argument, with Q^G in the role of C^G, yields that either $\left|Q^G\right| = 1$ or $\left|Q^G\right| = q+1$. It should be stressed that, differently from C, the possibility $\left|Q^G\right| = 1$ might occur. Indeed, Lemma 3.2(2) cannot be applied to Q as $H = (C, a)$, $\{Q\} = a \cap l$ and $Q \neq C$. Now, suppose that $l - (C^G \cup Q^G) \neq \emptyset$. Then there exists a point $Y \in l - (C^G \cup Q^G)$ such that $|G_Y|$ is odd. Moreover, $|G_Y| > q+1$, since $Y^G \subseteq l - (C^G \cup Q^G)$, and since $\left|l - (C^G \cup Q^G)\right| < q^2 - q - 1$ as $n < q^2$ and $\left|Q^G\right| \geq 1$. This leads to a contradiction by a direct inspection of the list given in [17, Hauptsatz II.8.27], since $|G_Y|$ is odd. Thus, $l = C^G \cup Q^G$. Since $\left|C^G\right| = q+1$, then either $n = q+1$ or $n = 2q+1$ according to whether $\left|Q^G\right| = 1$ or $\left|Q^G\right| = q+1$, respectively. So, we obtain a contradiction in each case by [23, Theorem 26], since G acts 2-transitively on C^G. As a consequence, G does not fix lines of Π.

(III) Either $n = q$ or $n = q^2$.

If $n < q^2$ and $n \neq q$, then G does not fix points or lines of Π by (II) and its dual. Furthermore, G does not fix triangles of Π, since G is simple as $q > 3$. So, G is irreducible on Π. Moreover, G contains involutory perspectivities by (I). This is impossible by [12, Lemma 5.1], since q is even and $q > 4$. Thus, either $n = q$ and hence $\Pi \cong \mathrm{PG}(2, q)$ by Theorem 2.2, or $n = q^2$. That is the assertions (2a) and (4a.iii) (of Theorem 1.1). This completes the proof. □

Once Theorem 1.1 has been proved, Theorem 1.2 is just a consequence of this one. Theorem 1.3 follows in turn by a combination of Theorem 1.2, of Theorems 2.1 and 3.3 of [6] and of Theorem 5.1 of [7].

Finally, we have the following *other* examples for Theorem 1.1 (these are not quoted examples in [15, Theorem A] or [13, Theorem 6.1] or [14, Theorem C]):

(1) Let $G \cong \mathrm{PSL}(2, 7)$ and let $\Gamma \cong \mathrm{PSL}(3, m^h)$, with $7 < m^h < 49$.

Assume that m^h is odd. If $G \leq \Gamma$, then $m^{3h} \equiv 1 \bmod 7$ by [1] and this case really occurs. Actually, $m^h = 9, 11, 23, 25, 29, 37$ or 43, as $7 < m^h < 49$.

Since the other cases are already quoted in [15] or [13] or [14], we may assume that $q = 25$ or 43. Hence $G \cong \mathsf{PSL}(2,7)$ acts $\Pi \cong \mathsf{PG}(2,25)$ or $\mathsf{PG}(2,43)$. In latter, clearly, the involutions are homologies. Furthermore, by Theorem 1.1, the group G is strongly irreducible (we do not need to have additional assumptions as in Theorem A of [15]).

Assume that $m^h = 32$ is even. Then $G \leq \Gamma$ by [8]. Hence, $G \cong \mathsf{PSL}(2,7)$ acts on $\Pi \cong \mathsf{PG}(2,32)$.

(2) Let $G \cong \mathsf{PSL}(2,9)$ and let $\Gamma \cong \mathsf{PSL}(3,m^h)$, with $9 < m^h < 81$.

Assume that m^h is odd. If $G \leq \Gamma$, then either $m^h \equiv 1, 19 \bmod 30$ or $m = 5$ and h even by [1] and these cases really occur. Actually, $m^h = 19, 31, 25, 49, 61$ or 79, as $9 < m^h < 81$. Since the other cases are already quoted in [15] or [13] or [14], we may assume that $q = 49$ or 79. While in latter the involutions are clearly homologies, in the former this follows by Theorem 2.6. Furthermore, it follows by Theorem 1.1 that, the group G is strongly irreducible (no additional assumptions are required, as in Theorem A of [15]).

(3) Let $G \cong \mathsf{PSL}(2,9)$. Then G is a subgroup of $\mathsf{PSL}(3,4)$ by [2]. Now, the group $\mathsf{PSL}(3,4)$, and hence $\mathsf{PSL}(2,9)$, leaves invariant a Desarguesian subplane of order 4 in a Desarguesian plane or a Figueroa plane of order 64 (see [5], [10])(so this is an example for the case (3b.iii) of Theorem 1.1.

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