The general structure of the projective planes admitting $\text{PSL}(2, q)$ as a collineation group

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Abstract

Projective planes of order $n$ admitting $\text{PSL}(2, q), q > 3$, as a collineation group are investigated for $n \leq q^2$. As a consequence, affine planes of order $n$ admitting $\text{PSL}(2, q), q > 3$, as a collineation group are classified for $n < q^2$ and $(q, n) \neq (5, 16)$. Finally, a complete classification of the translation planes order $n$ that admitting $\text{PSL}(2, q), q > 3$, as a collineation group is obtained for $n \leq q^2$.

Keywords: projective plane, collineation group, orbit

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1. Introduction and main results

A classical problem in finite geometry is classifying finite projective planes II of order $n$ admitting a collineation group $G$ isomorphic to $\text{PSL}(2, q)$. The first significant result related to this problem dates back to 1964 and is due to Lüneburg [21] and to Yaqub [26]. In their papers, the authors provide a characterization of the Desarguesian projective planes of order $n = q$. Some years later, Kantor [19], Hering [9], Hering and Walker [11, 12], Reifart and Stroth [25] extended the investigation to planes of more arbitrary order but with additional assumptions: $G$ does not fix points, lines or triangles of II and $G$ contains involutory perspectivities. In 1989, Moorhouse obtains significant progress along these lines in two ways: he classifies projective planes of order $n$ admitting $\text{PSL}(2, q)$ as a collineation group for $n < q$ and he investigates the structure of the planes of order $q^2$ for $q$ odd. In the second case, Moorhouse shows that II cannot be the projective extension of an affine plane admitting $\text{PSL}(2, q)$ as a collineation group, except for $q = 5$ or 9 which remain still unsolved. In particular, Moorhouse provides a new proof for $q$ odd of the characterization, due to Foulser
and to Johnson [6, 7], of the translation planes of order \( q^2 \) admitting \( \text{PSL}(2, q) \). In 1991, Dempwolff [3] obtains a complete characterization of the projective planes of order 16 admitting \( \text{PSL}(2, 7) \) as collineation group. In that paper, Dempwolff shows that, beside the Desarguesian plane of order 16, the Lorimer-Rahilly plane of order 16, the Johnson-Walker plane of order 16, and their duals also occur. A similar result for translation planes of order 16 was obtained by Johnson [18] in 1984. In 1994, Ho [13] and Ho-Gonçalves [15] investigate the projective planes of order \( n \) admitting \( G \) isomorphic to \( \text{PSL}(2, q) \) for \( q \) odd, under the assumption that \( G_P \neq \langle 1 \rangle \) for each point \( P \) of \( \Pi \). The authors prove that \( G \) does not fix points, lines or triangles of \( \Pi \). In particular, \( \Pi \) cannot be the projective extension of an affine plane that admits \( \text{PSL}(2, q) \) as a collineation group. They also obtain a characterization of the Desarguesian plane of order \( q \) under the assumption that \( G \) contains involutory homologies and that \( G_P \) has a particular order for each point \( P \) of \( \Pi \). Recently, Liu and Li [20] proved that the unique projective plane \( \Pi \) of order \( n \) admitting \( \text{PSL}(2, q) \) as transitive collineation group is \( \Pi \cong \text{PG}(2, 2) \) and the group is isomorphic to \( \text{PSL}(2, 7) \).

This paper focuses on the main problem cited above. In particular, the projective planes of order \( n \) admitting a collineation group \( G \) isomorphic to \( \text{PSL}(2, q) \), \( q > 3 \), for \( n \leq q^2 \), are investigated and the following results are obtained.

**Theorem 1.1.** Let \( \Pi \) be a projective plane of order \( n \) admitting a collineation group \( G \cong \text{PSL}(2, q) \), \( q > 3 \). If \( n \leq q^2 \), then one of the following occurs:

1. \( n < q \) and one of the following occurs:
   
   (a) \( n = 4 \), \( \Pi \cong \text{PG}(2, 4) \) and \( G \cong \text{PSL}(2, 5) \);
   
   (b) \( n = 2 \) or 4, \( \Pi \cong \text{PG}(2, 2) \) or \( \text{PG}(2, 4) \), respectively, and \( G \cong \text{PSL}(2, 7) \);
   
   (c) \( n = 4 \), \( \Pi \cong \text{PG}(2, 4) \) and \( G \cong \text{PSL}(2, 9) \).

2. \( n = q \), \( \Pi \cong \text{PG}(2, q) \) and one of the following occurs:
   
   (a) \( G \) fixes a line or a point and \( q \) is even;
   
   (b) \( G \) is strongly irreducible and \( q \) is odd.

3. \( q < n < q^2 \) and one of the following occurs:
   
   (a) \( G \) fixes a point or a line, and one of the following occurs:
   
   (i) \( n = 16 \) and \( G \cong \text{PSL}(2, 5) \);
   
   (ii) \( n = 16 \), \( \Pi \) is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and \( G \cong \text{PSL}(2, 7) \);
   
   (b) \( G \) fixes a subplane \( \Pi_0 \) of \( \Pi \), \( q \) is odd and one of the following occurs:
(i) \( n = 16, \Pi_0 \cong \text{PG}(2, 4) \) and \( G \cong \text{PSL}(2, 5) \);
(ii) \( \Pi_0 \cong \text{PG}(2, 2) \) or \( \text{PG}(2, 4) \), and \( G \cong \text{PSL}(2, 7) \);
(iii) \( \Pi_0 \cong \text{PG}(2, 4) \) and \( G \cong \text{PSL}(2, 9) \).

(c) \( G \) is strongly irreducible and \( q \) is odd;

(4) \( n = q^2 \) and one of the following occurs:

(a) \( G \) fixes a point or a line, and one of the following occurs:
   (i) \( n = 25 \) and \( G \cong \text{PSL}(2, 5) \);
   (ii) \( n = 81 \) and \( G \cong \text{PSL}(2, 9) \);
   (iii) \( n = q^2, q \) even, and \( G \cong \text{PSL}(2, q) \).

(b) \( G \) fixes a subplane \( \Pi_0 \) of \( \Pi \), \( q \) is odd and one of the following occurs:
   (i) \( n = q^2, \Pi_0 \cong \text{PG}(2, q) \) and \( G \cong \text{PSL}(2, q) \);
   (ii) \( n = 25, \Pi_0 \cong \text{PG}(2, 4) \) and \( G \cong \text{PSL}(2, 5) \);
   (iii) \( n = 81, \Pi_0 \cong \text{PG}(2, 4) \) and \( G \cong \text{PSL}(2, 9) \);
   (iv) \( n = 81, \Pi_0 \) is a Hughes plane of order 9 and \( G \cong \text{PSL}(2, 9) \);

(c) \( G \) is strongly irreducible.

The Theorem 1.1 under the additional assumptions \( n \leq q \), or \( n = q^2 \) with \( q \) odd yields the cases (1), (2), and (4) for \( q \) odd. So, we need to prove that (3) occurs when \( q < n < q^2 \), and either (4a.iii) or (4c) for \( n = q^2 \) and \( q \) even.

Examples corresponding to case (1) or (2) really occur (see [24] and [21, 26], respectively). Examples of the case (3a.i) occur in the Dempwolff plane of order 16 (see [7]), those of type (3a.ii) really occur (see [3]). Examples of the case (3b.i) occur in the Hall plane of order 16, those corresponding to the case (3b.ii) occur in the Desarguesian plane of order 16 when \( \Pi_0 \cong \text{PG}(2, 2) \) by [3]. Also, examples of the case (3b.iii) occur in the Desarguesian plane of order 64 or the Figueroa plane of order 64. See section 8 for a description of the latter. Furthermore, examples of the case (3c) occurs in the Desarguesian planes of prime order. In these cases \( G \cong \text{PSL}(2, q) \) with \( q = 5, 7 \) or 9 and the involutions in \( G \) are a homologies of \( \Pi \). For a description of these examples see [15] and [13]. While cases (4a.i) and (4a.ii) are open, examples of the case (4a.iii) typically occurs in the Desarguesian planes, in the Hall planes and in the Ott-Schaeffer planes (see [7]). The case (4b.i) occurs in the Desarguesian or Generalized Hughes planes (see [22]). Finally the cases (4b.ii), (4b.iii), (4b.iv) and (4c) are open. Other examples are obtained in section 8.

A special case of the previous theorem is the following which focuses on the projective extensions of affine planes of order \( n \) that admit \( G \cong \text{PSL}(2, q), q > 3 \),
as a collineation group when \( n \leq q^2 \). It should be stressed that it furnishes a complete classification of such affine planes, when \( n < q^2 \) and \( (q,n) \neq (5,16) \).

**Theorem 1.2.** Let \( \Pi \) be the projective extension of an affine plane of order \( n \) that admits a collineation group \( G \cong \text{PSL}(2,q) \), \( q > 3 \). If \( n \leq q^2 \), then one of the followings occurs:

1. \( n = q, \ q = 2^h, \ h > 1, \ II \cong \text{PG}(2,q) \) and \( G \cong \text{PSL}(2,q) \);
2. \( n = 16 \) or \( 25 \), and \( G \cong \text{PSL}(2,5) \);
3. \( n = 16 \), \( \Pi \) is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and \( G \cong \text{PSL}(2,7) \);
4. \( n = 81 \) and \( G \cong \text{PSL}(2,9) \);
5. \( n = q^2, \ q = 2^h, \ h > 1, \) and \( G \cong \text{PSL}(2,q) \).

Finally, the previous theorem leads to a complete classification of the projective extensions of translation planes of order \( n \) that admit a collineation group \( G \cong \text{PSL}(2,q) \), \( q > 3 \), for \( n \leq q^2 \). In particular, it represents an extension of the Foulser-Johnson Theorems [6] and [7], when \( G \cong \text{PSL}(2,q) \) and \( q \) is even.

**Theorem 1.3.** Let \( \Pi \) be the projective extension of a translation plane of order \( n \) that admits a collineation group \( G \cong \text{PSL}(2,q) \), \( q > 3 \). If \( n \leq q^2 \), then one of the following occurs:

1. \( n = q, \ q = 2^h, \ h > 1, \ G \cong \text{PSL}(2,q) \) and \( \Pi \cong \text{PG}(2,q) \);
2. \( n = 16 \), \( G \cong \text{PSL}(2,7) \) and \( \Pi \) is the Lorimer-Rahilly plane or the Johnson-Walker plane;
3. \( n = q^2, \ q = 2^h, \ h > 1, \ G \cong \text{PSL}(2,q) \) and \( \Pi \) is the Desarguesian or Hall plane of even order \( q^2 \), or the Ott-Schaeffer plane of order \( q^2 \), or the Dempwolff plane of order 16 (in this case \( q = 4 \)).

The paper is structured as follows. In section 2, we fix notation and introduce some geometrical and group-theoretical background. In section 3, we provide a reduction for the group-structure of \( G_P \), where \( P \) is a point of a line \( l \) of \( \Pi \) fixed by \( G \). A reduction is also provided for types and numbers of \( G \)-orbits on \( l \). The same is also made for \( G_m \), where \( m \) is a line of \( [Q] \) and \( Q \) is a point of \( \Pi \) fixed by \( G \). Sections 4, 5, 6 and 7 are devoted to the proof of Theorem 1.1 for \( q \equiv 1, 3, 5, 7 \mod 8 \), respectively. Finally, in section 8 the proofs of Theorems 1.1, 1.2 and 1.3 are completed and some examples are provided. In particular, in this section, the case \( q \) even is resolved.
2. The background

In this section, we introduce the background for the problem investigated and we state the group-theoretical theorems that are used in the proof of our main result. Furthermore some useful numerical and group-theoretical results are proved.

For what concerns finite groups and in particular the group PSL(2, q) the reader is referred to [4] and [17]. The necessary background about finite projective planes may be found in [16].

Let \( \Pi = (\mathcal{P}, \mathcal{L}) \) be a finite projective plane of order \( n \). If \( H \) is a collineation group of \( \Pi \) and \( P \in \mathcal{P} \) (\( l \in \mathcal{L} \)), we denote by \( H(P) \) (by \( H(l) \)) the subgroup of \( H \) consisting of perspectivities with centre \( P \) (axis \( l \)). Also, \( H(P, l) = H(P) \cap H(l) \). Furthermore, we denote by \( H(P, P) \) (by \( H(l, l) \)) the subgroup of \( H \) consisting of elations with centre \( P \) (axis \( l \)).

Let \( \Pi \) be a finite projective plane of order \( n \) admitting a collineation group \( G \) isomorphic to PSL(2, q), and assume that \( n \leq q^2 \). The following theorems deal with the case \( n < q \), \( n = q \) and \( n = q^2 \), respectively.

**Theorem 2.1 (Moorhouse).** If \( \Pi \) is a projective plane of order \( n < q \) admitting a collineation group \( G \) isomorphic to PSL(2, q), then \( \Pi \) is Desarguesian and \((n, q) = (2, 3), (2, 7), (4, 5), (4, 7) \) or \((4, 9) \). Moreover, each of the latter cases indeed occurs.

**Proof.** See [24, Theorem 1.1].

**Theorem 2.2 (Lüneburg-Yaqub).** If \( \Pi \) is a projective plane of order \( q \) admitting a collineation group \( G \) isomorphic to PSL(2, q), then \( \Pi \) is Desarguesian.

**Proof.** See [21] and [26].

**Theorem 2.3 (Moorhouse).** Suppose that a projective plane \( \Pi \) of order \( q^2 \) admits a collineation group \( G \) isomorphic to PSL(2, q), where \( q \) is odd. Then one of the following must hold:

1. \( G \) acts irreducibly on \( \Pi \);
2. \( q = 3 \) and \( G \) fixes a triangle but no point or line of \( \Pi \);
3. \( q = 5 \), \( \text{Fix}(G) \) consists of an antiflag \((X, l)\) and \( G \) has point orbits of length \( 5, 5, 6 \) and \( 10 \) on \( l \);
4. \( q = 9 \) and \( \text{Fix}(G) \) consists of a flag.

**Proof.** See [24, Theorem 1.2].
**Theorem 2.4 (Moorhouse).** Suppose that a projective plane $\Pi$ of order $q^2$ admits a collineation group $G$ isomorphic to $\text{PSL}(2, q)$ (where $q$ is odd), and that $G$ leaves invariant a subplane $\Pi_0$ of $\Pi$. If $q \neq 5, 9$, then $\Pi_0$ is a Desarguesian Baer subplane of $\Pi$.

**Proof.** See [24, Corollary 5.2(i)].

As a consequence of the previous theorems, it follows that we may consider projective planes $\Pi$ of order $n$ admitting a collineation group $G$ isomorphic to $\text{PSL}(2, q)$ for $q < n < q^2$ when $q$ is odd, and for $n < q^2$ and $n \neq q$ when $q$ is even.

Before starting our investigation, we introduce some tools that will be used throughout the paper. Let $P^G$ be an orbit on $l$, let $X$ be any subgroup of $G$ and let $\alpha$ be any element of $G$. Set $\text{Fix}_{P^G}(X) = \text{Fix}(X) \cap P^G$ and $\text{Fix}_{P^G}(\alpha) = \text{Fix}(\alpha) \cap P^G$. If $r^G$ is an orbit of lines of $\Pi$, set $\text{Fix}_{r^G}(X) = \text{Fix}(X) \cap r^G$ and $\text{Fix}_{r^G}(\alpha) = \text{Fix}(\alpha) \cap r^G$.

**Proposition 2.5 (Moorhouse).** Let $G$ be a collineation group of a finite projective plane $\Pi$ of order $n$, let $P \in l$ and let $H$ be a subgroup of $G$. Then

$$|\text{Fix}_{P^G}(H)| = \frac{|N_G(H)|}{|G_P|} \cdot |\{U \leq G_P : U \text{ is conjugate to } H \text{ in } G\}|.$$

**Proof.** See [24, relation (9)].

Note that (1) still works if we replace $\text{Fix}_{P^G}(H)$ with $\text{Fix}_{r^G}(X)$ and $G_P$ with $G_r$.

**Theorem 2.6 (Ho).** Let $G$ be a collineation group of a finite projective plane $\Pi$ of order $n$. Suppose that either $n$ is not a square or $n = m^2$ with $m \equiv 2$ or $3 \mod 4$. If $4 \mid |G|$, then $G$ contains an involutory perspectivity.

**Proof.** See [14, Theorem A].

As we will see, the following lemmas play a central role in section 4.

**Lemma 2.7.** Let $q$ be an even power of an odd prime, let $x$ be a positive integer, let $u$ be a positive divisor of $\frac{\sqrt{q} + 1}{2}$ and let $h = 2$ or $4$. Then the following hold:

1. The quadruple $(x, h, u, \sqrt{q}) = (1, 4, 1, 3)$ is the unique solution of the Diophantine equation

$$x\sqrt{q} = h \frac{\sqrt{q} - 1}{2u} - 1.$$
(II) The quadruple \((x, h, u, \sqrt{q}) = (1, h, h/2, \sqrt{q})\) for \(\sqrt{q} \equiv 3 \mod h\) is the unique solution of the Diophantine equation

\[ x\sqrt{q} = h\frac{\sqrt{q} + 1}{2u} - 1. \] (3)

**Proof.** Consider the Diophantine equation (2). Assume that \(h = 2\). Then (2) becomes \(x\sqrt{q} = \frac{\sqrt{q} - 1}{u} - 1\). Since \(u \geq 1\), then \(x\sqrt{q} \leq \sqrt{q} - 2\). Nevertheless, this is impossible, since \(x \geq 1\). So, no solutions arise for \(h = 2\).

Assume that \(h = 4\). Then (2) becomes

\[ x\sqrt{q} = 2\frac{\sqrt{q} - 1}{u} - 1. \] (4)

If \(u \geq 2\), then \(x\sqrt{q} \leq \sqrt{q} - 2\). Thus, we again obtain a contradiction, since \(x \geq 1\). Therefore, \(u = 1\). By substituting this value in (4), we obtain \(x\sqrt{q} = 2\sqrt{q} - 3\). This one has a unique solution \((x, \sqrt{q}) = (1, 3)\). Hence, \((x, h, u, \sqrt{q}) = (1, 4, 1, 3)\) is the unique solution of the Diophantine equation (2) for \(h = 4\). From this and bearing in mind that (2) has no solutions for \(h = 2\), we obtain the assertion (I).

Now, consider the Diophantine equation (3). Assume that \(h = 2\). Then (3) becomes

\[ x\sqrt{q} = \frac{\sqrt{q} + 1}{u} - 1. \] (5)

If \(u > 1\), then \(\frac{\sqrt{q} + 1}{u} - 1 < \sqrt{q} \leq x\sqrt{q}\). Thus, (5) has no solutions in this case. So, assume that \(u = 1\). By substituting this value in (5), we obtain \(x\sqrt{q} = \sqrt{q}\) and hence \(x = 1\). Therefore, we have proved that \((x, h, u, \sqrt{q}) = (1, 2, 1, \sqrt{q})\) is the unique solution of (3) for \(h = 2\).

Now, assume that \(h = 4\). Then (3) becomes

\[ x\sqrt{q} = 2\frac{\sqrt{q} + 1}{u} - 1. \] (6)

If \(u > 1\), then \(2\frac{\sqrt{q} + 1}{u} - 1 < \sqrt{q} \leq x\sqrt{q}\). Therefore, (6) has no solutions in this case. So, there are admissible solutions for (6) only for \(u \leq 2\). If \(u = 1\), then (6) becomes \(x\sqrt{q} = 2\sqrt{q} + 1\). This one has no solutions, since the first part is divisible by \(\sqrt{q}\), while the second is not. Thus, \(u = 2\). At this point, it is a straightforward computation to see that \((x, h, u, \sqrt{q}) = (1, 2, 1, \sqrt{q})\) for \(\sqrt{q} \equiv 3 \mod 4\) is a solution of (6) and hence of (3). From this and bearing in mind that \((x, h, u, \sqrt{q}) = (1, 2, 1, \sqrt{q})\) is the unique solution of (3) for \(h = 2\), we obtain the assertion (II).

**Lemma 2.8.** Let \(q\) be an even power of an odd prime, let \(x\) be a positive integer and let \(u_1\) and \(u_2\) be two positive divisors of \(\frac{\sqrt{q} - 1}{2}\). Furthermore, let \(h = 2\) or 4.
If $u_1 \leq u_2$, then $(x, h, u_1, u_2, \sqrt{q}) = (1, 2, 1, \frac{\sqrt{q}-1}{2}, \sqrt{q})$, $(3, 4, 1, 1, 5)$, $(1, 4, 3, 3, 7)$, or $(1, 4, 3, 5, 31)$ are the unique solutions of the Diophantine equation

\[ x\sqrt{q} = h\frac{\sqrt{q} - 1}{2u_1} + h\frac{\sqrt{q} - 1}{2u_2} - 1. \tag{7} \]

**Proof.** Multiplying by $2u_1u_2$ each term of (7), we have

\[ 2u_1u_2x\sqrt{q} = hu_2(\sqrt{q} - 1) + hu_1(\sqrt{q} - 1) - 2u_1u_2. \]

Now, collecting the terms with respect to $\sqrt{q}$, we obtain

\[ [h(u_1 + u_2) - 2u_1u_2x]\sqrt{q} = h(u_1 + u_2) + 2u_1u_2. \tag{8} \]

Since $h(u_1 + u_2) + 2u_1u_2 > 0$, then

\[ 2u_1u_2x < h(u_1 + u_2). \tag{9} \]

Assume that $h = 2$. Then $u_1u_2x < u_1 + u_2$ by (9). In particular, $u_1u_2 < u_1 + u_2$, as $x \geq 1$. This, in turn, yields $u_1u_2 < 2u_2$, since $u_1 \leq u_2$ by our assumption. Thus, $u_1 < 2$. That is $u_1 = 1$. Now, by substituting $h = 2$ and $u_1 = 1$ in (9), we obtain $2u_2x < 2(1 + u_2)$ and hence $x < 1 + 1/u_2$. Then $x = 1$, as $u_2 \geq 1$. By substituting the values $x = 1$, $h = 2$ and $u_1 = 1$ in (8), and then by elementary calculations of this one, we have $2\sqrt{q} = 2 + 4u_2$. Hence, $u_2 = \frac{\sqrt{q} - 1}{2}$. Consequently, $(x, h, u_1, u_2, \sqrt{q}) = (1, 2, 1, \frac{\sqrt{q}-1}{2}, \sqrt{q})$ is a solution of (7).

Assume that $h = 4$. Then

\[ u_1u_2x < 2(u_1 + u_2) \tag{10} \]

by (9).

If $x \geq 4$, then $2u_1u_2 < u_1 + u_2 \leq 2u_2$ by (10), since $u_1 \leq u_2$ by our assumption. This yields $u_1 < 1$, which is a contradiction.

If $x = 3$, then $3u_1u_2 < 2(u_1 + u_2)$ by (10). Since $u_1 \leq u_2$, we have $3u_1u_2 < 4u_2$, and hence $u_1 = 1$. Now, by substituting $(x, h, u_1) = (3, 4, 1)$ in (9), we obtain $u_2 < 2$. Actually, $u_2 = 1$. Finally, by substituting $(x, h, u_1, u_2) = (3, 4, 1, 1)$ in (8), we have $\sqrt{q} = 5$. So, $(x, h, u_1, u_2, \sqrt{q}) = (3, 4, 1, 1, 5)$ is a solution of (7).

If $x = 2$, then $u_1u_2 < u_1 + u_2$ by (10). By arguing as above, we obtain $u_1 = 1$. By substituting $(x, h, u_1) = (2, 4, 1)$ in (8), we have $4\sqrt{q} = 4 + 6u_2$ and hence $u_2 = \frac{2(\sqrt{q} - 1)}{3}$. Nevertheless, this contradicts the assumption $u_2 \mid \frac{\sqrt{q}-1}{2}$.

If $x = 1$, then $u_1u_2 < 4(u_1 + u_2)$ by (9). Thus, $u_1 < 8$, since $u_1 \leq u_2$. On the other hand, by substituting $h = 4$ and $x = 1$ in (7), we have

\[ \frac{\sqrt{q} + 1}{2} = \frac{\sqrt{q} - 1}{u_1} + \frac{\sqrt{q} - 1}{u_2}. \tag{11} \]
If \( u_1 = 1 \), no solutions arise, since
\[
\sqrt{q} - 1 < \frac{\sqrt{q} - 1}{u_1} + \frac{\sqrt{q} - 1}{u_2} = \frac{\sqrt{q} + 1}{2}
\]
and since \( \sqrt{q} \) is odd. So, \( u_1 \geq 2 \). Assume that \( u_1 \geq 4 \). Then \( u_2 \geq 4 \) as \( u_2 \geq u_1 \). Hence,
\[
\frac{\sqrt{q} + 1}{2} = \frac{\sqrt{q} - 1}{u_1} + \frac{\sqrt{q} - 1}{u_2} \leq \frac{\sqrt{q} - 1}{2}.
\]
Therefore, there are also no solutions in this case. Consequently, \( u_1 = 2 \) or 3. Assume that \( u_1 = 2 \). Then (11) becomes \( \frac{\sqrt{q} + 1}{2} = \frac{\sqrt{q} - 1}{u_2} \). This yields \( u_2 = \sqrt{q} - 1 \). Nevertheless, this cannot occur, since \( u_2 \mid \frac{\sqrt{q} - 1}{2} \) by our assumptions. Hence, \( u_1 = 3 \). Then \( u_2 = \frac{6(\sqrt{q} - 1)}{\sqrt{q} + 5} \) from (11). This yields \( \sqrt{q} + 5 \mid 36 \), since \( (\sqrt{q} + 5, \sqrt{q} - 1) \mid 6 \). As a consequence, \( \sqrt{q} = 7, 13 \) or 31. Then \( u_2 = 3, 4 \) or 5, respectively, since \( u_2 = \frac{6(\sqrt{q} - 1)}{\sqrt{q} + 5} \). Nevertheless, only the cases \( u_2 = 3 \) or 5 are admissible, since \( u_2 \mid \frac{\sqrt{q} - 1}{2} \) by our assumption. Actually, \( (x, h, u_1, u_2, \sqrt{q}) = (1, 4, 3, 3, 7) \) or \( (1, 4, 3, 5, 31) \) are solutions of (7). This completes the proof. 

**Lemma 2.9.** Let \( q \) be an even power of an odd prime, let \( x \) be a positive integer and let \( u_1 \) and \( u_2 \) be two positive divisors \( \frac{\sqrt{q} + 1}{2} \). Furthermore, let \( h = 2 \) or 4. If \( u_1 \leq u_2 \), then \( (x, h, u_1, u_2, \sqrt{q}) = (5, 4, 1, 1, 3), (3, 4, 1, 3, 5), (1, h, h, h, \sqrt{q}) \) and \( \sqrt{q} \equiv -1 \mod 2h \), or \( (1, 4, 3, 6, \sqrt{q}) \) and \( \sqrt{q} \equiv -1 \mod 12 \) are the unique solutions of the Diophantine equation
\[
x\sqrt{q} = h\frac{\sqrt{q} + 1}{2u_1} + h\frac{\sqrt{q} + 1}{2u_2} - 1. \quad (12)
\]

**Proof.** Multiplying by \( 2u_1u_2 \) each term of (12), we have
\[
2u_1u_2x\sqrt{q} = hu_2(\sqrt{q} + 1) + hu_1(\sqrt{q} + 1) - 2u_1u_2.
\]
Now, collecting the terms with respect to \( \sqrt{q} \), we obtain
\[
[2u_1u_2x - h(u_1 + u_2)]\sqrt{q} = h(u_1 + u_2) - 2u_1u_2. \quad (13)
\]
We treat the cases \( 2u_1u_2x - h(u_1 + u_2) \neq 0 \) and \( 2u_1u_2x - h(u_1 + u_2) = 0 \) separately. Assume the former occurs. Then \( \sqrt{q} = \frac{h(u_1 + u_2) - 2u_1u_2}{2u_1u_2x - h(u_1 + u_2)} \). That is
\[
\sqrt{q} = -\frac{h(u_1 + u_2) - 2u_1u_2}{h(u_1 + u_2) - 2u_1u_2x}. \quad (14)
\]
Note that \( h(u_1 + u_2) < 2u_1u_2 \) implies \( h(u_1 + u_2) < 2u_1u_2x \), since \( x \geq 1 \). Then, by (14), we have that \( h(u_1 + u_2) > 2u_1u_2 , h(u_1 + u_2) < 2u_1u_2x \) and \( x \geq 2 \). In particular, since \( x \geq 2 \) and \( \sqrt{q} \geq 3 \), from the first part of (13), we have
\[
[2u_1u_2x - h(u_1 + u_2)]\sqrt{q} \geq 3 [4u_1u_2 - h(u_1 + u_2)]. \quad (15)
\]
Combining (13) with (15), we obtain

$$3 \left[ 4u_1 u_2 - h(u_1 + u_2) \right] \leq h(u_1 + u_2) - 2u_1 u_2 .$$

Elementary calculations of the previous inequality yield $14u_1 u_2 \leq 4h(u_1 + u_2)$. That is

$$7u_1 u_2 \leq 2h(u_1 + u_2) .$$

Therefore, $7u_1 u_2 \leq 4h u_2$ and hence

$$1 \leq u_1 \leq \frac{4}{7} h .$$

since $u_1 \leq u_2$.

Assume that $h = 2$. Then $u_1 = 1$ by (17). So, $h(u_1 + u_2) - 2u_1 u_2 = 2$. Then $\sqrt{q} \mid 2$ by (13). Nevertheless, this is impossible, since $\sqrt{q}$ is a power of an odd prime. Hence, (12) has no solutions for $h = 2$.

Assume that $h = 4$. Then either $u_1 = 1$ or $u_1 = 2$ by (17). Assume the latter occurs, then $14u_2 \leq 8(2 + u_2)$ by (16). As a consequence, $u_2 \leq \frac{16}{7}$. On the other hand, $u_2 \geq 2$, since $u_1 = 2$ and $u_1 \leq u_2$. So, $2 \leq u_1 \leq u_2 \leq \frac{16}{7}$. Therefore, $u_1 = u_2 = 2$. Then $h(u_1 + u_2) - 2u_1 u_2 = 8$, since $h = 4$. Thus, $\sqrt{q} \mid 8$ by (13). Again, this is impossible, since $\sqrt{q}$ is a power of an odd prime. For this reason, we have $u_1 = 1$. Then (13) becomes

$$[(2x - 4)u_2 - 4]\sqrt{q} = 4 + 2u_2 .$$

Note that $x \geq 3$, otherwise the first part of (18) is negative while the second one is positive, as $u_2 \geq u_1 = 1$.

Assume that $x \geq 4$. Then $12(u_2 - 1) \leq 4 + 2u_2$ by (18), as $\sqrt{q} \geq 3$. This yields $u_2 = 1$ and $4 + 2u_2 = 6$. Then $\sqrt{q} = 3$ and $x = 5$ again by (18), since $\sqrt{q}$ is a power of an odd prime. So, $(x, h, u_1, u_2, \sqrt{q}) = (5, 4, 1, 1, 3)$ is the unique solution of (12) for $h = 4$ and $x \geq 4$.

Assume that $x = 3$. Then $(u_2 - 2)\sqrt{q} = u_2 + 2$ by (18). Now, collecting with respect to $u_2$, we have $(\sqrt{q} - 1)u_2 = 2(\sqrt{q} + 1)$. That is $u_2 = 2 + 4/(\sqrt{q} - 1)$. This Diophantine equation has solutions $(u_2, \sqrt{q}) = (4, 3)$ or $(3, 5)$. Actually, the former is not admissible, since $u_2 \mid \frac{\sqrt{q} + 1}{2}$ by our assumption. Hence, $(x, h, u_1, u_2, \sqrt{q}) = (3, 4, 1, 3, 5)$ is the unique solution of (12) for $h = 4$ and $x = 3$.

Assume that $2u_1 u_2 x - h(u_1 + u_2) = 0$. Then

$$h(u_1 + u_2) - 2u_1 u_2 = 0$$

by (13). As a consequence, $x = 1$.
If \( h = 2 \), then \( 2(u_1 + u_2) - 2u_1u_2 = 0 \) by (19). So, \( u_1 + u_2 = u_1u_2 \).

Now, it is plain to see that \( u_1 = u_2 = 2 \). Then \( \sqrt{q} \equiv -1 \mod 4 \), since \( u_1 \) and \( u_2 \) are positive divisors of \( \sqrt{q+1} \) by our assumption. Thus, by substituting \( (x, h, u_1, u_2) = (1, 2, 2, 2) \) in (13), we have that \( (x, h, u_1, u_2, \sqrt{q}) = (1, 2, 2, 2, \sqrt{q}) \) is a solution of (12).

If \( h = 4 \), then \( 4(u_1 + u_2) - 2u_1u_2 = 0 \) by (19). So, \( 2(u_1 + u_2) = u_1u_2 \). As \( u_2 \mid 2u_1 \) and \( u_1 \leq u_2 \), then either \( u_2 = u_1 \) or \( u_2 = 2u_1 \). Assume that \( u_2 = u_1 \). Then \( u_1 = u_2 = 4 \) by \( 2(u_1 + u_2) = u_1u_2 \). As a consequence, \( \sqrt{q} \equiv -1 \mod 8 \), since \( u_1 \) and \( u_2 \) are positive divisors of \( \sqrt{q+1} \) by our assumption. Thus, by substituting \( (x, h, u_1, u_2) = (1, 4, 4, 4) \) in (13), we have that \( (x, h, u_1, u_2, \sqrt{q}) = (1, 4, 4, 4, \sqrt{q}) \) is a solution of (12). Finally, assume that \( u_2 = 2u_1 \). Then \( u_1 = 3 \) and \( u_2 = 6 \). Now, since \( u_1 = 3 \) and \( u_2 = 6 \) are two positive divisors of \( \sqrt{q+1} \), we have that \( \sqrt{q} \equiv -1 \mod 12 \). Moreover, by substituting \( (x, h, u_1, u_2) = (1, 4, 3, 6) \) in (13), we see that \( (x, h, u_1, u_2, \sqrt{q}) = (1, 4, 3, 6, \sqrt{q}) \) is a solution of (12). This completes the proof. 

In Lemmas 2.8 and 2.9 the assumption \( u_1 \leq u_2 \) can be dropped. Indeed, if \( u_1 \geq u_2 \) we obtain the 'same' solutions for (7) and (12) but with the values of \( u_1 \) and \( u_2 \) exchanged.

**Lemma 2.10.** Let \( D \) be a dihedral group acting on a projective plane \( \Pi \). Assume that \( D \) fixes a line \( l \) and there exists a dihedral subgroup \( D_0 \) of \( D \) which fixes two distinct points on \( l \) and contains a non central involutory homology. Then one of the following occurs:

1. There exists a subgroup \( D_1 \) of \( D \), such that \( D_0 \leq D_1 \) and \([D : D_1] \leq 2\), fixing at least one point on \( l \);

2. \( D_0 \cong E_4 \).

**Proof.** Suppose that \( D \) fixes a line \( l \) of \( \Pi \) and that there exists a subgroup \( D_0 \) of \( D \) which fixes two distinct points \( X \) and \( Y \) on \( l \) and which contains an involutory homology. Set \( |D| = 2m \) and \( |D_0| = 2m_0 \), where \( m, m_0 > 1 \). Also, set \( D_0 = \langle \alpha, \beta \rangle \), where \( \alpha^{m_0} = \beta^2 = 1 \) and \( \alpha \beta = \alpha^{-1} \). We may assume that \( \beta \) is an involutory homology, since \( D_0 \) contains a non central one by our assumption. If \( a_\beta = l \) then \( \beta \in N \), where \( N \) is the kernel of the action of \( D \) on \( l \). Clearly, it holds that \( N \leq D \). Thus, \( N = D \) if \( m \) is odd, and \( N \cong D_m \) or \( N = D \) for \( m \) even, since \( D \) is dihedral and \( \beta \in N \). If we set \( D_1 = D_0N \), we obtain the assertion (1).

Assume that \( a_\beta \neq l \). Thus, either \( C_\beta = X \) and \( a_\beta \cap l = \{ Y \} \), or \( C_\beta = Y \) and \( a_\beta \cap l = \{ X \} \), since \( D_0 \) fixes two distinct points \( X \) and \( Y \) on \( l \) and since \( \beta \in D_0 \).
We may assume that $C_\beta = X$ and $\{Y\} = a_\beta \cap l$. Assume also that $m_0$ is odd. Then each involution in $D_0$ is a homology of center $X$ and axis intersecting $l$ in $Y$, since $D_0$ fixes $X$ and $Y$, with $X \neq Y$, and since $D_0$ contains a unique conjugate class of involutions as $m_0$ is odd.

If two distinct involutions in $D_0$ have distinct axes (passing through $Y$), then $D_0(X, X) \neq \{1\}$ by [16, Theorem 4.25], since each involution in $D_0$ is homology of center $X$. In particular, $D_0(X, X) \leq S(D_0)$, where $S(D_0)$ denotes the unique maximal (normal) cyclic subgroup of $D_0$. Therefore, $D_0(X, X) \triangleleft D$, since $S(D_0) \triangleleft S(D) \triangleleft D$ and $D$ is dihedral (actually, $D_0(X, X) = D_0(X, r)$ by [16, Theorem 4.14]). Thus, $D$ fixes $X$ and we again obtain the assertion (1).

If all involutions in $D_0$ have the same axis $a_\beta$, then $D_0 = D_0(X, a_\beta)$ as $m_0$ is odd. In particular, $S(D_0) = S(D_0)(X, a_\beta)$, with $S(D_0) \neq \{1\}$, as $m_0$ is odd and $m_0 > 1$. Thus, $D$ fixes $X$, $a_\beta$ and hence $Y$, where $\{Y\} = a_\beta \cap l$, since $S(D_0) \triangleleft D$. Hence, we obtain the assertion (1) also in this case.

Assume that $m_0$ is even and $m_0 > 2$. Thus, $\alpha^{m_0/2}$ is a homology by [19, Proposition 3.3], since $Z(D_0) = \langle \alpha^{m_0/2} \rangle$ (actually, $Z(D) = \langle \alpha^{m_0/2} \rangle$) and since $\beta$ is a homology. Set $\delta = \alpha^{m_0/2}$. If $C_\delta \in l$, then $D$ fixes $C_\delta$, since $\langle \delta \rangle = Z(D)$, $D$ being a dihedral group. Thus, $D$ still obtain the assertion (1).

Now, recall that $a_\beta \neq l$ by our assumption. Set $K = \langle \delta, \beta \rangle$. Then $K$ is a Klein group consisting of commuting involutory homologies whose vertices lie in the triangle $\{X, Y, C_\delta\}$, as $D_0$ fixes $X$ and $Y$ on $l$ and since $a_\beta \neq l$. Let $\rho \in D_0$ and consider $K^\rho$. Then $K^\rho = \langle \delta, \beta^\rho \rangle$ as $\delta$ is central in $D_0$. Furthermore, $K^\rho$ is still a Klein group consisting of commuting involutory homologies whose vertices lying in the triangle $\{X, Y, C_\delta\}$, since $D_0$ fixes $X$, $Y$, and $C_\delta$. Then $K^\rho = K$ by [19, Lemma 3.1]. Hence, $K \leq D_0$. Thus, $D_0 \simeq D_8$ as $m_0 > 2$ by our assumptions. Therefore, $\delta = \alpha^2$ and $K = \langle \alpha^2, \beta \rangle$. Since $\beta$ is an involutory $(X, a_\beta)$-homology with $a_\beta = C_\delta Y$, since $D_0$ fixes $X$ an $Y$ and $C_\delta$ and since $\alpha \in D_0$, then $\beta^\alpha$ is still an involutory $(X, a_\beta)$-homology. This is a contradiction, since $\beta^\alpha = \alpha^2 \beta$ and the collineation $\alpha^2 \beta$ is a $(Y, C_\delta X)$ homology lying in $K$, as $K$ is a Klein group consisting of commuting involutory homologies whose centres are the vertices of the triangle $\{X, Y, C_\delta\}$. Thus, $m_0 = 2$. That is, $D_0 \simeq E_4$. So, we have proved the assertion (2).

3. General reductions

In this section, we provide some reductions for the action of $G$ on $\Pi$. In particular, when $G$ fixes a line $l$ (resp. a point $Q$) of $\Pi$, we determine the admissible stabilizer of a point (resp. line) on $l$ (resp. on $[Q]$), the length of the corresponding $G$-orbit on $l$ (resp. on $[Q]$). Finally, we provide some upper bounds
for the number of some $G$-orbits of points on $l$ (resp. $G$-orbits of lines on $[Q]$).

**Lemma 3.1.** If $G \cong \text{PSL}(2, q)$, with $q$ odd and $q > 3$, does not fix points or lines of $\Pi$, then $G$ is irreducible on $\Pi$. Furthermore, one of the following occurs:

1. $G$ fixes a subplane $\Pi_0 \cong \text{PG}(2, m)$, where $(m, q) = (2, 7), (4, 7), (4, 9)$;
2. $G$ is strongly irreducible on $\Pi$.

**Proof.** Assume that $G$ does not fix lines or points of $\Pi$. Then $G$ does not fix triangles of $\Pi$, since $G$ is non abelian simple as $q > 3$. So, $G$ is irreducible on $\Pi$. Now, assume that $G$ fixes a subplane $\Pi_0$ of $\Pi$ of order $m$. Then $m < q$ by [16, Theorem 3.7], since $n < q^2$ by our assumption. Thus, $\Pi_0 \cong \text{PG}(2, m)$, where $(m, q) = (2, 7), (4, 5), (4, 7), (4, 9)$, by Theorem 2.1, as $q > 3$. Since $G$ acts irreducibly on $\Pi$, then it does the same on $\Pi_0$. Hence, the case $(m, q) = (4, 5)$ is ruled out, since $\text{PSL}(2, 5)$ fixes always a point or a line in $\text{PG}(2, 4)$. So, $(m, q) = (2, 7), (4, 7), (4, 9)$ and hence we obtain the assertion.

**Lemma 3.2.** The following holds:

1. If $G$ fixes a line $l$, then $G(l) = \langle 1 \rangle$ and hence $G$ acts faithfully on $l$.
2. If $G$ fixes a point $P$, then $G(P) = \langle 1 \rangle$ and hence $G$ acts faithfully on $[P]$.

**Proof.** Assume that $G$ fixes $l$. If $G(l, l) \neq \langle 1 \rangle$, then $G = G(l, l)$, since $G$ is simple as $q > 3$. Actually, $G = G(A, l)$ for some point $A \in l$ by [16, Theorem 4.14], since $G$ is non abelian. So, $|G| \mid n$ and hence $|G| < q^2$, as $n < q^2$, which is a contradiction. Thus, $G(l, l) = \langle 1 \rangle$. Now, assume that $G$ contains homologies of axis $l$. Each involution in $G$ of axis $l$ must have the same center, say $C$, otherwise $G(l, l) \neq \langle 1 \rangle$ by [16, Theorem 4.25], as $G$ fixes $l$. Therefore, $G$ fixes $C$ and hence $\langle 1 \rangle < G(C, l) < G$. Then $G = G(C, l)$, since $G$ is simple as $q > 3$. So, $|G| \mid n - 1$ and hence $|G| < q^2$ as $n < q^2$. Hence, we arrive at a contradiction. As a consequence, $G(l) = \langle 1 \rangle$ and hence $G$ acts faithfully on $l$. So, we have proved the assertion (1). Now, dualizing the previous proof, we obtain also the assertion (2).

**Lemma 3.3.** If $q > 3$ and $G$ fixes a line $l$ of $\Pi$, then the involutions in $G$ are Baer collineations of $\Pi$. In particular, $\sqrt{n} \equiv 0, 1 \mod 4$.

**Proof.** Let $\sigma$ be any involution of $G$. Assume that $\sigma$ is a $(C_\sigma, a_\sigma)$-perspectivity of $G$. Then $C_\sigma \in l$ and $a_\sigma \neq l$ by Lemma 3.2(1), since $G$ fixes $l$. Clearly, $C_\sigma(\sigma)$ fixes $C_\sigma$, the lines $l$ and $a_\sigma$ and hence the point $l \cap a_\sigma$ (note that the points $C_\sigma$ and $l \cap a_\sigma$ might coincide or not according to whether $n$ is even or odd, respectively). Hence, $C_\sigma(\sigma) \leq GC_\sigma$. Furthermore, $GC_\sigma < G$ by Lemma 3.2(2).
Then \( G_{C_\sigma} = C_G(\sigma) \), since \( C_G(\sigma) \) is maximal in \( G \), being \( C_G(\sigma) \cong D_{q^\pm 1} \) by [17, Hauptsatz II.8.27], according to whether \( q \equiv 3 \mod 4 \) or \( q \equiv 1 \mod 4 \), respectively. Then \( \sigma \) fixes exactly either \((q+3)/2\) points or \((q+1)/2\) points on \( C_G^G \) by (1) of Proposition 2.5, for either \( q \equiv 3 \mod 4 \) or \( q \equiv 1 \mod 4 \), respectively. Thus, \( \sigma \) fixes at least 3 points on \( l \) in each case as \( q > 3 \). This is a contradiction, since \( \sigma \) is \((C_\sigma, a_\sigma)\)-perspectivity of \( G \) with \( C_\sigma \in l \) and \( a_\sigma \neq l \). Thus, \( \sigma \) is a Baer collineation of \( \Pi \). Then each involution of \( G \) is a Baer collineation of \( \Pi \), since \( G \cong P\text{SL}(2, q) \) contains a unique conjugate class of involutions. This yields \( \sqrt{n} \equiv 0, 1 \mod 4 \) by Theorem 2.6.

**Lemma 3.4.** Let \( \Pi \) be a finite projective plane of order \( n \) and let \( G \cong P\text{SL}(2, q) \), \( q > 3 \), be a collineation group of \( \Pi \) fixing a line \( l \) of \( \Pi \). If \( P \in l \), then one the following occurs:

1. \( G_P = G \);
2. \( G_P \cong D_{q-1} \);
3. \( G_P \cong D_{q+1} \);
4. \( G_P \cong A_4 \) and \( q = 5, 7, 9, 11, 13, 17, 19 \);
5. \( G_P \cong A_5 \) and \( q = 5, 9, 11, 19, 25, 29, 31, 41, 49, 59, 61, 71, 79, 81, 89, 101, 109 \);
6. \( G_P \cong S_4 \) and \( q = 7, 9, 17, 23, 25, 31, 41 \);
7. \( G_P \cong P\text{SL}(2, \sqrt{q}) \);
8. \( G_P \cong P\text{GL}(2, \sqrt{q}) \);
9. \( G_P \cong E_{p_{m-\epsilon} Z_{p^{m-1}}} \), where \( 2 \epsilon | m \);  
10. \( G_P \cong F_{q^d} Z_4 \).

**Proof.** Note that \( n \leq (q - 1)^2 \), since \( n < q^2 \) and \( n \) is a square by Lemma 3.3. Since \( P^G \subset l \) and \( n + 1 \leq (q - 1)^2 + 1 \), then \( |P^G| \leq (q - 1)^2 + 1 \). That is \( q(q-1)(q-1) \leq 2(G_P) \leq (q-1)^2+1 \). Actually, \( q(q-1)(q-1) < (q-1)^2+1 \) and hence \( q(q-1)(q-1) < (q-1)^2+1 \). Then \( 2|G_P| \geq \frac{(q-1)(q+1)}{2|G_P|} \) and consequently \( 2|G_P| > q+1 \). So,

\[
|G_P| > \frac{q+1}{2}.
\]

(20)

Now, filtering the list of the proper subgroups of \( G \) given in [17, Hauptsatz II.8.27], with respect to (20) and bearing in mind [24, Lemma 2.8], when \( G_P \leq F_{p_{m-\epsilon} Z_{p^{m-1}}} \), we obtain the assertion.

Let \( P \in l \). We say that \( G_P \) is of type (i), where \( 1 \leq i \leq 10 \), if \( G_P \) is a group isomorphic to the \( i \)-th group of the list given in the previous lemma. Also, we say that the orbit \( P^G \) is of type (i) if \( G_P \) is of type (i). So, for example, \( P^G \) and
$G_P$ are of type (6) if $G_P \cong S_4$. Finally, we denote by $x_i$, the number of $G$-orbits on $l$ of type (i).

The $G$-orbits on $l$ are of type (i), with $1 \leq i \leq 8$, $i$ fixed, have the same length. Hence, they cover exactly $x_i |P^G|$ points on $l$, where $P^G$ is of type (i). The $G$-orbits on $l$ of type (9) or (10) might have different lengths depending on $e$ and $d$, respectively. Nevertheless, there exists at most one orbit on $l$ of type (9), as we will see in the following lemma (that is $x_9 \leq 1$). So, let us focus on the $G$-orbits of type (10) on the points of $l$ that they cover. Each $G$-orbit of type (10) has length $\frac{q^2 - 1}{2d}$ which depends on the particular divisor $d$ of $\frac{q-1}{2}$. Therefore, all the $G$-orbits on $l$ of type (10) cover exactly $\sum_{j=1}^{2x_10} \frac{q^2 - 1}{2d_j}$ points of $l$.

Set $S = \sum_{j=1}^{x_10} \frac{q^2 - 1}{2d_j}$. We introduce the following abbreviations for the $G$-orbits on $l$ of type (10): $S_1 = \sum_{j=1}^{x_10} \frac{q-1}{d_j}$ and $S_2, S_2', S_4, S_{2,4}$ (sum with the same summands $\frac{q-1}{d_j}$ but over $2 \mid d_j, 2 \mid d_j, 4 \mid d_j$ and $d_j \equiv 2 \mod 4$, respectively). In particular, we have the following relations $S = \frac{q+1}{2} S_1, S_1 = S_2 + S_2'$ and $S_2 = S_4 + S_{2,4}$.

When investigating the admissible orbital decomposition of $l$ under $G$, the following situation might arise (as we will see, in some cases it actually does): $G$ fixes at least a point $Q$ on $l$ and the admissible orbital decomposition of $G$ on set of lines of $[Q]$ it is easier to be investigated than the admissible one $l$, since the first one has some influences on the second one. In order to do so, we introduce further notation as follows.

If $G$ fixes a point $Q$, clearly, $G$ acts on $[Q]$. Now, consider $\Pi^*$, the dual of $\Pi$. The group $G$ acts on $\Pi^*$ fixing the line $[Q]$. Then we may apply Lemma 3.4 to $\Pi^*$. As a result, we obtain the same list of admissible groups with $G_m$, where $m$ is a point of $[Q]$. Then we may extend the notation previously introduced to the groups $G_m$. Hence, we say that $G_m$ is of type $(i)^*$, where $1 \leq i \leq 10$, if $G_m$ is a group isomorphic to the $i$-th group of the list given in Lemma 3.4. Now, going back to $\Pi$, we obtain the same list of admissible groups with $G_m$ in the role of $G_P$, where $m$ is a line of $[Q]$ and $Q$ is a point of $\Pi$ fixed by $G$. So, we are actually applying the dual of Lemma 3.4 referred to $G$-orbits of lines through a point $Q$ fixed by $G$. At this point, continuing with this notation, we say that the orbit $m^{G}$ is of type $(i)^*$ if the respective $G_m$ is of type $(i)$. So, for example, $m^G$ and $G_m$ are of type (6)* if $G_m \cong S_4$. Finally, we denote by $x_i^*$, the number of $G$-orbits on $[Q]$ of type $(i)^*$. In particular, since we might have $G$-orbits of type (10)*, it makes sense considering $S^* = \sum_{j=1}^{x_1^*} \frac{q^2 - 1}{2d_j}$ and hence $S_2^*, S_2', S_4^*, S_{2,4}^*$ with the same meaning of $S_2, S_2', S_4, S_{2,4}$, respectively, but referred to lines instead of points. As a consequence, we have $S^* = \frac{q+1}{2} S_1^*, S_1^* = S_2^* + S_2'^*$. It should be stressed that, the notation used
depends on the particular point $Q$ fixed by $G$ (the same could be made for $l$). So, it would be correct using $x_i^*(Q)$ instead of $x_i^*$. Nevertheless, we shall use the second notation, since it will be clear from the context which point we are focusing on.

**Lemma 3.5.** If $q > 9$, then the following hold:

1. $x_2 \leq 1$;
2. $x_3 \leq 1$;
3. $x_4 \leq 1$;
4. $x_5 \leq 3$;
5. $x_6 \leq 1$ for $q \neq 17$ and $x_6 \leq 2$ for $q = 17$;
6. $x_9 \leq 1$.

**Proof.** Assume that $l$ contains $x_i$ orbits of $G$ of type (i). Assume also that $2 \leq i \leq 6$ with $i$ fixed. Clearly, these $G$-orbits have same length. So, they cover exactly $x_i |P^G|$ points on $l$, where $P^G$ is any orbit of type (i). Therefore, $x_i |P^G| \leq n + 1$ and hence $x_i \frac{q(q^2 - 1)}{2|G_P|} \leq n + 1$. Now, arguing as in Lemma 3.4, we have $x_i \frac{q(q^2 - 1)}{2|G_P|} \leq (q - 1)^2 + 1$ and consequently

$$|G_P| > x_i \frac{q + 1}{2}.$$ 

Assume that $x_i \geq 2$. Then $|G_P| > q + 1$. This is a contradiction by Lemma 3.4. Thus, we have proved the assertion for $i = 2$ or 3.

Assume that $i = 4$. Hence, $G_P \cong A_4$. Then $q < 11$, as $|G_P| > q + 1$. Actually, we have $q \leq 9$, which is a contradiction by our assumption. So, $x_4 \leq 1$ and we obtain the assertion also in this case.

Assume that $i = 5$. Then $G_P \cong A_5$. If $x_5 \geq 4$, then $|G_P| > 2(q + 1)$. So $2(q + 1) < 60$. Hence, $q < 29$. Actually, $q = 11, 19$ or 25 by Lemma 3.4. Let $\sigma$ be an involution lying in $G_P$. By [4], there exists one conjugate class of involutions in $G$. If $q = 11$ or 19, then $C_G(\sigma) \cong D_{q+1}$ again by [4], since $q \equiv 3 \pmod{4}$. Therefore, using (1) of Proposition 2.5, we obtain that $\sigma$ fixes exactly $\frac{q+1}{4}$ points on $P^G$. As a consequence, $\sigma$ fixes at least $q + 1$ points on $l$, since $x_5 \geq 4$. Hence $\sqrt{n} \leq q$, since $\sigma$ is a Baer collineation of II by Lemma 3.3. So, we arrived at a contradiction, since $n < q^2$ by our assumptions. Thus, $q = 25$. In this case, $C_G(\sigma) \cong D_{q-1}$. Arguing as above, we see that $\sqrt{n} \geq q - 1$. Actually, $\sqrt{n} = q - 1$, since $\sqrt{n} < q$. That is $\sqrt{n} = 24$. Let $T$ be a Klein subgroup of $G$ such that $\sigma \in T$ and $T \subseteq G_P$. Then $N_G(T) \cong S_4$ by [4]. Furthermore, all Klein subgroups in $G_P \cong A_5$ are conjugate, since they are Sylow 2-subgroups of it. Then, using (1) of Proposition 2.5, we obtain that $T$ fixes exactly 2 points
on $P^G$. Hence, $\text{Fix}(T) \cap l \subset \text{Fix}(\sigma) \cap l$. Since $x_5 \geq 4$, then $T$ induces a Baer collineation on $\text{Fix}(\sigma)$. This is a contradiction, since $\sqrt{n} = 24$. Thus, $x_5 \leq 3$ and we obtain the assertion also in this case.

Assume that $i = 6$. So, $G_P \cong S_4$. Then $q + 1 < 24$ as $|G_P| > q + 1$. Therefore $x_6 = 2$ and $q = 17$ by Lemma 3.4. Hence, we have proved the assertion in this case.

Assume that $i = 9$. Let $P^G$ be a $G$-orbit on $l$ of type (9). Then $|P^G| = \frac{p^e(q^2-1)}{2(p^e-1)}$, where $q$ is a square $p^e | \sqrt{q}$ and $e \geq 1$ by Lemma 3.4. Clearly, $|P^G| > \frac{q^2-1}{2}$ and hence $x_9 \leq 1$, since $n + 1 \leq (q - 1)^2 + 1$. □

Clearly, we may consider the dual of Lemma 3.5. In other words, we may apply the previous lemma to $[Q]$ if $Q$ is a point fixed by $G$ on $\Pi$. So, we have $x_i^* \leq 1$ for $2 \leq i \leq 5$ or $i = 9$, and $x_6^* \leq 1$ for $q \neq 17$ and $x_6^* \leq 2$ for $q = 17$. We shall do the same for any lemma or proposition in the sequel whenever it is needed.

**Lemma 3.6.** Let $q > 9$. If $x_2 + x_3 > 0$, then the following hold:

1. $x_2 + x_3 = 1$;
2. $x_4 = 0$;
3. $x_5 \leq 2$ and if $x_5 > 0$, then $q = 11, 19, 25, 29, 31, 41, 49$;
4. $x_6 = 0$ for $q \neq 17$ and $x_6 \leq 1$ for $q = 17$;
5. $x_9 = 0$.

**Proof.** Assume $x_2 + x_3 > 0$. Let $P^G$ be an orbit on $l$ of type (2) or (3). If $P^G$ is of type (2), then $|P^G| = \frac{q(q-1)}{2}$, and if $P^G$ is of type (3), then $|P^G| = \frac{q(q+1)}{2}$. Hence, $|P^G| \geq \frac{q(q-1)}{2}$ in each case. Then $|l - P^G| = n + 1 - \frac{q(q-1)}{2}$. In particular, $|l - P^G| \leq (q-1)^2 + 1 - \frac{q(q-1)}{2}$ as $n + 1 \leq (q - 1)^2 + 1$. So $|l - P^G| \leq \frac{1}{2}(q^2 - 3q + 4)$. Assume there are $x_i$ orbits of $G$ of type (i) on $l - P^G$, where $2 \leq i \leq 6$ or $i = 9$, $i$ fixed. Let $Q^G$ be one of these orbits. It is a plain that, $x_i \cdot |Q^G| \leq |l - P^G|$ and hence $x_i \cdot |Q^G| \leq \frac{1}{2}(q^2 - 3q + 4)$. As a consequence, $|GQ| \geq x_i \cdot \frac{q(q-1)}{2}$. Easy computation, similar to that used in the first part of the proof of Lemma 3.5, yield the assertion, unless $i = 5$ and $q = 11$.

Assume that $i = 5$ and $q = 11$ and assume that $x_5 \geq 3$. Now, arguing in the second part of the proof of Lemma 3.5, we have that $\sqrt{n} + 1 \geq 3\frac{11}{2} - 1$ and $\sqrt{n}$ is an integer. Then $\sqrt{n} = 3$ and hence $\sqrt{n} = 9$. So, $3|Q^G| \leq 3^4 + 1 - \frac{11(11-1)}{2} = 81 + 1 - 33 = 49$ and hence $|Q^G| \leq 9$. Hence, we arrive at a contradiction, since $|Q^G| = \frac{2(q^2-1)}{120}$ and $q = 11$. Thus, we have proved the assertion in any case. □
Now, we recall some known facts about $G \cong \text{PSL}(2, q)$ which are useful hereafter. By [4], there exists a unique conjugate class of involutions in $G$ and there are either one or two conjugate class of Klein subgroups of $G$ according to whether $q \equiv 3, 5 \mod 8$ or $q \equiv 1, 7 \mod 8$, respectively. Let $\sigma$ be a representative of the involutions in $G$. Let $T_1$ and $T_2$ the representatives of the two conjugate classes of Klein subgroups of $G$. We may choose $T_1$ an $T_2$ in order to contain $\sigma$ (see [4] or [24]). Clearly, $T_1$ and $T_2$ are conjugate if $q \equiv 3, 5 \mod 8$. So, if $q \equiv 3, 5 \mod 8$, we shall just denote by $T$ the representative of the unique conjugate classes of Klein subgroups of $G$. Hence, by [4], the following admissible cases arise:

1. $q \equiv 1 \mod 8$. Then $C_G(\sigma) \cong D_{q-1}$ and $N_G(T_j) \cong S_4$, where $j = 1$ or 2;
2. $q \equiv 3 \mod 8$. Then $C_G(\sigma) \cong D_{q+1}$ and $N_G(T) \cong A_4$;
3. $q \equiv 5 \mod 8$. Then $C_G(\sigma) \cong D_{q-1}$ and $N_G(T) \cong A_4$;
4. $q \equiv 7 \mod 8$. Then $C_G(\sigma) \cong D_{q+1}$ and $N_G(T_j) \cong S_4$, where $j = 1$ or 2.

We investigate these cases separately.

4. The case $q \equiv 1 \mod 8$

This section is devoted to the cases $q \equiv 1 \mod 8$. By [4], there are two conjugate classes of subgroups isomorphic to $A_4$ (type (4)), to $A_5$ (type (5)), to $S_4$ (type (6)), to $\text{PSL}(2, \sqrt{q})$ (type (7)), to $\text{PGL}(2, \sqrt{q})$ (type (8)). Since there are two conjugate classes of subgroups of type (4) regarded as stabilizer of a point $P$ on $l$, we may extend our preceding notation as follows: we label the subgroups $G_P$ isomorphic to $A_4$ and belonging to the first conjugate class under $G$ to be of type (4a), while those belonging to the second one to be of type (4b). Moreover, $P^G$ is a $G$-orbit of type either (4a) or (4b) if the corresponding $G_P$ is of type (4a) or (4b), respectively. We denote by $x_{4a}$ and $x_{4b}$ the number of $G$-orbits on $l$ of type (4a) or (4b), respectively. Clearly, $x_4 = x_{4a} + x_{4b}$. Extending the previous notation, when $P^G$ and $G_P$ are of type $(i)$, for $4 \leq i \leq 8$, we actually say that they are of type $(ia)$ or $(ib)$ depending on the particular conjugate class under $G$ the group $G_P$ lies. Hence, we write $x_i = x_{ia} + x_{ib}$ for $4 \leq i \leq 8$.

The usual argument involving Proposition 2.5 yields the following table containing all the informations we need about the admissible stabilizers in $G$ of any point $P$ of $l$. It should also be stressed that the $G$-orbits of type (7), (8) or (9) might occur only when $q$ is a square.

For $\pm$ and $\mp$ read the upper sign if $\sqrt{q} \equiv 1 \mod 4$ and the lower sign if $\sqrt{q} \equiv 3 \mod 4$ (for $q$ square). This convention is followed throughout this section.
Recall that the Sylow $p$-subgroups of $G$ are elementary abelian. Furthermore, by [4], there are two conjugate classes of $p$-elements. Let $\rho_1$ and $\rho_2$ be the representatives of these two classes lying in a Sylow $p$-subgroup $S$ of $G$ which is normalized by $\sigma$. Since $\sigma$ acts as the inversion on $S$, then $\sigma$ normalizes $\langle \rho_1 \rangle$ and $\langle \rho_2 \rangle$ and hence $\langle \rho_1, \sigma \rangle \cong \langle \rho_2, \sigma \rangle \cong D_{2p}$. Again by [4], there is a unique conjugate class of elements of order for 4 in $G$. Let $\gamma$ be a representative of this class such that $\gamma^2 = \sigma$. By using (1) of Proposition 2.5, we obtain the following table.

The sign $\pm$ has the same meaning as above. In particular, the non negative integers $k_1$ and $k_2$ are such that $k_1 + k_2 = \frac{q-1}{d} \cdot \frac{r}{e}$, where $2e \mid m$ (see [24], Table IV* and related remarks).

It should be pointed out that Tables I and II, with types and entries in differ-
Table II

(We use the abbreviation $F$ for $\text{Fix}_{\rho_2}G$ in the top line of this table.)

| Type | $|F(\rho_1)|$ | $|F(\rho_2)|$ | $|F(\rho_1, \sigma)|$ | $|F(\rho_2, \sigma)|$ | $|F(\gamma)|$ |
|------|--------------|--------------|----------------|----------------|----------------|
| 1    | 1            | 1            | 1              | 1              | 1              |
| 2    | 0            | 0            | 0              | 0              | 1              |
| 3    | 0            | 0            | 0              | 0              | 0              |
| 4a   | 0            | 0            | 0              | 0              | 0              |
| 4b   | 0            | 0            | 0              | 0              | 0              |
| 5a   | 0            | 0            | 0              | 0              | 0              |
| 5b   | 0            | 0            | 0              | 0              | 0              |
| 6a   | 0            | 0            | 0              | 0              | $q^{-1}$       |
| 6b   | 0            | 0            | 0              | 0              | $q^{-1}$       |
| 7a   | $2\sqrt{q}$ | 0            | $1 \pm 1$      | 0              | $\sqrt{q} \pm 1$, $q \equiv_{16} 1$ 0, $q \equiv_{16} 9$ |
| 7b   | 0            | $2\sqrt{q}$ | $1 \pm 1$      | 0              | $\sqrt{q} \pm 1$, $q \equiv_{16} 1$ 0, $q \equiv_{16} 9$ |
| 8a   | $\sqrt{q}$  | 0            | 1              | 0              | $\sqrt{q} \pm 1$ |
| 8b   | 0            | $\sqrt{q}$  | 0              | 1              | $\sqrt{q} \pm 1$ |
| 9    | $k_1p^e$     | $k_2p^e$     | $k_1$          | $k_2$          | $\frac{q-1}{p^2-1}$, $p^e \equiv_4 1$ 0, $p^e \equiv_4 3$ |
| 10   | $q-1$        | $2d$         | $q-1$          | $2d$           | $q-1$          | $4d$ |

ent order, can be extracted from Tables III* and IV* of [24], respectively.

Now, if $G$ acts on $[Q]$, where $Q$ is any point of $\Pi$, then $[Q]$ consists of $G$-orbits of lines of type $(i)^*$ for $1 \leq i \leq 10$, following the notation introduced in section 3. As $G$ contains two conjugate classes of subgroups isomorphic to $A_4$ (type $(4)^*$), to $A_5$ (type $(5)^*$), to $S_4$ (type $(6)^*$), to $\text{PSL}(2, \sqrt{q})$ (type $(7)^*$), to $\text{PGL}(2, \sqrt{q})$ (type $(8)^*$), the distinction made for $G$-orbits of points of $\Pi$ inside a fixed type $(i)$ in subtypes $(ia)$ and $(ia)$ can be extended in $G$-orbits of lines of $\Pi$ in the following sense. Let $m$ be any line of $[Q]$ and assume that a subgroup $G_m$ of $G$ is isomorphic to $A_4$. We say that $G_m$ is either of type $(4a)^*$ or of type $(4b)^*$
depending on which of the two conjugate classes of subgroups isomorphic to $A_4$, the group $G_m$ lies. So, we denote by $x_{4a}^*$ and $x_{4b}^*$ the number of $G$-orbits on $[Q]$ of type $(4a)^*$ and $(4b)^*$, respectively. Clearly $x_4^* = x_{4a}^* + x_{4b}^*$. Extending the previous notation, when $m^G$ and $G_m$ are of type $(i)^*$, for $4 \leq i \leq 8$, we actually say that they are of type $(ia)^*$ or $(ib)^*$ depending on the particular conjugate class under $G$ the group $G_m$ lies. Hence, we write $x_i^* = x_{ia}^* + x_{ib}^*$ for $4 \leq i \leq 8$.

It is a plain that, at this point, we may use Tables I and II referred to $G$-orbits of lines of $\mathcal{P}$. So in this case, the first column containing types $(i)$ is replaced by types $(i)^*$ and $G_P$ is replaced by $G_m$. So, when we use Tables I and II referred to $G$-orbits of lines of $\mathcal{P}$ through some point fixed by $G$, we actually use the duals of Tables I and II, respectively.

The strategy of the proof in this section is the following. Assuming that $G$ fixes a line $l$ of $\mathcal{P}$, we show that each $T_j$ induces either a Baer collineation or a perspectivity of axis distinct from $l$ on $\text{Fix}(\sigma)$ (Lemma 4.2). We use this fact to show that $\gamma$, where $\gamma^2 = \sigma$, induces either the identity or a Baer collineation on $\text{Fix}(\sigma)$ (Lemma 4.4). Then, using Tables I and II, we show that, if the first case occurs, the group $T_j$ induces a homology on $\text{Fix}(\sigma)$ (Lemma 4.10). Nevertheless, this is impossible (Lemma 4.11). Thus $\gamma$ induces a Baer collineation on $\text{Fix}(\sigma)$. Again, Table I and II imply that each $T_j$ induces a Baer collineation on $\text{Fix}(\sigma)$ and on $\text{Fix}(\gamma)$ by Propositions 4.12 and 4.18, respectively. Thus, $G$ fixes necessarily a subplane of $\mathcal{P}$ of order $\sqrt{n}$ pointwise (Lemma 4.19), which is a contradiction (Proposition 4.21).

Recall that $\sigma$ is a Baer collineation of $\mathcal{P}$ by Lemma 3.3. Set $C = C_G(\sigma)$. Then $C$ acts on $\text{Fix}(\sigma)$ with kernel $K$. Hence, let $\bar{C} = C/K$. Clearly, $\langle \sigma \rangle \trianglelefteq K \trianglelefteq C$. Furthermore, either $K \trianglelefteq Z_{\bar{C}}$ or $K \cong D_{\bar{C}}$ or $K = C$, since $C \cong D_{q-1}$ and $q \equiv 1 \mod 8$. Now, we need to investigate the admissible structure of $K$ in order to show that $T_j$ cannot induce on $\text{Fix}(\sigma)$ either the identity or a perspectivity of axis $\text{Fix}(\sigma) \cap l$ for each $j = 1, 2$.

**Lemma 4.1.** If $\text{Fix}(T_j) \cap l = \text{Fix}(\sigma) \cap l$ for some $j = 1$ or $2$, then either $K \cong D_{\bar{C}}$ or $K = C$.

**Proof.** Assume that $\text{Fix}(T_1) \cap l = \text{Fix}(\sigma) \cap l$ and that $K \trianglelefteq Z_{\bar{C}}$. Then $\text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l$ by Table I, since $q > 9$. Set $l_0 = \text{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \text{Fix}(G) \cap l$.

Assume that $\bar{C} = \bar{C}(l_0, l_0)$. Then $T_1$ induces a perspectivity $\bar{\beta}_1$ of center $C_{\bar{\beta}_1}$ and axis $l_0$ on $\text{Fix}(\sigma)$. Suppose that $\bar{\beta}_1$ is an elation. Hence, $C_{\bar{\beta}_1} \in l$. Thus, $G$ fixes $C_{\bar{\beta}_1}$. So, $\text{Fix}(G) \cap [C_{\bar{\beta}_1}] = \text{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$, by dual of Table I, since $\text{Fix}(T_1) \cap [C_{\bar{\beta}_1}] = \text{Fix}(\sigma) \cap [C_{\bar{\beta}_1}]$. Therefore, $\bar{C} = \bar{C}(C_{\bar{\beta}_1}, l_0)$. Let $X \in l_0 - \{C_{\bar{\beta}_1}\}$. 

\[ \text{Fix}(X) \cap [C_{\bar{\beta}_1}] = \text{Fix}(\sigma) \cap [C_{\bar{\beta}_1}] \]
For each line $t \in [X] \cap \text{Fix}(\sigma)$, we have that $\sigma \in G_t$ but $G_t$ does not contain Klein groups. Clearly, $\text{Fix}(G_t) \subseteq \text{Fix}(\sigma)$. Actually, $\text{Fix}(G_t) = \text{Fix}(\sigma)$, since $\text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l$, since $\text{Fix}(G) \cap [C_\beta] = \text{Fix}(\sigma) \cap [C_\beta]$, and since $t \in \text{Fix}(G_t) \cap \text{Fix}(\sigma)$ and $t \notin [C_\beta]$. So, $\text{Fix}(G_t)$ is a Baer subplane of $\Pi$. Assume that $p \mid |G_t|$ and let $S_0$ be a Sylow $p$-subgroup of $G_t$. Then $\text{Fix}(S_0) = \text{Fix}(G_t)$, since $\text{Fix}(G_t)$ is a Baer subplane of $\Pi$. Furthermore, either $q$ is a square and $|S_0| = p^e$ with $p^e \mid \sqrt{q}$, or $|S_0| = q$ by dual of Table II. Assume the latter occurs. Then $q \mid n - \sqrt{n}$, since $S_0$ must be semiregular on $l - \text{Fix}(S_0)$, as $\text{Fix}(S_0) = \text{Fix}(\sigma)$ and $\text{Fix}(\sigma)$ is a Baer subplane of $\Pi$. This yields that either $q \mid \sqrt{n} - 1$ or $q \mid \sqrt{n}$, since $q$ is a prime power. This gives a contradiction, since $\sqrt{n} < q$ by our assumption. Thus, $q$ is a square and $|S_0| = p^e$ with $p^e \mid \sqrt{q}$. In particular, $G_t$ is of type (9)*. Moreover, $\text{Fix}(S_0) \cap [X] = \text{Fix}(\sigma) \cap [X]$, since $\text{Fix}(S_0) = \text{Fix}(\sigma)$. This yields $\text{Fix}_{\sigma^e}(S_0) = \text{Fix}_{\sigma^e}(\sigma)$ and hence $|\text{Fix}_{\sigma^e}(S_0)| = |\text{Fix}_{\sigma^e}(\sigma)|$, since $G$ fixes $X$. Then $k_1p^e = \frac{q^2 - 1}{2}$ by duals of Tables I and II, which is a contradiction. As a consequence, $(p, |G_t|) = 1$. Therefore, $G_t \cong D_q + 1$ by dual of Table I, since $\sigma \in G_t$ but $G_t$ does not contain Klein groups. Then $q + 1 \mid n - \sqrt{n}$, since $G_t$ must be semiregular on $l - \text{Fix}(G_t)$, as $\text{Fix}(G_t) = \text{Fix}(\sigma)$ and $\text{Fix}(\sigma)$ is a Baer subplane of $\Pi$. Furthermore, $K \leq G_t$. Thus $K = \langle \sigma \rangle$, since $\langle \sigma \rangle \leq K \leq Z_{\frac{q - 1}{2}}$ and since $G_t \cong D_q + 1$. So, $\tilde{C} \cong D_q + 1$ and hence $Z_{\frac{q - 1}{2}} \mid \sqrt{n}$, since $\tilde{C} = \tilde{C}(C_\beta, l_0)$. Actually, either $\sqrt{n} = \frac{q - 1}{2}$ or $\sqrt{n} = q - 1$, since $\sqrt{n} < q$ by our assumptions. On the other hand, $t^G \subset [X] \setminus \{l\}$, as $G$ fixes $X$. Then $n \geq \frac{q(q - 1)}{2}$, since $|t^G| = \frac{q(q - 1)}{2}$ as $G_t \cong D_q + 1$. Since $n \geq \frac{q(q - 1)}{2}$ and since $\frac{q(q - 1)}{2} > \left(\frac{q - 1}{2}\right)^2$, the case $\sqrt{n} = \frac{q - 1}{2}$ cannot occur. Hence, $\sqrt{n} = q - 1$. Then $q + 1 \mid (q - 1)(q - 2)$, since $q + 1 \mid n - \sqrt{n}$, being $G_t$ semiregular on $l - \text{Fix}(G_t)$. Since $(q + 1, q - 2) = 2$ and $(q + 1, q - 2) \mid 3$, then $q + 1 \mid 6$. This gives a contradiction, since $q > 9$ by our assumptions. Thus, $C(l_0, l_0) < \tilde{C}$.

Assume that $\tilde{C}(l_0, l_0) \neq \langle 1 \rangle$. Note that $\tilde{C}(Y, l_0) \neq \langle 1 \rangle$ for some point $Y \in \text{Fix}(\sigma) \setminus l_0$, since $\tilde{C}(l_0, l_0) < \tilde{C}$ and $\tilde{C} = \tilde{C}(l_0)$. In particular, $C(l_0, l_0) \leq Z_{\frac{q - 1}{2}}$, since $\tilde{C}(l_0, l_0) < \tilde{C}$ and $\tilde{C} \cong D_q + 1$. Actually, $\tilde{C}(l_0, l_0) = \tilde{C}(V, l_0) \cong Z_{\frac{q - 1}{2}}$ and $C(Y, l_0) \cong Z_2$ by [16, Theorems 4.14 and 4.25], since $C(l_0, l_0) \cong D_q + 1$, $q \equiv 1 \mod 8$ and $k$ is even. Let $u \in [V] \cap \text{Fix}(\sigma) \setminus \{l, VY\}$, then $u$ is fixed by $K$ and by $\tilde{C}(V, l_0)$. Therefore, $Z_{\frac{q - 1}{2}} \leq G_u$, where $Z_{\frac{q - 1}{2}} \leq C_G(\sigma)$. Since $G$ fixes $l \cap \text{Fix}(\sigma)$, since $q > 9$ and by dual of Lemma 3.4, we have that either $G_u \cong F_q.Z_{\frac{q - 1}{2}}$ or $G_u = C_G(\sigma)$ or $G_u = G$. The two latter cases cannot occur, since $C_u \cong Z_{\frac{q - 1}{2}}$. So, $G_u \cong F_q.Z_{\frac{q - 1}{2}}$ for each $u \in [V] \cap \text{Fix}(\sigma) \setminus \{l, VY\}$. Note also that $x_1^* = 1$, since $G$ fixes only the line $l$ through $V$, and $x_2^* \geq 1$ since $G_{VY} = C$. Actually, $x_2^* = 1$ by dual of Lemma 3.5(2). Moreover, $|\text{Fix}_{\sigma^e}(\sigma)| = 2$.
by dual of Table II. Then
\[ \sqrt{n} + 1 = 1 + \frac{q+1}{2} + S_2 \]  (21)
by dual of Table I, since \( x_1^* = x_2^* = 1 \) and since \( G_u \cong F_q.Z_{q^2-1} \) for each \( u \in [V] \cap \text{Fix}(\sigma) - \{ l, VY \} \). Let \( W \) be the Sylow \( p \)-subgroup of \( G \) normalized by \( \sigma \). Then, by (21), \( W \) fixes exactly \( 1 + \frac{1}{2}S_2 \) lines through \( V \), namely \( l \) and the lines lying in the \( G \)-orbits corresponding to stabilizer isomorphic to \( F_q.Z_{q^2-1} \). Furthermore, if \( R \in l - \text{Fix}(G) \), then \( G_R \) must have odd order, since \( |\text{Fix}(G) \cap l| = \sqrt{n} + 1 \) and since the involutions in \( G \) are Baer involutions of \( \Pi \) by Lemma 3.3. Then \( G_R \) must be of type (10) by Lemma 3.4. Henceforth, \( W \leq G_L \) for some point \( L \in R^G \). Consequently, \( W \) fixes at least \( \sqrt{n} + 2 \) points on \( l \) and at least \( 1 + \frac{1}{2}S_2 \) lines through \( V \). Thus, the \( p \)-elements in \( G \) cannot be planar. So, if \( Z \in \text{Fix}(G) \cap \{ l \} \), \( Z \neq V \), for each line \( r \in [Z] \cap \text{Fix}(\sigma) - \{ l, ZV \} \), the group \( G_r \) contains \( \sigma \) but does not contain Klein groups and \( (p, |G_l|) = 1 \). This implies that \( G_r \cong D_{q+1} \) by dual of Table I. Now, as \( K \leq G_r \) and \( (\sigma) \leq K \leq Z_{q^2-1} \), then \( K = \langle \sigma \rangle \).

As a consequence, \( \tilde{C}(V, l_0) \cong Z_{q^2-1} \), being \( k = |K| \). In particular, \( \tilde{C}(V, l_0) \) has even order. Nevertheless, this is a contradiction, since \( \tilde{C}(Y, l_0) \cong Z_2 \) and \( Y \in \text{Fix}(\sigma) - l_0 \). So, \( \tilde{C}(l_0, l_0) = 1 \).

Assume that \( \tilde{C} = C(Z, l_0) \) for some \( Z \in \text{Fix}(\sigma) - l_0 \). Let \( Q \in l_0 \) and \( m \in [Q] \cap \text{Fix}(\sigma) - \{ l, YQ \} \). Then \( \sigma \in G_m \) but \( G_m \) does not contain Klein groups. Therefore, by dual of Table I, we have that \( G_m \cong D_{q+1} \) or \( G_m \cong E_{p^m - \epsilon}Z_{p^m - 1} \) or \( G_m \cong F_q.Z_d \), since \( G \) fixes \( Q \). Thus, \( x^*_1 > 0 \) for either \( i = 3 \) or \( 9 \), since \( G \) acts on \( [Q] \). The cases \( i = 3 \) or \( 9 \) cannot occur by dual of Lemma 3.6(1) and (5), since \( x^*_2 > 0 \), as \( G_{ZQ} = C \cong D_{q-1} \). As a consequence, \( G_m \cong F_q.Z_d \). Let \( S \) be Sylow \( p \)-subgroup of \( G \) which is normalized by \( \sigma \). Then \( \text{Fix}(\sigma)^m(S) \cong 1 \) for each \( Q \in l_0 \) and \( m \in [Q] \cap \text{Fix}(\sigma) - \{ l, ZQ \} \). Assume that \( |\text{Fix}(\sigma)^m(S)| \geq 2 \) for some line \( m_1 \in [Q] \cap \text{Fix}(\sigma) - \{ l, ZQ_1 \} \) and for some point \( Q_1 \in l_0 \). Then \( S \) is planar, since \( |\text{Fix}(\sigma)^m(S)| \geq 1 \) for each other \( Q \in l_0 \) and each \( m \in [Q] \cap \text{Fix}(\sigma) - \{ l, ZQ \} \) and since \( \text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l \). Then \( S \) fixes a Baer subspace of \( l_0 \), since \( \text{Fix}(S) \cap l = \text{Fix}(\sigma) \cap l \) and \( |\text{Fix}(\sigma) \cap l| = \sqrt{n} + 1 \). Now, arguing as above with \( S \) in the role of \( S_0 \), we obtain a contradiction. Thus, \( |\text{Fix}(\sigma)^m(S)| = 1 \) for each \( Q \in l_0 \) and \( m \in [Q] \cap \text{Fix}(\sigma) - \{ l, ZQ \} \). Nevertheless, we still have a contradiction if \( |\text{Fix}(S) \cap [Q_2]| \geq 2 \) for some point \( Q_2 \in l_0 \). So, \( |\text{Fix}(S) \cap [Q]| = 1 \) for each point \( Q \in l_0 \). Consequently, \( G_{fQ} \cong F_q.Z_{q^2-1} \), where \( \{ fQ \} = \text{Fix}(S) \cap [Q] \) by the dual of Table II. Therefore, \( \sigma \) fixes exactly two lines in \( fQ \) for each \( Q \in l_0 \) by Table I. Actually, \( \sigma \) fixes exactly two lines in \( [Q] \cap \{ l, ZQ \} \) for each \( Q \in l_0 \), since \( |\text{Fix}(\sigma)^m(S)| \geq 1 \) for \( m \in [Q] \cap \text{Fix}(\sigma) - \{ l, ZQ \} \), while \( |\text{Fix}(S) \cap [Q]| = 1 \). So, \( \sqrt{n} + 1 = 9 \). On the other hand, by dual of Table II, we have \( \sqrt{n} - 1 = 2 \). That is \( \sqrt{n} = 9 \). On the other hand, by dual of Table II, we have \( \sqrt{n} + 1 \geq 1 + \frac{q+1}{2} + 2 \), since \( x^*_1 = 1 \) as \( G \) fixes only the \( l \) through \( Z \), since \( x^*_2 \geq 1 \).
as \( G_{ZQ} = C \) and since \( x_{10}^+ \geq 1 \) as \( G_{fQ} \cong F_q.Z_{q-1} \). Then \( \sqrt{n} \geq 11 \) since \( q \geq 17 \) being \( q \equiv 1 \mod 8 \) and \( q > 9 \). Hence, we arrive at a contradiction, since it was proved above that \( \sqrt{n} = 9 \). \( \square \)

**Lemma 4.2.** It holds that \( \text{Fix}(T_j) \cap l \subset \text{Fix}(\sigma) \cap l \) for each \( j = 1, 2 \).

**Proof.** Assume that \( K = C \). Let \( P \) be any point of \( l_0 \) and let \( r \) be any line of \( \{P\} - \{l\} \). Then \( C \leq G_r \). Since \( q > 9 \), then \( C \) is maximal in \( G \) and hence either \( G_r = C \) or \( G_r = G \). Assume that \( G_r = C \). Again by the maximality of \( C \) in \( G \), the line \( r \) is the unique one in \( r^G \) fixed by \( C \). Furthermore, \( x_2^+ = 1 \) dual of Lemma 3.5. Therefore, \( r \) is the unique line in \( \{P\} \) fixed by \( C \). So, the remaining lines are fixed by \( G \). Now, by repeating the previous argument for each point \( U \) of \( \text{Fix}(\sigma) \cap l \), we see that \( C \) fixes exactly one line of \( \{U\} \cap \text{Fix}(\sigma) \) and the remaining ones are fixed by \( G \). If \( \sqrt{n} > 2 \), then \( G \) is planar. Thus, \( \text{Fix}(G) = \text{Fix}(\sigma) \), since \( \text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l \) and \( \text{Fix}(G) \subseteq \text{Fix}(\sigma) \). Then \( G \) fixes \( r \), since \( r \in \text{Fix}(\sigma) \). This is a contradiction, since \( G_r = C \) by our assumptions. So, \( \sqrt{n} = 2 \) and \( n = 4 \), which is a contradiction, since \( q < n < q^2 \) and \( q > 9 \). As a consequence, \( G_r = G \). Now, by repeating the previous argument for each point of \( \text{Fix}(\sigma) \cap l \), we again obtain \( \text{Fix}(G) = \text{Fix}(\sigma) \). Thus, \( G \) fixes a Baer subplane of \( \Pi \). Then \( G \) is semiregular on \( l - \text{Fix}(G) \) and hence \( |G| = n - \sqrt{n} \). Hence, we arrive at a contradiction, since \( n < q^2 \).

Finally, assume that \( K \cong D_{q-1} \). We may also assume that \( T_1 \leq K \) and \( C \cong D_{q-1} \). Then \( \text{Fix}(T_1) \cap [B] = \text{Fix}(\sigma) \cap [B] \) for each point \( B \in l_0 \). As \( l_0 = \text{Fix}(G) \cap l \), then \( \text{Fix}(G) \cap [B] = \text{Fix}(\sigma) \cap [B] \) for each point \( B \in l_0 \) by dual of Table I. So, \( \text{Fix}(G) = \text{Fix}(\sigma) \) and we have a contradiction as above. Thus, \( \text{Fix}(T_1) \cap l \subset \text{Fix}(\sigma) \cap l \).

Now repeating the above arguments with \( T_2 \) in the role of \( T_1 \), we obtain \( \text{Fix}(T_2) \cap l \subset \text{Fix}(\sigma) \cap l \). \( \square \)

**Lemma 4.3.** If \( |\text{Fix}(\gamma) \cap l| \leq 2 \), then the following hold:

1. \( |\text{Fix}(\gamma) \cap l| = x_1 + x_2 = 1 \) or 2;
2. \( x_6 = 0 \);
3. \( x_7 > 0 \), if \( q \) is a square and \( q \equiv 9 \mod 16 \);
4. \( x_8 = 0 \);
5. \( x_9 > 0 \), if \( q \) is a square and \( p^e \equiv 3 \mod 4 \), where \( p^e \mid \sqrt{q} \);
6. \( S_4 = 0 \);
7. \( T_j \) induces a Baer involution on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \).

**Proof.** Assume that \( |\text{Fix}(\gamma) \cap l| \leq 2 \). Then \( \gamma \) induces an involutory perspectivity \( \tilde{\gamma} \) on \( \text{Fix}(\sigma) \) and hence \( |\text{Fix}(\gamma) \cap l| = 1 \) or 2. Clearly, \( C_{\tilde{\gamma}} \in l \cap \text{Fix}(\sigma) \) and
$a_q \neq l \cap \text{Fix}(\sigma)$. Set $\{X\} = a_q \cap l$. The points $C_\gamma$ and $X$ might coincide or not according to whether $\tilde{\gamma}$ is either an elation or a homology of $\text{Fix}(\sigma)$, respectively. Let $\tilde{\beta}_j$ be the involution induced on $\text{Fix}(\sigma)$ by the Klein subgroup $T_j$ containing $\sigma$ (and hence lying in $C_j$), $j = 1, 2$. As $\tilde{\gamma}$ is central in $C$, then $C$ fixes $C_\gamma$, $a_q$ and so $X$. Thus, $C$ does it. Therefore, $C \leq G_{C_\gamma}$ and $C \leq G_X$. Then, by Table II and since $q > 9$, we have that

1. $|\text{Fix}(\gamma) \cap l| = x_1 + x_2 = 1$ or 2;
2. $x_6 = 0$;
3. $x_7 > 0$ if $q$ is a square and $q \equiv 9 \mod 16$;
4. $x_8 = 0$;
5. $x_9 > 0$ if $q$ is a square and $p^e \equiv 3 \mod 4$, where $p^e | \sqrt{q}$;
6. $S_4 = 0$.

It remains to prove the assertion (7). If $C < G_{C_\gamma}$ and $C < G_X$. Then $G_{C_\gamma} = G_X = G$, since $C$ is maximal in $G$ as $q > 9$. As a consequence, $\text{Fix}(\gamma) \cap l = \text{Fix}(G) \cap l$. Assume that $\tilde{\beta}_1$ is an involutory $(C_{\tilde{\beta}_1}, a_{\tilde{\beta}_1})$-perspectivity. Then $C_{\tilde{\beta}_1} \in l$ and $a_{\tilde{\beta}_1} \neq l$ by Lemma 4.2. So, $C_{\tilde{\beta}_1} \in \{C_\gamma, X\}$, since $G_{C_\gamma} = G_X = G$. Therefore, $G$ fixes $C_{\tilde{\beta}_1}$. Hence, we arrive at a contradiction by dual of Lemma 4.2, since $\text{Fix}(T_1) \cap \{C_{\tilde{\beta}_1}\} = \text{Fix}(\sigma) \cap \{C_{\tilde{\beta}_1}\}$. Thus, $\tilde{\beta}_1$ is a Baer involution of $\text{Fix}(\sigma)$. The previous argument with $T_2$ in the role of $T_1$, yields that $\tilde{\beta}_2$ is also a Baer involution of $\text{Fix}(\sigma)$.

Assume there exists $Q \in \{C_\gamma, X\}$ such that $G_Q = C$. Then $\tilde{\beta}_j$ is a Baer involution of $\text{Fix}(\sigma)$ for each $j = 1, 2$, since $|\text{Fix}_{Q^e}(T_j)| = 3$ by Table I for each $j = 1, 2$. Therefore, $\tilde{\beta}_j$ is a Baer involution of $\text{Fix}(\sigma)$ for each $j = 1$ or 2 in any case. This completes the proof.

Lemma 4.4. It holds that $|\text{Fix}(\gamma) \cap l| \geq 3$.

Proof. Suppose that $|\text{Fix}(\gamma) \cap l| \leq 2$. Then either $|\text{Fix}(\gamma) \cap l| = 1$ or $|\text{Fix}(\gamma) \cap l| = 2$, as $G$ fixes $l$ and $\gamma$ induces and involution $\tilde{\gamma}$ on $\text{Fix}(\sigma)$.

Assume that $|\text{Fix}(\gamma) \cap l| = 1$. Then $\tilde{\gamma}$ is an involutory $(C_{\tilde{\gamma}}, a_{\tilde{\gamma}})$-elation of $\text{Fix}(\sigma)$ with $C_{\tilde{\gamma}} \in l \cap \text{Fix}(\sigma)$ and $a_{\tilde{\gamma}} \neq l \cap \text{Fix}(\sigma)$. Thus, $x_1 + x_2 = 1$ by Lemma 4.3(1). Moreover, by Table II in conjunction with Lemma 4.3(2)–(5), we have $|\text{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_4a + 2x_5a + 2x_7a$ and $|\text{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_4b + 2x_5b + 2x_7b$ for $\sqrt{q} \equiv 1 \mod 4$, and $|\text{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_4b + 2x_5b + 2x_7b$ and $|\text{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_4a + 2x_5a + 2x_7a$ for $\sqrt{q} \equiv 3 \mod 4$. Then

$$\sqrt{n} + 1 = x_1 + 3x_2 + 2x_4a + 2x_5a + 2x_7a$$

$$\sqrt{n} + 1 = x_1 + 3x_2 + 2x_4b + 2x_5b + 2x_7b$$

(22)  (23)
in each case, being \(|\text{Fix}(T_j) \cap l| = \sqrt{n} + 1\) for each \(j = 1, 2\) by Lemma 4.3(7). Now, summing up (22) and (23), we have

\[
\sqrt{n} + 1 = x_1 + 3x_2 + x_4 + x_5 + x_7.
\]

Bearing in mind that \(x_1 + x_2 = 1\), we actually obtain

\[
\sqrt{n} = 2x_2 + x_4 + x_5 + x_7.
\]  \(\text{(24)}\)

Assume that \(x_4 > 0\). Then \(q = 17\) and \(x_5 = 0\) by Lemma 3.4, since \(q \equiv 1 \text{ mod } 8\). Furthermore, \(x_4 = 1\) and \(x_2 = 0\) by Lemma 3.5(3) and Lemma 3.6(3), respectively. Then \(x_1 = 1\), as \(x_1 + x_2 = 1\) by the above argument. So, \(x_7 = \sqrt{n} - 1\) by (24). On the other hand, \(\sqrt{n} + 1 \geq 1 + x_7(\sqrt{n} + 1)\) by Table I. By substituting \(x_7 = \sqrt{n} - 1\) in the previous inequality and by elementary calculations of the inequality, we have \(\sqrt{n} + 1 \geq \sqrt{n} + 1\). Then \(\sqrt{n} = \sqrt{n} - 1\) and \(\sqrt{n} \equiv 3 \text{ mod } 4\), since \(\sqrt{n} < \sqrt{n}\) by our assumptions. Nevertheless, this contradicts [16, Theorem 13.18], since \(\gamma\) acts non trivially on the plane \(\text{Fix}(T_j)\) and since \(q > 9\). So, \(x_4 = 0\).

Assume that \(x_5 > 0\). If \(x_2 = 1\), then \(x_5 \leq 2\) and \(q = 25, 41\) or 49 by Lemma 3.6(4), since \(q \equiv 1 \text{ mod } 8\). Then \(n + 1 \geq \frac{q(q+1)}{2} + \frac{q(q^2-1)}{120}\), with \(n < q^2\), and \(n\) a fourth power. This is impossible, since \(q = 25, 41\) or 49. Then \(x_2 = 0\) and hence \(x_1 = 1\), since \(x_1 + x_2 = 1\). Then \(x_7 \geq \sqrt{n} - 3\) by (24), since \(x_5 \leq 3\) by Lemma 3.5(4). If \(\sqrt{n} > 3\), then \(x_7 > 0\). This implies that \(q\) is a square and \(q \equiv 9 \text{ mod } 16\) by Lemma 4.3(3). As a consequence, \(q = 25\). Then \(\sqrt{n} = 4\), since \(q < n < q^2\), \(n\) is a fourth power and \(\sqrt{n} > 3\). As \(n + 1 \geq 1 + x_5\frac{q(q^2-1)}{120}\) by Table I, where \(n = 4^4\) and \(q = 25\), then \(x_5 = 1\) and hence \(x_7 = 3\). Therefore, by Table I, \(n + 1 \geq x_1 + x_5\frac{q(q^2-1)}{120} + x_7\frac{q(q+1)}{2}\), where \(x_1 = x_5 = 1\) and \(x_7 = 3\) and \(q = 25\). That is \(n \geq 325\). Nevertheless, this contradicts the fact that \(n = 4^4\). Then \(\sqrt{n} \leq 3\) and hence \(n \leq 3^4\). Nevertheless, \(n \geq 130\), being \(n \geq \frac{q(q^2-1)}{120}\) with \(q \geq 25\) by Lemma 3.4. So, we again obtain a contradiction. Thus, \(x_5 = 0\).

Since \(x_4 = x_5 = 0\), then \(\sqrt{n} = 2x_2 + x_7\) by (24). If \(x_2 = 0\), then \(x_7 = \sqrt{n}\) and hence \(\sqrt{n} + 1 \geq 1 + \sqrt{n}(\sqrt{q} \pm 1)\) by Table I. Consequently, \(\sqrt{n} \geq \sqrt{q} \pm 1\). Actually, \(\sqrt{n} = \sqrt{q} - 1\), since \(\sqrt{n} < \sqrt{q}\) by our assumptions. At this point the above argument rules out this case. Then \(x_2 = 1\) and hence \(x_7 = \sqrt{n} - 2\). If \(x_7 > 0\), then \(\sqrt{n} > 2\) and hence

\[
\sqrt{n} + 1 \geq \frac{q + 1}{2} + (\sqrt{n} - 2)(\sqrt{q} \pm 1)
\]  \(\text{(25)}\)

by Table I. Note that \(\sqrt{n} + 1 < q + 1\). So \(\sqrt{n} + 1 > \frac{\sqrt{n}+1}{2} + (\sqrt{n} - 2)(\sqrt{q} \pm 1)\) by (25). Collecting with respect to \(\sqrt{n} + 1\), we have \(\sqrt{n} + 1 > 2(\sqrt{n} - 2)(\sqrt{q} \pm 1)\). Since \(\sqrt{n} > 2\), then \(\frac{\sqrt{n}}{2(\sqrt{n}-2)} < \sqrt{n}\) and therefore \(\sqrt{n} > (\sqrt{q} \pm 1)\). In particular,
\(\sqrt{n} > \sqrt{q} - 1\) in each case. On the other hand, \(\sqrt{n} < \sqrt{q}\), since \(n < q^2\) by our assumption. So, \(\sqrt{q} - 1 < \sqrt{n} < \sqrt{q}\), where \(\sqrt{q}\) is integer by Lemma 4.3(3), being \(x_7 > 0\). Clearly, this is a contradiction. Thus, \(x_7 = 0\). Then \(\sqrt{n} = 2\), since \(x_7 = \sqrt{n} - 2\). So, \(n = 16\). Nevertheless, \(n + 1 \geq \frac{q(q+1)}{2}\), as \(x_2 = 1\) and \(q > 9\), which is still a contradiction.

Assume that \(|\text{Fix}(\gamma) \cap l| = 2\). Then \(\tilde{\gamma}\) is an involutory \((C_\gamma, \sigma_\gamma)\)-homology of \(\text{Fix}(\sigma)\) with \(C_\gamma \in l \cap \text{Fix}(\sigma)\) and \(\sigma_\gamma \neq l \cap \text{Fix}(\sigma)\). Then \(x_1 + x_2 = 2\) by Lemma 4.3(1). Recall that \(\{X\} = \sigma_\gamma \cap l\) (clearly \(C_\gamma \neq X\)) and each \(T_j\) induces a Baer involution on \(\text{Fix}(\sigma)\) by Lemma 4.3(7). Then either \(x_1 = x_2 = 1\) or \(x_1 = 2\) and \(x_2 = 0\), since \(x_2 \leq 1\) by Lemma 3.5(1).

Assume that \(x_1 = x_2 = 1\). Arguing as above, we have \(\sqrt{n} + 1 = x_1 + 3x_2 + x_4 + x_5 + x_7\) by Table I in conjunction with Lemma 4.3(2)–(7). Actually, \(x_4 = 0\) by Lemma 3.5(3), since \(x_2 = 1\). Therefore,

\[
\sqrt{n} = 3 + x_5 + x_7, \quad (26)
\]
as \(x_1 = x_2 = 1\). If \(x_5 > 0\), then \(x_6 \leq 2\) and \(q = 25, 41, 49, 81\) or \(89\) by Lemma 3.6(4), since \(q \equiv 1\) mod 8. Then \(n + 1 \geq \frac{q(q+1)}{2} + \frac{q(q^2-1)}{120}\), with \(n < q^2\) and \(n\) a fourth power, which is a contradiction as above. Thus, \(x_5 = 0\). So, \(x_7 = \sqrt{n} - 3\) by (26). On the other hand, \(\sqrt{n} + 1 \geq 1 + \frac{q+1}{2} + x_7(\sqrt{q} + 1)\) by Table I, since \(x_1 = x_2 = 1\). Then

\[
\sqrt{n} + 1 \geq 1 + \frac{q+1}{2} + (\sqrt{n} - 3)(\sqrt{q} + 1), \quad (27)
\]
since \(x_7 = \sqrt{n} - 3\). If \(\sqrt{n} > 3\), then \(x_7 > 0\). Hence, \(q\) is a square and \(q \equiv 9\) mod 16 by Lemma 4.3(3). Thus the cases \(q = 41, 49, 81\) or \(89\) are ruled out. As a consequence, \(q = 25\). This yields \(\sqrt{n} < 5\), since \(n < q^2\) by our assumptions. Then \(\sqrt{n} = 4\), since \(\sqrt{n} > 3\). This is a contradiction, since \(\tilde{\gamma}\) is an involutory homology of \(\text{Fix}(\sigma)\). Therefore \(\sqrt{n} = 3\) and hence \(n = 3^4\). Then \(\frac{q(q+1)}{2} \leq 82\), since \(\frac{q(q+1)}{2} \leq n + 1\) by (27), being \(x_2 = 1\). This is still a contradiction, since \(q = 25, 41, 49, 81\) or \(89\).

Assume that \(x_1 = 2\) and \(x_2 = 0\). Recall that \(\tilde{\beta}_j\) is a Baer involution of \(\text{Fix}(\sigma)\). Hence \(|\text{Fix}(T_j) \cap l| = \sqrt{n} + 1\) for \(j = 1, 2\) by Lemma 4.3(7). Therefore,

\[
\sqrt{n} = 1 + x_4 + x_5 + x_7, \quad (28)
\]
arguing as above, as \(x_1 = 2\) and \(x_2 = 0\).

Assume that \(x_4 > 0\). Then \(x_4 = 1\) by Lemma 3.5(3). Then \(q = 17\) by Lemma 3.4, since \(q \equiv 1\) mod 8. Moreover, \(x_5 = 0\) again by Lemma 3.4, and \(x_7 = 0\) since \(q\) is a non square. So, \(\sqrt{n} = 2\) by (28). That is \(n = 16\). Nevertheless, this contradicts the fact that \(q < n\) by our assumptions. So, \(x_4 = 0\).
Assume that \( x_5 > 0 \), then \( q = 25, 41, 49, 81 \) or 89 by Lemma 3.4, since \( q \equiv 1 \mod 8 \). Furthermore, \( x_5 \leq 3 \) by Lemma 3.5(4). Thus, \( x_7 \geq \sqrt[3]{n} - 4 \) by (28). If \( \sqrt[3]{n} > 4 \), then \( x_7 > 0 \) and hence \( q \) is a square and \( q \equiv 9 \mod 16 \). Therefore, only the case \( q = 25 \) is admissible. Nevertheless, \( \sqrt[3]{n} < 5 \), since \( q < n < q^2 \) by our assumptions. This is a contradiction, since \( \sqrt[3]{n} > 4 \). As consequence, \( \sqrt[3]{n} = 4 \) and \( x_7 = 0 \). Then \( \sqrt[3]{n} = 16 \), and we again obtain a contradiction, since \( \gamma \) is an involutory homology of \( \text{Fix}(\sigma) \). So, \( x_5 = 0 \).

Since \( x_4 = x_5 = 0 \), then \( x_7 = \sqrt[3]{n} - 1 \) by (28). Now, bearing in mind that \( x_1 = 2, x_2 = x_4 = x_5 = 0 \) and \( x_7 = \sqrt[3]{n} - 1 \), we have \( \sqrt[3]{n} \geq 1 + (\sqrt[3]{n} - 1)(\sqrt[3]{q} \pm 1) \) by Table I. Therefore, \( \sqrt[3]{n} + 1 \geq (\sqrt[3]{q} \pm 1) \). Then \( \sqrt[3]{q} \equiv 3 \mod 4 \) and \( \sqrt[3]{n} = \sqrt[3]{q} - 2 \) or \( \sqrt[3]{q} + 1 \), since \( \sqrt[3]{n} < \sqrt[3]{q} \) by our assumptions. Actually, only the case \( \sqrt[3]{n} = \sqrt[3]{q} - 2 \) is admissible, since \( \sqrt[3]{n} \) is odd, as \( \gamma \) is an involutory homology of \( \text{Fix}(\sigma) \). Then \( x_7 = \sqrt[3]{q} - 3 \) by (28). Hence \( \sqrt[3]{n} = \sqrt[3]{q} - 2 \). Now, by substituting these values in \( \sqrt[3]{n} \geq 1 + (\sqrt[3]{n} - 1)(\sqrt[3]{q} - 1) \) (obtained by Table I), we actually obtain an equality. Thus, there are exactly two points on \( l \) fixed by \( G \) \((x_1 = 2) \) and the stabilizer in \( G \) of any of the remaining ones on \( l \cap \text{Fix}(\sigma) \) is isomorphic to \( \text{PSL}(2, \sqrt[3]{q}) \). Then \( S_2 = x_3 = x_9 = 0 \) by Table I. Therefore, \( S = \frac{q^2 + 1}{2} S_2' \), being \( S = \frac{q^2 + 1}{2} S_1 \) and \( S = S_2 + S_2' \). By this and by Table I, we have

\[
n + 1 = 2 + \frac{\sqrt[3]{q}(\sqrt[3]{q} - 3)}{2} (q + 1) + \frac{q + 1}{2} S_2',
\]

since \( x_1 = 2, x_2 = 0 \) by our assumption, since \( x_3 = x_4 = x_5 = x_9 = 0 \) and \( x_7 = \sqrt[3]{q} - 3 \) by the above argument, and since \( x_6 = x_8 = 0 \) by Lemma 4.3(2) and (4). Since \( q \equiv 1 \mod 8 \), then \( S_2' \) is even (see its definition) and hence \( q + 1 \mid n - 1 \). That is \( q + 1 \mid (\sqrt[3]{q} - 2)^4 - 1 \), since \( \sqrt[3]{n} = \sqrt[3]{q} - 2 \). Easy computations yield \( q + 1 \mid 40\sqrt[3]{q} - 8 \). As \( q + 1 \neq 40\sqrt[3]{q} - 8 \), then \( q + 1 \leq 80\sqrt[3]{q} - 16 \) and so \( \sqrt[3]{q} \leq 79 \). Actually, since \( \sqrt[3]{q} \leq 41 \), since \( (\sqrt[3]{q})^2 + 1 \leq 40\sqrt[3]{q} - 8 \). Now, it is straightforward computation to show that there are no \( \sqrt[3]{q} \), such that \( \sqrt[3]{q} \leq 41 \) and \( (\sqrt[3]{q})^2 + 1 \mid 40\sqrt[3]{q} - 8 \). Thus, we have proved the assertion.

Let \( C = C_G(\sigma) \) and let \( K \) and \( K^* \) be the kernels of the action of \( C \) on \( \text{Fix}(\sigma) \) and on \( \text{Fix}(\sigma) \cap l \), respectively. Clearly \( \langle \sigma \rangle \leq K \leq K^* \leq C \). Moreover, either \( K^* \leq Z_{4^{-1}} \) or \( K^* \cong D_{4^{-1}} \), or \( K^* = C \), since \( q \equiv 1 \mod 8 \). Actually, the cases \( K^* \cong D_{4^{-1}} \) or \( K^* = C \) are ruled out by Lemma 4.2. Then \( \langle \sigma \rangle \leq K \leq K^* \leq Z_{2^{-1}} \). Let \( \gamma \in C \) such that \( \gamma^2 = \sigma \). The previous lemma shows that either \( \gamma \in K^* \) or \( \gamma \) induces a Baer involution on \( \text{Fix}(\sigma) \). Now, we investigate these two configurations separately.

### 4.1. The collineation \( \gamma \in K^* \)

**Lemma 4.5.** If \( \text{Fix}(\gamma) \cap l = \text{Fix}(\sigma) \cap l \), then the following hold:
Thus, the assertions (1)–(4) follow by a direct inspection of the Tables I and II.

Proof. Assume that \( \text{Fix}(\gamma) \cap l = \text{Fix}(\sigma) \cap l \). Note that \( |\text{Fix}(\gamma) \cap l| = \sum_h |\text{Fix}_{P_h^G}(\gamma)| \) and \( |\text{Fix}(\sigma) \cap l| = \sum_h |\text{Fix}_{P_h^G}(\sigma)| \). Then \( |\text{Fix}_{P_h^G}(\gamma)| = |\text{Fix}_{P_h^G}(\sigma)| \) for each admissible \( P_h^G \) on \( l \), since \( \text{Fix}(\gamma) \cap l = \text{Fix}(\sigma) \cap l \) and \( |\text{Fix}_{P_h^G}(\gamma)| \leq |\text{Fix}_{P_h^G}(\gamma)| \). Thus, the assertions (1)–(4) follow by a direct inspection of the Tables I and II.

It remains to show the assertion (5). In order to do so, note that \( |\text{Fix}(T_1) \cap l| = x_1 + 2x_{7a} \) and \( |\text{Fix}(T_2) \cap l| = x_1 + 2x_{7b} \) for \( \sqrt{q} \equiv 1 \) mod 4, while \( |\text{Fix}(T_1) \cap l| = x_1 + 2x_{7a} \) and \( |\text{Fix}(T_2) \cap l| = x_1 + 2x_{7b} \) by Table I, since \( x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0 \) by (2). Therefore, we have proved the assertion (5).

\[ \square \]

**Lemma 4.6.** If \( \text{Fix}(\gamma) \cap l = \text{Fix}(\sigma) \cap l \) then one of the following occurs:

1. The group \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for either \( j = 1 \) or \( j = 2 \), and the following occur:
   
   a. \( x_1 = 0 \);
   
   b. \( x_{7a}, x_{7b} > 0 \). In particular, either \( x_{7a} = 1 \) or \( x_{7b} = 1 \).

2. The group \( T_j \) induces a Baer involution on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \), and the following occur:
   
   a. \( \sqrt{n} + 1 = x_1 + x_7 \);
   
   b. \( x_1 \geq 3 \);
   
   c. The collineation \( \gamma \) induces the identity on \( \text{Fix}(\sigma) \);
   
   d. The group \( G \) fixes a subplane of \( \Pi \) of order \( x_1 - 1 \).

Proof. Let \( \bar{\beta}_j \) be the involution induced on \( \text{Fix}(\sigma) \) by a Klein subgroup \( T_j \) containing \( \sigma \) (and hence lying in \( C \), \( j = 1, 2 \)). Assume that \( \bar{\beta}_1 \) is a \( (C_{\beta_j}, a_{\beta_j}) \)-perspectivity. Then \( C_{\beta_j} \in \text{Fix}(\sigma) \cap l \) and \( a_{\beta_j} \neq l \) by Lemma 4.2. Set \( \{X\} = a_{\beta_1} \cap l \). If \( \beta_1 \) is an elation \( \text{Fix}(\sigma) \), then \( C_{\beta_1} = X \) and hence \( |\text{Fix}(T_1) \cap l| = 1 \). Then \( x_1 = 1 \) and \( x_{7a} = 0 \), since \( |\text{Fix}(T_1) \cap l| = x_1 + 2x_{7a} \) by Lemma 4.5(5). So, \( G \) fixes \( C_{\beta_1} \), which is a contradiction by dual of Lemma 4.2, since \( \text{Fix}(T_1) \cap [C_{\beta_1}] = \text{Fix}(\sigma) \cap [C_{\beta_1}] \). Thus, \( \bar{\beta}_1 \) is a \( (C_{\beta_1}, a_{\beta_1}) \)-homology of \( \text{Fix}(\sigma) \). Then \( C_{\beta_1} \neq X \) and hence \( |\text{Fix}(T_1) \cap l| = 2 \). Therefore, either \( x_1 = 2 \) and \( x_{7a} = 0 \) or \( x_1 = 0 \).
and \( x_{7a} = 1 \), since \(|\text{Fix}(T_1) \cap l| = x_1 + 2x_{7a}\) by Lemma 4.5(5). Assume the former occurs. Then \( G \) fixes \( C_{\beta_1} \) and \( X \), which is a contradiction by the same argument as above. Consequently, \( x_1 = 0 \) and \( x_{7a} = 1 \). Moreover, \( x_{7b} > 0 \), since \(|\text{Fix}(T_1) \cap l| = x_1 + 2x_{7b}\), being \( x_1 = 0 \). The previous argument still works with \( T_2 \) in the role of \( T_1 \). Hence, we obtain the assertion (1a) and (1b).

Assume that \( T_j \) induces a Baer involution on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \). Then \(|\text{Fix}(T_j) \cap l| = \sqrt{n} + 1 \) for each \( j = 1, 2 \). Then \( \sqrt{n} + 1 = x_1 + 2x_{7a} \) or \( \sqrt{n} + 1 = x_1 + 2x_{7b} \) by Lemma 4.5(5). As a consequence,

\[
\sqrt{n} + 1 = x_1 + x_7, \tag{29}
\]

since \( x_7 = x_{7a} + x_{7b} \). Thus, we have proved the assertion (2a).

Now, note that \( \sqrt{n} + 1 \geq x_1 + x_7(\sqrt{q} \pm 1) \) by Table I. By composing this one with (29), we obtain

\[
\sqrt{n} + 1 \geq x_1 + (\sqrt{n} + 1 - x_1)(\sqrt{q} \pm 1). \tag{30}
\]

Assume that \( x_7 > 0 \). Thus \( q \equiv 1 \mod 16 \) by Lemma 4.5(3). If \( x_1 = 0 \), then \( x_7 = \sqrt{n} + 1 \) by (29) and hence \( \sqrt{n} + 1 \geq (\sqrt{n} + 1)(\sqrt{q} \pm 1) \) by (30). That is \( \sqrt{n} - 1 \geq (\sqrt{q} \pm 1) \geq \sqrt{q} - 1 \). This yields \( \sqrt{n} \geq \sqrt{q} \), which is a contradiction, since \( n < q^2 \). Then \( x_1 \geq 1 \). If \( x_1 = 1 \), then \( x_7 = \sqrt{n} \) by (29).

Furthermore, \( \sqrt{n} + 1 \geq 1 + \sqrt{n}(\sqrt{q} \pm 1) \) by (30). That is \( \sqrt{n} \geq (\sqrt{q} \pm 1) \). As \( \sqrt{n} < \sqrt{q} \) by our assumption, then \( \sqrt{q} \equiv 3 \mod 4 \) and \( \sqrt{n} = \sqrt{q} - 1 \). By substituting the determined values of \( x_1, x_7 \) and of \( \sqrt{n} \) in (30), we see that this one is satisfied as an equality. So, \( x_9 = 0 \) and \( S_4 = 0 \) by Table I. Therefore, \( n + 1 = 1 + \mathcal{S} + (\sqrt{q} - 3)(q + 1) \). As \( q + 1 \mid \mathcal{S} \), being \( \mathcal{S} = \frac{q+1}{q+1} \mathcal{S}_1 \) and being \( S_1 \) even by its definition, then \( q + 1 \mid n \). Consequently, \( q + 1 \mid (\sqrt{q} - 1)^4 \). Elementary calculations of the previous relation yield \( q + 1 \mid 4 \). Hence, we arrive at a contradiction, since \( q \) is a square as \( x_7 > 0 \). Therefore, \( x_1 \geq 2 \). Assume that \( x_1 = 2 \). Then \( x_7 = \sqrt{n} - 1 \) by (29). Furthermore, \( \sqrt{n} + 1 \geq 2 + (\sqrt{n} - 1)(\sqrt{q} \pm 1) \) by (30). This yields \( \sqrt{n} + 1 \geq \sqrt{q} \). As \( \sqrt{n} < \sqrt{q} \) by our assumption, then \( \sqrt{q} \equiv 3 \mod 4 \) and \( \sqrt{n} \geq \sqrt{q} - 2 \). Then either \( \sqrt{n} = \sqrt{q} - 1 \) or \( \sqrt{n} = \sqrt{q} - 2 \), again since \( \sqrt{n} < \sqrt{q} \). Actually, the case \( \sqrt{n} = \sqrt{q} - 1 \) is ruled out by the above argument. So, \( \sqrt{n} = \sqrt{q} - 2 \). This forces \( \sqrt{n} + 1 \geq 2 + (\sqrt{n} - 1)(\sqrt{q} \pm 1) \) to be an equality. As a consequence, \( x_9 = 0 \) and \( S_4 = 0 \) by Table II. Thus \( n + 1 = 2 + S_1 + (\sqrt{q} - 3)(q + 1) \) by Table I. As \( q + 1 \mid S_1 \), then \( q + 1 \mid n - 1 \). Hence \( q + 1 \mid (\sqrt{q} - 2)^4 - 1 \), since \( \sqrt{n} = \sqrt{q} - 2 \). Easy computations yield a contradiction. Therefore, \( x_1 \geq 3 \) for \( x_7 > 0 \). Actually, \( x_1 \geq 3 \) also for \( x_7 = 0 \) by (29), since \( \sqrt{n} \geq 2 \). Thus \( x_1 \geq 3 \) in each case, which is the assertion (2b).

Now, \( G \) and hence \( \gamma \) acts on \([X]\) for each point \( X \) of the \( x_1 \) ones fixed by \( G \) on \( l \). Then \( \gamma \) fixes at least 3 lines of \([X]\) for each point \( X \) of the \( x_1 \) ones fixed by
$G$ on $l$ by dual of Lemma 4.4. So, $\gamma$ induces the identity on $\text{Fix}(\sigma)$, since $x_1 \geq 3$ and since $\text{Fix}(\gamma) \cap l = \text{Fix}(\sigma) \cap l$. Thus, we have proved the assertion (2c).

Now, we may apply the dual of the above argument to $[X]$ in the role of $l$ for each point $X$ of the $x_1$ ones fixed by $G$ on $l$. This yields that $G$ fixes at least 3 lines through each of these $x_1$ points on $l$. Then $G$ is planar, which is the assertion (2d).

Proposition 4.7. The group $T_j$ induces a homology on $\text{Fix}(\sigma)$ for either $j = 1$ or $j = 2$, and the following hold:

1. $x_1 = 0$;
2. $x_{7a}, x_{7b} > 0$. In particular, either $x_{7a} = 1$ or $x_{7b} = 1$.

Proof. Assume that $T_j$ induces a Baer involution on $\text{Fix}(\sigma)$ for each $j = 1, 2$. Then $x_1 \geq 3$ and $\gamma$ induces the identity on $\text{Fix}(\sigma)$ by Lemma 4.6(2b) and (2c), respectively. Then, by Table I and by Lemma 4.6(2b), we have the following system of Diophantine equations:

\begin{align*}
\sqrt{n} + 1 &= x_1 + x_7 \\
\sqrt{n} + 1 &= x_1 + x_7(\sqrt{q} \pm 1) + x_9 \frac{q - 1}{p^e - 1} + S_4 \\
n + 1 &= x_1 + x_\ell \sqrt{q(q + 1)} + x_9 \frac{p^e q^2 - 1}{2(p^e - 1)} + \frac{q + 1}{2} S_1.
\end{align*}

Suppose that $x_7 = 0$. Then $\sqrt{n} + 1 = x_1$ by (31) and hence $o(\text{Fix}(G)) = \sqrt{n}$ by Lemma 4.6(2d). Thus, $\sqrt{n} + 1 = S_4 + x_9 \frac{q - 1}{p^e - 1} + \sqrt{n}$ by (32). As a consequence, $S_4 + x_9 > 0$. Let $\rho_t$, where $t = 1$ or 2, be the representatives the two conjugates of $p$-elements in $G$. Then $\rho_t$ is planar for each $t = 1, 2$ since $G$ is planar. In particular, $\rho_t$ fixes a Baer subplane of $\Pi$ by [16, Theorem 3.7], since $\text{Fix}(G) \subset \text{Fix}(\rho_t)$ and $o(\text{Fix}(G)) = \sqrt{n}$. Furthermore, $\text{Fix}(G) \subset \text{Fix}(\rho_t, \sigma) \subset \text{Fix}(\rho_t)$. Then either $\text{Fix}(\rho_t, \sigma) = \text{Fix}(G)$ or $\text{Fix}(\rho_t, \sigma) = \text{Fix}(\rho_t)$ again by [16, Theorem 3.7], since $\text{Fix}(G)$ is a Baer subplane of $\text{Fix}(\rho_t)$. Actually, $\text{Fix}(\rho_t, \sigma) = \text{Fix}(\rho_t)$, since $\text{Fix}(\rho_t) \subset \text{Fix}(\sigma)$ and $\text{Fix}(\sigma)$ is a Baer subplane of $\Pi$. Then $\frac{1}{2} = \sqrt{n} + 1 = S_4 + x_9 \frac{q - 1}{p^e - 1} + \sqrt{n}$ by Tables II and I, respectively. Hence, we arrive at a contradiction, since $S_4 = S_4$ and $S_4 > 0$. Thus, $x_7 > 0$.

Let us focus on the group $\langle \rho_t, \sigma \rangle$, $t = 1, 2$. Then $\langle \rho_t, \sigma \rangle$ is planar for each $t = 1, 2$, since $\text{Fix}(G) \subset \text{Fix}(\rho_t, \sigma)$. In particular, $o(\text{Fix}(\rho_t, \sigma)) = \sqrt{n} + 1$ by [16, Theorem 3.7], since $o(\text{Fix}(\rho_t, \sigma))$ is a proper subplane of $\text{Fix}(\sigma)$ as $x_7 > 0$, and since $\text{Fix}(\sigma)$ is a Baer subplane of $\Pi$. Moreover, by Table II,
\( o(\text{Fix}(\langle \rho_t, \sigma \rangle)) + 1 = x_1 + x_7 \varepsilon + x_9 k_t + \frac{1}{2} S_4 \), where \( \varepsilon = 2 \) or 0 according to whether \( \sqrt{q} \equiv 1 \mod 4 \) or \( \sqrt{q} \equiv 3 \mod 4 \), respectively. So,

\[
x_1 + x_7 \varepsilon + x_9 k_t + \frac{1}{2} S_4 \leq x_1 + x_7, \text{ for each } t = 1, 2, \tag{34}
\]

since \( o(\text{Fix}(\langle \rho_t, \sigma \rangle))) < \sqrt{n} + 1 \) and since \( \sqrt{n} + 1 = x_1 + x_7 \) by (31). It follows from (34) that \( \varepsilon = 0 \) and hence \( \sqrt{q} \equiv 3 \mod 4 \), as \( \varepsilon = 2 \) or 0 according to whether \( \sqrt{q} \equiv 1 \mod 4 \) or \( \sqrt{q} \equiv 3 \mod 4 \). Then summing up the two inequalities in (34) (one for \( t = 1 \) and the other for \( t = 2 \)) and then subtracting \( 2x_1 \) to the sum, we obtain

\[
S_4 + x_9 (k_1 + k_2) \leq 2x_7. \tag{35}
\]

Assume that \( S_4 + x_9 > 0 \). If \( x_9 = 0 \), then \( S_4 > 0 \). Moreover, \( S_4 \leq 2x_7 \) by (35). Since \( \text{Fix}(G) \) is a proper subplane of \( \text{Fix}(T_j) \), then \( (x_1 - 1)^2 \leq (x_1 - 1) + x_7 \) by [16, Theorem 3.7], and hence \( x_1 - 1 \leq x_7 \). Then \( x_1 - 1 + S_4 \leq 3x_7 \), since \( S_4 \leq 2x_7 \). Now, note that \( \sqrt{n} + 1 = x_1 + x_7 (\sqrt{q} - 1) + S_4 \) by (32), being \( x_9 = 0 \) and \( \sqrt{q} \equiv 3 \mod 4 \). This produces \( \sqrt{n} \leq 3x_7 + x_7 (\sqrt{q} - 1) \) as \( S_4 \leq 2x_7 \). Hence, \( x_7 \geq \frac{\sqrt{n}}{\sqrt{q} + 2} \). On the other hand, \( n + 1 \geq x_7 \sqrt{q}(q + 1) + 1 \) by (33), since \( x_1 \geq 1 \) (actually, \( x_1 \geq 3 \)). Now, by substituting \( x_7 \geq \frac{\sqrt{n}}{\sqrt{q} + 2} \) in the last inequality, we obtain \( n \geq \frac{\sqrt{n}}{\sqrt{q} + 2} \sqrt{q}(q + 1) \). Since \( \frac{q + 1}{\sqrt{q} + 2} = \sqrt{q} - 2 + \frac{5}{\sqrt{q} + 2} \), we actually obtain \( \sqrt{n} > (\sqrt{q} - 1)^2 \) and hence \( \sqrt{n} \geq (\sqrt{q})^2 \), since \( \sqrt{n} \) is a square. This is impossible, since \( n < q^2 \) by our assumptions. Therefore, \( x_9 > 0 \). Actually, \( x_9 = 1 \) by Lemma 3.5(6). Then \( 2x_7 \geq k_1 + k_2 \) by (35). Hence, \( x_7 \geq \frac{q - p^e}{2p^e(p^e - 1)} \), since \( k_1 + k_2 = \frac{q - p^e}{p^e(p^e - 1)} \). Recall that \( q = p^{2we}, w \geq 1 \). Hence \( x_7 \geq \frac{p^{2we} - p^e}{2p^e(p^e - 1)} \), Now, by substituting these value in \( \sqrt{n} \geq x_7 (\sqrt{q} - 1) + \frac{q - 1}{p^e - 1} \) which is obtained by (32), as \( x_1 \geq 1 \) and \( x_9 = 1 \) and \( \sqrt{q} \equiv 3 \mod 4 \), we have

\[
\sqrt{n} \geq \frac{p^{(2w-1)e - 1}}{2(p^e - 1)} (p^{we} - 1) + \frac{p^{2we} - 1}{p^e - 1}.
\]

Furthermore, since \( q = p^{2we} \) and \( \sqrt{q} \equiv 3 \mod 4 \), then \( w \) is odd. Assume that \( w \geq 3 \). Then \( \frac{p^{we - 1}}{p^e - 1} \geq p^{2e} + p^e + 1 \) and hence \( \frac{p^{we - 1}}{p^e - 1} > 2p^e \). Then \( \sqrt{n} > \frac{p^{2we - 1}}{p^e - 1} + (p^{(2w-1)e - 1})p^e \) and hence \( \sqrt{n} > p^{2we} \). Thus \( \sqrt{n} > q \), which is contradiction, since \( \sqrt{n} < q \) by our assumptions. Then \( w < 3 \) and hence \( w = 1 \), since \( w \) is odd. So, \( p^e = \sqrt{q} \) and hence \( p^e \equiv 3 \mod 4 \), since \( \sqrt{q} \equiv 3 \mod 4 \). This is a contradiction, by Lemma 4.5(4).

Finally, assume that \( S_4 = x_9 = 0 \). Then \( n + 1 \geq x_1 + x_7 \sqrt{q}(q + 1) \) by (33). If we subtract (31) from (32), and then (31) from \( n + 1 \geq x_1 + x_7 \sqrt{q}(q + 1) \), by bearing in mind that \( \sqrt{q} \equiv 3 \mod 4 \), we obtain

\[
\sqrt{n} - \sqrt{q} = x_7 (\sqrt{q} - 2) \tag{36}
\]

\[
n - \sqrt{q} \geq x_7 \left[ \sqrt{q}(q + 1) - 1 \right]. \tag{37}
\]
Now, combining (36) and (37), and bearing in mind that \( x_7 > 0 \), we obtain

\[
\frac{n - \sqrt[q]{n}}{\sqrt{q} - \sqrt[n]{n}} \geq \frac{\sqrt{q}(q + 1) - 1}{\sqrt{q} - 2}.
\]

(38)

Since \( n - \sqrt[q]{n} = (\sqrt[n]{n} - \sqrt[n]{n})(\sqrt[n]{n} + \sqrt[n]{n} + 1) \) and since \( \frac{\sqrt{q}(q+1)-1}{\sqrt{q}-2} > q \), then \( \sqrt[n]{n} + \sqrt[n]{n} + 1 > q \) by (38). Then \( (\sqrt[n]{n} + 1)^2 > q \) and hence \( \sqrt[n]{n} > \sqrt[q]{q} - 1 \), as \( q \) is a square. On the other hand, \( \sqrt[n]{n} < \sqrt[q]{q} \) by our assumptions. So, \( \sqrt[q]{q} - 1 < \sqrt[n]{n} < \sqrt[q]{q} \), with \( \sqrt[n]{n} \) and \( \sqrt[q]{q} \) integers. This is clearly a contradiction. At this point, the assertion easily follows by Lemma 4.6.

**Lemma 4.8.** The following hold:

1. \( x_9 = 0 \);
2. \( S_4 > 0 \);
3. Let \( h = 2 \) or 4. Then \( \frac{\sqrt[q]{q+1}}{h} | |K| \) for \( \sqrt[q]{q} \equiv \pm 1 \mod 8 \), respectively.

**Proof.** The group \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for at least one \( j = 1 \) or 2 by Proposition 4.7. Furthermore, \( x_1 = 0, x_7a, x_7b > 0 \) and either \( x_7a = 1 \) or \( x_7b = 1 \). We may assume that \( T_1 \) does and that \( x_7a = 1 \). Let \( \tilde{\beta}_1 \) is a \((C_{\tilde{\beta}_1}, a_{\tilde{\beta}_1})\)-homology induced by \( T_1 \) on \( \text{Fix}(\sigma) \). Set \( \{X\} = a_{\tilde{\beta}_1} \cap l \). Then, by Table I and by Lemma 4.5, we have

\[
\sqrt[n]{n} + 1 = S_4 + x_7(\sqrt[q]{q} \pm 1) + x_9 \frac{q - 1}{p^e - 1} \quad (39)
\]

\[
n + 1 = \frac{q + 1}{2} S_1 + x_7 \sqrt[q]{q}(q + 1) + x_9 \frac{p^e q^2 - 1}{2(p^e - 1)}. \quad (40)
\]

Since \( S_1 \geq S_4 \), we compose (39) and (40) obtaining

\[
n + 1 \geq \frac{q + 1}{2}(\sqrt[n]{n} + 1) + x_7 \frac{q + 1}{2}(\sqrt[q]{q} \pm 1) + x_9 \frac{q^2 - 1}{2}. \quad (41)
\]

In particular, \( n + 1 \geq \frac{q + 1}{2}(\sqrt[n]{n} + 1) + x_9 \frac{q^2 - 1}{2} \). Since \( q + 1 > \sqrt[n]{n} + 1 \), we have \( n + 1 > (\frac{q + 1}{2} + x_9 \frac{q^2 - 1}{2})(\sqrt[n]{n} + 1) \). Since \( \sqrt[n]{n} + 1 \) does not divide \( n + 1 \) or \( n \), then \( n - 1 \geq (\frac{q + 1}{2} + x_9 \frac{q^2 - 1}{2})(\sqrt[n]{n} + 1) \) being the second part an integer. Hence, dividing each term by \( \sqrt[n]{n} + 1 \), we obtain \( \sqrt[n]{n} \geq 1 + \frac{q + 1}{2} + x_9 \frac{q^2 - 1}{2} \). If \( x_9 \geq 1 \), then \( \sqrt[n]{n} \geq 1 + q \), which is a contradiction. Therefore, \( x_9 = 0 \) and we have proved the assertion (1).

Assume that \( S_4 = 0 \). Then \( x_7 = \frac{\sqrt[n]{n} + 1}{\sqrt[q]{q} \pm 1} \) by (39) and hence \( n + 1 = \frac{q + 1}{2} S_1 + \frac{\sqrt[n]{n} + 1}{\sqrt[q]{q} \pm 1} \sqrt[q]{q}(q + 1) \) by (40), as \( x_9 = 0 \). Note that \( \sqrt[q]{q} \pm 1 \mid \sqrt[n]{n} + 1 \), since \( x_7 \) is an integer, otherwise we would have a contradiction. In particular, \( n + 1 \geq \frac{\sqrt[n]{n} + 1}{\sqrt[q]{q} \pm 1} \sqrt[q]{q}(q + 1) \). As \( \sqrt[n]{n} \geq 2 \), we have \( \sqrt[n]{n} + 1 \mid n + 1 \). Furthermore, \( \sqrt[n]{n} + 1 \mid n \).
Hence, \( n - 1 \geq \sqrt[n]{q+1} \) since the second part is an integer. So, \( \sqrt{n} \geq 1 + \sqrt[\overline{q+1}]{(q+1)} \). If \( \sqrt[\overline{q}]{q} \equiv 3 \mod 4 \), then \( \sqrt{n} > q + 1 \), which is a contradiction by our assumptions. As a consequence, \( \sqrt[\overline{q}]{q} \equiv 1 \mod 4 \). Then \( \sqrt{n} \geq 1 + \frac{\sqrt[\overline{q}]{(q+1)}}{\sqrt{n}+1} \) and hence \( \sqrt{n} > (\sqrt[\overline{q}]{q} - 1)^2 \).

If \( x_7 > 2 \), then \( T_2 \) must induce a Baer collineation \( \text{Fix}(\sigma) \) and consequently \( \sqrt{n} \) must be an integer. Since \( n < q^2 \), then \( \sqrt{n} < \sqrt[\overline{q}]{q} \). Actually, \( \sqrt{n} \leq \sqrt[\overline{q}]{q} - 1 \), since \( \sqrt{n} \) is an integer. Therefore, \( \sqrt{n} \leq (\sqrt[\overline{q}]{q} - 1)^2 \) by squaring. This is a contradiction, since \( \sqrt{n} > (\sqrt[\overline{q}]{q} - 1)^2 \) by the above argument.

If \( x_7 = 2 \). Then \( \sqrt{n} = 2(\sqrt[\overline{q}]{q} + 1) - 1 \) by (39), since \( x_9 = 0 \), and since \( S_1 = 0 \) by our assumption. Hence \( \sqrt{n} = 2\sqrt[\overline{q}]{q} + 1 \). On the other hand, \( n \geq 2\sqrt[n]{(q+1)} - 1 \) by (40). By composing these inequalities, we have \( (2\sqrt[\overline{q}]{q} + 1)^2 \geq 2\sqrt[n]{(q+1)} - 1 \). Easy computations yield a contradiction, since \( q > 9 \). Thus, \( S_1 > 0 \) which is the assertion (2).

As \( x_{7a}, x_{7b} > 0 \), it follows that \( q \) is a square and \( q \equiv 1 \mod 16 \) by Lemma 4.5(3). So, either \( \sqrt[\overline{q}]{q} \equiv 1 \mod 8 \) and \( \sqrt[\overline{q}]{q} \equiv 7 \mod 8 \). Since \( T_1 \) fixes exactly two points on \( l \), by Table I, these ones must lie in either a \( G \)-orbit on \( l \) of type (7a) or in a \( G \)-orbit on \( l \) of type (7b) according to whether \( \sqrt[\overline{q}]{q} \equiv 1 \mod 8 \) or \( \sqrt[\overline{q}]{q} \equiv 7 \mod 8 \), respectively.

Assume that \( \sqrt[\overline{q}]{q} \equiv 1 \mod 8 \). Then \( C_{\beta_1}, X \in C^{G}_{\beta_1} \), where \( G_{C_{\beta_1}} \equiv \text{PSL}(2, \sqrt[\overline{q}]{q}) \), since \( T_1 \) fixes exactly two points in \( G \)-orbit of type (7a). In particular, \( G_{C_{\beta_1}} = G_X \), since \( G_{C_{\beta_1}} \vartriangleleft \text{PGL}(2, \sqrt[\overline{q}]{q}) \). Recall that \( C = C_G(\sigma) \) and that \( \overline{C} = C/K \), where \( K \) is the kernel of \( C \) on \( \text{Fix}(\sigma) \). Clearly, \( C \cong D_{q-1} \) and \( \beta_1 \in C \). Note that \( C_X = C \cap G_X = C_{G_X}(G) \) and hence \( C_X \cong D_{q-1} \), since \( G_X \cong \text{PSL}(2, \sqrt[\overline{q}]{q}) \). In particular, \( C_X = C_{X,C_{\beta_1}} \). Set \( C_0 = C_{X,C_{\beta_1}} \). Clearly, \( K \leq C_0 \) and \( \beta_1 \in C_0 \), where \( C_0 = C_0/K \). Then \( X \) and \( C_{\beta_1} \) are the unique points on \( \text{Fix}(\sigma) \cap l \) fixed by \( C_0 \), as \( \beta_1 \in C_0 \). Set \( h = |C_0| \). Then \( h \) is even, as \( \beta_1 \in C_0 \). If \( |C_0| > 4 \), then \( C_0 \) is dihedral and therefore exists a point \( Y \in l \) such that \( C_Y = C \) by Lemma 2.10. That is \( C \leq G_Y \). Then \( G_Y = C \), where \( C = C_G(\sigma) \), since \( C \) is maximal in \( G \) as \( q > 9 \), and since \( x_1 = 0 \) by Proposition 4.7(1). Nevertheless, this is a contradiction, since \( x_2 = 0 \) by Lemma 4.5(2). Thus, \( h = |C_0| \leq 4 \). Actually, either \( h = 2 \) or \( 4 \), since \( h \) is even. On the other hand, \( |C_0| = h |K| \). Hence, \( \frac{\sqrt[n]{q+1}}{h} \mid |K| \), where \( h = 2, 4 \), since \( C_0 \equiv D_{\sqrt[n]{q+1}} \) as \( C_0 = C_{G_X}(G) \) and \( G_X \equiv \text{PSL}(2, \sqrt[\overline{q}]{q}) \).

Now, repeating the previous argument for \( \sqrt[\overline{q}]{q} \equiv 7 \mod 8 \), we find that \( C_0 \equiv D_{\sqrt[n]{q+1}} \) and hence \( \frac{\sqrt[n]{q+1}}{h} \mid |K| \), where \( h = 2, 4 \). So, we have proved the assertion (3).

**Lemma 4.9.** If the \( p \)-elements are not planar, then \( S_1 = S_4 = 2(\sqrt[n]{q} + 1) \).
Proof. Let \( \rho_t, t = 1, 2 \), be the representatives the two conjugate classes of \( p \)-elements in \( G \). Since \( x_{7a}, x_{7b} > 0 \), the collineation \( \rho_t \) fixes at least \( 2\sqrt{q} \) points on \( l \) for each \( t = 1, 2 \) by Table II. Then \( \rho_t \) must fix at least \( 2\sqrt{q} \) lines on \( \Pi \) by [16, Theorem 13.3]. Since \( \rho_t \) cannot be planar, all these lines must concur to a unique point \( X_t \) of \( \Pi \). It is a plain that \( X_1 \) and \( X_2 \) might coincide. Let \( S \) be the Sylow \( p \)-subgroup of \( G \) containing \( \rho_t \) for each \( t = 1, 2 \). Clearly, \( \rho_t \) fixes \( X_t^S \) and at least \( 2\sqrt{q} \) lines through each point of \( X_t^S \), since \( S \) is abelian. Then \( |X_t^S| = 1 \), since \( \rho_t \) cannot be planar on \( \Pi \) by our assumption. Thus \( S \) fixes \( X_t \) for each \( t = 1, 2 \). Assume that \( X_t \in \Pi - l \) for at least one \( t = 1 \) or 2. As \( S_q > 0 \) by Lemma 4.8(2), then there exists a point \( Y \) on \( l \) fixed by \( S \). Hence \( S \) acts on \( X_tY - \{X_t, Y\} \). Assume that \( S_R \neq \langle 1 \rangle \) for some point \( R \in X_tY - \{X_t, Y\} \). Let \( \psi \in S_R, \psi \neq 1 \). Clearly, \( \psi \) fixes \( Y \) and \( R \) on \( \Pi - l \). Furthermore, \( \psi \) fixes at least \( 2\sqrt{q} \) points on \( l \). Indeed, \( \psi \) conjugate either to \( \rho_1 \) or \( \rho_2 \) and each of these collineations fixes at least \( 2\sqrt{q} \) points on \( l \) as \( x_{7a}, x_{7b} > 0 \). So, \( \psi \) is planar. This is impossible by our assumption. Thus, \( S \) is semiregular on \( X_tY - \{X_t, Y\} \). Hence, \( q \mid n - 1 \). That is \( n = aq + 1 \) for some positive integer \( a \). On the other hand,

\[
n + 1 = \frac{q + 1}{2}S_4 + x_7\sqrt{q}(q + 1) \tag{42}
\]

by (40) of Lemma 4.8, since \( x_9 = 0 \) by Lemma 4.8(1). Since \( S_1 \) is even, then \( q + 1 \mid n + 1 \) by (42). Then \( q + 1 \mid a - 1 \), since \( n = aq + 1 \). If \( a = 1 \), it follows that \( n = q + 1 \). As \( q = 1 \mod 8 \), we have that \( n = 2 \mod 4 \). Hence, we arrive at a contradiction by [16, Theorem 13.18], since \( q > 3 \). Thus, \( a > 1 \). Hence, \( a = \theta(q + 1) + 1 \), with \( \theta \geq 1 \). Therefore, \( n = \theta q(q + 1) + q + 1 \). This yields \( n > q^2 \), as \( \theta \geq 1 \), which is a contradiction. So, \( X_t \in l \) for each \( t = 1, 2 \). Then \( G_{X_t} \cong F_qZ_{d_t} \) by Table I, since \( S \) fixes \( X_t, |S| = q \), and since \( x_1 = 0 \) by Proposition 4.7(1). As a consequence, \( x_{10} > 0 \).

Note that \( \sigma \) normalizes \( S \) and it acts as the inversion on \( S \). Thus, \( \sigma \) normalizes \( \langle \rho_t \rangle \) for each \( t = 1, 2 \). If \( d_t \) is odd, then \( \sigma \) moves \( X_t \). Then \( \rho_t \) fixes at least \( 2\sqrt{q} \) lines through \( X_t \) and at least other \( 2\sqrt{q} \) ones though \( X_t \sigma \). As a consequence, \( \rho_t \) is planar on \( \Pi \). Nevertheless, this contradicts our assumptions. So, \( d_t \) must be even. This implies \( S_{2^t} = 0 \). Therefore, \( S_1 = S_4 \) by Lemma 4.5(1). In particular, by (42), we have \( n + 1 = \frac{q + 1}{2}S_4 + x_7\sqrt{q}(q + 1) \).

Let \( \tau_t \) be a line of \( [X_t] - \{l\} \) fixed by \( \rho_t \). Clearly each line of \( r_t^G \) intersect \( l \) in a (unique) point of \( X_t^G \), where \( G_{X_t} \cong F_qZ_{d_t} \) and \( d_t \) is even. In particular each element in \( \rho_t^G \) fixes at least one line of \( r_t^G \). Since the \( p \)-elements in \( G \) cannot be planar, then for each element \( \tau \) in \( \rho_t^G \) actually there exists a unique point \( Q_t \in X_t^G \) such that each line of \( \Pi \) fixed by \( \tau \) lies in \( [Q_t] \). As a consequence, each \( p \)-element in \( G \) fixes a subset of a pencil of lines concurrent to a point lying either in \( X_1^G \) or in \( X_2^G \). If \( x_{10} > 2 \), there exists a \( G \)-orbit of type (10), say \( Q^G \), such that \( S \) is semiregular on \( [Q] - \{l\} \). Thus \( q \mid n \) and hence \( n = bq \) where
\( b \geq 1 \). Then \( q + 1 \mid b - 1 \), since \( q + 1 \mid n + 1 \) arguing as above. As \( n > q \) by our assumptions, then \( b > 1 \) and hence \( b = f(q+1)+1 \). Therefore, \( n = f(q+1)+q \), which is a contradiction, since \( n < q^2 \) by our assumption. Thus, \( 0 < x_{10} \leq 2 \). That is \( x_{10} = 1 \) or \( 2 \).

Assume that \( x_{10} = 1 \). Then \( X_1 = X_2 \) and \( d_1 = d_2 \), since \( \rho_1 \) and \( \rho_2 \) lies in the same Sylow \( p \)-subgroup \( S \) of \( G \). Set \( X = X_1 = X_2 \) and \( d = d_1 = d_2 \). Then \( S_4 = \frac{n-1}{d} \). Now, recall that \( \sqrt[4]{\frac{n+1}{h}} \mid |K| \), \( h = 2, 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, by Lemma 4.8(3). Thus, \( \sqrt[4]{\frac{n+1}{h}} \mid |C_X| \), \( h = 2, 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, since \( K \leq C_X \). This fact, in conjunction with the fact that \( G_X \cong F_q \times Z_d \), yields \( C_X \cong Z_d \), where \( d = \sqrt[4]{\frac{n+1}{h}} u, h = 2, 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively. Here \( u \) is a positive divisor of \( d \). So, \( S_4 = \sqrt[4]{\frac{n+1}{u}} \) for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, since \( S_4 = \frac{n-1}{d} \). Now, let \( P \) be a point of \( l \) such that \( G_P \cong \text{PSL}(2, \sqrt[n]{\frac{q}{h}}) \) and let \( S_P = S \cap G_P \). Then \( S_P \) must be semiregular on \( [P] \), since \( P \not\in X^G \), since the lines fixed by any non trivial element in \( S \) lie in \( X^G \) and since \( S \) does not contain planar elements. Hence, \( \sqrt[n]{\frac{q}{h}} \mid n \) as \( |S_P| = \sqrt[n]{\frac{q}{h}} \). Then \( \sqrt[n]{\frac{q}{h}} \) does not contain elements. This fact, in conjunction with the fact that \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, yields \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8. Then \( (x, u, \sqrt[n]{\frac{q}{h}}) = (1, 2, 1, \sqrt[n]{\frac{q}{h}}) \) by Lemma 2.7(2). Therefore, \( S_4 = 2(\sqrt[n]{\frac{q}{h}} + 1) \). Since \( S_1 = S_4 \), we have the assertion.

Assume that \( x_{10} = 2 \). Then \( S_4 = \frac{n-1}{d_1} + \frac{n-1}{d_2} \) and \( X_1 \neq X_2 \) for \( x_{10} = 2 \), since \( S_1 = S_4 \). Now, recall that \( \sqrt[4]{\frac{n+1}{h}} \mid |K| \), \( h = 2, 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, by Lemma 4.8(3). Thus, \( \sqrt[4]{\frac{n+1}{h}} \mid |C_X| \), \( h = 2, 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, since \( K \leq C_X \). For each \( t = 1, 2 \). On the other hand, \( C_X_t \cong Z_{d_t} \), since \( G_{X_t} \cong F_q \times Z_{d_t} \) for each \( t = 1, 2 \). So, \( d_t = \frac{n-1}{h} u_t, h = 2, 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively. Here, \( u_t \) is a positive divisor of \( d_t \). Then \( S_4 = h \frac{n-1}{u_1} + h \frac{n-1}{u_2} \), where \( h = 2 \) or \( 4 \), for \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, since \( S_4 = \frac{n-1}{d_1} + \frac{n-1}{d_2} \). Arguing as above, we have \( \sqrt[n]{\frac{q}{h}} \not\equiv \frac{1}{2} S_4 - 1 \) by (42). Thus, \( \sqrt[n]{\frac{q}{h}} \not\equiv h \frac{n-1}{2u_1} + h \frac{n-1}{2u_2} - 1, \) where \( h = 2 \) or \( 4 \) \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, since \( S_4 = h \frac{n-1}{u_1} + h \frac{n-1}{u_2} \). Then there exists a positive integer \( x \) such that \( x \sqrt[n]{\frac{q}{h}} = h \frac{n-1}{2u_1} + h \frac{n-1}{2u_2} - 1, \) where \( h = 2 \) or \( 4 \) \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8, respectively, in particular, \( S_4 = 2(\sqrt[n]{\frac{q}{h}} + 1) \). Then \( x = 1 \) in any case by Lemma 2.8 and 2.9, since \( \sqrt[n]{\frac{q}{h}} \equiv \pm 1 \) mod 8. Therefore, \( S_4 = 2(\sqrt[n]{\frac{q}{h}} + 1) \). Since \( S_1 = S_4 \), we have the assertion.

**Lemma 4.10.** The group \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for each \( j = 1 \) or \( 2 \).

**Proof.** The group \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for either \( j = 1 \) or \( j = 2 \)
by Proposition 4.7. Assume that $T_1$ does it. We may also assume that $x_{7a} = 1$. Assume also that $T_2$ induces a Baer collineation on \( \text{Fix}(\sigma) \). Then by Table I in conjunction with Lemmas 4.5, 4.8 and Proposition 4.7, we have the following system of Diophantine equations:

\[
\begin{align*}
\sqrt{n} + 1 &= 2x_{7b} \\
\sqrt{n} + 1 &= S_4 + x_7(\sqrt{q} \pm 1) \\
(n + 1) &= \frac{q + 1}{2}S_1 + x_7\sqrt{q}(q + 1).
\end{align*}
\]  

(43) (44) (45)

By substituting $x_7 = 1 + x_{7b}$ in (43), we have that $x_7 = \frac{\sqrt{n} + 3}{2}$. Now, by substituting this value of $x_7$ in and (44), we obtain $\sqrt{n} + 1 = S_4 + (\sqrt{n} + 3)\frac{\sqrt{q} \pm 1}{2}$.

That is

\[
\sqrt{n} - 9 = (S_4 - 10) + (\sqrt{n} + 3)\frac{\sqrt{q} \pm 1}{2}.
\]  

(46)

Assume that $S_4 = 10$. Then $\sqrt{n} = 3 + \frac{\sqrt{q} \pm 1}{2}$ by (46). As $x_7 > 0$, being $x_7 = 1 + x_{7b}$, then $q \equiv 1 \text{ mod } 16$ by Lemma 4.5(3). This yields $\sqrt{q} \equiv 1, 7 \text{ mod } 8$. So, $\sqrt{n} \equiv 3 \text{ mod } 4$, which is a contradiction by Lemma 3.3. Hence, $S_4 \neq 10$. Nevertheless, $\sqrt{n} + 3 \mid S_4 - 10$ again by (46).

Assume that $S_4 < 10$, then $\sqrt{n} + 3 \mid 10 - S_4$. As $\sqrt{n} \geq 2$, then $\sqrt{n} + 3 \geq 5$ and therefore $10 - S_4 \geq 5$. That is $S_4 \leq 5$. Then $S_4 = 2$ or $4$, since $S_4$ is even by its definition and since $S_4 > 0$ by Lemma 4.8(2). Assume that $S_4 = 4$. It follows that $\sqrt{n} = 3 + x_7(\sqrt{q} \pm 1)$. As $x_7 > 0$, then $\sqrt{q} \equiv \pm 1 \text{ mod } 8$ by the above argument. Then $\sqrt{n} \equiv 3 \text{ mod } 4$, as $\sqrt{n} = 3 + x_7(\sqrt{q} \pm 1)$. Nevertheless, this contradicts Lemma 3.3. Therefore, $S_4 = 2$. Hence, $\sqrt{n} + 3 \mid 8$, as $\sqrt{n} + 3 \mid S_4 - 10$. Then $\sqrt{n} + 3 = 8$, since $\sqrt{n} \geq 2$. As a consequence, $\sqrt{n} = 5$. This yields $x_7 = 4$, as $x_7 = \frac{\sqrt{n} + 3}{2}$. Thus, 4($\sqrt{q} \pm 1$) = 24 by (44), as $S_4 = 2$. So, $\sqrt{q} \pm 1 = 6$. On the other hand, $\sqrt{q} \equiv 1 \equiv 0 \text{ mod } 8$ by the previous argument, as $x_7 > 0$. Nevertheless, this contradicts $\sqrt{q} \pm 1 = 6$.

Assume that $S_4 > 10$. Then $S_4 = \theta(\sqrt{n} + 3) + 10$ with $\theta \geq 1$. Assume $\theta$ is odd. Then $\theta(\sqrt{n} + 3) = S_4 - 10$ and hence $\sqrt{n} - 3 = \theta + \frac{\sqrt{q} \pm 1}{2}$. Note that $\sqrt{n} = (3 + \theta) + \frac{\sqrt{q} \pm 1}{2}$ is even, as $\theta$ is odd. Thus $\sqrt{n}$ is even, which is a contradiction, since $T_1$ induces a homology on $\text{Fix}(\sigma)$. Then $\theta$ is even and hence $\theta \geq 2$, as $\theta \geq 1$. Since $\sqrt{n} + 1 = 2x_{7b}$, then $S_4 > 4x_{7b}$.

Let $\rho_t$, $t = 1$ or $2$, be the representative of the two conjugate classes $p$-elements in $G$. Suppose that $\rho_t$ is planar for either $t = 1$ or $t = 2$. Then $o(\text{Fix}(\rho_t)) + 1 = \frac{1}{2}S_1 + x_{7v}2\sqrt{q}$ by Table II, where $v = a$ for $t = 1$ and $v = b$ for $t = 2$, since $x_1 = 0$ by Proposition 4.7, since $x_8 = 0$ by Lemma 4.5(2) and since $x_9 = 0$ by Lemma 4.8(1). Clearly, $\sigma$ acts on $\text{Fix}(\rho_t)$, since $\sigma$ inverts $\rho_t$. Furthermore, it follows from Table II that $|\text{Fix}(\langle \rho_t, \sigma \rangle)| \cap l| = \frac{1}{2}S_4 + x_{7v}\varepsilon$,
where \( \varepsilon \) is either 2 or 0 according to whether \( \sqrt{q} \equiv 1 \mod 4 \) or \( \sqrt{q} \equiv 3 \mod 4 \), respectively. Since \( S_4 > 10 \), then \( |\text{Fix}(\langle \rho_1, \sigma \rangle) \cap l| > 3 \). On the other hand, \( |\text{Fix}(\langle \rho_t, \sigma \rangle) \cap l| < \frac{1}{2} S_1 + x_{7b} \sqrt{q} \). So, \( \langle \rho_t, \sigma \rangle \) induces a Baer collineation on \( \text{Fix}(\rho_t) \). Therefore, \( \langle \rho_t, \sigma \rangle \) is planar. In particular, \( \text{Fix}(\langle \rho_t, \sigma \rangle) \) is a subplane of \( \text{Fix}(\sigma) \) of order \( \frac{1}{2} S_1 + x_{7b} \varepsilon - 1 \), where \( \varepsilon = a \) for \( t = 1 \) and \( \varepsilon = b \) for \( t = 2 \). Then \( \frac{1}{2} S_1 + x_{7b} \varepsilon \leq \sqrt{q} + 1 \) by \([16, \text{Theorem 3.7}]\), since \( \text{Fix}(\sigma) \) is a Baer subplane of \( \Pi \). This yields \( \frac{1}{2} S_1 + x_{7b} \varepsilon \leq 2x_{7b} \) by \((43)\). In particular, \( \frac{1}{2} S_1 \leq 2x_{7b} \) and hence \( S_4 \leq 4x_{7b} \). Hence, we arrive at a contradiction, since \( S_4 > 4x_{7b} \) by the above argument. Thus, \( G \) cannot contain \( p \)-planar elements. Then \( S_1 = S_4 = 2(\sqrt{q}+1) \) by Lemma \(4.9\). This yields \( x_7 = \frac{\sqrt{n-1}}{\sqrt{q}-1} - 2 \) by \((44)\). By substituting these values of \( S_4 \) and \( x_7 \) in \((45)\), we have

\[
n + 1 = (q + 1)(\sqrt{q} + 1) + \left( \frac{\sqrt{n-1}}{\sqrt{q}-1} - 2 \right) \sqrt{q}(q+1).
\]

By elementary calculations of this one, we have

\[
n - q = \frac{\sqrt{n} - \sqrt{q}}{\sqrt{q}-1} \sqrt{q}(q+1).
\]

Thus, \( \sqrt{n} + \sqrt{q} \geq q + 1 \) and \( \sqrt{n} \geq q - \sqrt{q} + 1 \). On the other hand, \( \sqrt{n} \leq (\sqrt{q} - 1)^2 \) since \( n < q^2 \), \( q \) is a square and \( n \) is a fourth power, since \( T_2 \) induces a Baer collineation on \( \text{Fix}(\sigma) \) by our assumption. So, we obtain a contradiction, since \( q - \sqrt{q} + 1 > (\sqrt{q} - 1)^2 \). Hence, \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for each \( j = 1 \) or 2.

**Proposition 4.11.** The collineation \( \gamma \) induces a Baer collineation on \( \text{Fix}(\sigma) \) and hence \( K \cong Z_{\frac{q+1}{2}} \).

**Proof.** The group \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for each \( j = 1 \) or 2 by Lemma \(4.10\). Then \( x_7 = x_{7b} = 1 \) by Table 1, since \( x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0 \) by Lemma \(4.5(2)\), since \( x_1 = 0 \) by Proposition \(4.7(1)\), and since \( x_9 = 0 \) by Lemma \(4.8(1)\). Therefore, \( x_7 = 2 \). Then we obtain the following system of Diophantine equations:

\[
\sqrt{n} + 1 = S_4 + 2(\sqrt{q} + 1) \tag{47} \\
n + 1 = \frac{q+1}{2} S_1 + 2\sqrt{q}(q+1). \tag{48}
\]

Since \( S_1 \geq S_4 \), then \( n + 1 \geq \frac{q+1}{2} S_4 + 2\sqrt{q}(q+1) \) by \((48)\). Now, composing this inequality with \((47)\), we obtain

\[
n + 1 \geq \frac{q+1}{2}(\sqrt{n} + 1) + (q + 1)(\sqrt{q} + 1) \tag{49}
\]
and hence \( n + 1 > \left( \frac{q + 1}{2} + \sqrt{q + 1} \right) (\sqrt{n} + 1) \), since \( \sqrt{n} + 1 < q + 1 \). As \( \sqrt{n} \geq 2 \), we have \( n + 1 \neq 1 \) or \( n + 1 \). Furthermore, \( \sqrt{n} + 1 \neq n \). So, \( n - 1 \geq \left( \frac{q - 1}{2} + \sqrt{q - 1} \right) (\sqrt{n} + 1) \). Dividing each term by \( \sqrt{n} + 1 \) in the previous inequality, we obtain \( \sqrt{n} - 1 > \frac{q - 1}{2} + \sqrt{q - 1} \). This implies \( \sqrt{n} - 1 > \frac{q - 1}{2} + \sqrt{q - 1} \) and therefore

\[
\sqrt{n} + 1 > (\sqrt{q} + 1) \left( \frac{\sqrt{q} + 1}{2} + 1 \right). \tag{50}
\]

Let \( \rho_t, t = 1 \) or \( 2 \), be the representatives of the two conjugate classes \( p \)-elements in \( G \). Suppose that \( \rho_t \) is planar. Then \( o(\text{Fix}(\rho_t)) + 1 = \frac{1}{2}S_1 + 2\sqrt{q} \) by Table II, since \( x_{7a} = x_{7b} = 1 \). Clearly, \( \sigma \) acts on \( \text{Fix}(\rho_t) \), since \( \sigma \) inverts \( \rho_t \). Again by Table II, we have \( |\text{Fix}(\langle \rho_t, \sigma \rangle) \cap l| = \frac{1}{2}S_4 + \varepsilon \), where \( \varepsilon \) is either 2 or 0 according to whether \( \sqrt{q} \equiv 1 \) mod 4 or \( \sqrt{q} \equiv 3 \) mod 4, respectively.

Assume that \( \sqrt{q} \geq 5 \). Then \( \sqrt{n} + 1 > 6 (\sqrt{q} + 1) \) and hence \( S_4 > 4 (\sqrt{q} + 1) \) by (47). In particular, \( S_4 > 8 \). Then \( |\text{Fix}(\langle \rho_t, \sigma \rangle) \cap l| > 3 \). On the other hand, \( |\text{Fix}(\langle \rho_t, \sigma \rangle) \cap l| < \frac{1}{2}S_1 + 2\sqrt{q} \). Hence, \( \langle \rho_t, \sigma \rangle \) induces a Baer collineation on \( \text{Fix}(\rho_t) \). Then \( \left( \frac{1}{2}S_4 + \varepsilon - 1 \right)^2 \leq \frac{1}{2}S_1 + 2\sqrt{q} \) by [16, Theorem 3.7]. Note that \( \left( \frac{1}{2}S_4 + \varepsilon - 1 \right)^2 > S_4 \), as \( S_4 > 8 \). So, \( S_4 < \frac{1}{2}S_4 + 2\sqrt{q} - 1 \). Hence, \( S_4 < 4\sqrt{q} - 2 \). On the other hand, we proved \( S_4 > 4 (\sqrt{q} + 1) \). Combining these two inequalities involving \( S_4 \), we obtain \( \sqrt{q} \equiv 3 \) mod 4 and \( 4\sqrt{q} - 2 > S_4 > 4 (\sqrt{q} + 1) \). Therefore, \( S_4 = 4\sqrt{q} - 3 \), which is a contradiction, since \( S_4 \) must be even.

Assume that \( \sqrt{q} \leq 4 \). Recall that the upper sign if \( \sqrt{q} \equiv 1 \) mod 4 and the lower sign if \( \sqrt{q} \equiv 3 \) mod 4. This yields \( q = 25 \) or \( 49 \), since \( q \) is odd and \( q > 9 \). Actually, only the case \( q = 49 \) is admissible, since \( q \equiv 1 \) mod 16 by Lemma 4.5(3), being \( x_7 > 0 \). Now, by substituting \( q = 49 \) in (49), we have \( \sqrt{n} \geq 35 \). Hence \( \leq \sqrt{n} < 49 \), since \( \sqrt{n} < \sqrt{q} \) by our assumptions. Furthermore, \( \frac{q - 1}{2} \mid n + 1 \) by (48). That is \( 25 \mid n + 1 \), since \( q = 49 \). Now, filtering the list \( 35 \leq \sqrt{n} < 49 \) with respect to the conditions \( 25 \mid n + 1 \), and \( \sqrt{n} \) odd, as the \( T_j \) induces a homology on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \), we obtain \( \sqrt{n} = 43 \). Nevertheless, this contradicts Lemma 3.3. As a consequence, the \( p \)-elements in \( G \) cannot be planar. Then \( S_4 = 2 (\sqrt{q} + 1) \) by Lemma 4.9. This is still a contradiction, since \( S_4 > 4 (\sqrt{q} + 1) \) by the above argument, being \( q > 9 \).

4.2. The collineation \( \gamma \) induces a Baer involution on \( \text{Fix}(\sigma) \)

Proposition 4.12. The group \( T_j \) induces a Baer collineation on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \).

Proof. Recall that \( C = C_G(\sigma) \) and let \( K \) and \( K^* \) be the kernels of the action of \( C \) on \( \text{Fix}(\sigma) \) and on \( \text{Fix}(\sigma) \cap l \), respectively. In particular, \( K \leq Z_{\frac{q+1}{2}} \), since
\( \gamma \) induces a Baer involution \( \tilde{\gamma} \) on \( \text{Fix}(\sigma) \) by Proposition 4.11. Let \( \beta_j \) be the involution induced on \( \text{Fix}(\sigma) \) by a Klein subgroup \( T_j \) containing \( \sigma \) (and hence lying in \( C \)) for \( j = 1, 2 \).

Suppose that \( K \cong \mathbb{Z}_{\frac{q-1}{2}} \). Assume also that \( \beta_j \) is an involutory \( (C_{\beta_j}, a_{\beta_j}) \)-perspectivity. Then \( C_{\beta_j} \in l \cap \text{Fix}(\sigma) \) and \( a_{\beta_j} \not\in l \) by Lemma 4.2. Thus, \( K \leq G_{C_{\beta_j}} \).

This implies \( C \leq G_{C_{\beta_j}} \), since the collineation \( \tilde{\gamma} \) fixes \( C_{\beta_j} \) as \( \tilde{\gamma} \) centralizes \( \beta_j \) and since \( K \cong \mathbb{Z}_{\frac{q-1}{2}} \). Note that \( N_G(T_j) \cap C \cong D_8 \), where \( N_G(T_j) \cong S_4 \).

Then \( N_G(T_j) \leq G_{C_{\beta_j}} \), since \( |\text{Fix}(T_j) \cap l| = 1 \) or 2, since \( C_{\beta_j} \in \text{Fix}(T_j) \cap l \) and since \( C \leq G_{C_{\beta_j}} \). So, \( G \) fixes \( G_{C_{\beta_j}} \), since \( \langle C, N_G(T_j) \rangle \leq G_{C_{\beta_j}} \) and \( G = \langle C, N_G(T_j) \rangle \). Hence, we arrive at a contradiction by dual of Lemma 4.2, since \( \text{Fix}(T_j) \cap [G_{C_{\beta_j}}] = \text{Fix}(\sigma) \cap [G_{C_{\beta_j}}] \).

Suppose that \( K < \mathbb{Z}_{\frac{q-1}{2}} \). Then \( C/K \cong D_{2m} \) with \( m \equiv 0 \mod 4 \), as \( q \equiv 1 \mod 8 \). If \( n \) is odd, then each involution induced on \( \text{Fix}(\sigma) \) by a Klein subgroup \( T_j \) containing \( \sigma \) is a Baer involution by [19, Proposition 3.3], since \( \tilde{\gamma} \) is a Baer involution of \( \text{Fix}(\sigma) \). Thus, we have proved the assertion for \( n \) odd.

Assume that \( n \) is even. Assume also that \( \beta \) is an involutory \( (C_{\beta}, a_{\beta}) \)-elation of \( \text{Fix}(\sigma) \). As \( \text{Fix}(T_1) \cap l = \{ C_{\beta_1} \} \), then \( N_G(T_1) \leq G_{C_{\beta_1}} \), where \( N_G(T_1) \cong S_4 \) as \( q \equiv 1 \mod 8 \). Clearly, \( G_{C_{\beta_1}} < G \), otherwise, we would have a contradiction by the above argument. Then either \( G_{C_{\beta_1}} \cong S_4 \) and \( q \equiv 9 \mod 16 \) or \( G_{C_{\beta_1}} \cong \text{PGL}(2, \sqrt{q}) \) by Table I, since \( |\text{Fix}(T_1) \cap l| = 1 \). If \( G_{C_{\beta_1}} \cong S_4 \), then \( q = 25 \) or 41 by Lemma 3.4, since \( q \equiv 9 \mod 16 \). So, \( \frac{|C_{\beta_1}|}{G_{C_{\beta_1}}} = \frac{q(q^2-1)}{48} \). Then \( \frac{q(q^2-1)}{48} \leq n+1 < q^2+1 \), since \( C_{\beta} \cong l \). Furthermore, \( n \) is a fourth power by Proposition 4.11, and \( n \) is even. This is a contradiction, since \( q = 25 \) or 41. Thus, \( G_{C_{\beta}} \cong \text{PGL}(2, \sqrt{q}) \).

Now, since \( |\text{Fix}(\gamma) \cap l| = \sqrt{q} + 1 \) and \( |\text{Fix}(T_1) \cap l| = 1 \), then

\[
\sqrt{q} + 1 = x_8 \frac{1}{2} (\sqrt{q} \pm 1) + S_4,
\]

where \( x_8a_1 = 1 \). If \( x_8 \geq 2 \), then \( \sqrt{q} + 1 \geq \sqrt{q} \pm 1 \). Then \( \sqrt{q} \equiv 3 \mod 4 \) and hence \( \sqrt{q} \geq \sqrt{q} - 2 \), since \( \sqrt{q} < \sqrt{q} \). Actually, \( \sqrt{q} = \sqrt{q} - 1 \), since \( \sqrt{q} \) is even as \( \beta_1 \) is an involutory \( (C_{\beta_1}, a_{\beta_1}) \)-elation of \( \text{Fix}(\sigma) \). Therefore, \( \sqrt{q} \equiv 2 \mod 4 \) as \( \sqrt{q} \equiv 3 \mod 4 \). Then \( \sqrt{q} = 2 \) by [16, Theorem 13.18], since \( \beta_1 \) acts non trivially on \( \text{Fix}(\tilde{\gamma}) \). As a consequence, \( \sqrt{q} = 3 \). Nevertheless, this is a contradiction, since \( q > 9 \) by our assumptions. Then \( x_8 = x_{8a_1} = 1 \) and hence \( |\text{Fix}(T_2) \cap l| = 3 \) by Table I. Thus, \( T_2 \) induces a Baer collineation on \( \text{Fix}(\sigma) \). Therefore, \( \sqrt{q} + 1 = 3 \).

Now, by substituting \( x_8 = 1 \) and \( \sqrt{q} = 2 \) in (51), we obtain \( \frac{1}{2} (\sqrt{q} \pm 1) + S_4 = 3 \). As a consequence, \( S_4 = 0 \) or \( S_4 = 2 \), since \( S_4 \) is even. If the latter occurs, then \( \sqrt{q} \pm 1 = 2 \). Nevertheless, we again obtain a contradiction, since \( q > 9 \). So, \( S_4 = 0 \) and \( \sqrt{q} \pm 1 = 6 \). Consequently, \( \sqrt{q} = 5 \) or 7 and \( n = 2^4 \), which is
a contradiction, since \( q < n \) by our assumptions. Hence, we have proved
the assertion when the order \( n \) of \( \Pi \) is even. This completes the proof.

Lemma 4.13. The following occur:

\begin{enumerate}
\item \( x_4 = x_5 = x_6 = 0 \);
\item if \( x_7 > 0 \), then \( q \equiv 9 \mod 16 \);
\item \( x_8 \leq 1 \);
\item if \( x_9 > 0 \), then \( p^e \equiv 3 \mod 4 \).
\end{enumerate}

Proof. Recall that \( \gamma \) induces a Baer collineation of \( \text{Fix}(\sigma) \) by Proposition 4.11.
Therefore, \( n \) is a fourth power. Clearly, \( \sqrt[n]{n} \geq 2 \).

(1) Note that \( q \geq 17 \), since \( q \equiv 1 \mod 8 \) and \( q > 9 \). Then \( \sqrt[n]{n} > 2 \), since \( q < n \) by our assumption. Assume that \( \sqrt[n]{n} = 3 \) or \( 7 \). Thus, \( \text{Fix}(\sigma) \) has order 9 or 49, respectively. Furthermore, the group induced by \( C_G(\sigma) \) on
\( \text{Fix}(\sigma) \) has order divisible by 4 and each its involution is Baer collineation of
\( \text{Fix}(\sigma) \) by Propositions 4.11 and 4.12. Nevertheless, this is a contradiction
by Theorem 2.6, since \( \sqrt[n]{n} \equiv 3 \mod 4 \). Thus \( \sqrt[n]{n} \geq 4 \) and \( \sqrt[n]{n} \neq 7 \). Moreover, the case \( \sqrt[n]{n} \neq 6 \) by [16, Theorem 3.6]. Hence, \( \sqrt[n]{n} \geq 4 \) and \( \sqrt[n]{n} \neq 6, 7 \).

Then \( q > 17 \), since \( q < n < q^2 \). Therefore, \( x_4 = 0 \) by Lemma 3.4, since
\( q \equiv 1 \mod 8 \).

Assume that \( x_6 > 0 \). Then \( q = 25 \) or 41 by Lemma 3.4(4), since \( q \equiv 1 \mod 8 \)
and \( q \neq 17 \). Then \( \sqrt[n]{n} = 4 \), since \( q < n < q^2 \), since \( n \) is a fourth power with
\( \sqrt[n]{n} \geq 4 \) and \( \sqrt[n]{n} \neq 6 \). On the other hand, \( n + 1 \geq \frac{q(q^2 - 1)}{48} \) by Table I, as \( x_6 > 0 \). Thus, either \( n \geq 325 \) or \( n \geq 1435 \) according to whether \( q = 25 \) or 41, respectively. This is impossible, since \( n = 4^q \). So, \( x_6 = 0 \).

Assume that \( x_5 > 0 \). Then \( n + 1 \geq \frac{q(q^2 - 1)}{120} \) by Table I. Hence, \( \frac{q(q^2 - 1)}{120} x_5 - 1 \leq n < q^2 \). Furthermore, \( q = 25, 41, 49, 81 \) or 89 by Lemma 3.4(5), since \( q \equiv 1 \mod 8 \).

In addition, \( n \) is a fourth power with \( \sqrt[n]{n} \geq 4 \) and \( \sqrt[n]{n} \neq 6, 7 \) by the
above argument. Thus, \( (q, n) = (25, 4^4) \) or \( (41, 5^4) \) or \( (89, 9^4) \). Moreover,
\( x_5 = 1 \) in each of these cases.

Assume that \( (q, n) = (41, 5^4) \). Let \( S \cong Z_{41} \) which is normalized by \( \sigma \).
Since \( n = 5^4 \), then \( n + 1 \equiv 11 \mod 41 \) and \( n^2 \equiv 10 \mod 41 \). Hence, \( S \) is
planar. In particular, \( o(\text{Fix}(S)) = 10 + 041 \), where \( \theta \geq 0 \). Actually, \( \theta = 0 \)
by [16, Theorem 3.7], since \( n = 5^4 \). Therefore, \( o(\text{Fix}(S)) = 10 \). Since \( \sigma \)
normalizes \( S \), it acts on \( \text{Fix}(S) \). Note that \( \sigma \) must act trivially on \( \text{Fix}(S) \),
otherwise we would have a contradiction by [16, Theorem 13.18]. Thus,
\( \text{Fix}(S) \subset \text{Fix}(\sigma) \). So, we arrive at a contradiction by [16, Theorem 3.7],
since \( o(\text{Fix}(S)) = 10 \), while \( o(\text{Fix}(\sigma)) = 25 \).

Assume that \( (q, n) = (89, 9^4) \). Let \( U \leq G \) such that \( U \cong Z_{89} \). Since \( n = 9^4 \),
then \( n + 1 \equiv 65 \mod 89 \) and \( n^2 \equiv 64 \mod 89 \). Hence, \( U \) is planar. In
particular, \( o(Fix(U)) = 64 + \lambda 89 \), where \( \lambda \geq 0 \). Actually, \( \lambda = 0 \) by [16, Theorem 3.7], since \( n = 9^4 \). Thus, \( o(Fix(U)) = 64 \). Let \( V \leq N_G(U) \) such that \( V \cong Z_{11} \). Clearly, \( V \) acts on \( Fix(U) \). Since \( 65 \equiv 10 \mod 11 \) and \( 64^2 \equiv 4 \mod 11 \), then \( V \) fixes a subplane of \( Fix(U) \) of order 9 at least. Then \( Fix(U) \subseteq Fix(V) \), otherwise we would have a contradiction by [16, Theorem 3.7], since \( o(Fix(U)) = 64 \). If \( Fix(U) \subset Fix(V) \), we obtain a contradiction by [16, Theorem 3.7], since \( o(Fix(U)) = 64 \), while \( o(Fix(V)) \leq 81 \) as \( n = 9^4 \). Then \( Fix(U) = Fix(V) \) and hence \( o(Fix(V)) = 64 \). Clearly, \( V \cong Z_{11} \) must be semiregular on \( l - Fix(V) \). So, \( 11 \mid |l - Fix(V)| \). This is a contradiction, since \( |l - Fix(V)| = 6497 \), as \( n = 9^4 \).

Assume that \((q, n) = (25, 4^4)\) and \( x_5 = 1 \). Let us focus on the action of the involution \( \sigma \) of \( \Pi \). Clearly, \( \sigma \) fixes exactly 17 points on \( l \), since it induces a Baer collineation on \( \Pi \) and \( n = 4^4 \). Let \( X^G \) the orbit of type (5). Then \( |Fix_{X^G}(\sigma)| = 6 \) by Table I. Hence, \( \sigma \) fixes exactly 11 points on \( l - X^G \). If \( Y^G \) is an on orbit on \( l \) of type (3), then \( Y^G \subseteq l - X^G \). Furthermore, \( |Fix_{Y^G}(\sigma)| = 12 \) again by Table I. Nevertheless, this is a contradiction. Thus, \( x_3 = 0 \). Then each admissible non trivial \( G \)-orbit on \( l \) has length divisible by 13. Indeed, one can compute each length orbit on \( l \) using Table I for \( q = 25 \). Therefore, \( 13 \mid |l - (l \cap Fix(G))| \). That is \( 13 \mid n - x_1 \), since \( |l \cap Fix(G)| = x_1 \). Then \( x_1 \geq 10 \), since \( n = 256 \) and \( 257 \equiv 10 \mod 13 \). So, \( \gamma \), where \( \gamma^2 = \sigma \), fixes at least 10 points on \( l \). This contradicts the facts that \( \gamma \) fixes exactly 5 points on \( l \) by Lemma 4.11, being \( n = 4^4 \).

(2) Assume that \( x_7 \geq 0 \) and \( q \equiv 1 \mod 16 \). Then, by Table II, the collineation \( \gamma \) fixes at least \( x_7(\sqrt{q} \pm 1) \) points on \( l \cap Fix(\sigma) \) according to whether \( \sqrt{q} \equiv 1 \mod 4 \) or \( \sqrt{q} \equiv 3 \mod 4 \), respectively. Then \( \sqrt{n} + 1 \geq x_7(\sqrt{q} \pm 1) \). On the other hand, \( \sqrt{n} < \sqrt{q} \) by our assumption. Hence, \( \sqrt{n} + 1 \leq \sqrt{q} \). By composing, we have \( x_7(\sqrt{q} \pm 1) \leq \sqrt{q} \). Actually, \( x_7(\sqrt{q} \pm 1) \leq \sqrt{q} - 1 \). So, \( x_7 = 1 \) and \( \sqrt{n} \equiv 3 \mod 4 \). Therefore, \( \sqrt{n} = \sqrt{q} - 2 \). Let \( P^{G^2} \) the \( G \)-orbit of type (7). Note that \( Fix(\gamma) \cap l = Fix_{P^{G^2}}(\gamma) \), since \( \sqrt{n} = \sqrt{q} - 2 \). Hence, \( x_1 = x_2 = x_3 = 0 \) by Table I, being \( x_4 = x_5 = x_6 = 0 \) by part (1). This yields \( \sqrt{n} + 1 = 2x_7 \) again by Table I, where \( x_7 = 1 \), since \( T_1 \) induces a Baer collineation on \( Fix(\sigma) \). Nevertheless, we again obtain a contradiction, since \( \sqrt{n} \geq 2 \). Thus, we have proved the assertion (1).

(3) Assume that \( x_8 \geq 2 \). Then, by Table II, the collineation \( \gamma \) fixes at least \( \sqrt{q} \pm 1 \) points on \( Fix(\sigma) \cap l \) according to whether \( \sqrt{q} \equiv 1 \mod 4 \) or \( \sqrt{q} \equiv 3 \mod 4 \), respectively. Then \( \sqrt{n} + 1 \geq \sqrt{q} \pm 1 \). If \( \sqrt{q} \equiv 1 \mod 4 \), then \( \sqrt{n} + 1 \geq \sqrt{q} + 1 \) and hence \( \sqrt{n} \geq \sqrt{q} \). Nevertheless, this contradicts our assumption. Hence, \( \sqrt{q} \equiv 3 \mod 4 \), then \( \sqrt{n} \geq \sqrt{q} - 2 \). Then either \( \sqrt{n} = \sqrt{q} - 1 \) or \( \sqrt{n} = \sqrt{q} - 2 \), since \( \sqrt{n} < \sqrt{q} \) and \( q \) is a square. Assume that \( \sqrt{n} = \sqrt{q} - 1 \). Then \( \sqrt{n} = 2 \mod 4 \) as \( \sqrt{q} \equiv 3 \mod 4 \). Let
\[ C = C_G(\sigma) \] and recall that \( K \trianglelefteq \mathbb{Z}_{2^{q-1}} \), where \( K \) is the kernel of the action of \( C \) on \( \text{Fix}(\sigma) \). Thus, \( 4 \mid |C| \), where \( \bar{C} = C/K \). Note also that each involution in \( \bar{C} \) is a Baer collineation of \( \text{Fix}(\sigma) \). Indeed, each involution in \( \bar{C} \) is induced either by \( \gamma \) or by the \( T_j \) for each \( j = 1, 2 \), and all these ones are Baer collineations of \( \text{Fix}(\sigma) \) by Propositions 4.11 and 4.12, respectively. Nevertheless, this is impossible by Theorem 2.6.

Assume that \( \sqrt{n} = \sqrt{q} - 2 \). Let \( P_1^G \) and \( P_2^G \) be two distinct orbits on \( l \) both of type \( (9) \). Since \( \sqrt{n} + 1 = \sqrt{q} - 1 \), we have \( \sqrt{n} + 1 = 2x_7a + 3x_8a + x_{8b} \) by Table I, since \( x_1 = x_2 = 0 \) by the previous argument, since \( x_4 = x_5 = x_6 = 0 \) by part (1), and being \( \sqrt{q} \equiv 3 \pmod{4} \). Hence \( \sqrt{n} + 1 = 2x_7a + 3x_8a + x_{8b} \). Arguing as above with \( T_2 \) in the role of \( T_1 \), we obtain \( \sqrt{n} + 1 = 2x_7a + x_{8b} + 3x_{8b} \) (see Table I). Summing up these two equations and bearing in mind that \( x_7 = x_7a + x_{7b} \) and \( x_8 = x_{8a} + x_{8b} \), we have \( 2(\sqrt{n} + 1) = 2x_7 + 4x_8 \). Hence, \( \sqrt{n} + 1 = x_7 + 2x_8 \). As \( x_8 = 2 \), then \( x_7 = \sqrt{n} - 5 \). That is \( x_7 = \sqrt{q} - 7 \), as \( \sqrt{n} = \sqrt{q} - 2 \). On the other hand, we have \( \sqrt{q}(q + 1)x_7 + x_8\sqrt{q + 1} \leq n + 1 \) again by Table I. That is \( \sqrt{q}(q + 1)(\sqrt{q} - 7) + \sqrt{q}(q + 1) \leq (\sqrt{q} - 2)^4 + 1 \), since \( x_7 = \sqrt{q} - 7 \), \( x_8 = 2 \) and \( \sqrt{n} = \sqrt{q} - 2 \). Easy computations yield a contradiction, since \( q > 9 \).

Thus, we have proved the assertion (3).

(4) Suppose that \( p^e \equiv 1 \pmod{4} \) and \( x_0 > 0 \), then \( \gamma \) fixes \( \frac{q-1}{p^{2w}} \) points on \( \text{Fix}(\sigma) \cap l \), where \( q = p^{2w} \), \( w \geq 1 \), by Table II and following remark. Then \( \sqrt{n} + 1 \geq \frac{q-1}{p^{2w}} \) and hence \( \sqrt{n} + 1 \geq p^{2w} + 1 \), as \( \frac{q-1}{p^{2w}} \geq p^{2w} + 1 \). That is \( \sqrt{n} + 1 \geq \sqrt{q} + 1 \). So, \( n \geq q^2 \). Therefore, we arrive at a contradiction by our assumption. Then the assertion (4) follows by Table I.

Lemma 4.14. It holds that \( \sqrt{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8 \).

Proof. Assume that \( \sqrt{q} \equiv 1 \pmod{4} \). Then \( |\text{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_7a + 3x_{8a} + x_{8b} \) by Table I, since \( x_4 = x_5 = x_6 = 0 \) by Lemma 4.13(1). Then \( \sqrt{n} + 1 = x_1 + 3x_2 + 2x_7a + 3x_{8a} + x_{8b} \), since \( T_1 \) induces a Baer involution on \( \text{Fix}(\sigma) \) by Proposition 4.12. Arguing as above with \( T_2 \) in the role of \( T_1 \), we have \( |\text{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_7b + x_{8a} + 3x_{8b} \) and hence \( \sqrt{n} + 1 = x_1 + 3x_2 + 2x_7b + x_{8a} + 3x_{8b} \). Summing up, the two relations involving \( \sqrt{n} + 1 \), we obtain \( \sqrt{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8 \), as \( x_7 = x_7a + x_{7b} \) and \( x_8 = x_{8a} + x_{8b} \).

Assume that \( \sqrt{q} \equiv 3 \pmod{4} \). Then, arguing as above, we obtain \( |\text{Fix}(T_1) \cap l| = x_1 + 3x_2 + 2x_7a + x_{8a} + 3x_{8b} \) and \( |\text{Fix}(T_2) \cap l| = x_1 + 3x_2 + 2x_7a + 3x_{8a} + x_{8b} \). Thus, the role of \( T_1 \) and \( T_2 \), in term of fixed points, are exchanged. This yields \( \sqrt{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8 \) as above.

\[ \square \]
Let \( H_j = \langle T_j, \gamma \rangle \) for each \( j = 1, 2 \). Clearly, \( H_j \cong D_8 \), since \( \gamma^2 = \sigma, T_j \cong E_4, \sigma \in T_j \) and \( H_j \leq C_G(\sigma) \). By [4], two cases arise:

1. \( q \equiv 1 \mod 16 \). In this case, \( H_1 \) and \( H_2 \) are the representatives of the two distinct conjugate classes under \( G \). Moreover, \( N_G(H_j) \cong D_{16} \) for each \( j = 1, 2 \);

2. \( q \equiv 9 \mod 16 \). In this case, the dihedral subgroups of order 8 are Sylow 2-subgroups of \( G \) and hence they are conjugate. In particular, \( H_1 = H_2 \). Set \( H = H_1 \), then \( N_G(H) = H \cong D_8 \).

**Lemma 4.15.** One of the following occurs:

1. \( q \equiv 1 \mod 16 \) and one of the following occurs:
   
   (a) \( H_j \) induces the identity on \( \text{Fix}(\gamma) \) for each \( j = 1, 2 \), \( G \) fixes a subplane of order \( \sqrt[n]{8} \) and \( S_4 = x_2 = x_7 = x_8 = 0 \);
   
   (b) \( H_j \) induces a perspectivity of axis \( \text{Fix}(\gamma) \cap l \) on \( \text{Fix}(\gamma) \) for each \( j = 1, 2 \). Furthermore, \( x_1 = \sqrt[n]{8} + 1 \) and \( x_2 = x_7 = x_8 = 0 \);
   
   (c) \( H_j \) induces a perspectivity on \( \text{Fix}(\gamma) \) of axis distinct from \( \text{Fix}(\gamma) \cap l \) for each \( j = 1, 2 \). In particular, \( x_1 + x_2 + x_8 = 1, 2 \);
   
   (d) \( H_j \) induces a Baer involution on \( \text{Fix}(\gamma) \) for each \( j = 1, 2 \) and hence \( x_1 + x_2 + x_8 = \sqrt[n]{8} + 1 \).

2. \( q \equiv 9 \mod 16 \) and one of the following occurs:

   (a) \( H \) induces the identity on \( \text{Fix}(\gamma) \), \( G \) fixes a subplane of order \( \sqrt[n]{8} \) and \( S_4 = x_2 = x_7 = x_8 = 0 \);

   (b) \( H \) induces a perspectivity on \( \text{Fix}(\gamma) \) of axis \( \text{Fix}(\gamma) \cap l \). Furthermore, \( x_1 = \sqrt[n]{8} + 1 \) and \( x_2 = x_7 = x_8 = 0 \);

   (c) \( H \) induces a perspectivity on \( \text{Fix}(\gamma) \) of axis distinct from \( \text{Fix}(\gamma) \cap l \) and \( x_1 + x_2 + x_8 = 1, 2 \);

   (d) \( H \) induces a Baer involution on \( \text{Fix}(\gamma) \) and hence \( x_1 + x_2 + x_8 = \sqrt[n]{8} + 1 \).

**Proof.** Assume that \( q \equiv 1 \mod 16 \). In this case, \( H_1 \) and \( H_2 \) are the representatives of the two distinct conjugate classes under \( G \) of dihedral subgroups of order 8. Moreover, \( N_G(H_j) \cong D_{16} \) for each \( j = 1, 2 \). In particular, the unique \( G \)-orbits on \( l \) on which \( H \) fixes points are those of type (1),(2),(8) by Tables I and II, since \( x_6 = x_7 = 0 \) by Lemma 4.13(1) and (2) as \( q \equiv 1 \mod 16 \). Clearly, \( |\text{Fix}_{Q^G}(H_j)| = 1 \) if \( Q^G \) is of type (1). Since \( H_j \leq C_G(\sigma) \) for each \( j = 1, 2 \), and since two subgroups of \( G \) isomorphic to \( D_8 \) are conjugate in \( G \) if they are conjugate \( C_G(\sigma) \) by [4, §246], then, by Proposition 2.5, \( |\text{Fix}_{Q^G}(H_j)| = 1 \) if \( Q^G \) is of
type (2). Also, $|\text{Fix}_{Q^G}(H_j)| = 1$ if $Q^G$ is of type (8) by Proposition 2.5. Indeed, in this case, $G_Q \cong \text{PGL}(2, \sqrt{q})$ and hence $|\text{Fix}_{Q^G}(H_j)| = |G_Q|/16$ for each $j = 1, 2$ again by [4, §246]. Thus, $|\text{Fix}(H_j) \cap l| = x_1 + x_2 + x_8$ for each $j = 1, 2$. Assume that $\text{Fix}(H_j) \cap l = \text{Fix}(\gamma) \cap l$ for each $j = 1, 2$. Then $x_1 + x_2 + x_8 = \sqrt{n} + 1$ since $\gamma$ induces a Baer collineation on $\text{Fix}(\sigma)$ by Proposition 4.11. On the other hand, $\sqrt{n} + 1 = x_1 + 3x_2 + 2x_8$ by Lemma 4.14 (note that $x_7 = 0$ by Lemma 4.13(2) as $q \equiv 1 \pmod{16}$). Hence $x_1 + x_2 + x_8 = x_1 + 3x_2 + 2x_8$. This yields $x_2 = x_8 = 0$ and $\sqrt{n} + 1 = x_1$. Then we obtain the assertion (1a) or (1b) according to whether $H_j$ induces the identity or a perspectivity on $\text{Fix}(\gamma) \cap l$ on $\text{Fix}(\gamma)$, respectively. At this point, the assertions (1b)–(1c) easily follow.

Assume that $q \equiv 9 \pmod{16}$. Then $H$ is a Sylow 2-subgroup of $G$ and $N_G(H) = H$. In particular, the unique $G$-orbits on $l$ on which $H$ fixes points are those of type (1),(2),(8) by Table I and II. Indeed, $x_6 = 0$ by Lemma 4.13(1). Also, $x_7 = 0$. Namely, if $P^G$ is of type (7), we have $G_P \cong \text{PSL}(2, \sqrt{q})$, where $\sqrt{q} \equiv 3, 5 \pmod{8}$, as $q \equiv 9 \pmod{16}$. Hence, $8 \nmid |G_P|$. Now, by Proposition 2.5, we obtain that $H$ fixes 1 point for each $G$-orbit on $l$ of type (1),(2) or (8), since $H$ is a Sylow 2-subgroup of $G$ and $N_G(H) = H$. Therefore, $|\text{Fix}(H_j) \cap l| = x_1 + x_2 + x_8$. Assume that $\text{Fix}(H_j) \cap l = \text{Fix}(\gamma) \cap l$. Then $x_1 + x_2 + x_8 = \sqrt{n} + 1$, since $\gamma$ induces a Baer collineation on $\text{Fix}(\sigma)$ by Proposition 4.11. At this point, the same argument as $q \equiv 1 \pmod{16}$ can be applied to obtain the assertions (2a)–(2d).

**Lemma 4.16.** If $H_j$ induces a perspectivity on $\text{Fix}(\gamma)$ of axis distinct from $\text{Fix}(\gamma) \cap l$ for each $j = 1, 2$, then $x_2 = x_8 = 0$.

**Proof.** Assume that $H_j$ induces a perspectivity on $\text{Fix}(\gamma)$ of axis distinct from $\text{Fix}(\gamma) \cap l$ for each $j = 1, 2$. We treat the cases $q \equiv 1 \pmod{16}$ and $q \equiv 9 \pmod{16}$ at the same time, bearing in mind that $H_j = H$ when the latter occurs. Hence, $x_1 + x_2 + x_8 = 1$ or 2 by Lemma 4.15.

**$x_2 = 0$.** Assume that $x_2 > 0$. Then $x_2 = 1$ by Lemma 3.5(1). If $x_1 = x_8 = 0$, then

\begin{align}
\sqrt{n} &= 2 + x_7 \\
\sqrt{n} + 1 &\geq \frac{q(q + 1)}{2} + (\sqrt{q} \pm 1)x_7
\end{align}

by Lemma 4.14 and Table I, respectively. So, $x_7 = \sqrt{n} - 2$ by (52). Now, by substituting this value in (53) and then elementary computations of this one, we obtain

\begin{align}
(\sqrt{q} \pm 1)(\sqrt{n} - 2) + \frac{(q - 9)}{2} &\leq \sqrt{n} - 4.
\end{align}
As \( q > 9 \), then \((\sqrt{q} + 1)(\sqrt{n} - 2) < \sqrt{n} - 4\). As \( x_2 = 1 \) and \( x_1 = x_8 = 0 \), then \( H \) must induce an elation on \( \text{Fix}(\gamma) \). Thus \( \sqrt{n} \) must be even. So, the case \( \sqrt{n} = \sqrt{q} - 2 \) is ruled out as \( q \) is odd. Therefore, \( \sqrt{n} = \sqrt{q} - 1 \). Nevertheless, this case cannot occur by [16, Theorem 13.18], since \( \sqrt{n} \equiv 2 \mod 4 \) and \( \sqrt{n} > 2 \), as \( \sqrt{q} \equiv 3 \mod 4 \) with \( \sqrt{q} > 3 \), and since \( H \) induces a non trivial involutory collineation on \( \text{Fix}(\gamma) \). Then either \( x_1 = 1 \) and \( x_8 = 0 \) or \( x_1 = 0 \) and \( x_8 = 1 \), since \( x_1 + x_2 + x_8 \leq 2 \) and \( x_2 = 1 \).

If \( x_1 = 1 \) and \( x_8 = 0 \), then
\[
\sqrt{n} = 3 + x_7 \quad (55)
\]
\[
\sqrt{n} + 1 \geq 1 + \frac{q(q + 1)}{2} + (\sqrt{q} \pm 1)x_7 \quad (56)
\]
by Lemma 4.14 and Table I, respectively. If \( x_7 = 0 \), then \( \sqrt{n} = 3 \) by (55). Consequently, \( \frac{q(q + 1)}{2} \leq 9 \) by (56). A contradiction, since \( q > 9 \). Thus \( x_7 > 0 \) and hence \( q \) is a square. In particular, \( x_7 = \sqrt{n} - 3 \) by (55). Then
\[
(q \pm 1)(\sqrt{n} - 3) + \frac{(q - 17)}{2} \leq \sqrt{n} - 9, \quad (57)
\]
combining \( x_7 = \sqrt{n} - 3 \) with (56). As \( q \) is an odd square number and \( q > 9 \), then \( q \geq 25 \) and hence \( \frac{q(q + 1)}{2} > 0 \). This yields \((\sqrt{q} \pm 1)(\sqrt{n} - 3) < \sqrt{n} - 9 \) by (57). If \( \sqrt{q} \equiv 1 \mod 4 \), then \( \sqrt{n} > \sqrt{q} - 2 \). Then \( \sqrt{n} = \sqrt{q} - 1 \), as \( \sqrt{n} < \sqrt{q} \). If \( \sqrt{q} \equiv 3 \mod 4 \), then \( \sqrt{n} > \sqrt{q} - 3 \) and hence either \( \sqrt{n} = \sqrt{q} - 1 \) or \( \sqrt{n} = \sqrt{q} - 2 \). As \( x_1 = x_2 = 1 \) and \( x_8 = 0 \), then \( H \) must induces a homology on \( \text{Fix}(\gamma) \). Thus, \( \sqrt{n} \) must be odd. Then only \( \sqrt{n} = \sqrt{q} - 2 \) is really admissible as \( q \) is odd. Now, by substituting this value in (57) and bearing in mind that \( \sqrt{q} \equiv 3 \mod 4 \), we obtain a contradiction, since \( q > 9 \).

If \( x_1 = 0 \) and \( x_8 = 1 \), then
\[
\sqrt{n} = 4 + x_7 \quad (58)
\]
\[
\sqrt{n} + 1 \geq \frac{q(q + 1)}{2} + (\sqrt{q} \pm 1)x_7 + \sqrt{q} \quad (59)
\]
by Lemma 4.14 and Table I, respectively. Then \( x_7 = \sqrt{n} - 4 \). If \( \sqrt{n} = 4 \), then \( x_7 = 0 \). Now, by substituting theses values in (59), we have \( \frac{q(q + 1)}{2} + \sqrt{q} \leq 17 \). Nevertheless, this yields contradiction, since \( q \geq 25 \) as \( q \) is an odd square number and \( q > 9 \). Then \( \sqrt{n} > 4 \) and hence \( x_7 > 0 \). Note also that \( q \neq 25 \), since \( q < n < q^2 \) with \( q = 25 \), and since \( n \) a fourth power with \( \sqrt{n} > 4 \). \( q \geq 49 \). Indeed, \( q \) is an odd square number and \( q > 9 \). Now, by substituting \( x_7 = \sqrt{n} - 4 \) in (59), we obtain
\[
\frac{(q + 1)}{2} + (\sqrt{q} \pm 1)(\sqrt{n} - 4) + (\sqrt{q} - 1) \leq \sqrt{n} \quad (60)
\]
and hence
\[
(\sqrt{q} - 1)(\sqrt{n} - 4) + \frac{(q - 31)}{2} \leq \sqrt{n} - 16. \quad (61)
\]
As \(q \geq 49\), then \(\frac{(q - 31)}{2} > 0\). Moreover, by (61), we have \((\sqrt{q} - 1)(\sqrt{n} - 4) < \sqrt{n} - 16\) and hence \(\sqrt{n} + 4 > \sqrt{q} - 1\). That is \(\sqrt{n} > \sqrt{q} - 5\). Then \(\sqrt{n} = \sqrt{q} - \theta\), where \(1 \leq \theta \leq 4\), as \(\sqrt{n} < \sqrt{q}\). As \(x_2 = x_8 = 1\) and \(x_1 = 0\), it follows that \(H\) must induce a homology on \(\text{Fix}(\gamma)\). Thus, \(\sqrt{n}\) must be odd. Therefore, we actually have either \(\sqrt{n} = \sqrt{q} - 2\) or \(\sqrt{n} = \sqrt{q} - 4\), as \(q\) is odd. Nevertheless, these cases cannot occur. Indeed, if we substitute each of them in (60), we obtain a contradiction.

\(x_8 = 0\). Assume that \(x_8 > 0\). Then \(x_8 = 1\) by Lemma 4.13(3). The previous point implies \(x_2 = 0\). Thus, either \(x_1 = 0\) or \(x_1 = 1\) as \(x_1 + x_2 + x_8 \leq 2\) and \(x_8 = 1\). Assume that \(x_1 = 0\). Then
\[
\sqrt{n} = 1 + x_7
\]
\[
\sqrt{n} + 1 \geq (\sqrt{q} \pm 1)x_7 + \sqrt{q}
\]
by Lemma 4.14 and Table I, respectively. It follows that \(x_7 = \sqrt{n} - 1\) and hence \((\sqrt{q} \pm 1)(\sqrt{n} - 1) + \sqrt{q} - 2 \leq \sqrt{n} - 1\) by (62) and (63). This yields \(\sqrt{n} + 1 > \sqrt{q} \pm 1\). So, \(\sqrt{n} > \sqrt{q} \pm 1 - 1\). Then \(\sqrt{q} \equiv 3\) mod 4 and \(\sqrt{n} > \sqrt{q} - 2\), as \(\sqrt{n} < \sqrt{q}\). That is \(\sqrt{n} = \sqrt{q} - 1\). Therefore \(\sqrt{n} \equiv 2\) mod 4 and \(\sqrt{n} > 2\), as \(\sqrt{q} \equiv 3\) mod 4 and \(\sqrt{q} > 3\). Nevertheless, this is a contradiction by [16, Theorem 13.18], since \(H\) induces a non trivial involutory collineation on \(\text{Fix}(\gamma)\).

If \(x_1 = 1\), then
\[
\sqrt{n} = 2 + x_7
\]
\[
\sqrt{n} + 1 \geq \sqrt{q} + 1 + (\sqrt{q} \pm 1)x_7
\]
by Lemma 4.14 and Table I, respectively. If \(\sqrt{n} = 2\), then \(x_7 = 0\). By substituting these values in (65), we obtain \(\sqrt{q} \leq 4\). Then \(\sqrt{q} = 3\), since \(\sqrt{q}\) is odd. Hence, we arrive at a contradiction, since \(q > 9\) by our assumptions. Then \(\sqrt{n} > 2\) and hence \(x_7 = \sqrt{n} - 2\) by (64). Again combining the previous equation with (65), we have
\[
(\sqrt{q} \pm 1)(\sqrt{n} - 2) + \sqrt{q} - 4 \leq \sqrt{n} - 4. \quad (66)
\]
This yields \(\sqrt{n} + 2 > \sqrt{q} \pm 1\). So, \(\sqrt{n} > \sqrt{q} \pm 1 - 2\). Then \(\sqrt{q} \equiv 3\) mod 4 and \(\sqrt{n} > \sqrt{q} - 3\) as \(\sqrt{n} < \sqrt{q}\). Consequently, either \(\sqrt{n} = \sqrt{q} - 1\) or \(\sqrt{n} = \sqrt{q} - 2\). As \(x_1 = x_8 = 1\) and \(x_2 = 0\), then \(H\) must induce an involutory homology on \(\text{Fix}(\gamma)\). Thus, \(\sqrt{n}\) must be odd. Therefore, the case \(\sqrt{n} = \sqrt{q} - 1\) is ruled out, as \(q\) is odd. Hence, \(\sqrt{n} = \sqrt{q} - 2\). Now, by
substituting this value in (66) and bearing in mind that $\sqrt{q} \equiv 3 \mod 4$, we obtain an equality. Then $S_2 = S_4 = 0$, since $\sqrt{n} + 1 \geq (\sqrt{q} \pm 1)x_7 + \sqrt{q} + 1 + S_2$ by Table I. It follows that $|\text{Fix}(\gamma) \cap l| = 1 + \frac{\sqrt{q} - 1}{2}$ by Table II, since $x_1 = x_8 = 1$ and $S_4 = 0$ by the previous argument, since $q \equiv 9 \mod 16$ for $x_7 > 0$ by Lemma 4.13(2), and since $p^e \equiv 3 \mod 4$ for $x_9 > 0$ by Lemma 4.13(4) (note that the collineation $\gamma$ does not fix points on the $G$-orbits on $l$ of type (7) for $q \equiv 9 \mod 16$ or (9) for $p^e \equiv 3 \mod 4$ by Table II). That is $\sqrt{n} = \frac{\sqrt{q} - 1}{2}$, as $\gamma$ induces a Baer collineation on $\text{Fix}(\sigma)$. This is a contradiction, since $\sqrt{n} = \sqrt{q} - 2$.

Lemma 4.17. The group $H_j$ induces either the identity or a Baer involution on $\text{Fix}(\gamma)$ for each $j = 1, 2$.

Proof. Assume that $H_j$ induces a perspectivity on $\text{Fix}(\gamma)$. We treat the cases $q \equiv 1 \mod 16$ and $q \equiv 9 \mod 16$ at the same time, bearing in mind that $H_j = H$ when the latter occurs. If the axes of the perspectivities induced on $\text{Fix}(\gamma)$ by $H_j$ are distinct from $\text{Fix}(\gamma) \cap l$ for each $j = 1, 2$, then

$$\sqrt{n} + 1 = x_7 + x_1$$  
$$\sqrt{n} + 1 \geq (\sqrt{q} \pm 1)x_7 + x_1$$  

by Lemma 4.14 and Table I, respectively, since $x_2 = x_8 = 0$ by Lemma 4.16. In particular, either $x_1 = 1$ or $x_1 = 2$, since $x_1 + x_2 + x_8 = 1$ or 2 by Lemma 4.15 and being $x_2 = x_8 = 0$.

Assume that $x_1 = 1$. Then $x_7 = \sqrt{n}$ and hence $(\sqrt{q} \pm 1)\sqrt{n} + 1 \leq \sqrt{n} + 1$ by (67) and (68). By calculations of the previous inequality, we have $\sqrt{n} \geq \sqrt{q} \pm 1$. It follows that $\sqrt{q} \equiv 3 \mod 4$ and $\sqrt{n} = \sqrt{q} - 1$, as $\sqrt{n} < \sqrt{q}$. Then $q \geq 49$ and hence $\sqrt{n} \geq 6$, as $\sqrt{q} \equiv 3 \mod 4$ and $\sqrt{q} > 3$. Moreover, $\sqrt{n} \equiv 2 \mod 4$ and $\sqrt{n} > 2$, as $\sqrt{q} \equiv 3 \mod 4$ and $\sqrt{q} > 3$, respectively. Nevertheless, this contradicts [16], Theorem 13.18, since $H_j$ induces a non trivial involutory collineation on $\text{Fix}(\gamma)$.

Assume that $x_1 = 2$. Then $x_7 = \sqrt{n} - 1$ by (67). Now, by substituting this value in (68), we obtain $(\sqrt{q} \pm 1)(\sqrt{n} - 1) + 2 \leq \sqrt{n} + 1$ and so

$$(\sqrt{q} \pm 1)(\sqrt{n} - 1) \leq \sqrt{n} - 1.$$  

This yields $\sqrt{n} + 1 \geq \sqrt{q} \pm 1$ and hence $\sqrt{n} \geq \sqrt{q} \pm 1 - 1$. Then $\sqrt{q} \equiv 3 \mod 4$ and $\sqrt{n} \geq \sqrt{q} - 2$, as $\sqrt{n} < \sqrt{q}$. Therefore, either $\sqrt{n} = \sqrt{q} - 1$ or $\sqrt{n} = \sqrt{q} - 2$. As $x_1 = 2$ and $x_2 = x_8 = 0$, then $H_j$ must induces a homology on $\text{Fix}(\gamma)$. Thus, $\sqrt{n}$ must be odd. Then we actually have $\sqrt{n} = \sqrt{q} - 2$, as $q$ is odd. Now, by substituting this value in (69) and bearing in mind that $\sqrt{q} \equiv 3 \mod 4$, we have $(\sqrt{q} - 1)(\sqrt{q} - 2) + 2 \leq (\sqrt{q} - 2)^2 + 1$. It is a straightforward computation to see that the previous inequality is impossible.
Assume that \( H_j \) induces a perspectivity on \( \text{Fix}(\gamma) \) of axis \( \text{Fix}(\gamma) \cap l \). Then \( x_1 = \sqrt[n]{n} + 1 \) again by Lemma 4.15. Thus \( \text{Fix}(G) \cap l = \text{Fix}(\gamma) \cap l \). Now, dualizing the above argument, we obtain that \( |\text{Fix}(H_j) \cap [P]| \geq 3 \) for each point \( P \in \text{Fix}(\gamma) \cap l \) and for each \( j = 1, 2 \). Nevertheless, this is impossible, since the \( H_j \) induces a perspectivity on \( \text{Fix}(\gamma) \) of axis \( \text{Fix}(\gamma) \cap l \). At this point, the assertion follows by Lemma 4.15.

**Proposition 4.18.** The group \( H_j \) induces a Baer involution on \( \text{Fix}(\gamma) \) for each \( j = 1, 2 \).

**Proof.** The group \( H_j \) induces either the identity or a Baer involution on \( \text{Fix}(\gamma) \) for each \( j = 1, 2 \) by Lemma 4.17. Assume the former occurs. Then \( G \) fixes a subplane of order \( \sqrt[n]{n} \) and \( S_4 = x_2 = x_7 = x_8 = 0 \) by Lemma 4.15. That is \( \text{Fix}(G) = \text{Fix}(\gamma) \). Then

\[
\sqrt[n]{n} = \frac{q - 1}{2} x_3 + \frac{q - 1}{p^e - 1} x_9 + S_{2,4}
\]

(70)

and

\[
n = \sqrt[n]{n} + \frac{q(q - 1)}{2} x_3 + \frac{p^e(q^2 - 1)}{2(p^e - 1)} x_9 + \frac{q + 1}{2} S_1
\]

(71)

by Table I. Assume that \( x_3 > 0 \). Then \( x_3 = 1 \) by Lemma 3.5(2). Hence, let \( P \in l \) such that \( G_P \cong D_{q+1} \). Then \( G_P \) fixes exactly one point \( P^G \), since it is maximal in \( G \). Thus, \( \text{Fix}(G_P) \cap l = \{P\} \cup \text{Fix}(G_P) \cap l \). Furthermore, \( G_P \) is planar, since \( \text{Fix}(G) \subset \text{Fix}(G_P) \) and \( G \) is planar. Actually \( \text{Fix}(G) \subset \text{Fix}(G_P) \subset \text{Fix}(\sigma) \). So, we arrive at a contradiction by [16, Theorem 3.7], since \( o(\text{Fix}(G)) = \sqrt[n]{n} \) and \( o(\text{Fix}(\sigma)) = \sqrt[n]{n} \). Then \( x_3 = 0 \) and hence \( x_9 + S_{2,4} > 0 \) by (70).

Let \( \rho_t \), where \( t = 1, 2 \) be the representatives of the two conjugate classes of \( p \)-elements in \( G \). Since \( x_9 + S_{2,4} > 0 \), then \( \text{Fix}(G) \subset \text{Fix}(\langle \rho_t \rangle) \) for each \( t = 1, 2 \). It follows that the group \( \langle \rho_t \rangle \) fixes a Baer subplane of \( \Pi \) for each \( t = 1, 2 \) by [16, Theorem 3.7], since \( o(\text{Fix}(G)) = \sqrt[n]{n} \). Clearly, \( \sigma \) inverts \( \rho_t \) for each \( t = 1, 2 \). Furthermore, \( \text{Fix}(G) \subset \text{Fix}(\langle \rho_t, \sigma \rangle) \subset \text{Fix}(\langle \rho_t \rangle) \), since \( x_9 + S_{2,4} > 0 \) (see Table II). This still contradicts [16, Theorem 3.7], since \( o(\text{Fix}(G)) = \sqrt[n]{n} \) and \( o(\text{Fix}(\rho_t)) = \sqrt[n]{n} \) for each \( t = 1, 2 \). Thus, the group \( H_j \) induces a Baer involution on \( \text{Fix}(\gamma) \) for each \( j = 1, 2 \).

**Lemma 4.19.** The group \( G \) fixes a subplane of \( \Pi \) of order \( \sqrt[n]{n} \) pointwise.

**Proof.** By Proposition 4.18 and by Lemma 4.15, we have \( \sqrt[n]{n} + 1 = x_1 + x_2 + x_8 \). Recall that \( x_2 \leq 1 \) by Lemma 3.5(1) and \( x_8 \leq 1 \) by Lemma 4.13(3). Since \( \sqrt[n]{n} \geq 2 \), then \( x_1 \geq 1 \). Assume that \( x_1 \leq 2 \). Hence either \( x_1 = 1 \) or \( x_1 = 2 \). Then we have the following admissible triples \( (x_1, x_2, x_8) = (1, 1, 1), (2, 0, 1), \)
(2, 1, 0), (2, 1, 1), since \( \sqrt[3]{n} \geq 2 \). Furthermore, \( \sqrt[3]{n} + 1 = x_1 + 3x_2 + x_7 + 2x_8 \) by Lemma 4.14. Thus,

\[
(x_1 + x_2 + x_8 - 1)^2 = x_1 + 3x_2 + x_7 + 2x_8 - 1 .
\]

(72)

By substituting the values found of \((x_1, x_2, x_8)\) in (72), we see that \((x_1, x_2, x_8) = (1, 1, 1)\) is ruled out.

If \((x_1, x_2, x_8) = (2, 0, 1)\), then \( \sqrt[3]{n} = 2 \) and \( x_7 = 1 \). So, \( \sqrt[3]{n} + 1 \geq 2 + \frac{\sqrt[3]{7} + 1}{2} + \sqrt[3]{q} \) by Table I. It follows that \( \sqrt[3]{q} 2 \leq 15 \) as \( \sqrt[3]{n} = 16 \). This yields \( 3\sqrt[3]{q} \leq 31 \). That is \( \sqrt[3]{q} = 5 \), 7 or 9, as \( \sqrt[3]{q} > 3 \). Therefore, \( q = 5^2 \), 72 or 92. Nevertheless, only the case \( q = 5^2 \) is admissible, since it must be \( q \equiv 3 \) mod 16 by Lemma 4.13(2), being \( x_7 = 1 \). If \( x_3 > 0 \), then \( n + 1 \geq \frac{q(q-1)}{2} \) by Table I. Nevertheless, this is impossible, since \( n = 2^8 \), while \( q = 5^2 \). Then the length of any admissible non trivial \( G \)-orbit on \( l \) is divisible by \( \frac{q + 1}{2} \) by Table I, since \( x_4 = x_5 = x_6 = 0 \) by Lemma 4.13(1). Thus, \( \frac{q + 1}{2} \) must divide \( |l - \text{Fix}(G)| \). That is \( \frac{q + 1}{2} | n + 1 - x_1 \), being \( |l - \text{Fix}(G)| = n + 1 - x_1 \). Hence, we arrive at a contradiction, since \( \frac{q + 1}{2} = 13 \), as \( q = 5^2 \), while \( n + 1 - x_1 = 2^8 - 1 \), as \( n = 2^8 \) and \( x_1 = 2 \).

If \((x_1, x_2, x_8) = (2, 1, 0)\), then \( \sqrt[3]{n} = 2 \) and \( x_7 = 1 \). As a consequence, \( \frac{q + 1}{2} + \sqrt[3]{q} \leq 15 \). So, we again obtain a contradiction, since \( q \equiv 3 \) mod 16 and \( q > 9 \).

Finally, assume that \((x_1, x_2, x_8) = (2, 1, 1)\). Then \( \sqrt[3]{n} = 3 \) and \( x_7 = 3 \). Then \( \sqrt[3]{n} + 1 \geq 2 + \frac{q + 1}{2} + 3\sqrt[3]{\frac{q + 1}{2}} + \sqrt[3]{2}, \) which is a contradiction. Then \( x_1 \geq 3 \) and hence \( G \) fixes at least 3 points on \( l \). Let \( P_1, P_2, P_3 \) three distinct points on \( l \) which are fixed by \( G \). Now, repeating all the arguments with \([P_i]\) in the role of \( l \), for each \( i = 1, 2, 3 \), we see that \( G \) fixes at least three lines at least 3 lines, including \( l \), on \([P_i]\) for each \( i = 1, 2, 3 \). Thus, \( G \) is planar on \( \Pi \). In particular, \( \text{Fix}(G) \) is a subplane of \( \text{Fix}(H) \) of order \( x_1 - 1 \).

Assume that \( \text{Fix}(G) \subset \text{Fix}(H) \). This yields \( x_2 + x_8 > 0 \), since \( \sqrt[3]{n} + 1 = x_1 + x_2 + x_8 \). Then, by [16, Theorem 3.7], either

\[
(x_1 - 1)^2 = x_1 + x_2 + x_8 - 1, \text{ or} \]

(73)

\[
(x_1 - 1)^2 + (x_1 - 1) \leq x_1 + x_2 + x_8 - 1 ,
\]

(74)

since \( o(\text{Fix}(G)) = x_1 - 1 \) and \( o(\text{Fix}(H)) = x_1 + x_2 + x_8 - 1 \). If \( x_2 = 1 \), then either \((x_1 - 1)^2 = x_1 + x_8 \) or \((x_1 - 1)^2 + (x_1 - 1) \leq x_1 + x_8 \). Note that \( x_8 \leq 1 \) by Lemma 4.13(2). Assume that \( x_8 = 0 \). Then either \((x_1 - 1)^2 = x_1 \) or \((x_1 - 1)^2 \leq 1 \).

This yields a contradiction in any case, as \( x_1 \geq 3 \). Then \( x_8 = 1 \). Thus, either \((x_1 - 1)^2 = x_1 + 1 \) or \((x_1 - 1)^2 \leq 2 \). Actually, only the former occurs and hence \( x_1 = 3 \). Therefore, \( o(\text{Fix}(H)) = 4 \). Then \( \sqrt[3]{n} = 4 \) and hence \( n = 4^8 \). In particular, \( x_3 = 0 \) by Lemma 3.6(1), as \( x_2 = 1 \). Thus the length of any admissible non trivial \( G \)-orbit on \( l \) is divisible by \( \frac{q + 1}{2} \) (see Table I). Therefore,
\[ \frac{n+1}{2} \mid n + 1 - x_1, \text{ as } |l - \text{Fix}(G)| = n - x_1. \] That is \[ \frac{n+1}{2} \mid 4^8 - 2, \text{ as } n = 4^8 \]
and \( x_1 = 3 \). Now, it is a plain that \( 4^8 - 2 \) has no divisors of the form \( \frac{n+1}{2} \) with \( q \) an even power of an odd prime. Hence, \( x_2 = 0 \). Then \( x_8 = 1 \), as \( x_8 \leq 1 \) and \( x_2 + x_8 > 0 \). Now, by substituting the couple \((x_2, x_8) = (0,1) \) in \((73)\) and \((74)\), we have either \((x_1 - 1)^2 = x_1 \) or \((x_1 - 1)^2 + (x_1 - 1) \leq x_1 \). While the first equation has no solutions, the second one yields \( x_1 = 4 \). Therefore, \( x_2 = 2 \). Actually, \( x_1 = 2 \), as \( x_1 \geq 2 \), being \( \sqrt[n]{q} = x_1 \). Therefore, \( n = 2^8 \). Now, recall that \( x_3 \leq 1 \) by Lemma 3.5(2). If \( x_3 = 0 \), then \( \frac{n+1}{2} \mid n + 1 - x_1 \) arguing as above. Then \( \frac{n+1}{2} \mid 2^8 - 1 \), as \( n = 2^8 \) and \( x_1 = 2 \). Easy computations show that \( q = 13^2 \), since \( q \) is an even power of a prime and \( q \geq 11 \). Recall that \( x_8 = 1 \). Since \( \gamma \) fixes a subplane of \( \Pi \) of order \( \sqrt[n]{q} \), since \( 13^2 \equiv 9 \mod 16 \) and by Table I, we see that \( \sqrt[n]{q} + 1 \geq 7x_8 = 7 \). Nevertheless, this is a contradiction, since \( \sqrt[n]{q} = 4 \). So, \( x_3 = 1 \). Then the length of any admissible non trivial \( G \)-orbit on \( l \) is divisible by \( \frac{n+1}{2} \), unless this one is of type (3) by Table I, since \( x_4 = x_5 = x_6 = 0 \) by Lemma 4.13(1). Thus, \( \frac{n+1}{2} \mid n + 1 - x_1 - \frac{q(q-1)}{2} \). This yields \( \frac{n+1}{2} \mid n + 1 - x_1 \) and hence \( \frac{n+1}{2} \mid 2^8 - 2 \). Since \( q \equiv 1 \mod 8 \), then \( \frac{n+1}{2} \) is odd. Consequently, \( \frac{n+1}{2} \mid 2^8 - 1 \). Actually, \( \frac{n+1}{2} = 2^7 - 1 \), as \( 2^7 - 1 \) is prime and \( \frac{n+1}{2} > 1 \). So, \( q = 253 \). So, we arrive at a contradiction, since \( q \) must be a square as \( x_8 = 1 \). Thus, \( \text{Fix}(G) = \text{Fix}(H) \). Therefore, we have proved the assertion. \( \Box \)

**Lemma 4.20.** If \( q > 9 \), then the group \( G \) does not fix lines of \( \Pi \).

**Proof.** Assume that \( G \) fixes a subplane of \( \Pi \) of order \( \sqrt[n]{q} \) pointwise. Assume that \( x_i > 0 \) for either \( i = 2 \) or \( 3 \). Then \( x_i = 1 \) for either \( i = 2 \) or \( 3 \) by Lemma 3.6(1).

Hence, let \( P \in l \) such that \( G_P \cong D_{q^2} \). Then \( G_P \) fixes exactly one point \( P^G \), since it is maximal in \( G \). Thus, \( \text{Fix}(G_P) \cap l = \{ P \} \cup \text{Fix}(G_P) \cap l \). Furthermore, \( G_P \) is planar, since \( \text{Fix}(G) \subseteq \text{Fix}(G_P) \) and \( G \) is planar. In particular, \( o(\text{Fix}(G_P)) = \sqrt[n]{q} + 1 \). Moreover, \( \sqrt[n]{q} \leq \sqrt[n]{q} + 1 \) by [16, Theorem 3.7], since \( \text{Fix}(G) \subseteq \text{Fix}(G_P) \).

Nevertheless, this is a contradiction. Therefore, \( x_2 = x_3 = 0 \). In addition, \( x_4 = x_5 = x_6 = 0 \) by Lemma 4.13(1). So, we have the following system of Diophantine equations:

\[
\sqrt[n]{q} = \sqrt[n]{x_7 + 2x_8} \tag{75}
\]

\[
\sqrt[n]{q} = \sqrt[n]{\frac{\sqrt[n]{q} + 1}{2} x_8 + S_4} \tag{76}
\]

By subtracting (76) from (75), we obtain

\[
x_7 = \left[ \frac{\sqrt[n]{q} + 1}{2} - 2 \right] x_8 + S_4 \tag{77}
\]

Let \( \rho_t \) be the representative of the two conjugate classes of \( p \)-elements in \( G \) for \( t = 1, 2 \). We may assume that \( \rho_t \), for each \( t = 1, 2 \), lie in the Sylow
$p$-subgroup $S$ of $G$ normalized by $\sigma$. Then $\rho_t$ is planar, since $\text{Fix}(G) \subset \text{Fix}(\rho_t)$ for each $t = 1, 2$. In particular, by Table II,

$$o(\text{Fix}(\rho_t)) \geq \sqrt[n]{p} + x_2 \sqrt{q} + x_8 \sqrt[q]{q} + \frac{1}{2} S_1.$$  

(78)

Assume that $x_8 > 0$. So $q$ is a square. Actually, $x_8 = 1$ by Lemma 4.13(3). Then $o(\text{Fix}(\rho_t)) > \sqrt[n]{p}$ and hence $o(\text{Fix}(\rho_t)) \geq \sqrt[n]{p}$ by [16, Theorem 3.7], since $o(\text{Fix}(\rho)) = \sqrt[n]{p}$ and $\text{Fix}(G) \subset \text{Fix}(\rho_t)$. If $o(\text{Fix}(\rho)) = \sqrt[n]{p}$, it follows that $\sqrt[n]{p} > \sqrt{q}$, as $o(\text{Fix}(\rho_t)) \geq \sqrt[n]{p} + x_2 \sqrt{q} + x_8 \sqrt[q]{q}$ with $\sqrt[n]{p} \geq 2$ and $x_8 = 1$. Nevertheless, this contradicts the assumption $n < q$. Thus, $o(\text{Fix}(\rho_t)) > \sqrt[n]{p}$. Nevertheless, $o(\text{Fix}(\rho_t)) \leq \sqrt[n]{p}$ by [16, Theorem 3.7]. Note that $x_7 \geq \sqrt[n]{q} - 1 = 2$ by (77), since $x_8 = 1$. By substituting $x_7 \geq \sqrt[n]{q} - 1 = 2$ in (78), we obtain

$$o(\text{Fix}(\rho_t)) \geq \sqrt[n]{p} + 2\sqrt[p]{(\sqrt[n]{q} - 1)^2} + \sqrt{q} + \frac{1}{2} S_1.$$  

(79)

Then $2 + 2\sqrt[p]{(\sqrt[n]{q} - 1)^2} + \sqrt{q} + \frac{1}{2} S_1 \leq \sqrt[n]{p}$, since $o(\text{Fix}(\rho_t)) \leq \sqrt[n]{p}$ and $\sqrt[n]{p} \geq 2$. By elementary calculations of the previous inequality, we obtain $\sqrt[n]{p} \geq q + \sqrt[q]{q} - 3\sqrt[q]{q} + 2 + \frac{1}{2} S_1$. Assume that $\sqrt[q]{q} \equiv 3 \mod 4$. Hence, $\sqrt[n]{p} \geq q - 4\sqrt[q]{q} + 2 + \frac{1}{2} S_1$. That is, $\sqrt[n]{p} \geq (\sqrt[q]{q} - 2)^2$. So, $\sqrt[n]{p} \geq \sqrt[q]{q} - 2$. Then $\sqrt[n]{p} = \sqrt[q]{q} - 1$, since $\sqrt[n]{p} < \sqrt[q]{q}$, as $n < q$ by our assumption. Note that $\sqrt[n]{p} \equiv 2 \mod 4$ and $\sqrt[n]{p} \geq 2$, since $\sqrt[q]{q} \equiv 3 \mod 4$ and $q > 9$. Nevertheless, this yields a contradiction by [16, Theorem 13.18], since $H_j$ acts non trivially on $\text{Fix}(\gamma)$ by Proposition 4.18 and since $o(\text{Fix}(\gamma)) = \sqrt[n]{p}$. Hence, $\sqrt[q]{q} \equiv 1 \mod 4$. Then $\sqrt[n]{p} \geq (\sqrt[q]{q} - 1)^2 + \frac{1}{2} S_1$ by (79) as $\sqrt[n]{p} \geq 2$. Actually, $\sqrt[n]{p} = (\sqrt[q]{q} - 1)^2$ and $S_1 = 0$, since $\sqrt[n]{p} < \sqrt[q]{q}$ being $n < q^2$ by our assumption, and being $n$ a fourth power and $q$ as square. That is $\sqrt[n]{p} = \sqrt[q]{q} - 1$. Now note that $S_4 = 0$, since $S_1 \geq S_4 \geq 0$ and since $S_1 = 0$. Then $x_7 = \sqrt[q]{q} - 1$ by (77), since $\sqrt[n]{p} \equiv 1 \mod 4$ and $x_8 = 1$. Now, by substituting $x_7 = \sqrt[q]{q} - 1$, $x_8 = 1$ and $\sqrt[n]{p} = \sqrt[q]{q} - 1$ in (75), we obtain $\sqrt[q]{q} - 1 = \sqrt[n]{p} + \frac{1}{2} S_1$ by (75), since also $x_8 = 0$. So $S_4 > 0$. In particular, $\sqrt[n]{p} = \sqrt[q]{q} + S_4$ by (76).

Finally, let us consider the subgroup $W$ of $G$, where $W = S(\langle \gamma \rangle)$ and $S$ is the Sylow $p$-subgroup of $G$ normalized by $\sigma$ and hence by $\gamma$. Then $W$ fixes at least a point $Q$ on $l$ since $S_4 > 0$. Hence, let $G^Q$ be an orbit of type (10). Clearly, $G^Q \cong F_{j_0},Z_{d_0}$ where $d_0 \equiv 0 \mod 4$. In particular, $|\text{Fix}_{G^Q}(W)| = \frac{q - 1}{2d} - 1$ by (1) of Proposition 2.5. Thus, the number of points coming out from $G$-orbits on $l$ of type (10) which are fixed by $W$ are exactly $\sum d_0 = 0 \mod 4 \frac{q - 1}{2d}$. These turn out to be $\frac{1}{2} S_4$ as $S_4 = \sum d_0 = 0 \mod 4 \frac{q - 1}{2d}$. Then $|\text{Fix}(W) \cap l| = \sqrt[n]{p} + 1 + \frac{1}{2} S_4$, since...
\[|\text{Fix}(G) \cap l| = \sqrt[3]{n} + 1 \text{ by Lemma 4.19.} \] Furthermore, \( W \) is planar, since \( \text{Fix}(G) \subset \text{Fix}(W) \). On the other hand, \( \text{Fix}(W) \subset \text{Fix}(\gamma) \), since \( \text{o}((\text{Fix}(W)) = \sqrt[3]{n} + \frac{1}{3} S_4 \) and \( \text{o}((\text{Fix}(\gamma)) = \sqrt[3]{n} + S_4 \) being \( S_4 > 0 \). Therefore, \( \text{Fix}(G) \subset \text{Fix}(W) \subset \text{Fix}(\gamma) \), where \( \text{o}((\text{Fix}(G)) = \sqrt[3]{n} \) and \( \text{o}((\text{Fix}(\gamma)) = \sqrt[3]{n} \). Nevertheless, this contradicts [16, Theorem 3.7]. Thus, \( G \) does not fix lines of \( \Pi \).

**Proposition 4.21.** Let \( \Pi \) be a projective plane of order \( n \) admitting a collineation group \( G \cong \text{PSL}(2, q) \). If \( q < n < q^2 \) and \( q \equiv 1 \mod 8 \), then \( G \) cannot fix lines of \( \Pi \).

**Proof.** Assume that \( G \) fixes a line \( l \) of \( \Pi \). Then \( q \leq 9 \) by Lemma 4.20. Actually, \( q = 9 \), since \( q \equiv 1 \mod 8 \). Then \( n = 16, 25, 36, 49, 64 \), since \( q < n < q^2 \), with \( n \) a square by Lemma 3.3. The case \( n = 36 \) and \( n = 49 \) are ruled out by Lemma 3.3. Thus, \( n = 16, 25 \) or 64.

Assume that \( n = 16 \). Let \( P^G \) be a non trivial orbit on \( l \). Then \( |P^G| \leq 17 \). Then \( G_P \) is isomorphic either to \( Z_9.Z_4 \) or to \( S_4 \) or to \( A_5 \). If \( G_P \cong Z_9.Z_4 \), then \( |P^G| = 10 \). In particular, \( G \) acts 2-transitively on \( P^G \), which contradicts [23, Theorem 1], since \( n = 16 \). If \( G_P \cong A_5 \), then \( |P^G| = 15 \) and hence \( |l - P^G| = 11 \). Let \( Q \in l - P^G \). Then \( |Q^G| \leq 11 \), since \( |l - P^G| = 11 \). Clearly, \( G_Q \not\cong Z_9.Z_4 \) by the previous argument. Furthermore, \( G_Q \not\cong S_4 \), otherwise \( |Q^G| = 15 \). Thus, \( G_Q \cong A_5 \). Therefore, \( |l - (P^G \cup Q^G)| = 5 \). Then \( G \) fixes \( l - (P^G \cup Q^G) \) pointwise, since the minimal primitive permutation representation degree of \( G \cong \text{PSL}(2, 9) \) is 6. So, any involution in \( G \) fixes at least 8 points on \( l \). Hence, we arrive at a contradiction, since each involution in \( G \) is a Baer collineation of \( \Pi \) by Lemma 3.3 and since \( n = 16 \). Thus, \( G_P \cong S_4 \). Then \( |P^G| = 15 \) and hence \( |l - P^G| = 2 \). So, \( G \) fixes \( l - P^G \) pointwise. Set \( \{X, Y\} = l - P^G \). It follows that \( G_r \cong S_4 \) for some line \( r \in [X] \) and \( G_u \cong S_4 \) for some line \( u \in [Y] \) by dual of the above argument, since \( G \) acts on \([X]\) and on \([Y]\) fixing two lines through each of them (clearly, \( l \) is one of them). Therefore, \( G \) fixes a triangle \( \Delta = \{X, Y, Z\} \). Let \( \rho_1 \) and \( \rho_2 \) are the representatives of the 3-elements in \( G \). We may assume that the lie in the Sylow 3-subgroup of \( G \) normalized by \( \sigma \). As a consequence, \( \sigma \) inverts each of them. Since \( G_P \cong S_4 \), then one of them fixes exactly 3 points on \( P^G \), 3 on \( XZ \) and 3 on \( YZ \) by Table IV* of [24], since \( q = 9 \).

We may assume that \( \rho_1 \) does it. Hence, \( \rho_1 \) fixes a Baer subplane of \( \Pi \), since \( \rho_1 \) fixes exactly 3 points on \( P^G \) and the points \( X \) and \( Y \). So, \( \text{o}((\text{Fix}(\rho_1)) = 4 \). The involution \( \sigma \) acts on \( \text{Fix}(\rho_1) \), since it inverts \( \rho_1 \). Note that \( \langle \rho_1, \sigma \rangle \) does not fix point on \( P^G \) by Table IV* of [24], since \( G_P \cong S_4 \) and \( q = 9 \). Therefore, \( \sigma \) fixes exactly 2 points on \( \text{Fix}(\rho_1) \cap l \), namely \( X \) and \( Y \). So, \( \sigma \) induces a homology on \( \text{Fix}(\rho_1) \). Nevertheless, this is impossible, since \( \text{o}((\text{Fix}(\rho_1)) = 4 \).

Assume that \( n = 25 \) or 64. Assume also that \( \text{Fix}(T_j) \cap l = \text{Fix}(\sigma) \cap l \) for some \( j = 1 \) or 2. Then \( \text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l \) by Table III* of [24], since
$q = 9$. Therefore, for each point $A \in l - \text{Fix}(G)$, the group $G_A$ has odd order. Then $G_A \cong E_9$ by Table III* of [24], since $G \cong \text{PSL}(2,9)$. Hence $|A^G| = 40$ for each point $A \in l - \text{Fix}(G)$. Then $40 | |l - \text{Fix}(G)|$. That is $40 | n - \sqrt{n}$, since $\text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l$ and $|\text{Fix}(\sigma) \cap l| = \sqrt{n} + 1$. This is a contradiction, since $n = 25$ or 64. Thus, $|\text{Fix}(T_j) \cap l| = 2$ or 1 for each $j = 1, 2$, according to whether $n = 25$ or 64, respectively. Therefore, $T_j$ induces a non trivial perspectivity $\beta_j$ on $\text{Fix}(\sigma)$ for each $j = 1, 2$. Clearly, $T_1$ and $T_2$ are subgroups of $C_G(\sigma) \cong D_8$. Furthermore, $C_G(\sigma)$ acts on $\text{Fix}(\sigma)$ inducing a subgroup $\bar{C}$ isomorphic either to $E_4$ or to $Z_2$. In each case $\beta_1 \in \bar{C}$ and $\beta_1 \neq 1$, since $\beta_1$ is a non trivial perspectivity of $\text{Fix}(\sigma)$. Then $\bar{C}$ fixes $C_{\beta_1}$, since $\beta_1$ is central in $\bar{C}$. So, $C_G(\sigma)$ fixes $C_{\beta_1}$. That is $C_G(\sigma) \leq G_{C_{\beta_1}}$. Let $U \leq N_{C_{\beta_1}}(T_1)$ such that $U \cong A_4$. Then $U$ fixes $\text{Fix}(T_1) \cap l$ pointwise, since $T_1 \triangle U$, $U \cong A_4$ and $|\text{Fix}(T_1) \cap l| = 1$ or 2. Then $U \leq G_{C_{\beta_1}}$ and therefore $\langle C_G(\sigma), U \rangle \leq G_{C_{\beta_1}}$. Note that $\langle C_G(\sigma), U \rangle \cong S_4$, since $\langle C_G(\sigma), U \rangle \leq N_{G(T_1)}$ as $G \cong \text{PSL}(2,9)$. So, either $G_{C_{\beta_1}} \cong S_4$ or $G_{C_{\beta_1}} = G$ by Table III* of [24] as $q = 9$. Actually, the case $G_{C_{\beta_1}} \cong S_4$ cannot occur, since $|\text{Fix}(T_j) \cap l| = 3$ with $C_{C_{\beta_1}}$, while we proved that $|\text{Fix}(T_j) \cap l| = 2$ or 1 for each $j = 1, 2$. As a consequence, $G_{C_{\beta_1}} = G$. This implies that $G$ acts on $[C_{\beta_1}]$ and $\text{Fix}(T_1) \cap [C_{\beta_1}] = \text{Fix}(\sigma) \cap [C_{\beta_1}]$, which is a contradiction by dual of the above argument. Thus, we have proved the assertion. \hfill $\Box$

**Theorem 4.22.** Let $\Pi$ be a projective plane of order $n$ admitting $G \cong \text{PSL}(2, q)$ as a collineation group. If $n \leq q^2$, $q \equiv 1 \mod 8$, then one of the following occurs:

1. $n < q$, $\Pi \cong \text{PG}(2, 4)$ and $G \cong \text{PSL}(2, 9);$  
2. $n = q$, $\Pi \cong \text{PG}(2, q)$ and $G$ is strongly irreducible on $\Pi$;  
3. $q < n < q^2$, one of the following occurs:  
   a. $G$ is strongly irreducible on $\Pi$;  
   b. $G \cong \text{PSL}(2, 9)$ fixes a proper subplane $\Pi_0 \cong \text{PG}(2, 4)$ of $\Pi$;  
4. $n = q^2$ and one of the following occurs:  
   a. $G$ is strongly irreducible on $\Pi$;  
   b. $n = 81$ and $G \cong \text{PSL}(2, 9)$ fixes a point and line of $\Pi$;  
   c. $G$ fixes a subplane $\Pi_0$ of $\Pi$. Furthermore, either $\Pi_0 \cong \text{PG}(2, q)$ is a Baer subplane of $\Pi$, or $n = 81$, $\Pi_0$ is the Hughes plane of order 9 and $G \cong \text{PSL}(2, 9)$.

**Proof.** If $n < q$ or $n = q$, the assertions (1) and (2) easily follow by Theorems 2.1 and 2.2, respectively. If $q < n < q^2$, the group $G$ does not fix lines or points
of II by Proposition 4.21 and its dual. At this point, the assertion (3a) and (3b) easily follow by Lemma 3.1, since $q \equiv 1 \mod 8$. The assertions (4a) and (4b) follow by Theorem 2.3. Finally, the assertion (4c) follows by Theorem 2.4. □

Clearly, Theorem 1.1 easily follows from Theorem 4.22 when $q \equiv 1 \mod 8$.

5. The case $q \equiv 3 \mod 8$

In this section, we deal with the case $q \equiv 3 \mod 8$. Recall that there exists a unique conjugate class of involutions and one of Klein subgroups of $G$. Let $\sigma$ be an involution of $G$ and let $T$ be a representative of this class containing $\sigma$. As pointed out at the end of section 3, we have $C_G(\sigma) \cong D_{q+1}$ and $N_G(T) \cong A_4$.

We filter the list given in Lemma 3.4 with respect to the condition $q \equiv 3 \mod 8$. For each of the resulting groups, we find its corresponding index in $G$. Thus, we determine the length of the orbit $P^G$, with $P$ a point of $l$, when $G_P$ is isomorphic to one of these groups. Next, for each of these groups, using (1) of Proposition 2.5, we obtain the number of points fixed by $\sigma$, and by $T$, in the orbit $P^G$. All these informations are displayed in the following table.

<table>
<thead>
<tr>
<th>Type</th>
<th>$G_P$</th>
<th>$[G : G_P]$</th>
<th>$\text{Fix}_{P^G}(\sigma)$</th>
<th>$\text{Fix}_{P^G}(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$D_{q-1}$</td>
<td>$\frac{q(q+1)}{2}$</td>
<td>$\frac{q+1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$D_{q+1}$</td>
<td>$\frac{q(q-1)}{2}$</td>
<td>$\frac{q+3}{2}$</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>$A_4$</td>
<td>$\frac{q^2-1}{24}$</td>
<td>$\frac{q+1}{4}$</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>$A_5$</td>
<td>$\frac{q^4-1}{120}$</td>
<td>$\frac{q+1}{4}$</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>$F_{q,Z}$</td>
<td>$\frac{q^2-1}{2d}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Recall that the $G$-orbits of type (10) on $l$ cover exactly $S$ points of $l$, where $S = \sum_{j=1}^{\infty} \frac{q^2-1}{2d_j}$. Recall also that $S_1 = \sum_{j=1}^{\infty} \frac{q^2-1}{2d_j}$ and $S_2$, $S_2'$, $S_4$, $S_2, S_4$ (sum with the same summands $\frac{q^2-1}{d_j}$ but over $2 \mid d_j$, $2 \nmid d_j$, $4 \mid d_j$ and $d_j \equiv 2 \mod 4$, respectively). Note that $S_2 = S_{2,4} = S_4 = 0$, since $q \equiv 3 \mod 8$. Hence, $S_1 = S_2'$ and $S = \frac{q+1}{2} S_2'$.

Finally, if $G$ fixes a point $Q$ and acts on $[Q]$, we may focus on the $G$-orbits of lines in $[Q]$. So, following the notation introduced in section 4, we obtain a
Furthermore, either $2$ in section acts on of type (10) for each point which point we are focusing on. In particular, since we might have $G$-orbits of type $(10)^*$, it makes sense considering $S^* = \sum_{j=1}^{10} \frac{n^2-1}{24}$ and hence $S^*, S^*_2, S^*_4, S^*_2, S^*_4$ with the same meaning of $S_2, S_2, S_4, S_2, S_4$, respectively, but referred to lines instead of points. Clearly, $S^*_2 = S^*_2 = S^*_4 = S^*_4 = 0$, since $q \equiv 3 \mod 8$.

Note that $\sigma$ is a Baer collineation of $\Pi$ by Lemma 3.3. Set $C = C_G(\sigma)$. Then $C$ acts on $\operatorname{Fix}(\sigma)$ with kernel $K$. Hence, let $\bar{C} = C/K$. Clearly, $\langle \sigma \rangle \leq K \leq C$. Furthermore, either $K \leq Z_{4+1}$ or $K = C$, since $C \cong D_{q+1}$ and $q \equiv 3 \mod 8$.

We need to investigate the admissible structure of $K$ in order to show that $T$ induces a Baer collineation on $\operatorname{Fix}(\sigma)$.

**Lemma 5.1.** If $\operatorname{Fix}(T) \cap l = \operatorname{Fix}(\sigma) \cap l$, then $K = C$.

**Proof.** Assume that $\operatorname{Fix}(T) \cap l = \operatorname{Fix}(\sigma) \cap l$ and that $K \leq Z_{4+1}$. Then $\operatorname{Fix}(G) \cap l = \operatorname{Fix}(\sigma) \cap l$ by Table III, since $q > 9$. Set $l_0 = \operatorname{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \operatorname{Fix}(G) \cap l$. In particular, $\bar{C} \cong D_{4+1}$, where $k = |K|$, since $C \cong D_{q+1}$. On the other hand, $\bar{C}$ is the semidirect product of $\bar{C}(l_0, l_0)$ with $\bar{C}(Y, l_0)$ for some point $Y \in \operatorname{Fix}(\sigma) - l_0$ by [16, Theorem 4.25].

Assume that $\bar{C}(l_0, l_0) \neq \langle 1 \rangle$. If $\bar{C}(l_0, l_0) = \bar{C}$, then $\bar{C} = \bar{C}(V, l_0)$, for some point $V \in l_0$ by [16, Theorem 4.14], since $\bar{C} \cong D_{4+1}$ and $q \equiv 3 \mod 8$. Hence for each point $X \in l_0 - \{V\}$ and for each line $t \in [X] \cap \operatorname{Fix}(\sigma)$, we have that $\sigma \in G_t$ but $G_t$ does not contain Klein groups. Then, by dual of Table III, we have that $G_t \cong D_{q-1}$, since $G$ fixes $X$. Moreover, $K \leq G_t$. Thus, $K = \langle \sigma \rangle$, since $\langle \sigma \rangle \leq K \leq Z_{4+1}$. Therefore, $\bar{C} \cong D_{4+1}$ and hence $\frac{q+1}{2} | \sqrt{n}$ as $\bar{C} = \bar{C}(V, l_0)$. Actually, $\sqrt{n} = \frac{q+1}{2}$, since $\sqrt{n} < q$ by our assumptions. Then $\sqrt{n} \equiv 2 \mod 4$, since $q \equiv 3 \mod 8$. Hence, we arrive at a contradiction by Lemma 3.3. So, $\bar{C}(l_0, l_0) < \bar{C}$. Then $\bar{C}(l_0, l_0) \leq Z_{4+1}$, since $\bar{C}(l_0, l_0) \leq \bar{C}$, $\bar{C} \cong D_{4+1}$ and $q \equiv 3 \mod 8$. Actually, $\bar{C}(l_0, l_0) = \bar{C}(V, l_0) \cong Z_{4+1}$ and $\bar{C}(Y, l_0) \cong Z_2$ by [16, Theorems 4.14 and 4.25].

Let $u \in [V] \cap \operatorname{Fix}(\sigma) - \{l, VY\}$. Then $u$ is fixed by $K$ and by $\bar{C}(V, l_0)$. Therefore, $Z_{4+1} \leq G_u < C_G(\sigma)$. Thus either $G_u = C_G(\sigma)$ or $G_u = G$ by dual of Lemma 3.4, since $G$ fixes $l_0$ and $q > 9$. We again obtain a contradiction, since $G_u < C_G(\sigma)$. Hence, $\bar{C}(l_0, l_0) = \langle 1 \rangle$.

Assume that $\bar{C} = \bar{C}(Y, l_0)$ for some point $Y \in \operatorname{Fix}(\sigma) - l_0$. Let $Q \in l_0$ and let $m \in [Q] \cap \operatorname{Fix}(\sigma) - \{l, YQ\}$. Then $\sigma \in G_m$ but $G_m$ does not contain Klein groups. Then $G_m \cong D_{q+1}$ by dual of Table III. Thus, $x^*_2 \geq 1$. Furthermore,
$$x_3^* \geq 1,$$ since $$G_{YQ} = C$$ and $$C \cong D_{q+1}$$ as $$q \equiv 3 \mod 8.$$ So, $$x_2^* + x_3^* \geq 2.$$ This is a contradiction by dual of Lemma 3.6(1), as $$q > 9.$$ \hfill \Box

**Lemma 5.2.** Fix($$T$$) $$\cap$$ $$l$$ $$\subset$$ Fix($$\sigma$$) $$\cap$$ $$l$$.

**Proof.** Assume that Fix($$T$$) $$\cap$$ $$l$$ $$=$$ Fix($$\sigma$$) $$\cap$$ $$l$$. Then $$K = C$$ by Lemma 5.1. As a consequence Fix($$T$$) $$=$$ Fix($$\sigma$$). Let $$P$$ be any point of Fix($$\sigma$$) $$\cap$$ $$l$$ and let $$r$$ be any line of $$[P]$$ $$\setminus$$ $$\{l\}$$. Then $$C \leq G_r.$$ Since $$q > 9,$$ then $$C$$ is maximal in $$G$$ and hence either $$G_r = C$$ or $$G_r = G$$. If the former occurs, then $$|$$Fix($$\sigma$$)||$$ = 3$$ and $$|$$Fix($$\sigma$$)||$$ = \frac{q+3}{2}$$ by dual of table III. Hence $$|$$Fix($$\sigma$$)||$$ > |$$Fix($$\sigma$$)||$$ as $$q > 9.$$ A contradiction, since Fix($$T$$) $$=$$ Fix($$\sigma$$). Hence $$G_r = G$$ for any point $$P$$ of Fix($$\sigma$$) $$\cap$$ $$l$$ and for any line $$r$$ of $$[P]$$ $$\setminus$$ $$\{l\}$$. Thus Fix($$G$$) $$=$$ Fix($$\sigma$$), since Fix($$G$$) $$\cap$$ $$l$$ $$=$$ Fix($$\sigma$$) $$\cap$$ $$l$$ and Fix($$G$$) $$\subset$$ Fix($$\sigma$$). So, $$G$$ fixes a Baer subplane of II. Then $$G$$ is semiregular on $$l$$ $$\setminus$$ Fix($$G$$) and hence $$|G|$$ | $$n - \sqrt{n}$$, which is a contradiction. Thus, we have proved the assertion. \hfill \Box

The previous lemma rules the possibility for $$T$$ to induce either the identity or a perspectivity of axis Fix($$\sigma$$) $$\cap$$ $$l$$ on Fix($$\sigma$$). Hence, $$T$$ induces either a perspectivity of axis distinct from Fix($$\sigma$$) $$\cap$$ $$l$$ or a Baer involution on Fix($$\sigma$$). The following lemma shows that only the second case is admissible.

**Lemma 5.3.** The group $$T$$ induces a Baer collineation on Fix($$\sigma$$).

**Proof.** The group $$T$$ induces an involution $$\bar{\beta}$$ on Fix($$\sigma$$) by Lemma 5.2. Assume that $$\bar{\beta}$$ is an involutory ($$C_\beta, a_\beta$$)-perspectivity on Fix($$\sigma$$). Then $$C_\beta \in$$ $$l$$ and $$a_\beta \neq$$ $$l$$ again by Lemma 5.2, since $$G$$ fixes $$l$$. Then Fix($$T$$) $$\cap$$ $$l$$ $$=$$ 1 or 2, where \begin{align*}
\sqrt{n} + 1 & = x_1 + \frac{q+1}{2}x_2 + \frac{q+1}{4}x_4 + \frac{q+1}{4}x_5 \\
n + 1 & = x_1 + \frac{q(q+1)}{2}x_2 + \frac{q(q^2-1)}{24}x_4 + \frac{q(q^2-1)}{120}x_5 + S .
\end{align*}

Suppose that $$\bar{\beta}$$ is an involutory ($$C_\beta, a_\beta$$)-elation of Fix($$\sigma$$). Then Fix($$T$$) $$\cap$$ $$l$$ $$=$$ \{$$C_\beta$$\}, since $$C_\beta \in$$ $$l$$ and $$a_\beta \neq$$ $$l$$ by the above argument. Thus, $$x_1 + x_4 + x_5 = 1$$, since Fix($$T$$) $$\cap$$ $$l$$ $$=$$ $$x_1 + x_4 + x_5$$. Clearly, $$G$$ cannot fix $$C_\beta$$, otherwise we have a contradiction by dual of Lemma 5.2, since Fix($$T$$) $$\cap$$ $$[C_\beta]$$ $$=$$ Fix($$\sigma$$) $$\cap$$ $$[C_\beta]$$. Consequently $$x_1 = 0$$ and $$x_4 + x_5 = 1$$, since $$x_1 + x_4 + x_5 = 1$$. Then either $$x_4 = 1$$ and $$x_1 = x_5 = 0$$, or $$x_5 = 1$$ and $$x_1 = x_4 = 0$$.

Assume that $$x_4 = 1$$ and $$x_1 = x_5 = 0$$. Then $$x_2 = 0$$ by Lemma 3.6(3) being $$q \neq 17$$. Moreover, \begin{align*}
\sqrt{n} + 1 & = \frac{q+1}{4} \text{ by (80).}
\end{align*}

Hence, \begin{align*}
\sqrt{n} & = \frac{q-3}{4} \text{. Since } \frac{q-3}{4} < \frac{q}{4} \text{, then}
\end{align*}
Assume that \( x_5 = 1 \) and \( x_1 = x_4 = 0 \). Then \( q = 11, 19, 59 \) by Lemma 3.4, since \( q \equiv 3 \mod 8 \). If \( x_2 = 1 \), then \( \sqrt{n} + 1 = \frac{q+1}{4} + \frac{q+1}{2} \) (80). Therefore, \( \sqrt{n} = \frac{3q+1}{4} \). Actually, \( q \neq 59 \) by Lemma 3.6(6). If \( q = 19 \), then \( \sqrt{n} = 14 \). Nevertheless, this case cannot occur by Lemma 3.3. Hence, \( q = 11 \) and \( \sqrt{n} = 8 \). This contradicts the fact that \( n \geq 65 \), since \( n + 1 \geq \frac{q(q+1)}{2} \) by (81), as \( x_2 = 1 \). Thus, \( x_2 = 0 \). Then \( \sqrt{n} = \frac{q+3}{4} \), by (80), as \( x_1 = x_2 = x_4 = 0 \). This is impossible for \( q = 11 \) or 19 by the above argument. As a consequence, \( q = 59 \) and \( \sqrt{n} = 14 \). Nevertheless, this case cannot occur by Lemma 3.3.

Suppose that \( \tilde{\beta} \) is an involutory \((C_{\tilde{\beta}}, a_{\tilde{\beta}})\)-homology of \( \text{Fix}(\sigma) \). Again, \( C_{\tilde{\beta}} \in l \) and \( a_{\tilde{\beta}} \neq l \) by the above argument. Hence, \( \lfloor \text{Fix}(T) \cap l \rfloor = 2 \). Then \( x_1 + x_4 + x_5 = 2 \), since \( \lfloor \text{Fix}(T) \cap l \rfloor = x_1 + 3x_3 + x_4 + x_5 \) and \( x_2 = 0 \). It follows that, \( x_1 \leq 1 \). Therefore, \( x_4 + x_5 \geq 1 \), since \( G \) cannot fix \( C_{\tilde{\beta}} \) and since \( \text{Fix}(G) \subset \text{Fix}(T) \). Thus, either \( x_1 = x_4 = 1 \) and \( x_5 = 0 \), or \( x_1 = x_4 = 1 \) and \( x_5 = 0 \), or \( x_1 = 0 \) and \( x_4 + x_5 = 2 \).

Assume that \( x_1 = x_4 = 1 \) and \( x_5 = 0 \). Then \( x_2 = 0 \) by Lemma 3.6(3). So, \( \sqrt{n} = \frac{q+1}{4} \) by (80). Furthermore, \( q = 11 \) or 19 by Lemma 3.4(4), since \( q \equiv 3 \mod 8 \). We obtain a contradiction as above, since \( \frac{q+1}{4} > \sqrt{q} \) being \( \frac{q+1}{4} > \frac{q^2-3}{4} \).

Assume that \( x_1 = x_4 = 1 \) and \( x_5 = 0 \). Then \( q = 11, 19, 59 \) by Lemma 3.4(5), since \( q \equiv 3 \mod 8 \). If \( x_2 = 1 \), then \( \sqrt{n} = \frac{q+1}{4} + \frac{q+1}{2} \) by (80) and hence \( \sqrt{n} = \frac{3q+1}{4} \). Furthermore, \( q \neq 59 \) by Lemma 3.6(6). If \( q = 11 \), then \( \sqrt{n} = 9 \). In addition, \( S = 4 \) by (81), since \( x_1 = x_2 = x_5 = 1 \). This is impossible, since \( \frac{q+1}{2} = 6 \) must divide \( S \) by the definition of this one. Hence, \( q = 19 \) and \( \sqrt{n} = 15 \), which is a contradiction by Lemma 3.3. Thus, \( x_2 = 0 \) and \( \sqrt{n} = \frac{q+1}{4} \). If \( q = 59 \), then \( \sqrt{n} = 15 \) and we have a contradiction by the previous argument. Consequently, \( q = 11 \) or 19. Moreover, \( \sqrt{n} = \frac{q+1}{4} \) by (80), since \( x_1 = x_2 = x_4 = 0 \) and \( x_5 = 1 \). Nevertheless this cannot occur by the above argument, since \( q = 11 \) or 19.

Finally, assume that \( x_1 = 0 \) and \( x_4 + x_5 = 2 \). Then \( \sqrt{n} + 1 = \frac{q+1}{2} x_2 + \frac{q+1}{2} \) by (80). If \( x_2 \geq 1 \), then \( \sqrt{n} \geq q \). Nevertheless, this cannot occur by our assumption. So, \( x_2 = 0 \) and hence \( \sqrt{n} = \frac{q+1}{2} - 1 \). That is \( \sqrt{n} = \frac{q-1}{2} \). If \( x_4 > 0 \), then \( \left( \frac{q-1}{2} \right)^2 + 1 \geq \frac{q(q^2-1)}{24} \) by (81), where \( q = 11 \) or 19 by Lemma 3.4(4), as \( q \equiv 3 \mod 8 \). Easy computations yield a contradiction. Therefore, \( x_4 = 0 \) and \( x_5 = 2 \), since \( x_4 + x_5 = 2 \). Then \( n + 1 = \frac{q(q^2-1)}{60} + S \) by (81), where \( n = \left( \frac{q-1}{2} \right)^2 \). It follows that, \( S = \left( \frac{q-1}{2} \right)^2 + 1 - \frac{q(q^2-1)}{60} \). In particular, \( q = 11, 19, 59 \) by Lemma 3.4(5), since \( q \equiv 3 \mod 8 \). Easy computation yield \( S = 4 \) or \(-32 \) or \(-2580 \). So the cases \( q = 19 \) or 59 are ruled out, since \( S \geq 0 \) by the definition of this one. Hence \( q = 11 \) and \( S = 4 \). Nevertheless, this case cannot occur, since \( \frac{q+1}{2} = 6 \) must divide \( S \) again by the definition of this one. Thus, \( T \) induces a
Baer collineation on \( \text{Fix}(\sigma) \).

**Lemma 5.4.** For each point \( P \in l \), the group \( G_P \) cannot be isomorphic either to \( A_4 \) or to \( A_5 \).

**Proof.** The group \( T \) induces a Baer collineation on \( \text{Fix}(\sigma) \) by Lemma 5.3. So, \( |\text{Fix}(T) \cap l| = \sqrt[5]{n} + 1 \). By Table III, we have the following system of Diophantine equations:

\[
\sqrt[5]{n+1} = x_1 + 3x_3 + x_4 + x_5 \tag{82}
\]

\[
\sqrt[5]{n+1} = x_1 + \frac{q+1}{2} x_2 + \frac{q+3}{2} x_3 + \frac{q+1}{4} x_4 + \frac{q+1}{4} x_5 \tag{83}
\]

\[
n + 1 = x_1 + \frac{q(q+1)}{2} x_2 + \frac{q(q-1)}{2} x_3 + \frac{q(q^2-1)}{24} x_4 + \frac{q(q^2-1)}{120} x_5 + S. \tag{84}
\]

Assume that \( x_4 > 0 \). Then \( x_4 = 1 \) by Lemma 3.5(3). Consequently \( x_2 = x_3 = 0 \) by Lemma 3.6(3). Furthermore, \( q = 11 \) or 19 by Lemma 3.4, since \( q \equiv 3 \) mod 8. If \( q = 11 \), then either \( \sqrt[5]{n} = 2 \) or 3, since \( n < q^2 \) by our assumption.

On the other hand, \( n + 1 \geq \frac{q(q^2-1)}{24} \), since \( x_4 = 1 \). Thus the case \( \sqrt[5]{n} = 2 \) cannot occur. Hence, \( \sqrt[5]{n} = 3 \). Then \( x_1 + x_5 = 3 \) and \( x_1 + 3x_5 = 7 \) by (82) and (83), since \( x_2 = x_3 = 0 \) and \( x_4 = 1 \). Thus, \( x_1 = 1 \) and \( x_5 = 2 \). So, \( S = 59 \) by (84), which is a contradiction, since \( \frac{q+1}{2} = 6 \) must divide \( S \) by the definition of this one. As a consequence \( q = 19 \) and hence \( \sqrt[5]{n} = 3 \) or 4, since \( q < n < q^2 \). Nevertheless, this contradicts the fact that \( n + 1 \geq \frac{q(q^2-1)}{24} \), being \( x_4 = 1 \). Therefore \( x_4 = 0 \).

Assume that \( x_5 > 0 \). Then \( q = 11, 19 \) or 59 by Lemma 3.4, since \( q \equiv 3 \) mod 8. If \( x_3 > 0 \), then \( x_3 = 1 \) by Lemma 3.5(2). Furthermore, \( x_2 = 0 \) and \( q \neq 59 \) by Lemma 3.6(2) and (3). Thus, \( x_1 = 0 \) and \( x_5 = 1 \) by (82), since \( x_4 = 0 \) and \( x_5 > 0 \). Now, by substituting \( x_1 = x_2 = x_4 = 0 \) and \( x_3 = x_5 = 1 \) in (83), we obtain \( \sqrt[5]{n} = \frac{3(q+1)}{4} \). Then \( \sqrt[5]{n} = 9 \) for \( q = 11 \) and \( \sqrt[5]{n} = 15 \) for \( q = 19 \).

The latter is ruled out by Lemma 3.3. Hence, \( \sqrt[5]{n} = 9 \) and \( q = 11 \), which is a contradiction, since \( n + 1 \geq \frac{q(q-1)}{2} \) as \( x_3 = 1 \). So, \( x_3 = 0 \).

Now, assume that \( x_2 > 0 \). Then \( x_2 = 1 \) by Lemma 3.5(1). Then \( q \neq 59 \) by Lemma 3.6(3). Then \( \sqrt[5]{n} + 1 \geq \frac{q+1}{2} + \frac{q+1}{4} \) by (83), as \( x_2, x_5 > 0 \). Therefore \( \frac{2q+1}{4} \leq \sqrt[5]{n} < \sqrt{q} \). Then \( \sqrt[5]{n} = 9 \) for \( q = 11 \) and \( \sqrt[5]{n} = 16 \) for \( q = 19 \).

Assume the former occurs. Then \( x_1 + x_5 = 4 \) and \( x_1 + 3x_5 = 4 \) by (82) and (83), respectively, since \( x_2 = x_4 = 0 \). Consequently, \( x_5 = 0 \). Hence, we arrive at a contradiction by our assumptions. Thus, \( \sqrt[5]{n} = 16 \) for \( q = 19 \). Then \( x_1 + x_5 = 5 \) and \( x_1 + 3x_5 = 7 \) by (82) and (83), respectively, since \( x_2 = x_4 = 0 \). Since \( x_1 \) and \( x_5 \) must be integers, the previous equation have no solutions. As a consequence, \( x_2 = x_3 = 0 \).

Now, subtracting (82) from (83), we obtain \( \sqrt[5]{n} - \sqrt[5]{n} = \frac{q-3}{4}x_5 \), since \( x_2 = x_3 = x_4 = 0 \). Easy computations for \( q = 11, 19 \) or 59, being \( 0 < x_5 \leq 3 \).
by Lemma 3.5(4), show that the admissible solutions for $\sqrt{n} - \sqrt{n} = \frac{2q - 3}{2}$ and $x_1 = \sqrt{n} + 1 - x_5$ are $(q, \sqrt{n}, x_1, x_5) = (11, 2, 2, 1), (11, 3, 1, 3), (19, 4, 2, 3)$ and $(59, 7, 5, 3)$. Now, by substituting these values in (84) and bearing in mind that $x_2 = x_3 = x_4 = 0$, we obtain $S = 4, 48, 84$ or $-2736$, respectively. The case $(q, \sqrt{n}, x_1, x_5) = (59, 7, 5, 3)$ cannot occur, since it must be $S \geq 0$. Furthermore, $4q + 1$ must divide $S$ by the definition of this one. Then also the cases $(q, \sqrt{n}, x_1, x_5) = (11, 2, 2, 1)$ and $(19, 4, 2, 3)$ cannot occur. Thus, $(q, \sqrt{n}, x_1, x_5) = (11, 3, 1, 3)$ and $S = 48$. Let $Y_G^G, h = 1, 2, 3$, the three distinct orbits of type (5) on $l$. Then $|Y_G^G| = 11$ for each $h = 1, 2, 3$ and hence $|l - \cup_{h=1}^{3} Y_G^G| = 48$.

As $S = 48$, then $x_{10} > 0$. Hence, let $X \subset l$ such that $G_X \leq Z_{11}.Z_5$. Then $G_X \cong Z_{11}.Z_5$, since $|X^G| \leq 48$ as $X^G \subset l - \cup_{h=1}^{3} Y_G^G$. Thus, each orbit of type (10) has length 12. Therefore, $x_{10} = 3$, since $S = 48$. In particular, $G$ acts 2-transitively on each of the orbits of type (10). Let $A$ be a subgroup of $G$ such that $A \cong Z_{11}$. Then $A$ fixes exactly 1 point in each of the three $G$-orbits of type (10), since $G$ acts 2-transitively on each of them. Hence, $A$ fixes exactly 4 points on $l$ by Table III, since $x_1 = 1$ and $x_5 = x_{10} = 3$. This is impossible, since $A$ must fix at least 5 points on $l$ and since $n + 1 = 5 \mod 11$, being $n = 3^4$. So, $x_4 = x_5 = 0$ and we have proved the assertion.

**Proposition 5.5.** Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $G \cong \text{PSL}(2, q)$, $q > 3$. If $q < n < q^2$ and $q \equiv 3 \mod 8$, then $G$ does not fix lines of $\Pi$.

**Proof.** Assume that $G$ fixes a line $l$ of $\Pi$. Note that $q > 9$, since $q \equiv 3 \mod 8$ and $q > 3$. Now, $|\text{Fix}(T) \cap l| = \sqrt{n} + 1$ by Lemma 5.3. Furthermore, $x_4 = x_5 = 0$ by Lemma 5.4. Then, by Table III, we have

\begin{align*}
\sqrt{n} + 1 &= x_1 + 3x_3 \\
\sqrt{n} + 1 &= x_1 + \frac{q + 1}{2} x_2 + \frac{q + 3}{2} x_3 \\
n + 1 &= x_1 + \frac{q(q + 1)}{2} x_2 + \frac{q(q - 1)}{2} x_3 + S.
\end{align*}

Assume that $x_3 > 0$. Then $x_3 = 1$ and $x_2 = 0$ by Lemma 3.6(1). Thus, $\sqrt{n} = x_1 + 2$ and $\sqrt{n} = x_1 + \frac{q + 3}{2}$ by (85) and (86), respectively. By composing these equations, we have $(x_1 + 2)^2 = x_1 + \frac{q + 1}{2}$ and hence $x_1^2 + x_1 - \frac{q - 7}{2} = 0$. If $x_1 \leq 2$, it is easily seen that, $(n, x_1, q) = (3^4, 1, 11)$ or $(4^4, 2, 19)$. Let $B^G$ be the $G$-orbit on $l$ of type (3). Then $l - (\text{Fix}(G) \cup B^G) \neq \emptyset$. Moreover, it consists of $G$-orbits of type (10). Then $q + 1 \mid n + 1 - x_1 - |B^G|$, since $|l - (\text{Fix}(G) \cup B^G)| = n + 1 - x_1 - |B^G|$ and since each $G$-orbit of type (10) has length divisible by $q + 1$. Hence, we arrive at a contradiction in any case, since $|B^G| = 55$ for $(n, x_1, q) = (3^4, 1, 11)$ and $|B^G| = 171$ for $(n, x_1, q) = (4^4, 1, 19)$. Thus, $x_1 \geq 3$
for $x_3 > 0$. Actually, $x_1 \geq 3$ also for $x_3 = 0$, since $\sqrt[4]{n} \geq 2$. So, $x_1 \geq 3$ in any case. Thus, $G$ fixes at least 3 points on $l$.

Let $Q$ be any of the points fixed by $G$ on $l$. Clearly, $|\text{Fix}(T) \cap [Q]| = \sqrt[4]{n} + 1$ by Lemma 5.3. Applying the dual of Lemma 5.4, we obtain that $G_4$ cannot be isomorphic either to $A_4$ or to $A_5$ for each $r \in [Q] - \{l\}$. Therefore, $x_4^r = x_5^r = 0$. Consequently, we obtain the same system of Diophantine equations as (85), (86) and (87) but referred to $[Q]$ and hence with the $x_i^r$ in the role of the $x_i$. At this point, the above argument yields that $G$ fixes at least 3 lines (including $l$) through any point $Q$ of $\text{Fix}(G) \cap l$. Thus, $G$ fixes a subplane of $\Pi$ pointwise, as $|\text{Fix}(G) \cap l| \geq 3$. In particular, $o(\text{Fix}(G)) = x_1 - 1$.

Assume that $\text{Fix}(G) \subset \text{Fix}(T)$. Then either $\sqrt[4]{n} = (x_1 - 1)^2$ or $\sqrt[4]{n} \geq (x_1 - 1)^2 + (x_1 - 1)$ by [16, Theorem 3.7], since $T$ induces a Baer collineation on $\text{Fix}(\sigma)$. Furthermore, there must be a $G$-orbit of type (3) on $l$. So $\sqrt[4]{n} = x_1 + 2$ by (85). It follows that, either $x_1 + 2 = (x_1 - 1)^2$ or $x_1 + 2 \geq (x_1 - 1)^2 + (x_1 - 1)$. Easy computations show that, no one of them occurs, since $x_1 \geq 3$. Hence, $\text{Fix}(G) = \text{Fix}(T)$. Thus, $G$ fixes a subplane of $\Pi$ of order $\sqrt[4]{n}$. This forces $x_3 = 0$ which yields $\sqrt[4]{n} + 1 = x_1$ in (85). Then $x_2 > 0$ by (86). Actually, $x_2 = 1$ by Lemma 3.5(1). So, $\sqrt[4]{n} - \sqrt[4]{n} = q^\frac{q + 1}{2}$ by (86). If $S = 0$, then $n - \sqrt[4]{n} = q(q + 1)$ by (87). Note that $n - \sqrt[4]{n} = (\sqrt[4]{n} - \sqrt[4]{n})(\sqrt[4]{n} + \sqrt[4]{n} + 1)$. As $n - \sqrt[4]{n} = q(q + 1)$ and $\sqrt[4]{n} - \sqrt[4]{n} = q^\frac{q + 1}{2}$, then $q(q + 1) = q^\frac{q + 1}{2}(\sqrt[4]{n} + \sqrt[4]{n} + 1)$. By elementary calculations of the previous equality, we obtain $\sqrt[4]{n} + \sqrt[4]{n} = q - 1$. Thus, $\sqrt[4]{n} \mid q - 1$. On the other hand, $\sqrt[4]{n} \mid q + 1$, since $\sqrt[4]{n} - \sqrt[4]{n} = q^\frac{q + 1}{2}$. So, $\sqrt[4]{n} = 2$. Then $q = 3$, since $\sqrt[4]{n} - \sqrt[4]{n} = q^\frac{q + 1}{2}$, which is a contradiction by our assumptions. Therefore, $S > 0$. Then a Sylow $p$-subgroup $S$ of $G$ fixes at least one point on $l - \text{Fix}(G)$. Consequently, $\text{Fix}(S)$ is a Baer subplane of $\Pi$ by [16, Theorem 3.7], since $\text{Fix}(G) \subset \text{Fix}(S)$ and since $\text{Fix}(G)$ fixes a subplane of $\Pi$ of order $\sqrt[4]{n}$. It follows that, $S$ is semiregular on $l - \text{Fix}(S)$ and $q \mid n - \sqrt[4]{n}$, since $|S| = q$. This yields that, either $q \mid \sqrt[4]{n} - 1$ or $q \mid \sqrt[4]{n}$, as $q$ is a prime power. Hence, $\sqrt[4]{n} \geq q$ in any case, which is a contradiction by our assumptions. As a consequence, $G$ does not fix lines of $\Pi$.

**Theorem 5.6.** Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $G \cong PSL(2, q)$, with $q \equiv 3 \mod 8$ and $q > 3$. If $n \leq q^2$, then one of the following occurs:

1. $n = q$, $\Pi \cong PG(2, q)$ and $G$ is strongly irreducible on $\Pi$;
2. $q < n < q^2$ and $G$ is strongly irreducible on $\Pi$;
3. $n = q^2$ and one of the following occurs:
   
   a. $G$ is strongly irreducible on $\Pi$;
(b) $G$ fixes a Baer subplane $\Pi_0 \cong \text{PG}(2, q)$ of $\Pi$.

Proof. No cases arise for $n < q$ by Theorem 2.1, as $q > 3$. If $n = q$, the assertions (1) easily follows by Theorem 2.2. If $q < n < q^2$, the group $G$ does not fix lines or points of $\Pi$ by Proposition 5.5 and its dual. Now, the assertion (2) follows in this case by Lemma 3.1, since $q \equiv 3 \mod 8$ and $q > 3$. When $n = q^2$, the assertions (3a) and (3b) follow by Theorem 2.3 and Corollary 2.4, respectively.

Finally, when $q \equiv 3 \mod 8$, Theorem 1.1 easily follows from Theorem 5.6.

### 6. The case $q \equiv 5 \mod 8$

Recall that $\sigma$ and $T$ are the representatives of the unique conjugate class of involutions and Klein subgroups of $G$, respectively. Recall also that $T$ is chosen such that $\sigma \in T$. Furthermore, $C_G(\sigma) \cong D_{q-1}$ and $N_G(T) \cong A_4$. We filter the list given in Lemma 3.4 with respect to the condition $q \equiv 5 \mod 8$. Now, arguing as in the beginning of the previous section, we obtain the following table.

| Type | $G_P$ | $[G : G_P]$ | $|\text{Fix}_{G_P}(\sigma)|$ | $|\text{Fix}_{G_P}(T)|$ |
|------|-------|-------------|----------------|------------------|
| 1    | $G$   | 1           | 1              | 1                |
| 2    | $D_{q-1}$ | $\frac{q(q+1)}{2}$ | $\frac{q+1}{2}$ | 3                |
| 3    | $D_{q+1}$ | $\frac{q(q-1)}{2}$ | $\frac{q-1}{2}$ | 0                |
| 4    | $A_4$  | $\frac{q(q^2-1)}{24}$ | $\frac{q-1}{4}$ | 1                |
| 5    | $A_5$  | $\frac{q(q^2-1)}{120}$ | $\frac{q-1}{4}$ | 1                |
| 10   | $F_{q,d}$ | $\frac{q^2-1}{2d}$ | $\frac{q-1}{d}$ | $2 \mid d$       |
|      |       |             | 0              | $2 \mid d$       |

By section 3, the $G$-orbits of type (10) on $l$ cover exactly $S$ points of $l$, where $S = \sum_{j=1}^{d} \frac{q^2-1}{2d_j}$. Moreover, $S_1 = \sum_{j=1}^{d} \frac{q-1}{d_j}$ and $S_2$, $S_2'$, $S_4$, $S_2, S_4$ (sum with the same summands $\frac{q-1}{d_j}$ but over $2 \mid d_j$, $2 \nmid d_j$, $4 \mid d_j$ and $d_j \equiv 2 \mod 4$, respectively). In particular, $S = \frac{q+1}{2} S_1$. Note also that, $S_4 = 0$, since $q \equiv 5 \mod 8$.

If $G$ fixes a point $Q$ and acts on $[Q]$, we may focus on the $G$-orbits of lines in $[Q]$. So, following the notation introduced in section 4, we obtain a table,
namely the dual of Table IV, where type (i)\(^*\) replaces (i), the group \(G_m\) replaces \(G_p\) and \(m^G\) replaces \(P^G\). Here, \(m\) is any line of \([Q]\). Recall that, we denote by \(x_i\) the number of \(G\)-orbits on \([Q]\) of type \((i)\)\(^*\). As mentioned in section 4, we write \(x_i\) instead of \(x_i(Q)\), even if the second notation would be correct. It will be clear from the context which point we are focusing on. In particular, since we might have \(G\)-orbits of type \((10)\)\(^*\), it makes sense considering \(S^* = \sum_j x_{10j} \frac{q^2 - 1}{2d_j}\) and hence \(S_2^*, S_2'^*, S_4^*, S_{2,4}^*\) with the same meaning of \(S_2, S_2', S_4, S_{2,4}\), respectively, but referred to lines instead of points. Clearly, \(S_4^* = 0\), since \(q \equiv 5 \text{ mod } 8\).

The collineation \(\sigma\) is a Baer collineation of \(\Pi\) by Lemma 3.3. Set \(C = C_G(\sigma)\). Then \(C\) acts on \(\text{Fix}(\sigma)\) with kernel \(K\). Hence, let \(\tilde{C} = C / K\). Clearly, \(\langle \sigma \rangle \leq K \leq C\). Furthermore, either \(K \leq Z_{\frac{q - 1}{2}}\) or \(K = C\), since \(C \cong D_{q-1}\) and \(q \equiv 5 \text{ mod } 8\).

As we will see, we need to investigate the structure of \(K\) in order to show that \(T\) induces a Baer collineation on \(\text{Fix}(\sigma)\).

**Lemma 6.1.** If \(\text{Fix}(T) \cap l = \text{Fix}(\sigma) \cap l\), then \(K = C\).

**Proof.** Assume that \(\text{Fix}(T) \cap l = \text{Fix}(\sigma) \cap l\) and that \(K \leq Z_{\frac{q - 1}{2}}\). Then \(\text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l\) by Table IV, since \(q > 9\). Set \(l_0 = \text{Fix}(\sigma) \cap l\). Then \(\tilde{C} = \tilde{C}(l_0)\), since \(l_0 = \text{Fix}(G) \cap l\). In particular, \(\tilde{C} \cong D_{q-1}\), where \(k = |K|\), since \(K \leq Z_{\frac{q - 1}{2}}\) and \(C \cong D_{q-1}\). On the other hand, \(\tilde{C}\) is the semidirect product of \(\tilde{C}(l_0, l_0)\) with \(\tilde{C}(Y, l_0)\) for some point \(Y \in \text{Fix}(\sigma) - l_0\) by \([16, \text{Theorem 4.25}]\).

Assume that \(\tilde{C}(l_0, l_0) \neq \{1\}\). Assume that also that \(\tilde{C}(l_0, l_0) = \tilde{C}\). Then \(\tilde{C} = \tilde{C}(V, l_0)\), for some point \(V \in l_0\) by \([16, \text{Theorem 4.14}]\), since \(\tilde{C} \cong D_{q-1}\), \(\langle \sigma \rangle \leq K \leq Z_{\frac{q - 1}{2}}\) and \(q \equiv 5 \text{ mod } 8\). Hence, for each point \(X \in l_0 - \{V\}\) and for each line \(t \in [X] \cap \text{Fix}(\sigma)\), we have that \(\sigma \in G_t\) but \(G_t\) does not contain Klein groups. Then, by dual of Table IV, we have that either \(G_t \cong D_{q+1}\) or \(G_t \cong F_q, Z_d\) with \(d\) even, since \(G\) fixes \(X\). Clearly, \(K \leq G_t\) and \(\langle \sigma \rangle \leq K \leq Z_{\frac{q - 1}{2}}\). Thus, \(K = \langle \sigma \rangle\), since \(2 \mid |G_t|\) but \(4 \nmid |G_t|\) as \(q \equiv 5 \text{ mod } 8\). Therefore, \(\tilde{C} \cong D_{\frac{q - 1}{2}}\) and \(\sqrt{n} = \frac{q - 1}{2}\) | \(\sqrt{n}\). Actually, either \(\sqrt{n} = \frac{q - 1}{2}\) or \(\sqrt{n} = q - 1\), since \(\sqrt{n} < q\) by our assumptions. If \(\sqrt{n} = \frac{q - 1}{2}\), then \(\sqrt{n} = 2 \text{ mod } 4\) as \(q \equiv 5 \text{ mod } 8\). This is a contradiction by Lemma 3.3. So \(\sqrt{n} = q - 1\). Note that, either \(G_t \cong D_{q+1}\) or \(G_t \cong F_q, Z_d\) with \(d = d(t)\) even, for each line \(t \in [X] \cap \text{Fix}(\sigma)\) such that \(t \neq l\). Moreover, \(\text{Fix}(T) \cap [V] = \text{Fix}(\sigma) \cap [V]\), since \(\tilde{C} = \tilde{C}(V, l_0)\). Then \(\text{Fix}(G) \cap [V] = \text{Fix}(\sigma) \cap [V]\) by dual of Table IV, since \(q > 9\). Thus either \(|G_r|\) is odd, or \(2 \mid |G_r|\) but \(4 \nmid |G_r|\) for each \(r \in [X] - \text{Fix}(\sigma)\). Consequently, either \(G_t \cong D_{q+1}\) or \(G_t \cong F_q, Z_d\), with \(d = d(r)\), for each \(r \in [X] - \text{Fix}(\sigma)\) by dual of Table IV, since \(G\) fixes \(X\). In this case \(d = d(r)\) might be also odd. Therefore, \([X]\) consists of \(G\)-orbits of type \((1)\)\(^*\), \((3)\)\(^*\) or \((10)\)\(^*\). Then, again by dual of Table IV, we have

\[
n = \frac{q(q - 1)}{2} x_3^* + \frac{q + 1}{2} S_1^*,
\]

(88)
since \( x_1^* = 1 \) (\( G \) fixes \( l \)) and since \( S^* = \frac{q+1}{2} S_1^* \). Actually,

\[
(q - 1)^2 = \frac{q(q - 1)}{2} x_3^* + \frac{q + 1}{2} S_1^*,
\]

since \( \Pi \) has order \( (q - 1)^2 \). Hence \( \frac{q+1}{2} | (q - 1)^2 - \frac{q(q - 1)}{2} x_3^* \), where \( x_3^* \leq 1 \) by dual of Lemma 3.5(2). If \( x_3^* = 0 \), by elementary calculations of the last divisibility relation, we obtain \( \frac{q+1}{2} | 4 \). So, we arrive at a contradiction, since \( q \equiv 5 \mod 8 \). Thus, \( x_3^* = 1 \). So, \( q + 1 | q^2 - 3q + 1 \) by \( \frac{q+1}{2} | (q - 1)^2 - \frac{q(q - 1)}{2} \). This is impossible, since \( q + 1 \) is even while \( q^2 - 3q + 1 \) is odd. Then \( C(l_0, l_0) < C \) and \( \tilde{C}(Y, l_0) \neq \langle 1 \rangle \) for some point \( Y \in \text{Fix}(\sigma) - l_0 \). It follows that, \( \tilde{C}(l_0, l_0) \leq Z_{q-1} \), since \( \tilde{C} \cong D_{q-1} \). Actually, \( \tilde{C}(l_0, l_0) = \tilde{C}(V, l_0) \cong Z_{q-1} \), and \( \tilde{C}(Y, l_0) \cong Z_q \) by \([16, \text{Theorems 4.14 and 4.25}]. \) Let \( R \in l_0 - \{V\} \) and set \( f = RY \). Clearly, \( \tilde{C}(Y, l_0) \) fixes \( f \). Then \( D_k \leq G_f \), where \( k \) is an even divisor of \( \frac{q+1}{2} \), since \( \langle \sigma \rangle \leq K \leq G_f \). If \( k = 2 \), then \( \tilde{C}(V, l_0) \cong Z_{q-1} \) and \( \frac{q+1}{2} | \sqrt{n} \), arguing as above. So, \( \sqrt{n} \) is even, as \( q \equiv 5 \mod 8 \), which is a contradiction, since \( \tilde{C}(Y, l_0) \) consists of an involutory relation. Therefore, \( k > 2 \). Hence, \( 4 | 2k \) with \( k > 2 \). As a consequence, \( C \leq G_f \) by dual of Table IV. Then \( \tilde{C} \) fixes \( f \). Hence, we obtain a contradiction, since \( \tilde{C}(l_0, l_0) = \tilde{C}(V, l_0) \neq \langle 1 \rangle \), while \( f = RY \) with \( R \in l_0 - \{V\} \). Thus, \( \tilde{C}(l_0, l_0) = \langle 1 \rangle \).

Assume that \( \tilde{C} = \tilde{C}(Y, l_0) \) for some point \( Y \in \text{Fix}(\sigma) - l_0 \). Let \( Q \in l_0 \) and \( m \in [Q] \cap \text{Fix}(\sigma) - \{l, QY\} \). Then \( \sigma \in G_m \) but \( G_m \) does not contain Klein subgroups of \( G \). By dual of table IV, either \( G_m \cong D_{q+1} \) or \( G_m \cong F_q \cdot Z_d \). So,

\[
\sqrt{n} = \frac{q + 1}{2} x_2^* + \frac{q - 1}{2} x_3^* + S_2^* \quad \text{(90)}
\]

Note that, \( x_2^* > 0 \), as \( G_{QY} = C \). Then \( x_2^* = 1 \) by dual of Lemma 3.5(2). Hence, \( x_3^* = 0 \) by dual of Lemma 3.6(1). Furthermore, \( T \) fixes exactly 3 points on \( QY \cdot G \). Thus, \( \sqrt{n} = 2 \), since \( T \) must induce either a perspectivity or the identity on \( \text{Fix}(\sigma) \) as \( \text{Fix}(T) \cap l = \text{Fix}(\sigma) \cap l \). On the other hand, \( \sqrt{n} \geq \frac{q+1}{2} \) by (90) as \( x_2^* = 1 \). This yields \( \sqrt{n} \geq 5 \), being \( q > 9 \), which is a contradiction, since we proved that \( \sqrt{n} = 2 \).

**Lemma 6.2.** Fix(\( T \)) \( \cap l \subset \text{Fix}(\sigma) \cap l \).

**Proof.** Assume that \( \text{Fix}(T) \cap l = \text{Fix}(\sigma) \cap l \). Then \( K = C \) by Lemma 6.1. Thus, \( \text{Fix}(T) = \text{Fix}(\sigma) \). Let \( P \) be any point of \( \text{Fix}(\sigma) \cap l \) and let \( r \) be any line of \( [P] - \{l\} \). Then \( C \leq G_r \). Since \( q > 9 \), then \( C \) is maximal in \( G \) and hence either \( G_r = C \) or \( G_r = G \). If the former occurs, then \( |\text{Fix}_{r, \sigma}(T_1)| = 3 \) and \( |\text{Fix}_{r, \sigma}(\sigma)| = \frac{q+1}{2} \) by dual of Table V. Hence, \( |\text{Fix}_{r, \sigma}(\sigma)| > |\text{Fix}_{r, \sigma}(T_1)| \) as \( q > 9 \). This is a contradiction, since \( \text{Fix}(T_1) = \text{Fix}(\sigma) \). So, \( G_r = G \) for any point
3.4 Proof. The group $T$ induces a non trivial involution $\beta$ on $\text{Fix}(\sigma)$ by Lemma 6.2. Assume that $\beta$ is an involutory $(C_3, a_3)$-perspectivity on $\text{Fix}(\sigma)$. Then $a_3 \not\equiv l$ again by Lemma 6.2. Thus, $|\text{Fix}(T) \cap l| = 1$ or 2. Therefore, $x_1 + 3x_2 + x_4 + x_5 = 1$ or 2, since $|\text{Fix}(T) \cap l| = x_1 + 3x_2 + x_4 + x_5$ by table IV. Clearly $x_2 = 0$. Hence, $x_1 + x_4 + x_5 = 1$ or 2. Furthermore, by table IV, we have the following system of Diophantine equations:

\[
\sqrt{n} + 1 = x_1 + \frac{q - 1}{2} x_3 + \frac{q - 1}{4} x_4 + \frac{q - 1}{4} x_5 + S_2
\]

(91)

\[
n + 1 = x_1 + \frac{q(q - 1)}{2} x_3 + \frac{q(q^2 - 1)}{24} x_4 + \frac{q(q^2 - 1)}{120} x_5 + S.
\]

(92)

Assume $|\text{Fix}(T) \cap l| = 1$. Then $\beta$ is an involutory $(C_3, a_3)$-elation of $\text{Fix}(\sigma)$ with $C_3 \in l$ and $a_3 \not\equiv l$. So $\text{Fix}(T) \cap l = \{C_3\}$ and $x_1 + x_4 + x_5 = 1$. Clearly, $G$ cannot fix $C_3$, otherwise we obtain a contradiction by dual of Lemma 6.2, since $\text{Fix}(T) \cap [C_3] = \text{Fix}(\sigma) \cap [C_3]$. Consequently, $x_1 = 0$ and $x_4 + x_5 = 1$.

Assume that $x_4 = 1$ and $x_5 = 0$. Then $x_3 = 0$ by Lemma 3.6(3) and $q = 13$ by Lemma 3.4. So, either $\sqrt{n} = 10$ or 12, since $\frac{q(q+1)}{2} \leq n < q^2 + 1$ with $n$ an even square number. The former is ruled out by Lemma 3.3. Hence $\sqrt{n} = 12$. Since $x_1 = x_3 = x_5 = 0$, since $\frac{4q+1}{2} \mid S$ and since $\frac{4q+1}{2}$ divides the orbit of type (4) as $q = 13$, then $\frac{q+1}{2} \mid n + 1$ by (92). This cannot occur, since $\frac{q+1}{2} = 7$ while $\sqrt{n} = 12$.

Assume that $x_4 = 0$ and $x_5 = 1$. Then $q = 29, 61, 101, 109$ by Lemma 3.4, since $q \equiv 5 \mod 8$. If $x_3 = 1$, then $q = 29$ by Lemma 3.6(4). Let $QG$ be an orbit of type (3). Clearly, $|QG| = \frac{q(q-1)}{2}$. Now, let $RG$ be an orbit of type (5), then $|RG| = \frac{q(q^2-1)}{120}$. Since $QG \cup RG \subseteq l$, it follows that, $n+1 \geq \frac{q(q-1)}{2} + \frac{q(q^2-1)}{120}$. Then $n \geq 608$, since $\frac{q(q-1)}{2} + \frac{q(q^2-1)}{120} = 609$, being $q = 29$. So, $24 < \sqrt{n} < 29$, since $n < q^2$ and $q = 29$. Actually, $\sqrt{n} = 26$ cannot occur by Lemma 3.3. Therefore, $\sqrt{n} = 28$, since $\sqrt{n}$ must be even. Thus, $S = 176$, since $n = 608 + S$ by (92), since $x_1 = x_4 = 0, x_3 = x_5 = 1$ and $q = 29$. Then $S = 176$, as $\sqrt{n} = 28$. Hence $\frac{4q+1}{2} \mid S = 176$, as $\frac{4q+1}{2} \mid S$, which is a contradiction, since $q = 29$. Consequently, $x_3 = 0$. Then $n + 1 = \frac{q(q^2-1)}{120} + S$ by (92), since $x_1 = x_3 = x_4 = 0$ and $x_5 = 1$. If $S = 0$, then $n = \frac{q(q^2-1)}{120} - 1$ with $q = 29, 61, 101, 109$. We again obtain a contradiction, since $n$ must be a square. Thus, $S > 0$. Since $\frac{q+1}{2} \mid S$, then
\[ \frac{q+1}{2} \mid n + 1 - \frac{q(q^2-1)}{120}, \text{ since } n + 1 = \frac{q(q^2-1)}{120} + S. \text{ Moreover, } q = 29, 61, 101, 109, \text{ and } \sqrt{\frac{q(q^2-1)}{120}} - 1 < n < q^2, \text{ with } n \text{ an even square number. Easy computations show that, only the case } \sqrt{n} = 98 \text{ and } q = 101 \text{ is admissible. Nevertheless, it cannot occur by Lemma 3.3.} \]

Assume that \(|\text{Fix}(T) \cap l| = 2\). Then \(\tilde{\beta}\) is an involutory \((C_\beta, a_\beta)\)-homology of \(\text{Fix}(\sigma)\) with \(C_\beta \in l \) and \(a_\beta \neq l\). Furthermore, \(x_1 + x_4 + x_5 = 2\), since \(|\text{Fix}(T) \cap l| = x_1 + 3x_2 + x_4 + x_5 \) with \(x_2 = 0\).

Assume that \(x_4 > 0\). Then \(x_4 = 1\) by Lemma 3.5(3). Then \(x_2 = x_3 = 0\) by Lemma 3.6(3). Moreover, \(q = 13\) by Lemma 3.4, since \(q \equiv 5 \mod 8\). So, \(\sqrt[4]{n} = 2 \) or \(3\), since \(q < n < q^2\) by our assumption. On the other hand, \(n + 1 \geq \frac{q(q^2-1)}{24}\), with \(q = 13\), since \(x_4 = 1\). Hence, we arrive at a contradiction. Thus, \(x_4 = 0\) and either \(x_1 = x_5 = 1\), or \(x_1 = 0 \) and \(x_5 = 2\), since \(x_1 \leq 1\) and \(x_1 + x_4 + x_5 = 2\). In order to make easier the analysis of these two cases, we are going to show that \(x_3 = 0\).

Assume that \(x_3 > 0\). Then \(x_3 = 1\) by Lemma 3.5(3). So, \(q = 29\) by Lemma 3.6(4), since \(x_5 \geq 1\). Therefore, \(24 < \sqrt{n} < 29\), arguing as above. Then \(\sqrt{n} = 25 \) or \(27\), since \(\sqrt{n}\) is odd. Actually, the case \(\sqrt{n} = 27\) cannot occur by Lemma 3.3, as \(\sqrt{n} \equiv 3 \mod 4\). Thus, \(\sqrt{n} = 25\). Let \(X_1^G\) and \(X_2^G\) be the orbits on \(l\) of type \((3)\) and \((5)\), respectively, as \(x_3 = 1 \) and \(x_5 \geq 1\). Then \(|X_1^G| = 406\) and \(|X_2^G| = 203\) as \(q = 29\). Since \(\sqrt{n} = 25\), we have \(|l - X_1^G - X_2^G| = 17\). As the minimal primitive permutation representation of \(G \cong \text{PSL}(2, 29)\) is \(30\), the group \(G\) fixes \(l - X_1^G - X_2^G\) pointwise. As a consequence, \(x_1 = 17\) and \(x_5 = 1\), since \(|l - X_1^G - X_2^G| = 17\). This is impossible, since we saw \(x_1 \leq 1\). So, \(x_3 = 0\).

Now, assume that \(x_1 = x_5 = 1\). Thus, \(n = \frac{q(q^2-1)}{120} + S\) by (92), as \(x_3 = x_4 = 0\). Furthermore, \(q = 29, 61, 101, 109\) by Lemma 3.4, since \(q \equiv 5 \mod 8\). Clearly, \(n > \frac{q(q^2-1)}{120}\), since \(n\) is a square by Lemma 3.3, while \(\frac{q(q^2-1)}{120}\) is not for these numerical values of \(q\). Therefore, \(S > 0\). Since \(\frac{q+1}{2} \mid S\), then \(\frac{q+1}{2} \mid n - \frac{q(q^2-1)}{120}\), where \(q = 29, 61, 101, 109\), and \(\sqrt{\frac{q(q^2-1)}{120}} < \sqrt{n} < q\) with \(\sqrt{n}\) odd and hence \(\sqrt{n} \equiv 1 \mod 4\) by Lemma 3.3. Easy computations show that no cases arise.

Assume that \(x_1 = 0\) and \(x_5 = 2\). Therefore, \(n + 1 \geq \frac{q(q^2-1)}{60}\) by (92), as \(x_3 = x_4 = 0\). Furthermore, \(q = 29, 61, 101, 109\) by Lemma 3.4, since \(q \equiv 5 \mod 8\). Actually, the cases \(q = 61, 101, 109\) cannot occur, since they do not satisfy \(\frac{q(q^2-1)}{60} \leq 1 \leq n < q^2\). Thus, \(q = 29\) and hence \(405 \leq n < 29^2\). Actually, either \(n = 21^2\) or \(25^2\), since \(\sqrt{n} \equiv 1 \mod 4\) by Lemma 3.3. Then \(S = 36\) or \(220\) by (92), respectively, since \(x_1 = x_3 = x_4 = 0\), \(x_5 = 2\) and \(q = 29\). This leads to a contradiction, since \(\frac{q+1}{2} = 15\) must divide \(S\) by the definition of this one. Hence, we have proved the assertion.
Lemma 6.4. For each point \( P \in l \), the group \( G_P \) cannot be isomorphic either to \( A_4 \) or to \( A_5 \).

Proof. The group \( T \) induces a Baer collineation on \( \text{Fix}(\sigma) \) by Lemma 6.3. Thus, \( |\text{Fix}(T) \cap l| = \sqrt[3]{n} + 1 \). Then, by Table IV, we have the following system of Diophantine equations:

\[
\sqrt[3]{n} + 1 = x_1 + 3x_2 + x_4 + x_5 
\]

(93)

\[
\sqrt{n} + 1 = x_1 + \frac{q + 1}{2} x_2 + \frac{q - 1}{2} x_3 + \frac{q - 1}{4} x_4 + \frac{q - 1}{4} x_5 + S_2 
\]

(94)

\[
n + 1 = x_1 + \frac{q(q + 1)}{2} x_2 + \frac{q(q - 1)}{2} x_3 + \frac{q(q^2 - 1)}{24} x_4 + \frac{q(q^2 - 1)}{120} x_5 + S. 
\]

(95)

Assume that \( x_4 > 0 \). Then \( x_4 = 1 \) by Lemma 3.5(3). So, \( x_2 = x_3 = 0 \) by Lemma 3.6(3). Furthermore, \( q = 13 \) by Lemma 3.4, since \( q \equiv 5 \mod 8 \). Thus, \( \sqrt[3]{n} = 2 \) or 3, since \( q \leq n < q^2 \) by our assumption. Hence, we obtain a contradiction, since \( n + 1 \geq \frac{q(q^2 - 1)}{24} \) for \( q = 13 \), being \( x_4 = 1 \). Therefore, \( x_4 = 0 \).

Assume that \( x_5 > 0 \). Then \( q = 29, 61, 101, 109 \) by Lemma 3.4, since \( q \equiv 5 \mod 8 \). If \( x_2 + x_3 > 0 \), then \( x_2 + x_3 = 1, x_5 \leq 2 \) and \( q = 29 \) by Lemma 3.6(2) and (4). Moreover, \( \sqrt{n} = 3, 4 \) or 5, since \( q < n < q^2 \) and since \( n \) is a fourth power by Lemma 6.3. Let \( Q^G \) be an orbit of type either (2) or (3), as \( x_2 + x_3 = 1 \). Then \( |Q^G| = \frac{q(q + 1)}{2} \), respectively. Now, let \( R^G \) be an orbit of type (5) as \( x_5 > 0 \), then \( |R^G| \geq \frac{q(q - 1)}{120} \). Since \( Q^G \cup R^G \subseteq l \), then \( n + 1 \geq \frac{q(q + 1)}{2} + \frac{q(q^2 - 1)}{120} \). In particular, \( n + 1 = 638 \) or 609, since \( \frac{q(q^2 - 1)}{120} \) is a fourth power, it is a contradiction, since \( \sqrt{n} = 3, 4 \) or 5 cannot occur when \( Q^G \) is of type (2), only \( \sqrt{n} = 5 \) is admissible when \( Q^G \) is of type (3). In this case, since \( |l - (Q^G \cup R^G)| = 17 \) and since the minimal primitive permutation representation of degree of \( \text{PSL}(2, 29) \) is 30, the group \( G \) fixes \( l - (Q^G \cup R^G) \) pointwise. Hence, \( x_1 = 17 \) as \( |l - (Q^G \cup R^G)| = 17 \) when \( Q^G \) is of type (3). So \( \sqrt{n} \geq 16 \) as \( \sqrt{n} + 1 = x_1 + 3x_2 + x_5 \), which is a contradiction, being \( \sqrt{n} = 5 \) by the above argument. Thus, \( x_2 = x_3 = 0 \).

Since \( \frac{q(q^2 - 1)}{120} \leq n + 1 < q^2 + 1 \) and since \( n \) is a fourth power, it is a straightforward calculation to show that, \( q, n = (29, 4^4), (29, 5^4) \), or \( (61, 7^4) \), or \( (101, 10^4) \). Assume that \( q \neq 29 \). Let \( u \) be an odd prime divisor of \( q + 1 \). In particular, \( u = 31 \) when \( q = 61 \) and \( u = 17 \) when \( q = 101 \). Note that, \( u \mid S \), since \( \frac{q - 1}{2} \mid S \). Furthermore, \( u \mid \frac{q(q^2 - 1)}{120} \). Then \( u \mid n + 1 - x_1 \). Indeed, we have \( n + 1 - x_1 = \frac{q(q^2 - 1)}{120} x_5 + S \) by (95), since \( x_2 = x_3 = x_4 = 0 \). Hence, \( n + 1 \equiv x_1 \mod u \). This yields \( x_1 \equiv 15 \mod 31 \) for \( q = 61 \) and \( x_1 \equiv 5 \mod 31 \) for \( q = 101 \). Since \( 0 < x_1 < \sqrt{n} + 1 \) and \( \sqrt{n} + 1 < u \) in each case, then \( x_1 = 15 \) for \( q = 61 \) and \( x_1 = 5 \) for \( q = 101 \). This is a contradiction, since \( x_1 + x_5 = \sqrt{n} + 1 \).
with $\sqrt[4]{n} \leq 10$. Thus, $(q, n) = (29, 4^4), (29, 5^4)$. Then $n + 1 = x_1 + 203x_5 + S$ by (95), since $x_2 = x_3 = x_4 = 0$. Assume that $S = 0$. Then $n + 1 = x_1 + 203x_5$. Since $\sqrt[4]{n} + 1 = x_1 + x_5$ by (93), then $n + 1 = \sqrt[4]{n} + 1 + 202x_5$. Therefore, $202 \mid n - \sqrt[4]{n}$, as $x_5 > 0$, which is a contradiction, since $\sqrt[4]{n} = 4$ or 5. Hence, $S > 0$. Actually, $S = n - \sqrt[4]{n} - 202x_5$. If $\sqrt[4]{n} = 4$, then $x_5 = 1$ and $S = 50$. If $\sqrt[4]{n} = 5$, then $x_5 \leq 3$. Furthermore, $S = 418, 216, or 19$, for $x_5 = 1, 2, or 3$, respectively. On the other hand, $15 \mid S$, since $\frac{q + 1}{2} = 15$ and since $\frac{q + 1}{2} \mid S$ by the definition of $S$, in each case. So, we obtain a contradiction in any case.

**Proposition 6.5.** Let $\Pi$ be a projective plane of order $n$ that admits a collineation group $G \cong \text{PSL}(2, q)$ fixing a line $l$. If $q < n < q^2$ and $q \equiv 5 \mod 8$, then $\Pi$ has order 16 and $G \cong \text{PSL}(2, 5)$.

**Proof.** Suppose that $G$ fixes a line $l$ of $\Pi$. Assume that $q = 5$. Then $5 < n < 5^2$ by our assumptions. Actually, $n = 16$, since $n$ must be a square and $\sqrt[4]{n} \equiv 0, 1 \mod 4$ by Lemma 3.3. Thus, we have proved the assertion (1).

Assume that $q > 5$. Actually, $q > 9$, since . Recall that, $|\text{Fix}(T) \cap l| = \sqrt[4]{n} + 1$ by Lemma 6.3, and that for each point $P \in l$, the group $G_P$ cannot be isomorphic either to $A_4$ or to $A_5$ by Lemma 6.4. So, $x_4 = x_5 = 0$. Hence, by table IV, we have

\[
\sqrt[4]{n} + 1 = x_1 + 3x_2
\]

\[
\sqrt[4]{n} + 1 = x_1 + \frac{q + 1}{2}x_2 + \frac{q - 1}{2}x_3 + S_2
\]

\[
n + 1 = x_1 + \frac{q(q + 1)}{2}x_2 + \frac{q(q - 1)}{2}x_3 + S.
\]

Assume that $x_2 > 0$. Then $x_2 = 1$ by Lemma 3.5(1) and therefore $x_3 = 0$ by Lemma 3.6(2). It follows that, $\sqrt[4]{n} = x_1 + 2$ and $\sqrt[4]{n} = x_1 + \frac{q + 1}{2} + S_2$ by (96) and (97), respectively. By elementary calculations of the previous equations, we obtain $(x_1 + 2)^2 + 1 = x_1 + \frac{q + 1}{2} + S_2$. So, $x_1 + 3x_1 = \frac{q - 3}{2} + S_2$. If $x_1 = 0$, then $q = 9$, since $S_2 \geq 0$. This is a contradiction, since $q \equiv 5 \mod 8$. If $x_1 = 1$, then $\sqrt[4]{n} = 3$ and hence $n = 81$. Moreover, $n + 1 \geq x_1 + \frac{q(q + 1)}{2}$ by (98), being $x_2 = 1$. This is a contradiction, since $\frac{q(q + 1)}{2} \geq 91$ as $q \geq 13$, while $n = 81$. If $x_1 = 2$, then $\sqrt[4]{n} = 4$ and therefore $q > 13$, since $q < n < q^2$ by our assumptions. Thus, $q \geq 29$, since $q \equiv 5 \mod 8$. Again, $n + 1 \geq x_1 + \frac{q(q + 1)}{2}$, with $\frac{q(q + 1)}{2} \geq 435$, being $q \geq 29$. This is impossible, since $n = 4^4$. Thus, $x_1 \geq 3$ for $x_2 > 0$. Note that, $x_1 \geq 3$ also for $x_2 = 0$, since $\sqrt[4]{n} + 1 \geq 3$. So, $x_1 \geq 3$ in any case. Thus, $G$ fixes always at least 3 points on $l$.

Let $P$ be any of these points and let $r$ be any line of $[P] \setminus \{l\}$. Applying the dual of Lemma 6.4, we obtain that, $G_r$ cannot be isomorphic either to $A_4$ or to $A_5$ for each line $r \in [P] \setminus \{l\}$. Hence, $x_4^r = x_5^r = 0$. By dual of Table IV, we
obtain the same system of Diophantine equations as (96), (97) and (98) but referred to \( |P| \) and with the \( x_1^* \) in the role of \( x_i \). Now, we may repeat the above argument showing that, \( G \) fixes at least 3 lines (including \( l \)) through any point \( P \) of \( \text{Fix}(G) \cap l \). Thus \( G \) fixes a subplane of \( \Pi \) pointwise, as \( |\text{Fix}(G) \cap l| \geq 3 \). In particular, \( o(\text{Fix}(G)) = x_1 - 1 \).

Assume that \( \text{Fix}(G) \subset \text{Fix}(T) \). Then either \( \sqrt[n]{n} = (x_1 - 1)^2 \) or \( \sqrt[n]{n} \geq (x_1 - 1)^2 + (x_1 - 1) \) by [16, Theorem 3.7], since \( T \) induces a Baer collineation on \( \text{Fix}(\sigma) \). Furthermore, there must be a \( G \)-orbit on \( l \) of type (2) by (96). So, \( \sqrt[n]{n} + 1 = x_1 + 2 \). Hence, either \( x_1 + 2 = (x_1 - 1)^2 \) or \( x_1 + 2 \geq (x_1 - 1)^2 + (x_1 - 1) \). Easy computations show that no one of them occurs, since \( x_1 \geq 3 \). Consequently, \( \text{Fix}(G) = \text{Fix}(T) \). This yields \( x_2 = 0 \) and \( \sqrt[n]{n} + 1 = x_1 \).

Assume that \( S = 0 \). Then \( x_3 > 0 \) by (97), as \( x_2 = 0 \). Actually, \( x_3 = 1 \) by Lemma 3.5(2). So (96), (97) and (98), respectively, become

\[
\begin{align*}
\sqrt[n]{n} + 1 &= x_1 \quad \text{(99)} \\
\sqrt[n]{n} + 1 &= x_1 + \frac{q - 1}{2} \quad \text{(100)} \\
n + 1 &= x_1 + \frac{q(q - 1)}{2} \quad \text{(101)}
\end{align*}
\]

Then \( \sqrt[n]{n} - \sqrt[n]{n} = \frac{q - 1}{2} \) combining (99) with (100), and \( n - \sqrt[n]{n} = \frac{(q - 1)^2}{2} \) combining (100) with (101). Finally, combining these ones, we have \( n + \sqrt[n]{n} = q - 1 \).

Then \( n + \sqrt[n]{n} = 2(\sqrt[n]{n} - \sqrt[n]{n}), as \sqrt[n]{n} - \sqrt[n]{n} = \frac{q - 1}{2} \). Now, dividing by \( \sqrt[n]{n} \), we obtain \( (\sqrt[n]{n})^3 - 2\sqrt[n]{n} + 3 = 0 \) which has no integer solutions. Therefore, \( S > 0 \).

Let \( S \) be a Sylow \( p \)-subgroup of \( G \) normalized by \( \sigma \) and let \( X \in l \) such that \( S \leq G_X \) (such a point does exist, as \( S > 0 \)). Then either \( G_X \cong F_{q^2}, \mathbb{Z}_{d_X} \), with \( d_X \mid \frac{q - 1}{2} \), or \( G_X = G \) by Table V. Then \( S \) fixes a Baer subplane of \( \Pi \), since \( \text{Fix}(G) \subset \text{Fix}(S) \), since \( o(\text{Fix}(G)) = \sqrt[n]{n} \) and since \( S > 0 \). Recall that \( S_1 = \sum_{j=1}^{x_1} \frac{q - 1}{d_j} \) and that \( S_2 \) and \( S_2' \) are the sum with the same summands \( \frac{q - 1}{d_j} \) but over \( 2 \mid d_j \) and \( 2 \nmid d_j \), respectively. Note that, \( d_X = d_h \) for some \( 1 \leq h \leq x_{10} \). Then \( |\text{Fix}_{X \cap G}(S)| = \frac{q - 1}{2d_h} \) by Proposition 2.5, since \( N_G(S) = S.Z_{q-1} \). Thus, the number of points coming out from \( G \)-orbits on \( l \) of type (10) which fixed by \( S \) are exactly \( \sum_{j=1}^{x_1} \frac{q - 1}{d_j} \). These turn out to be \( \frac{1}{2}S_1 \) as \( S_1 = \sum_{j=1}^{x_10} \frac{q - 1}{d_j} \). Therefore, \( o(\text{Fix}(S)) + 1 = x_1 + \frac{1}{2}S_1 \). So, \( \sqrt[n]{n} + 1 = x_1 + \frac{1}{2}S_1 \), since \( \text{Fix}(S) \) is a Baer subplane of \( \Pi \). Then \( x_1 + \frac{1}{2}S_1 = x_1 + \frac{q - 1}{2}x_3 + S_2 \), since \( \sqrt[n]{n} + 1 = x_1 + \frac{q - 1}{2}x_3 + S_2 \) by (97). As a consequence

\[
S_1 = (q - 1)x_3 + 2S_2.
\]

Assume that \( x_3 > 0 \). Then \( x_3 = 1 \) by Lemma 3.5(2). Then \( S_1 \geq q - 1 \) and hence \( S \geq \frac{q^2 - 1}{2} \), since \( S = \frac{q + 1}{2}S_1 \). Now, by substituting \( S \geq \frac{q^2 - 1}{2} \) in (98) and
bearing in mind $x_3 = 1$, we obtain $n + 1 \geq \frac{q(q - 1)}{2} + \frac{q^2 - 1}{2}$. On the other hand, $n \leq (q - 1)^2$ since $n < q^2$ and $n$ is a square. Then $(q - 1)^2 + 1 \geq \frac{q(q - 1)}{2} + \frac{q^2 - 1}{2}$, which is a contradiction.

Assume that $x_3 = 0$. Then $S_1 = 2S_2$ by (102). Note that, $S_1 > 0$, as $S = \frac{q+1}{2}S_1$ and $S > 0$. As a consequence, $S_2 > 0$ being $S_1 = 2S_2$ and $S_1 = S_2 + S_2$. Now, we focus on the points on $l$ fixed by $S(\sigma)$. If $S(\sigma)$ fixes a point $Q$ on $l$, then $G_Q$ is either of type (1) or of type (10). So, $S(\sigma)$ fixes at least $x_1$ points on $l$. Furthermore, if $Q^G$ is of type (10), then $|\text{Fix}_{Q^G}(S(\sigma))| = \frac{q-1}{2d_j}$ for $d_j$ even and $0$ for $d_j$ odd by Proposition 2.5. Therefore, the number of points coming out from $G$-orbits on $l$ of type (10) which fixed by $S(\sigma)$ are exactly $\sum_{d_j=0}^{\frac{q-1}{2d_j}}$. These turn out to be $\frac{1}{2}S_2$ as $S_2 = \sum_{d_j=0}^{\frac{q-1}{2d_j}}$. It follows that, $S(\sigma)$ fixes exactly $x_1 + \frac{1}{2}S_2$ on $l$. Hence, $\sigma$ fixes exactly $x_1 + \frac{1}{2}S_2$ points on $\text{Fix}(S) \cap l$. Then $\sigma$ induces a Baer collineation on $\text{Fix}(S)$, since $x_1 + \frac{1}{2}S_2 \geq 3$, since $o(\text{Fix}(S)) + 1 = x_1 + \frac{1}{2}S_1$ with $S_1 > S_2 > 0$. Consequently, $\sqrt{7n+1} = x_1 + \frac{1}{2}S_2$, since $\text{Fix}(S)$ is a Baer subplane of $\Pi$. On the other hand, $\sqrt{7n+1} = x_1$ by (99). Hence, $x_1 + \frac{1}{2}S_2 = x_1$. This yields $S_2 = 0$. Thus, $S = 0$, since $S_1 = 2S_2$ and $S = \frac{q+1}{2}S_1$. This is a contradiction, since $S > 0$. So, $G$ does not fix lines of $\Pi$. \hfill \Box

Corollary 6.6. Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $G \cong \text{PSL}(2,5)$. If $n < 25$ and $G$ fixes a subplane $\Pi_0$ of $\Pi$, then $\Pi_0 \cong \text{PG}(2,4)$ and $n = 16$.

Proof. Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $G \cong \text{PSL}(2,5)$. Assume that $n < 25$ and that $G$ fixes a subplane $\Pi_0$ of $\Pi$ of order $m$. Clearly, $m < 5$ by [16, Theorem 3.7]. Then $\Pi_0 \cong \text{PG}(2,4)$ by Theorem 2.1. In particular, $G$ fixes a secant $l$ of $\Pi_0$ which is the kernel of the line oval of $\Pi_0$ left invariant by $G$ itself. Then $n = 16$ by Proposition 6.5. \hfill \Box

Theorem 6.7. Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $G \cong \text{PSL}(2, q)$ with $q \equiv 5 \mod 8$. If $n \leq q^2$, then one of the following occurs:

1. $n < q$, $\Pi \cong \text{PG}(2,4)$ and $G \cong \text{PSL}(2,5)$;
2. $n = q$, $\Pi \cong \text{PG}(2,q)$ and $G$ is strongly irreducible on $\Pi$;
3. $q < n < q^2$ and one of the following occurs:
   a. $G$ is strongly irreducible on $\Pi$;
   b. $n = 16$ and $G \cong \text{PSL}(2,5)$ fixes a point, or a line of $\Pi$ or subplane $\Pi_0 \cong \text{PG}(2,4)$;
4. $n = q^2$ and one of the following occurs:
(a) $G$ is strongly irreducible on $\Pi$;
(b) $G$ fixes a subplane $\Pi_0$ of $\Pi$. In particular, if $q \neq 5$, then $\Pi_0 \cong \text{PG}(2, q)$ is a Baer subplane of $\Pi$.

Proof. If $n \leq q$, the assertions (1) (2) easily follows by Theorems 2.1 and 2.2, respectively. If $q < n < q^2$, then either the assertion (3b) or group $G$ does not fix lines or points of $\Pi$ by Proposition 6.5 and its dual. If the latter occurs, the assertion (3a) easily follows by Lemma 3.1, since $q \equiv 5 \mod 8$ and by Corollary 6.6. Finally, if $n = q^2$, the assertions (4a) and (4b) and follow by Theorems 2.3 and 2.4, respectively.

At this point, Theorem 1.1 easily follows, when $q \equiv 5 \mod 8$, from Theorem 6.7.

7. The case $q \equiv 7 \mod 8$

Assume that $q \equiv 7 \mod 8$. Recall that $\sigma$ is a representative of the unique conjugate class of involution in $G$, and that $T_1$ and $T_2$ are the representatives of the two conjugate classes of Klein subgroups of $G$. In particular, $T_1$ and $T_2$ are chosen in order to contain $\sigma$. Furthermore, $C_G(\sigma) \cong D_{q+1}$ and $N_G(T_j) \cong S_4$ for each $j = 1$ or $2$.

We filter the list given in Lemma 3.4 with respect to the condition $q \equiv 7 \mod 8$. Then, for each point $P \in l$, either $G_P = G$ (type (1)), or $G_P \cong D_{q-1}$ (type (2)), or $G_P \cong D_{q+1}$ (type (3)), $G_P \cong A_5$ (type (5)), or $G_P \cong S_4$ (type (6)) or $G_P \cong F_{q, 2}$, where $d \mid \frac{q-1}{2}$ and $d$ odd (type (10)). Note that there are two conjugate classes of subgroups isomorphic to $A_5$ and two ones of subgroups isomorphic to $S_4$ by [4]. So, following the notation introduced in section 4, there are admissible subgroups of type (5a) and (5b), and admissible ones of type (6a) and (6b). Hence, $x_i = x_{ia} + x_{ib}$ for $i = 5$ or 6. The usual argument, involving Proposition 2.5, yields the table on the next page containing all the required informations about the admissible $G_P$.

The numbers $S$, $S_1$, $S_2$, $S_2'$, $S_{2,4}$ and $S_4$ have the usual meaning. In particular, $S_2 = S_{2,4} = S_4 = 0$, since $q \equiv 7 \mod 8$. Consequently, $S_1 = S_{2'}$ and $S = \frac{q+1}{2} S_{2'}$.

As in the preceding sections, we may consider the dual of table V, that is the table referred to the $G$-orbits of lines through some point $Q$ of $\Pi$ fixed by $G$. In particular, we might have $G$-orbits of lines of type $(ia)^*$ and $(ib)^*$ for $i = 5$ or 6, and it makes sense considering $S^*$, $S_1^*$, $S_2^*$, $S_{2,4}^*$ and $S_4^*$. Similarly to
above, we have $S_2^* = S_{2,4}^* = S_4^* = 0$, since $q \equiv 7 \mod 8$, and hence $S_1^* = S_2^*$ and $S^* = \frac{q+1}{2} S_2^*$.

Note that $\sigma$ is a Baer collineation of $\Pi$ by Lemma 3.3. Set $C = C_G(\sigma)$. Then $C$ acts on $\text{Fix}(\sigma)$ with kernel $K$. Hence, let $\bar{C} = C/K$. Clearly, $\langle \sigma \rangle \leq K \leq C$. Furthermore, either $K \leq Z_{\frac{q+1}{2}}$ or $K \cong D_{\frac{q+1}{2}}$ or $K = C$, since $C \cong D_{q+1}$ and $q \equiv 7 \mod 8$. As we will see, we need to investigate the admissible structure of $K$ in order to show that $T_j$ induces a Baer collineation on $\text{Fix}(\sigma)$ for each $j = 1$ or 2.

**Lemma 7.1.** If $\text{Fix}(T_j) \cap l = \text{Fix}(\sigma) \cap l$ for some $j = 1$ or 2, then either $K \cong D_{\frac{q+1}{2}}$ or $K = C$.

**Proof.** Assume that $\text{Fix}(T_1) \cap l = \text{Fix}(\sigma) \cap l$ and that $K \leq Z_{\frac{q+1}{2}}$. Then $\text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l$ by Table V, since $q > 9$. Set $l_0 = \text{Fix}(\sigma) \cap l$. Then $\bar{C} = \bar{C}(l_0)$, since $l_0 = \text{Fix}(G) \cap l$. In particular, $\bar{C} \cong D_{\frac{q+1}{2}}$, where $k = |K|$, $k$ is even and $k \mid \frac{q+1}{2}$. Then the group $\bar{C}$ is the semidirect product of $\bar{C}(l_0, l_0)$ with $\bar{C}(Y, l_0)$ for some point $Y \in \text{Fix}(\sigma) - l_0$ by [16, Theorem 4.25].

Assume that $\bar{C}(l_0, l_0) \neq \langle 1 \rangle$. If $\bar{C}(l_0, l_0) = \bar{C}$, then either $\bar{C} \cong E_4$ and $K \cong Z_{\frac{q+1}{4}}$ or $\bar{C} = \bar{C}(V, l_0)$ for some point $V \in l_0$ by [16, Theorem 4.14], since $C \cong D_{q+1}$ and $q \equiv 7 \mod 8$. Suppose the former occurs. Let $R_i$, $i = 1, 2$ or 3, be the (unique) points on $l_0$, such that $\bar{C}(R_i, l_0) \neq \langle 1 \rangle$. Actually, $\bar{C}(R_i, l_0) \cong Z_2$ for each $i = 1, 2, 3$. So, there are at least two points among the $R_i$, $i = 1, 2$ or 3, say $R_2$ and $R_3$, such that $C_h \cong D_{\frac{q+1}{2}}$ for each line $h \in [R_i] \sim \{l\}$.

| Type | $G_P$ | $[G : G_P]$ | $|\text{Fix}_{PG}(\sigma)|$ | $|\text{Fix}_{PG}(T_1)|$ | $|\text{Fix}_{PG}(T_2)|$ |
|------|-------|-------------|-----------------|-----------------|-----------------|
| 1    | $G$   | 1           | 1               | 1               | 1               |
| 2    | $D_{q-1}$ | $\frac{q(q+1)}{2}$ | $\frac{q+1}{2}$ | 0               | 0               |
| 3    | $D_{q+1}$ | $\frac{q(q+1)}{2}$ | $\frac{q+3}{2}$ | 3               | 3               |
| 5a   | $A_5$  | $\frac{q(q^2-1)}{120}$ | $\frac{q+1}{4}$ | 2               | 0               |
| 5b   | $A_5$  | $\frac{q(q^2-1)}{120}$ | $\frac{q+1}{4}$ | 0               | 2               |
| 6a   | $S_4$  | $\frac{q(q^2-1)}{48}$ | $\frac{3(q+1)}{8}$ | 4, $q \equiv 16 \ 15$ | 4, $q \equiv 16 \ 15$ |
|      |       |             |                 | 1, $q \equiv 16 \ 7$ | 3, $q \equiv 16 \ 7$ |
| 6b   | $S_4$  | $\frac{q(q^2-1)}{48}$ | $\frac{3(q+1)}{8}$ | 4, $q \equiv 16 \ 15$ | 4, $q \equiv 16 \ 15$ |
|      |       |             |                 | 3, $q \equiv 16 \ 7$ | 1, $q \equiv 16 \ 7$ |
| 10   | $F_{q, Z_d}$ | $\frac{q^2-1}{2d}$ | 0               | 0               | 0               |
$i = 2, 3$, since $K \cong Z_{q+1}$ and $C \cong D_{q+1}$. As a consequence, $D_{q+1} \leq G_h$ for each $h \in ([R_2] \cup [R_3]) - \{l\}$. Now, since $G$ fixes the $R_i$, $i = 2, 3$, we may filter the groups listed in the dual of Lemma 3.4 with respect to the condition $D_{q+1} \leq G_h$. Easy computation show that either $G_h = C$ or $G_h = G$, which is a contradiction, since $C_h \cong D_{q+1}$. Therefore, $\tilde{C} = \tilde{C}(V, l_0)$ for some point $V \in l_0$ by \cite[Theorem 4.14]{G}. Thus, for each point $X \in l_0 - \{V\}$ and for each line $t \in [X] \cap \text{Fix}(\sigma)$, we have $\sigma \in G_t$, but $G_t$ does not contain Klein groups. Then $G_t \cong D_{q-1}$ by dual of table $V$, since $G$ fixes $X$. Assume there exists $u \in [X] \cap \text{Fix}(\sigma)$ such that $G_u \cong D_{q-1}$. Clearly, $K \leq G_u$. Then $K = \langle \sigma \rangle$, since $\langle \sigma \rangle \cong Z_{q+1}$, so, $\tilde{C} \cong D_{q+1}$ and hence $\sqrt{n} = \frac{q+1}{2}$, since $\sqrt{n} < q$ by our assumptions. On the other hand, $u^G \subset [X] - \{l\}$ as $G$ fixes $X$. Then $n \geq \frac{q(q+1)}{2}$ since $u^G = \frac{q(q+1)}{2}$ as $G_u \cong D_{q-1}$. Then $(\frac{q+1}{2})^2 \geq 2\frac{q(q+1)}{2}$, since $\sqrt{n} = \frac{q+1}{2}$. This contradicts the fact that $q > 9$. Thus, $\tilde{C}(l_0, l_0) < \tilde{C}$. Then $\tilde{C}(l_0, l_0) \leq Z_{q+1}$, since $\tilde{C} \cong D_{q+1}$. Actually, $\tilde{C}(l_0, l_0) = \tilde{C}(V, l_0) \cong Z_{q+1}$ and $\tilde{C}(Y, l_0) \cong Z_2$ by \cite[Theorems 4.14 and 4.25]{G}. Let $s \in [V] - \{l, VY\}$, then $s$ is fixed by $K$ and by $\tilde{C}(V, l_0)$. Therefore, $G_s \cap C \cong Z_{q+1}$. It follows that $Z_{q+1} \leq G_s$. Then either $G_s = C_G(\sigma)$ or $G_s = G$ by dual of Lemma 3.4, since $G$ fixes $l_0$, since $q > 9$. This is a contradiction, since $G_s \cap C \cong Z_{q+1}$. Hence, $\tilde{C}(l_0, l_0) = \{1\}$.

Assume that $\tilde{C} = \tilde{C}(Y, l_0)$ for some point $Y \in \text{Fix}(\sigma) - l_0$. Let $Q \in \text{Fix}(\sigma) \cap l$ and let $m \in [Q] \cap \text{Fix}(\sigma) - \{l, VQ\}$. Then $\sigma \in G_m$ but $G_m$ does not contain Klein groups. So, $G_m \cong D_{q-1}$ by dual of table $V$, since $G$ fixes $Q$. Therefore, $x_2^s \geq 1$. Furthermore, $x_3^s \geq 1$, since $G_{VQ} = C$. Thus, $x_2^s + x_3^s \geq 2$, which is a contradiction by dual of Lemma 3.6(1), being $q > 9$.

\textbf{Lemma 7.2.} It holds that $\text{Fix}(T_j) \cap l \subset \text{Fix}(\sigma) \cap l$ for each $j = 1, 2$.

\textbf{Proof.} Assume that $\text{Fix}(T_1) \cap l = \text{Fix}(\sigma) \cap l$. Then either $K \cong D_{q+1}$ or $K = C$ by Lemma 7.1.

Assume that $K = C$. Then $\text{Fix}(T_1) = \text{Fix}(\sigma)$. Let $P$ be any point of $\text{Fix}(\sigma) \cap l$ and let $r$ be any line of $[P] - \{l\}$. So, $C \leq G_r$. Since $q > 9$, then $C$ is maximal in $G$ and hence either $G_r = C$ or $G_r = G$. If the former occurs, then $|\text{Fix}_{r,C}(T_1)| = 3$ and $|\text{Fix}_{r,C}(\sigma)| = \frac{q+3}{2}$ by dual of table $V$. Therefore, $|\text{Fix}_{r,C}(\sigma)| > |\text{Fix}_{r,C}(T_1)|$ as $q > 9$. This is impossible, since $\text{Fix}(T_1) = \text{Fix}(\sigma)$. Thus, $G_r = G$ for any point $P$ of $\text{Fix}(\sigma) \cap l$ and for any line $r$ of $[P] - \{l\}$. Consequently, $\text{Fix}(G) = \text{Fix}(\sigma)$, since $\text{Fix}(G) \cap l = \text{Fix}(\sigma) \cap l$ and $\text{Fix}(G) \subset \text{Fix}(\sigma)$. Thus, $G$ fixes a Baer subplane of $\Pi$. Then $G$ is semiregular on $l - \text{Fix}(G)$ and hence $|G| = n - \sqrt{n}$, which is impossible.

Assume that $K \cong D_{q+1}$. Then $D_{q+1} \leq G_f$ for each line $f$ of $\text{Fix}(\sigma) - \{l\}$. Therefore, either $G_f = \tilde{C}$ or $G_f = G$ by dual of Lemma 3.4, being $q \equiv 7 \mod 8$.
and \( q > 9 \). Now, the above argument yields \( \text{Fix}(G) = \text{Fix}(\sigma) \) and we again obtain a contradiction. Thus, \( \text{Fix}(T_1) \cap \ell \subset \text{Fix}(\sigma) \cap \ell \).

Now, repeating the above argument with \( T_2 \) in the role of \( T_1 \), we obtain \( \text{Fix}(T_2) \cap \ell \subset \text{Fix}(\sigma) \cap \ell \). Hence, we have proved the assertion. \( \square \)

**Lemma 7.3.** The group \( T_j \) induces a Baer collineation on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \).

**Proof.** The group \( T_j \) induces an involution \( \overline{\beta}_j \) on \( \text{Fix}(\sigma) \) for each \( j = 1, 2 \) by Lemma 7.2. Assume that \( \overline{\beta}_1 \) is a \((C_{\overline{\beta}_1}, a_{\overline{\beta}_1})\)-elation of \( \text{Fix}(\sigma) \). Then \( C_{\overline{\beta}_1} \subset \ell \) and \( a_{\overline{\beta}_1} \neq \ell \) again by Lemma 7.2. Hence \( \text{Fix}(T_1) \cap \ell = \{ C_{\overline{\beta}_1} \} \). Thus \( N_G(T_1) \leq G_{C_{\overline{\beta}_1}} \), where \( N_G(T_1) \cong S_4 \). Then either \( G_{C_{\overline{\beta}_1}} = N_G(T_1) \) or \( G_{C_{\overline{\beta}_1}} = G \) by table \( V \).

Actually \( G_{C_{\overline{\beta}_1}} = G \) cannot occur, otherwise we have a contradiction by dual of Lemma 6.2, since \( \text{Fix}(T_1) \cap [C_{\overline{\beta}_1}] = \text{Fix}(\sigma) \cap [C_{\overline{\beta}_1}] \). Hence \( G_{C_{\overline{\beta}_1}} = N_G(T_1) \) and hence \( x_6 > 0 \). Actually \( x_{6a} = 1 \) and \( q \equiv 7 \mod 16 \) by table \( V \). Moreover, \( x_1 = x_3 = x_{5a} = x_{6b} = 0 \) again by table \( V \). Also, \( q = 23 \) and \( x_5 = 0 \) by Lemma 3.4(6), and \( x_2 = 0 \) by Lemma 3.6(4). Therefore \( \text{Fix}(\sigma) \cap \ell = \text{Fix}(\sigma) \cap C_{\overline{\beta}_1}^G \) and hence \( \sqrt{n} + 1 = \frac{3(q+1)}{8} \) again by table \( V \) being \( x_{6a} = 1 \). That is \( \sqrt{n} = 8 \), as \( q = 23 \). Thus also \( T_2 \) must induce an elation on \( \text{Fix}(\sigma) \). Nevertheless \( T_2 \) fixes exactly 3 points on \( C_{\overline{\beta}_1}^G \) by table \( V \), since \( G_{C_{\overline{\beta}_1}} = S_4 \) and \( q \equiv 7 \mod 16 \). Then \( T_2 \) fixes exactly 3 points on \( \text{Fix}(\sigma) \cap \ell \) as \( \text{Fix}(\sigma) \cap \ell = \text{Fix}(\sigma) \cap C_{\overline{\beta}_1}^G \). This is a contradiction, since \( T_2 \) induces an elation on \( \text{Fix}(\sigma) \) and \( \sqrt{n} = 8 \).

Assume that \( \overline{\beta}_1 \) is a \((C_{\overline{\beta}_1}, a_{\overline{\beta}_1})\)-homology of \( \text{Fix}(\sigma) \). Again \( C_{\overline{\beta}_1} \subset \ell \) and \( a_{\overline{\beta}_1} \neq \ell \) by Lemma 7.2. Set \( \{ X \} = a_{\overline{\beta}_1} \cap \ell \). Hence \( \text{Fix}(T_1) \cap \ell = \{ C_{\overline{\beta}_1}, X \} \). Let \( \overline{\gamma} \) be the collineation induced by \( \gamma \) on \( \text{Fix}(\sigma) \), where \( \gamma \in G \) and \( \gamma^2 = \sigma \) (clearly such a element does exist in \( G \), since \( q \equiv 7 \mod 8 \)). Then either \( \overline{\gamma} = 1 \) or \( \overline{\gamma} \) is a Baer involution or a involutory perspectivity. Nevertheless \( \overline{\gamma} \) centralizes \( \overline{\beta}_1 \) in each cases. Then \( \overline{\gamma} \) fixes \( C_{\overline{\beta}_1}, a_{\overline{\beta}_1}^2 \) and hence \( X \). Thus \( N_G(T_1) \leq G_{C_{\overline{\beta}_1}} \) and \( N_G(T_1) \leq G_X \). Similar argument to that used above yields \( G_{C_{\overline{\beta}_1}} < G \) and hence \( G_{C_{\overline{\beta}_1}} = N_G(T_1) \) by table \( V \), since \( N_G(T_1) \cong S_4 \). Thus \( x_{6a} > 0 \). Then \( x_6 = x_{6a} = 1 \) by Lemma 3.5(5) as \( q \equiv 7 \mod 16 \). Hence \( G_X = G \). Therefore \( x_1 = 1 \), since \( \text{Fix}(G) \cap \ell \subset \text{Fix}(T_1) \cap \ell \). Furthermore, \( q = 23 \) and \( x_5 = 0 \) by Lemma 3.4(6), and \( x_2 = 0 \) by Lemma 3.6(4). Finally, \( \sqrt{n} + 1 = \frac{3(q+1)}{8} x_{6a} + x_1 \) by table \( V \), where \( x_1 = x_{6a} = 1 \). That is \( \sqrt{n} = \frac{3(q+1)}{8} \). Then \( n = 81 \) as \( q = 23 \). On the other hand \( n + 1 \geq \frac{9(q^2-1)}{48} + 1 \) again by table \( V \). Hence, \( \left[ \frac{3(q+1)}{8} \right]^2 \geq \frac{9(q^2-1)}{48} \), which is a contradiction, since \( q = 23 \). Thus, \( T_1 \) induces a Baer involution on \( \text{Fix}(\sigma) \).

Arguing as above, with \( T_2 \) in the role of \( T_1 \), we have that \( T_2 \) induces a Baer involution on \( \text{Fix}(\sigma) \). Hence, we have proved the assertion. \( \square \)
**Lemma 7.4.** For each point \( P \in \ell \) the group \( G_P \) cannot be isomorphic either to \( S_4 \) or to \( A_5 \).

**Proof.** Assume that \( x_6 > 0 \). Then \( x_6 = 1 \) by Lemma 3.5(5), being \( q \equiv 7 \) mod 8. We may assume that \( x_6 = x_{6a} = 1 \) without loss of generality (see Table V).

Let \( Q \in \ell \) such that \( QG \cong S_4 \). Then \( |Q^G| = \frac{q(q^2-1)}{48} \) and hence \( n \geq \frac{q(q^2-1)}{48} - 1 \), as \( Q^G \subset \ell \). Also, \( q = 23 \) or 31 by Lemma 3.4. Easy computations show that \( \sqrt{n} = 4 \) for \( q = 23 \) and \( \sqrt{n} = 5 \) for \( q = 31 \), since \( \frac{q(q^2-1)}{48} - 1 \leq n < q^2 \) with \( n \) a fourth power by Lemma 7.3. In both cases \( n + 1 - \frac{q(q^2-1)}{48} < q + 1 \). It follows that \( |l - Q^G| < q + 1 \), since \( |l - Q^G| = n + 1 - \frac{q(q^2-1)}{48} \). Then \( G \) fixes \( l - Q^G \) pointwise, since the minimal primitive permutation representation of \( G \) is \( q + 1 \), being \( q = 23 \) or 31. That is \( x_1 = |l - Q^G| \). If \( q = 23 \), then \( o(Fix(T_1)) = x_1 \) and \( o(Fix(T_2)) = x_1 + 3 \) by Table V, since \( x_6 = 1 \) and \( q \equiv 7 \) mod 16. Nevertheless, \( o(Fix(T_1)) = o(Fix(T_2)) \) by Lemma 7.3. Hence, we arrive at a contradiction. As a consequence, \( q = 31 \). Thus, \( x_1 = 6 \). Therefore, \( \sqrt{n} + 1 \geq 10 \) as \( \sqrt{n} + 1 \geq x_1 + 4x_{6a} \). This is impossible, since \( \sqrt{n} = 5 \). So, \( x_6 = 0 \).

Assume that \( x_5 > 0 \). Since \( T_1 \) and \( T_2 \) fix Baer subplanes of \( Fix(\sigma) \), we have \( \sqrt{n} + 1 = x_1 + 3x_3 + 2x_{5a} \) and \( \sqrt{n} + 1 = x_1 + 3x_3 + 2x_{5b} \) by Table V, since \( x_6 = 0 \). Then \( \sqrt{n} + 1 = x_1 + 3x_3 + x_5 \) summing up these two equations and by bearing in mind that \( x_5 = x_{5a} + x_{5b} \). Hence, by Table V, we have

\[
\sqrt{n} + 1 = x_1 + 3x_3 + x_5
\]

(103)

\[
\sqrt{n} + 1 = x_1 + \frac{q+1}{2}x_2 + \frac{q+3}{2}x_3 + \frac{q+1}{4}x_5
\]

(104)

\[
n + 1 = x_1 + \frac{q(q+1)}{2}x_2 + \frac{q(q-1)}{2}x_3 + \frac{q(q^2-1)}{120}x_5 + S.
\]

(105)

Note that \( q = 31, 71 \) or 79 by Lemma 3.4. Assume that \( x_2 + x_3 > 0 \). Then \( x_2 + x_3 = 1, 0 < x_5 \leq 2 \) and \( q = 31 \) by Lemma 3.6(2) and (4). Therefore, \( \sqrt{n} = 3, 4 \) or 5, since \( q < n < q^2 \) being \( n \) a fourth power by Lemma 7.3. On the other hand, \( n + 1 \geq \frac{q(q-1)}{2} + \frac{q(q^2-1)}{120} \) by (105), since \( x_2 + x_3 = 1 \). That is \( n + 1 \geq 713 \), as \( q \geq 31 \). Nevertheless, this is a contradiction, since \( n \leq 5^4 \). Thus, \( x_2 = x_3 = 0 \). Then \( \sqrt{n} + 1 = x_1 + x_5 \) and \( \sqrt{n} + 1 = x_1 + \frac{2x+1}{4}x_5 \) by (103) and (104). By elementary calculations of the previous equations, we obtain \( \sqrt{n} - \sqrt{n} - \frac{2x+1}{4}x_5 = 0 \), where \( x_5 \leq 3 \) by Lemma 3.5(4), and where \( q = 31, 71 \) or 79. It is a straightforward computation to see that, no integer solutions arise. Thus, \( x_5 = x_6 = 0 \) and we obtain the assertion. \( \square \)

**Proposition 7.5.** Let \( \Pi \) be a projective plane of order \( n \) admitting a collineation group \( G \cong PSL(2, q) \) fixing a line \( \ell \). If \( q < n < q^2 \) and \( q \equiv 7 \) mod 8 then \( \Pi \) is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and \( G \cong PSL(2, 7) \).
Proof. Suppose that $G$ fixes a line $l$ of $\Pi$. Assume that $q > 7$. Hence, $q > 9$ as $q \equiv 7 \mod 8$. Recall that $|\text{Fix}(T_j) \cap l| = \sqrt{n} + 1$ by Lemma 7.3, and that for each point $P \in l$, the group $G_P$ cannot be isomorphic either to $A_4$ or to $A_5$ by Lemma 7.4. Then, by Table V, we have the following system of Diophantine equations:

\begin{align}
\sqrt{n} + 1 &= x_1 + 3x_3 \quad \text{(106)} \\
\sqrt{n} + 1 &= x_1 + \frac{q + 1}{2}x_2 + \frac{q + 3}{2}x_3 \quad \text{(107)} \\
n + 1 &= x_1 + \frac{q(q + 1)}{2}x_2 + \frac{q(q - 1)}{2}x_3 + S. \quad \text{(108)}
\end{align}

If $x_3 > 0$, then $x_3 = 1$ by Lemma 3.5(2). Furthermore, $x_2 = 0$ by Lemma 3.6(2). Then $\sqrt{n} = x_1 + 2$ and $\sqrt{n} = x_1 + \frac{q+3}{2}$. By composing these equations, we have $(x_1 + 2)^2 = x_1 + \frac{q+1}{2}$, and hence $x_1^2 + x_1 - \frac{q+3}{2} = 0$. If $x_1 \leq 2$, it is easily seen that $(n, x_1, q) = (2^4, 0, 7)$ as $q \equiv 7 \mod 8$. Nevertheless, $n + 1 \geq 21$ by (108) as $x_3 = 1$. So, $x_1 \geq 3$ for $x_3 > 0$. Actually, $x_1 \geq 3$ also for $x_3 = 0$ by (106), since it must be $\sqrt{n} \geq 2$. Consequently, $x_1 \geq 3$ in each case. Thus, $G$ fixes at least 3 points on $l$.

Let $P$ be any of these points. Applying the dual of Lemma 7.4, we obtain that the group $G_r$ cannot be isomorphic either to $A_4$ or to $A_5$ for each $r \in [P] - \{l\}$. Thus, $x_4^* = x_5^* = 0$. By dual of Table VI, we obtain the same system of Diophantine equations as (106), (107) and (108) but referred to $[P]$ and with the $x_i^*$ in the role of $x_i$. At this point, we may repeat the above argument showing that $G$ fixes at least 3 lines (including $l$) through any point $P$ of $\text{Fix}(G) \cap l$. So, $G$ fixes a subplane of $\Pi$ pointwise, as $|\text{Fix}(G) \cap l| \geq 3$. In particular, $o(\text{Fix}(G)) = x_1 - 1$. Now, we may use the same argument of Theorem 5.5, with (106), (107) and (108) in the role of (85), (86) and (87), respectively, in order to obtain that $G$ fixes a subplane of $\Pi$ of order $\sqrt{n}$. Hence, we have a contradiction.

Assume that $q \leq 7$. Actually, $q = 7$, since $q \equiv 7 \mod 8$. Then either $n = 16$ or $25$, since $q < n < q^2$ and since $\sqrt{n} \equiv 0, 1 \mod 4$ by Lemma 3.3. Assume that $q = 25$. Let $\varphi$ be any element in $G$ of order 7. Then $\varphi$ fixes at least 5 points on $l$ and 2 on $\Pi - l$, as $n + 1 \equiv 5 \mod 7$ and $n^2 \equiv 2 \mod 7$. Thus, $o(\text{Fix}(\varphi)) = 4 + 7\theta$, where $\theta \geq 0$. Actually, $\theta = 0$ by [16, Theorem 3.7], since $n = 25$. So, $o(\text{Fix}(\varphi)) = 4$. Note that $N_G(\langle \varphi \rangle) = \langle \varphi, \psi \rangle$, where $o(\psi) = 3$ and $\psi$ normalizes $\langle \varphi \rangle$. Also, $N_G(\langle \varphi \rangle)$ is the unique maximal subgroup of $G$ containing $\varphi$. Therefore, for each point $Q \in \text{Fix}(\varphi) \cap l$, either $G_Q = \langle \varphi \rangle$ or $G_Q = \langle \varphi, \psi \rangle$ or $G_Q = G$. Assume that $G_B = \langle \varphi \rangle$ for some $B \in \text{Fix}(\varphi) \cap l$. Then $|B^G| = 24$. Thus, $l$ consists of $B^G$ and of 2 points fixed by $G$ as $n + 1 = 26$. Consequently, any involution fixes exactly 2 points on $l$, namely those fixed by $G$, since $|B^G| = 24$ and $|G| = 168$. Hence, the involutions are homologies of $\Pi$, which is a contradiction by Lemma 3.3. It follows that either $G_Q = \langle \varphi, \psi \rangle$ or
$G_Q = G$ for or each $Q \in \text{Fix}(\varphi) \cap l$. Nevertheless, $\text{Fix}(\varphi) \cap l = \text{Fix}(\langle \varphi, \psi \rangle) \cap l$. So, $|\text{Fix}(N_G(\langle \varphi, \psi \rangle)) \cap l| = 5$. Assume that $|\text{Fix}(\langle \varphi, \psi \rangle) \cap l - \text{Fix}(G) \cap l| \geq 3$. Since this group is maximal, then there are at least 3 orbits of length 8. Therefore, $l$ consists of three $G$-orbits each of length 8 and of 2 points fixed by $G$. Thus, any involution fixes exactly 2 points on $l$ and we again obtain a contradiction by Lemma 3.3. It follows that $|\text{Fix}(N_G(\langle \varphi, \psi \rangle)) \cap l - \text{Fix}(G) \cap l| \leq 2$ and hence $|\text{Fix}(G) \cap l| \geq 3$. Now, we may repeat the above argument with $[X]$ in the role of $l$ for each point $X \in \text{Fix}(G) \cap l$. This yields $|\text{Fix}(G) \cap [X]| \geq 3$ for each $X \in \text{Fix}(G) \cap l$. Then $G$ is planar, since $|\text{Fix}(G) \cap l| \geq 3$. Therefore, $o(\text{Fix}(G)) \geq 2$.

Now, let $\beta$ be any involution of $G$. Then $o(\text{Fix}(\beta)) = 5$ by Lemma 3.3 as $n = 25$. Note that $\text{Fix}(G) \subset \text{Fix}(\beta)$, since $\varphi$ and $\beta$ fix exactly 4 and 6 points on $l$, respectively. So, we have a contradiction by [16, Theorem 3.7], since $o(\text{Fix}(\beta)) = 5$ while $2 \leq o(\text{Fix}(G)) < 5$. Thus, $n = 16$. Then either $\Pi$ is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or one of their duals by [3]. Hence, we have proved the assertion.

**Theorem 7.6.** Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $G \cong \text{PSL}(2, q)$, with $q \equiv 7 \mod 8$. If $n \leq q^2$, then one of the following occurs:

1. $n < q$, $\Pi \cong \text{PG}(2, 2)$ or $\text{PG}(2, 4)$ and $G \cong \text{PSL}(2, 7)$;
2. $n = q$ and $\Pi \cong \text{PG}(2, q)$;
3. $q < n < q^2$ and one of the following occurs:
   - (a) $G$ is strongly irreducible on $\Pi$;
   - (b) $n = 16$, $\Pi$ is the Lorimer-Rahilly plane or the Johnson-Walker plane, or one of their duals, and $G \cong \text{PSL}(2, 7)$;
   - (c) $G \cong \text{PSL}(2, 7)$ fixes a subplane of $\Pi$ isomorphic either to $\text{PG}(2, 2)$ or to $\text{PG}(2, 4)$;
4. $n = q^2$ and one of the following occurs:
   - (a) $G$ is strongly irreducible on $\Pi$;
   - (b) $G$ fixes a Desarguesian Baer subplane $\Pi_0$ of $\Pi$.

**Proof.** If $n \leq q$, the assertions (1) and (2) easily follow by Theorems 2.1 and 2.2, respectively. If $q < n < q^2$, then either the assertion (3b) or the group $G$ fixes lines or points of $\Pi$ by Proposition 5.5 and its dual. If the latter occurs, the assertions (3a) and (3c) easily follow by Lemma 3.1, since $q \equiv 7 \mod 8$. Finally, the assertions (4a) and (4b) follow by Theorems 2.3 and 2.4, respectively. □
Now, Theorem 1.1, when \( q \equiv 7 \mod 8 \), easily follows from Theorem 7.6.

8. Concluding proofs and other examples

Proof of Theorem 1.1. Let \( \Pi \) be a projective plane of order \( n \) admitting a collineation group \( G \cong \text{PSL}(2,q) \), \( q > 3 \). Assume that \( n \leq q^2 \). If \( q \) is odd, the assertion of Theorem 1.1 easily follows by Theorems 4.22, 5.6, 6.7 and 7.6 for \( q \equiv 1, 3, 5, 7 \mod 8 \), respectively. It remains to investigate the case \( q \) even in order to complete the proof of the theorem. Hence, assume that \( G \cong \text{PSL}(2,q) \), with \( q = 2^h \), \( h > 1 \). Since \( \text{PSL}(2,4) \cong \text{PSL}(2,5) \) and since we have already dealt with this case in Theorem 6.7, we may assume that \( q > 4 \).

(I) If \( n < q^2 \) the involutions in \( G \) are perspectivities of \( \Pi \).

Assume that \( n < q^2 \). Assume also that the involutions in \( G \) are Baer collineations of \( \Pi \). Let \( H \) be an elementary abelian subgroup of \( G \) of order \( q \). Then \( H \) fixes a point \( X \) of \( \Pi \), since \( n^2 + n + 1 \) is odd. Furthermore, each non trivial element in \( H \) fixes exactly \( \sqrt{n} + 1 \) lines through \( X \), since \( H - \{1\} \) consists of involutions. Then \( q \mid (q - 1)(\sqrt{n} + 1) + (n + 1) \) by [16, Result 1.14]. Hence, \( q \mid \sqrt{n} - 1 \). Thus, either \( q \mid \sqrt{n} - 1 \) or \( q \mid \sqrt{n} \), since \( q = 2^h \), \( h > 1 \). So, \( \sqrt{n} \geq q \) and therefore \( n \geq q^2 \) in any case. This is a contradiction, since \( n < q^2 \) by our assumption. Thus, the involutions in \( G \) are perspectivities of \( \Pi \), since \( G \cong \text{PSL}(2,q) \) contains a unique conjugate class of involutions by [4].

(II) If \( n < q^2 \) and \( n \neq q \), then \( G \) does not fix lines of \( \Pi \).

Assume that \( n < q^2 \), \( n \neq q \). Assume also that \( G \) fixes a line \( l \) of \( \Pi \). Let \( H \) be as above.

Suppose that \( n \) is even. Then \( H = H(C,C) \) for some point \( C \in l \) by (I), since \( H \) is an elementary abelian 2-group fixing \( l \) and since \( H(l) = \{1\} \) by Lemma 3.2(1). So, \( H \leq G_C \). Furthermore, \( G_C < G \) by Lemma 3.2(2). Then \( G_C \leq H.Z_d \), where \( d \mid q - 1 \) by [17, Hauptsatz II.8.27]. Note that \( H \) fixes exactly \( \frac{q-1}{d} \) points in \( C^G \) by (1) of Proposition 2.5. Nevertheless, \( H \) fixes exactly 1 point on \( l \). Then \( \frac{q-1}{d} = 1 \) and hence \( G_C \cong H.Z_{q-1} \). In particular, \( |C^G| = q + 1 \). Thus, \( n \geq q \). Actually, \( n > q \), since \( n \neq q \) by our assumptions. In addition, since \( H \) is a Sylow 2 subgroup of \( G \), then each Sylow 2 subgroup of \( G \) fixes exactly 1 point on \( l \) which lies in \( C^G \). Therefore, \( G_X \) has odd order for each point \( X \in l - C^G \). Such points do exist as \( n > q \). Moreover, \( |X^G| < q^2 - q \), since \( X^G \subseteq l - C^G \), and since \( |l - C^G| < q^2 - q \) as \( n < q^2 \). This yields \( |G_X| > q + 1 \) with \( |G_X| \) odd.
Hence, we arrive at a contradiction by a direct inspection of the list given in [17, Hauptsatz II.8.27].

Suppose that \( n \) is odd. Then \( H \) consists of homologies of \( \Pi \) by (I), since \( H \) is an elementary abelian 2-group. In particular, \( H = H(C, a) \), where \( C \in l, a \neq l \) by [19, Lemma (3.1)], since \( H(l) = \{1\} \) by Lemma 3.2(1). Set \( \{Q\} = a \cap l \).

Clearly, \( Q \neq C \). Arguing as above, we have \( G_C \leq H.Z_d \), where \( d \mid q - 1 \). Consequently, \( H \) fixes exactly \( \frac{q-1}{d} \) points in \( C^G \). Nevertheless, \( H \) fixes exactly 2 points on \( l \). Then \( \frac{q-1}{d} = 1 \), as \( q \) is even. Therefore, \( |C^G| = q + 1 \). Thus, \( n \geq q \), since \( C^G \subseteq l \). Actually, \( n > q \), since \( n \neq q \) by our assumptions. In particular, \( Q \notin C^G \). The above argument, with \( Q^G \) in the role of \( C^G \), yields that either \( |Q^G| = 1 \) or \( |Q^G| = q + 1 \). It should be stressed that, differently from \( C \), the possibility \( |Q^G| = 1 \) might occur. Indeed, Lemma 3.2(2) cannot be applied to \( Q \) as \( H = (C, a) \), \( \{Q\} = a \cap l \) and \( Q \neq C \). Now, suppose that \( l - (C^G \cup Q^G) \neq \emptyset \).

Thus there exists a point \( Y \in l - (C^G \cup Q^G) \) such that \( |G_Y| \) is odd. Moreover, \( |G_Y| > q + 1 \), since \( Y^G \subseteq l - (C^G \cup Q^G) \), and since \( |l - (C^G \cup Q^G)| < q^2 - q - 1 \) as \( n < q^2 \) and \( |Q^G| \geq 1 \). This leads to a contradiction by a direct inspection of the list given in [17, Hauptsatz II.8.27], since \( |G_Y| \) is odd. Thus, \( l = C^G \cup Q^G \). Since \( |C^G| = q + 1 \), then either \( n = q + 1 \) or \( n = 2q + 1 \) according to whether \( |Q^G| = 1 \) or \( |Q^G| = q + 1 \), respectively. So, we obtain a contradiction in each case by [23, Theorem 26], since \( G \) acts 2-transitively on \( C^G \). As a consequence, \( G \) does not fix lines of \( \Pi \).

(III) Either \( n = q \) or \( n = q^2 \).

If \( n < q^2 \) and \( n \neq q \), then \( G \) does not fix points or lines of \( \Pi \) by (II) and its dual. Furthermore, \( G \) does not fix triangles of \( \Pi \), since \( G \) is simple as \( q > 3 \). So, \( G \) is irreducible on \( \Pi \). Moreover, \( G \) contains involutory perspectivities by (I). This is impossible by [12, Lemma 5.1], since \( q \) is even and \( q > 4 \). Thus, either \( n = q \) and hence \( \Pi \cong \text{PG}(2, q) \) by Theorem 2.2, or \( n = q^2 \). That is the assertions (2a) and (4a.iii) (of Theorem 1.1). This completes the proof.

Once Theorem 1.1 has been proved, Theorem 1.2 is just a consequence of this one. Theorem 1.3 follows in turn by a combination of Theorem 1.2, of Theorems 2.1 and 3.3 of [6] and of Theorem 5.1 of [7].

Finally, we have the following other examples for Theorem 1.1 (these are not quoted examples in [15, Theorem A] or [13, Theorem 6.1] or [14, Theorem C]):

1. Let \( G \cong \text{PSL}(2, 7) \) and let \( \Gamma \cong \text{PSL}(3, m^h) \), with \( 7 < m^h < 49 \).

Assume that \( m^h \) is odd. If \( G \leq \Gamma \), then \( m^{3h} \equiv 1 \mod 7 \) by [1] and this case really occurs. Actually, \( m^h = 9, 11, 23, 25, 29, 37 \) or 43, as \( 7 < m^h < 49 \).
Since the other cases are already quoted in [15] or [13] or [14], we may assume that $q = 25$ or 43. Hence $G \cong \text{PSL}(2, 7)$ acts $\Pi \cong \text{PG}(2, 25)$ or $\text{PG}(2, 43)$. In latter, clearly, the involutions are homologies. Furthermore, by Theorem 1.1, the group $G$ is strongly irreducible (we do not need to have additional assumptions as in Theorem A of [15]).

Assume that $m^h = 32$ is even. Then $G \leq \Gamma$ by [8]. Hence, $G \cong \text{PSL}(2, 7)$ acts on $\Pi \cong \text{PG}(2, 32)$.

(2) Let $G \cong \text{PSL}(2, 9)$ and let $\Gamma \cong \text{PSL}(3, m^h)$, with $9 < m^h < 81$.

Assume that $m^h$ is odd. If $G \leq \Gamma$, then either $m^h \equiv 1, 19 \mod 30$ or $m = 5$ and $h$ even by [1] and these cases really occur. Actually, $m^h = 19, 31, 25, 49, 61$ or 79, as $9 < m^h < 81$. Since the other cases are already quoted in [15] or [13] or [14], we may assume that $q = 49$ or 79. While in latter the involutions are clearly homologies, in the former this follows by Theorem 2.6. Furthermore, it follows by Theorem 1.1 that, the group $G$ is strongly irreducible (no additional assumptions are required, as in Theorem A of [15]).

(3) Let $G \cong \text{PSL}(2, 9)$. Then $G$ is a subgroup of $\text{PSL}(3, 4)$ by [2]. Now, the group $\text{PSL}(3, 4)$, and hence $\text{PSL}(2, 9)$, leaves invariant a Desarguesian sub-plane of order 4 in a Desarguesian plane or a Figueroa plane of order 64 (see [5], [10])(so this is an example for the case (3b.iii) of Theorem 1.1.

References


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