Elation groups of the Hermitian surface $H(3,q^2)$ over a finite field of characteristic 2

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Abstract

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathcal{I})$ be a finite generalized quadrangle (GQ) having order (s,t). Let p be a point of $\mathcal{S}$. A *whorl* about p is a collineation of $\mathcal{S}$ fixing all the lines through p. An *elation* about p is a whorl that does not fix any point not collinear with p, or is the identity. If $\mathcal{S}$ has an elation group acting regularly on the set of points not collinear with p we say that $\mathcal{S}$ is an elation generalized quadrangle (EGQ) with base point p. The following question has been posed: Can there be two non-isomorphic elation groups about the same point p? In this presentation, we show that there are exactly two (up to isomorphism) elation groups of the Hermitian surface $H(3,q^2)$ over a finite field of characteristic 2.

Keywords: generalized quadrangles, elation groups, Hermitian surface

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1. Introduction

The focus of this article is $H(3,q^2)$, the Hermitian surface in three-dimensional projective space over the field GF(q^2), where $q=2^e$. The first results were discovered by Tim Penttila, for $q=2$ and $q=4$, using the software package Magma [5]. In this paper we give a constructive proof for any $q=2^e$. We introduce generalized quadrangles with some basic definitions.

Let $\mathcal{P}$ and $\mathcal{B}$ be two non-empty sets, called points and lines, with an incidence relation $\mathcal{I}$ such there are two positive integers s and t satisfying

(1) Each point is incident with $t+1$ lines; any two points are mutually incident with at most one line.

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(2) Each line is incident with $s+1$ points; any two lines are mutually incident with at most one point.

(3) Given a line L and a point x not incident with L there is a unique point y and a unique line M such that $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

Such a collection $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a *generalized quadrangle of order* (s,t) written GQ(s,t). The *dual* of a GQ(s,t) is the GQ(t,s) obtained by interchanging the roles of points and lines. Furthermore, any theorem or definition given for a GQ can be dualized by interchanging the words points and lines. It will therefore be assumed that whenever a definition or theorem is given, its dual has also been given.

Two points incident with a common line are said to be *collinear* and two lines incident with a common point are *concurrent*. If x and y are collinear we use the notation $x \sim y$. Similarly, if L and M are concurrent we denote this $L \sim M$.

If X is a set of points (respectively, lines) of $\mathcal{S}$, then $X^\perp$ denotes the set of all points collinear (resp., lines concurrent) with everything in X. If $X = \{x\}$ is a singleton set, it is common to write $X^\perp$ as $x^\perp$. The set $X^{\perp\perp}$, is the set of all points collinear (resp., lines concurrent) with all of $X^\perp$. By convention $x \in x^\perp$.

Let x, y be two noncollinear points of a GQ(s,t). We say that $\{x,y\}$ is a *regular pair* provided $|\{x,y\}^{\perp\perp}| = t+1$; that is, if $|\{x,y\}^{\perp\perp}|$ is as large as possible. If x is a point such that for every y, with $x \not\sim y$, we have$|\{x,y\}^{\perp\perp}| = t+1$, then we say x is a *regular point*. A set $\{x,y,z\}$ of pairwise non-collinear points is called a *triad* of points. If $\{x,y,z\}$ is a triad of points, then all points in $\{x,y,z\}^\perp$ are called *centers*.

Recall that a GQ is *classical* if it is isomorphic to a GQ (or its dual) that can be embedded in a projective space.

2. Elation generalized quadrangles

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a GQ(s,t), $s \geq 1$, $t \geq 1$, and let $p \in \mathcal{P}$ be a point of $\mathcal{S}$. A *whorl* about p is a collineation of $\mathcal{S}$ that leaves invariant each line incident with p. If there is a group of whorls acting transitively on the points not collinear with p we say that p is a *center of transitivity*. Let θ be a whorl about p. If $\theta = \mathrm{id}$ or if θ fixes no point of $\mathcal{P} \setminus p^\perp$, then θ is an *elation* about p. If there is a group G of elations about p acting regularly on $\mathcal{P} \setminus p^\perp$, we say $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* p. We will often denote this quadrangle as $(\mathcal{S}^{(p)}, G)$, or simply $\mathcal{S}^{(p)}$. A *skew-translation* GQ (STGQ), is an EGQ $(\mathcal{S}^{(p)}, G)$ where G contains a full group of symmetries about p. All known GQ with parameters (q^2, q) are in fact STGQ. Moreover, a result

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obtained by X. Chen [2] and independently by D. Hachenberger [3] states that an STGQ must have both s and t powers of the same prime. If G is abelian we say $\mathcal{S}^{(p)}$ is a *translation* generalized quadrangle, denoted TGQ.

Let G be a group with order s^2t. Then let $F = \{A_0, A_1, \ldots, A_t\}$ be a family of $t+1$ subgroups of G, each with order s, and let $F^* = \{A_0^*, A_1^*, \ldots, A_t^*\}$ be another family of $t+1$ subgroups of G, each having order st where $A_i \leq A_i^*$ for each $0 \leq i \leq t$.

Using the group G we define a coset geometry, which we denote $\mathcal{S}^{(\infty)}$, as follows. There are three types of points; (i) elements $g \in G$, (ii) cosets $A_i^* g$, (iii) a symbol (∞). There are two types of lines; (i) cosets $A_i g$, (ii) symbols $[A_i]$. Incidence is as follows; the symbol (∞) is incident with the $t+1$ lines of type (ii), the s cosets of A_i^* are the other s points on a line $[A_i]$, each point $A_i^* g$ is incident with lines corresponding to the cosets $A_i h$ that are completely contained in the coset $A_i^* g$, the remaining points on a line $A_i h$ are the group elements contained in the coset $A_i h$.

Theorem 2.1. *Let G be a group of order s^2t and let $F = \{A_0, A_1, \ldots, A_t\}$ be a family of $t+1$ subgroups, each with order s, and let $F^* = \{A_0^*, A_1^*, \ldots, A_t^*\}$ be another family of $t+1$ subgroups, each having order st where $A_i \leq A_i^*$ for each $0 \leq i \leq t$. Then if we build the coset geometry $\mathcal{S}^{(\infty)}$ as prescribed above, $\mathcal{S}^{(\infty)}$ is a* GQ*, having order (s,t), if and only if properties $K1$ and $K2$ hold, where*

$$K1 : A_j A_i \cap A_k = \{\mathrm{id}\} \ \textit{for all distinct } i, j, k\,,$$
$$K2 : A_j^* \cap A_i = \{\mathrm{id}\} \ \textit{for all } i \neq j\,.$$

In the previous theorem, we call F a 4*-gonal family* of G, and $\{G, F, F^*\}$ is called a *Kantor family*.

It is also well known that the set F^* is completely determined by the elements in F. Define $\Omega = \cup\{A_i : 0 \leq i \leq t\}$, then $A_i^* = A_i \cup \{A_i g : A_i g \cap \Omega = \emptyset\}$. The next theorem is also well known.

Theorem 2.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a* GQ(s,t)*. If G is an elation group about a point p, q a point in $\mathcal{P} \setminus p^\perp$, and $\{p,q\}^\perp = \{x_0, \ldots, x_t\}$, for $0 \leq i \leq t$, let A_i be the stabilizer of the line through q and x_i, and A_i^* be the stabilizer of the point x_i. Then $F = \{A_i : 0 \leq i \leq t\}$ is a 4-gonal family of G and the coset geometry $\mathcal{S}^{(\infty)}$ obtained from this Kantor family $\{G, F, F^*\}$ is a* GQ *isomorphic to $\mathcal{S}$.*

Theorem 2.3 (S.E. Payne and K. Thas). *Let $\mathcal{S}$ be a* GQ *and let H be a group of whorls about the point x acting transitively on the set $X = P \setminus \{x\}^\perp$. The set of elations in H does not form a group if and only if (at least) one of the following conditions is satisfied:*

(1) *There is a $j \geq 2$ for which $|\mathrm{Fix}(\sigma)| = j$ for some $\sigma \in H$.*

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(2) *There is a proper thick sub-GQ of $\mathcal{S}$ containing x (and all the lines through x) fixed pointwise by a non-identity element of H.*

3. Elation groups of $H(3, q^2)$

For this paper we will assume that $q = 2^e$, $\mathbb{F} = \mathsf{GF}(q)$, and as usual the $\mathbb{F}$-trace function is defined as

$$\mathrm{tr}(\alpha) = \sum_{i=0}^{e-1} \alpha^{2^i}.$$

We then choose $\delta \in \mathsf{GF}(q)$ with $\mathrm{tr}(\delta) = 1$, and let ζ be a root of the polynomial $x^2 + x + \delta$. Put $\mathbb{F}^2 = \{a + b\zeta : a, b \in \mathsf{GF}(q)\}$; a quadratic extension of $\mathbb{F}$.

Lemma 3.1. *The element $\zeta^q = \zeta + 1$ is also a root of the polynomial, and if $\alpha = a + b\zeta$, then $\mathrm{tr}(b) = \mathrm{tr}(\alpha + \alpha^q)$.*

Let $\mathcal{S} = H(3, q^2)$ be the Hermitian surface in projective 3-space. Its construction is well known. Consider the projective space $\mathsf{PG}(V)$, where V is a 4-dimensional vector space over $\mathbb{F}^2$. Without loss of generality we choose the Hermitian form $H : V \times V \mapsto \mathbb{F}$ where

$$H(\bar{x}, \bar{y}) = x_1 y_4^q + x_2 y_3^q + x_3 y_2^q + x_4 y_1^q.$$

The set of all absolute points and totally isotropic lines of $\mathsf{PG}(3, q^2)$ forms the Hermitian surface $H(3, q^2)$. This is a $\mathsf{GQ}(q^2, q)$.

Theorem 3.2 ([4]). *Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a GQ of order (s, t), $s, t > 1$, with s and t powers of the same prime p. Suppose (∞) is a regular point that is a center of transitivity, and let W_∞ be the full group of whorls about the point (∞). Let S_p be a Sylow$_p$ subgroup of W_∞. Then we have*

1. *$|S_p| = s^2 t$, or*
2. *$p = 2$, $|S_2| = 2s^2 t$, and $\mathcal{S}$ contains a proper thick sub-GQ of order t isomorphic to the symplectic GQ, denoted $W(t)$; consequently, $s = t^2$.*

Corollary 3.3. *Let $q = 2^e$ and $\mathcal{S} = H(3, q^2)$. Every point of $\mathcal{S}$ is a regular point, and for each point p of $\mathcal{S}$, if S_2 is a Sylow $_2$ subgroup of the entire group of whorls about p, then $|\mathcal{S}_2| = 2q^5$.*

Consider the point $p = (0, 0, 0, 1)$. If $\mathcal{P}$ is the set of all points of $H(3, q^2)$, then $\mathcal{P} \backslash p^\perp$ is the set of q^5 points $(1, \alpha, \beta, \mu) \in \mathsf{PG}(3, q^2)$ satisfying $\mu + \mu^q + \alpha\beta^q + \alpha^q\beta =$

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0. The group of matrices

$$E_p = \left\{ \begin{pmatrix} 1 & \alpha & \beta & \mu \\ 0 & 1 & 0 & \beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathsf{GL}(4, q^2) : \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0 \right\}$$

is an elation group about p, as this group acts regularly on the set of points in $\mathcal{P} \setminus \{p^\perp\}$ (the E_p-orbit of $(1,0,0,0)$ is the set of points $(1,\alpha,\beta,\mu)$ where $\mu + \bar{\mu} + \alpha\bar{\beta} + \bar{\alpha}\beta = 0$) and fixes every line through p. It is often more convenient to represent this group as the set of triples

$$E_p = \{[\alpha,\beta,\mu] : \alpha,\beta,\mu \in \mathbb{F}^2 \text{ and } \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0\}$$

with group operation

$$[\alpha,\beta,\mu] * [\alpha',\beta',\mu'] = [\alpha+\alpha', \beta+\beta', \mu+\mu'+\alpha\beta'^q+\beta\alpha'^q]\,.$$

Next define the Hermitian preserving involution $\phi : \mathsf{PG}(3,q^2) \to \mathsf{PG}(3,q^2)$: $(x,y,z,w) \mapsto (x^q,y^q,z^q,w^q)$. Then ϕ induces a collineation that is a whorl about the point p. If we adjoin ϕ to E_p we form the group

$$W_2 = \{[\alpha,\beta,\mu] \circ \phi^i : \alpha,\beta,\mu \in \mathbb{F}^2 \text{ and } \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0\}$$

with group operation being composition of maps. That is, for $g = [\alpha,\beta,\mu] \circ \phi^i$ and $g' = [\alpha',\beta',\mu'] \circ \phi^j$ we have

$$g * g' = [\alpha + \alpha'^{q^i}, \beta + \beta'^{q^i}, \mu + \mu'^{q^i} + \alpha\beta'^{q^{i+1}} + \beta\alpha'^{q^{i+1}}] \circ \phi^{i+j}\,.$$

Then $|W_2| = 2q^5$ and Theorem 3.2 gaurantees W_2 is a Sylow$_2$ subgroup of the group of whorls about p.

Theorem 3.4. *The set of all elations about p does not form a group.*

Proof. Since $\mathrm{Fix}(\phi) \supset \{(1,a,b,c) : a,b,c \in \mathbb{F}\}$, Theorem 2.3 guarantees that the set of all elations about p does not form a group. □

Theorem 3.5. *The only elements in W_2 that fix any points not collinear with $p = (0,0,0,1)$ are the conjugates of ϕ.*

Proof. Suppose that x is a point opposite p that is fixed by ϕ. As $E_p \leq W_2$ the size of the orbit of x under W_2 is exactly q^5. By the orbit-stabilizer theorem the size of the stabilizer of x in W_2 is 2; i.e., $|{W_2}_x| = 2$. Therefore, ${W_2}_x = \{\mathrm{id}, \phi\}$. Now choose any point y opposite p. Because E_p acts regularly on points not collinear with p, there is a unique $g \in M_p$ such that $y^g = x$. So $(y)^{g\phi g^{-1}} = y$. Thus $g\phi g^{-1} \in {W_2}_y$ and by the orbit-stabilizer theorem we get ${W_2}_y = \{\mathrm{id}, g\phi g^{-1}\}$. □

Corollary 3.6. *A subgroup $E \leq W_2$, with $|E| = q^5$, is an elation group of $H(3, q^2)$ if and only if E contains no conjugates of ϕ.*

As usual, for $g, h \in W_2$, let $[g, h] = g^{-1}h^{-1}gh$; the commmutator of g, h. Denote the commutator subgroup of W_2 as W_2'. We observe that each conjugate of ϕ is in the coset of the commutator subgroup containing ϕ. That is,

$$g\phi g^{-1} = g\phi g^{-1}\phi^{-1}\phi = [g^{-1}, \phi] \cdot \phi \, .$$

Then since $W_2' = \{[a, b, c] : a, b, c \in \mathbb{F}_q\}$ we see that $W_2' \neq W_2' \cdot \phi$.

We can choose the following $2q^2$ distinct coset representatives of the factor group W_2/W_2':

$$\left\{[\alpha, \beta, 0] \circ \phi^i : \alpha = 0 + ai, \beta = 0 + bi, i = 0, 1\right\} .$$

Put $\overline{W_2} = W_2/W_2'$ and represent its elements as triples;

$$\overline{W_2} = \{(a, b, i) : a, b \in \mathbb{F}_q, i = 0, 1\} .$$

Lemma 3.7 ([6]). *Let G be a finite p-group where $\Phi(G)$ is the Frattini subgroup. Then $\Phi(G) = G'G^p$ where G' is the commutator subgroup and G^p is the subgroup of G generated by all p^{th} powers. Moreover, $G/\Phi(G)$ is a vector space over* GF(p).

Theorem 3.8. *The factor group $\overline{W_2}$ is a vector space over* GF(2).

Proof. Every non-identity element in W_2 has order two or four. Moreover, if $|g| = 4$ it is easy to show that $g^2 \in S_2'$. It follows that $W_2^2 \leq W_2'$, and $W_2' = \Phi(W_2)$. □

We can treat $\mathbb{F}$ as an e-dimensional vector space over GF(2). Then for a fixed $\zeta \in \mathbb{F}^*$, the map

$$\mathrm{tr}_\zeta(x) = \sum_{i=0}^{e-1} (\zeta x)^{2^i}$$

is a linear functional; i.e., $\mathrm{tr}_\zeta : \mathbb{F} \mapsto$ GF(2). Letting ζ vary over $\mathbb{F}^*$ gives us exactly $q - 1$ non-zero linear functionals on $\mathbb{F}$, and for each $\zeta \in \mathbb{F}^*$ there corresponds a unique hyperplane in $\mathbb{F}$ giving $q - 1$ distinct hyperlplanes. But every hyperplane of $\mathbb{F}$ corresponds to a non-zero vector in $\mathbb{F}$. Hence $\{\mathrm{tr}_\zeta : \zeta \in \mathbb{F}^*\}$ gives us all linear functionals from $\mathbb{F}$ to GF(2).

Next, for some $\zeta \in \mathbb{F}$ define the map $T_{\zeta,\sigma} : \overline{W_2} \to$ GF(2) as $T_{\zeta,\sigma}(a, b, i) = \mathrm{tr}_\zeta(a) + \mathrm{tr}_\sigma(b) + i$. This is a linear functional from $\overline{W_2}$ onto GF(2). The kernel is a hyperplane and it is easy to see that the vector $(0, 0, 1)$ is not in the kernel of $T_{\zeta,\sigma}$. This gives us q^2 hyperplanes of S_2/S_2' without the vector $(0,0,1)$ and these correspond to subgroups of W_2 with order q^5 that do not contain any conjugates of ϕ. That is, these hyperplanes correspond to elation groups about the point p. If we then define the map $T^*_{\zeta,\sigma}(a,b,i) = \mathrm{tr}_\zeta(a) + \mathrm{tr}_\sigma(b)$ we get the other q^2 hyperplanes of $\overline{W_2}$, each one containing the vector $(0,0,1)$. Each of these hyperplanes corresponds to a subgroup of W_2 of order q^5 that contains conjugates of ϕ.

We have shown that there are q^2 elation groups about the point $p = (0,0,0,1)$. Furthermore, $\mathsf{P\Gamma U}(4,q^2)$ is transitive on the points of $H(3,q^2)$ and this holds for all points on the Hermitian surface. We summarize in the following theorem.

Theorem 3.9. *Given a point p of $H(3,q^2)$, there are exactly q^2 elation groups of $H(3,q^2)$ about p.*

We represent these groups in W_2 as follows. For some $\zeta \in \mathbb{F}$ and $\alpha \in \mathbb{F}^2$, define the map $T_\zeta(\alpha) = \mathrm{tr}_\zeta(\alpha + \alpha^q) \in \mathsf{GF}(2)$. Then using Observation 3.1 we see that for each pair $\zeta,\sigma \in \mathbb{F}$ we get an elation group

$$E_{\zeta,\sigma} = \left\{ [\alpha,\beta,\mu] \circ \phi^{T_\zeta(\alpha)+T_\sigma(\beta)} : \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0 \right\}.$$

We refer to $E_{0,0}$ as the *familiar* group and the other q^2-1 groups as *exotic*.

Theorem 3.10. *All of the q^2-1 exotic elation groups are pairwise isomorphic.*

Proof. We represent $E_{1,0}$ as a subgroup of $\mathsf{P\Gamma U}(4,q^2)$, the entire stabilizer of $H(3,q^2)$. Recall that if $A,B \in \mathsf{PGU}(4,q^2)$ and $\sigma,\sigma' \in \mathrm{Aut}(\mathbb{F}^2)$, then in $\mathsf{P\Gamma U}(4,q^2)$ we have the group product

$$(A\circ\sigma) * (B\circ\sigma') = \left(A \cdot B^{\sigma^{-1}}\right) \circ (\sigma\cdot\sigma'),$$

where $B^{\sigma^{-1}} = [b_{ij}^{\sigma^{-1}}]$.

It is well known that $|\mathsf{P\Gamma U}(4,q^2)| = 2e\cdot q^6(q+1)^2(q^3+1)(q^4-1)$. We first look at that particular Sylow_2 subgroup of $\mathsf{PGU}(4,q^2)$ which is the subgroup of all upper-triangular matrices with ones on the diagonal. We denote this group as M_2. For a matrix M to be in $\mathsf{PGU}(4,q^2)$ it must satisfy $MB(M^q)^T = B$, where B is the bilinear form associated with $H(3,q^2)$. Therefore,

$$M_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & b^q + a^q d \\ 0 & 0 & 1 & a^q \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} ab^q + a^q b + c + c^q = 0 \\ d = d^q \end{array} \right\}.$$

This subgroup has order q^6 and is the stabilizer of the flag (L,p), where L is the line given by $x_1 = x_2 = 0$ and $p = (0,0,0,1)$. If we adjoin each field automorphism whose order is a power of two we get a Sylow_2 subgroup of $\mathsf{P\Gamma U}(4,q^2)$. The normalizer in $\mathsf{PGU}(4,q^2)$ of M_2, which we denote by N_{M_2}, is the group of all upper-triangular matrices.

$$N_{M_2} = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & w^{-q} & r & y^q w + r x^q \\ 0 & 0 & w & x^q w^{-q} \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} xy^q + x^q y + z + z^q = 0 \\ rw^q = wr^q \\ w \neq 0 \end{array} \right\}.$$

If we adjoin all the automorphisms of $\mathbb{F}^2$ this gives us the normalizer, which we denote as $N^*_{M_2}$, of the Sylow_2 subgroup of $\mathsf{P\Gamma U}(4,q^2)$. Moreover, $|N^*_{M_2}| = 2e \cdot q^6(q^2-1)$. We next show that the index of the normalizer of an exotic elation group in $N^*_{M_2}$ is q^2-1. That is, we show that the normalizer of an exotic elation group in $N^*_{M_2}$ has order $2e \cdot q^6$.

First note that all of the field automorphisms normalize the elation group $E_{1,0}$. So we consider the normalizer of $E_{1,0}$ in N_{M_2}. Choose the element

$$g = M \circ \phi^{T_1(\alpha)} = \begin{pmatrix} 1 & \alpha & \beta & \mu \\ 0 & 1 & 0 & \bar\beta \\ 0 & 0 & 1 & \bar\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \phi^{T_1(\alpha)}$$

in $E_{1,0}$ and choose an arbitrary matrix $A \in N_{M_2}$, so that $A \cdot g \cdot A^{-1} = B \circ \phi^{T_1(\alpha)}$, where $B = AM\left(A^{-1}\right)^{q^{T_1(\alpha)}}$. First, when $T_1(\alpha) = 1$ we have $B_{21} = w^{q-q^2} = w^{q-1} = 1$ if and only if $w \in \mathbb{F}$. Then, with $w \in \mathbb{F}$ we have

$$B_{12} = w(x^{q^{T_1(\alpha)}} + x + \alpha)\,.$$

It follows that

$$\begin{aligned} T_1[w(x^{q^{T_1(\alpha)}} + x + \alpha)] &= T_1[w(x^{q^{T_1(\alpha)}} + x) + w\alpha] \\ &= T_1[w(x^{q^{T_1(\alpha)}} + x)] + T_1(w\alpha) = T_1(w\alpha) \end{aligned}$$

for all $\alpha \in \mathbb{F}^2$, and that $w = 1$. Then, given $w = 1$, we must have $B_{23} = r + r^{q^{T_1(\alpha)}} = 0$, and $r \in \mathbb{F}$. Given these conditions on w and r we have $AgA^{-1} =$

$C \circ \phi^{T_1(\alpha)}$ where

$$C = \begin{pmatrix} 1 & x^{q^{T_1(\alpha)}} + x + \alpha & \begin{matrix} r(x^{q^{T_1(\alpha)}} + x + \alpha) \\ + y + y^{q^{T_1(\alpha)}} + \beta \end{matrix} & * \\ 0 & 1 & 0 & \begin{matrix} r(\bar{x}^{q^{T_1(\alpha)}} + \bar{x} + \bar{\alpha}) \\ + \bar{y} + \bar{y}^{q^{T_1(\alpha)}} + \bar{\beta} \end{matrix} \\ 0 & 0 & 1 & \bar{x}^{q^{T_1(\alpha)}} + \bar{x} + \bar{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$* = \mu + z + z^{q^{T_1(\alpha)}} + \bar{\beta}x + \beta\bar{x}^{q^{T_1(\alpha)}} + \bar{\alpha}^y + \bar{\alpha}^{q^{T_1(\alpha)}} y + \alpha\bar{y}^{q^{T_1(\alpha)}} + x\bar{y}^{q^{T_1(\alpha)}} + \bar{x}^{q^{T_1(\alpha)}} y^{q^{T_1(\alpha)}} + x^{q^{T_1(\alpha)}} \bar{y}^{q^{T_1(\alpha)}}.$$

So $AgA^{-1} \in E_{1,0}$ for all choices x, y, z satisfying $x\bar{y} + y\bar{x} + z + \bar{z} = 0$, and the normalizer of $E_{1,0}$ in $N^*_{M_2}$ has order $2e \cdot q^6$. Hence, there are $q^2 - 1$ conjugates of $E_{1,0}$ in $N^*_{M_2}$. It follows that all $q^2 - 1$ exotic elation groups are conjugate and hence isomorphic. □

From now put $E = E_{1,0}$ and $T = T_1$. Further, we will denote an element $[\alpha, \beta, \mu] \circ \phi^{T(\alpha)}$ as simply $[\alpha, \beta, \mu]$. Let $\bar{x} = x^q$, and if we define the map $\widehat{\alpha} : x \mapsto x^{\widehat{\alpha}} = x^{q^{T(\alpha)}}$ we have the following binary operation in E.

$$[\alpha, \beta, \mu] * [\alpha', \beta', \mu'] = [\alpha + \alpha'^{\widehat{\alpha}}, \beta + \beta'^{\widehat{\alpha}}, \mu + \mu'^{\widehat{\alpha}} + \alpha(\bar{\beta'})^{\widehat{\alpha}} + \beta(\bar{\alpha'})^{\widehat{\alpha}}].$$

Moreover, straightforward computations give us

$$[\alpha, \beta, \mu]^{-1} = [\alpha^{\widehat{\alpha}}, \beta^{\widehat{\alpha}}, \mu^{\widehat{\alpha}} + (\bar{\alpha}\beta)^{\widehat{\alpha}} + (\bar{\beta}\alpha)^{\widehat{\alpha}}].$$

The following properties of the function $\widehat{\alpha}$ will help with computations.

Lemma 3.11. *Suppose $a, b \in \mathbb{F}^2$ and $c \in \mathbb{F}$. Then for any $x \in \mathbb{F}^2$ we have*

(i) $x^{\widehat{0}} = x$.

(ii) $x^{\widehat{a}} = x^{\widehat{\bar{a}}}$.

(iii) $x^{\widehat{c}} = x$.

(iv) $x^{\widehat{(a^{\widehat{b}})}} = x^{\widehat{a}}$.

(v) $x^{\widehat{a+b}} = \left(x^{\widehat{a}}\right)^{\widehat{b}} = \left(x^{\widehat{b}}\right)^{\widehat{a}}$.

Theorem 3.12. *Each of the exotic elation groups of $H(3,q^2)$ is not isomorphic to the familiar example.*

Proof. In the familiar example the center of the group is equal to the commutator subgroup and $E_{0,0}$ has nilpotency class 2. However, each of the exotic elation groups has nilpotency class 3. To see this choose $g = [\alpha, \beta, \mu]$ and $h = [\alpha', \beta', \mu']$ in E and compute $[g, h]$. Using the facts in Observation 3.11 we arrive at $[g, h] = [a, b, c]$, where

$$a = \left(\alpha + \alpha^{\widehat{\alpha'}}\right)^{\widehat{\alpha}} + \left(\alpha' + \alpha'^{\widehat{\alpha'}}\right)^{\widehat{\alpha'}},$$
$$b = \left(\beta + \beta^{\widehat{\alpha'}}\right)^{\widehat{\alpha}} + \left(\beta' + \beta'^{\widehat{\alpha'}}\right)^{\widehat{\alpha'}}.$$

Now consider the following cases:

(i) $\widehat{\alpha} = \widehat{\alpha'} = 1$: Then $[g, h] = [0, 0, c]$ with $c \in \mathbb{F}$.

(ii) $\widehat{\alpha} = \widehat{\alpha'} = q$: Then $[g, h] = [\alpha + \bar{\alpha} + \alpha' + \bar{\alpha'}, \beta + \bar{\beta} + \beta' + \bar{\beta'}, c]$ with $c \in \mathbb{F}$.

(iii) $\widehat{\alpha} = 1$ and $\widehat{\alpha'} = q$: Then $[g, h] = [\alpha + \bar{\alpha}, \beta + \bar{\beta}, c]$ with $c \in \mathbb{F}$.

(iv) $\widehat{\alpha} = q$ and $\widehat{\alpha'} = 1$: Then $[g, h] = [\alpha' + \bar{\alpha'}, \beta' + \bar{\beta'}, c]$ with $c \in \mathbb{F}$.

Clearly, in case (ii) we have $\widehat{a} = 1$, where $a = (\alpha + \alpha') + \overline{(\alpha + \alpha')}$. So the possible values taken by the first coordinate in this case are a subset of the possible values of the first coordinate in cases (iii) and (iv). Then since the relative trace function $\mathrm{tr}_q : x \mapsto x + x^q$ is a homomorphism from $\mathbb{F}^2 \to \mathbb{F}$, it follows that there are $q/2$ possible choices for the first coordinate of any commutator. It is then easy to see that for $[g, h] = [a, b, c]$ the coordinate b can take on all possible elements of $\mathbb{F}$. Next choose, $\alpha, \alpha', \beta' \in \mathbb{F}$, and we get $[g, h] = [0, 0, \alpha'(\beta + \bar{\beta})]$, and we have all of $\mathbb{F}$ as possible entries in the third coordinate c. Then since the product of two commutators is again a commutator we get $E' = \{[\alpha + \bar{\alpha}, b, c] : b, c \in \mathbb{F};\ \alpha \in \mathbb{F}^2$ and $\widehat{\alpha} = 1\}$ and $|E'| = q^3/2$.

Finally, using case (iv) from above, we see that if $h \in E'$, then $[g, h]$ is in $Z(E)$ and E has nilpotency class 3. □

We summarize the above with the following main result.

Theorem 3.13. *Let $\mathcal{S} = H(3, q^2)$ and p be any point of $\mathcal{S}$. Then, up to isomorphism, there are exactly two elation groups about p.*

We also observe that $Z(E) = \{[0, 0, c] : c \in \mathsf{GF}(q)\} = Z(E_{0,0})$.

4. Looking forward

Consider the important theorem due to R.C. Bose.

Theorem 4.1 (Bose). *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (q^2, q). Then every set of three pairwise non-concurrent lines has exactly $q + 1$ transversals.*

We can now give the following definition.

Definition 4.2. A GQ $\mathcal{S}$ with parameters (q^2, q) has Property (G) at the point p provided the following holds. Let L_1 and M_1 be distinct lines incident with the point p. Let M_1, M_2, M_3, M_4 be distinct lines and L_1, L_2, L_3, L_4 be distinct lines for which $L_i \sim M_j$ whenever $i + j \leq 7$. Then $L_4 \sim M_4$.

One of the most powerful recent results characterizing flock-GQ by Property (G) was given by J. A. Thas in [7], with one missing case in characteristic two, which was completed by M. Brown in [1].

Theorem 4.3 (J. A. Thas and M. Brown). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ(q, q^2), $q > 1$, and assume that $\mathcal{S}$ satisfies Property (G) at some line l. Then $\mathcal{S}$ is the dual of a flock-GQ.*

Theorem 4.4 (S.E. Payne and K. Thas). *Let $\mathcal{S}(\mathcal{F})$ be a non-classical flock generalized quadrangle of order (q^2, q), $q > 1$, q even. Then the set of all elations about (∞) does form a group.*

The previous theorems should be helpful in characterizing any EGQ admitting an exotic elation group. This is the most obvious question. Another problem worth considering is the following.

Question 4.5. *Are there any non-classical EGQ admitting one of the exotic elation groups?*

Another problem worth considering is the following.

Question 4.6. *If G is an elation group, is there a bound on the nilpotency class of G?*

We should remark that a very different and independent approach to the problem of whether an EGQ admits a non-classical elation group with base point p appears in a paper by K. Thas [8]. The author offers the following conjecture.

Conjecture 4.7. *Let $\mathcal{S} = (\mathcal{S}^{(p)}, G)$ be an EGQ with Kantor-family $\{G, F, F^*\}$ and suppose that, for some $A^* \in F^*$, A^* is not normal in G. Then $\mathcal{S}$ (which has order (s, t)) has non-isomorphic (full) elation groups, and $\mathcal{S}$ has a sub-GQ of order $(s/t, t)$ fixed pointwise by some non-trivial collineation (possibly under some mild extra assumption).*

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