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Elation groups of the Hermitian surface $H(3, q^2)$ over a finite field of characteristic 2

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Abstract

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a finite generalized quadrangle (GQ) having order (s, t) . Let p be a point of \mathcal{S} . A *whorl* about p is a collineation of \mathcal{S} fixing all the lines through p . An *elation* about p is a whorl that does not fix any point not collinear with p , or is the identity. If \mathcal{S} has an elation group acting regularly on the set of points not collinear with p we say that \mathcal{S} is an elation generalized quadrangle (EGQ) with base point p . The following question has been posed: Can there be two non-isomorphic elation groups about the same point p ? In this presentation, we show that there are exactly two (up to isomorphism) elation groups of the Hermitian surface $H(3, q^2)$ over a finite field of characteristic 2.

Keywords: generalized quadrangles, elation groups, Hermitian surface

MSC 2000: 51E12

1. Introduction

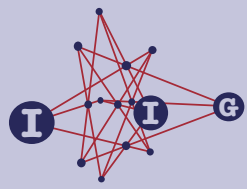
The focus of this article is $H(3, q^2)$, the Hermitian surface in three-dimensional projective space over the field $\text{GF}(q^2)$, where $q = 2^e$. The first results were discovered by Tim Penttila, for $q = 2$ and $q = 4$, using the software package Magma [5]. In this paper we give a constructive proof for any $q = 2^e$. We introduce generalized quadrangles with some basic definitions.

Let \mathcal{P} and \mathcal{B} be two non-empty sets, called points and lines, with an incidence relation \mathcal{I} such there are two positive integers s and t satisfying

- (1) Each point is incident with $t+1$ lines; any two points are mutually incident with at most one line.

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- (2) Each line is incident with $s + 1$ points; any two lines are mutually incident with at most one point.
- (3) Given a line L and a point x not incident with L there is a unique point y and a unique line M such that $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

Such a collection $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a *generalized quadrangle of order (s, t)* written $\text{GQ}(s, t)$. The *dual* of a $\text{GQ}(s, t)$ is the $\text{GQ}(t, s)$ obtained by interchanging the roles of points and lines. Furthermore, any theorem or definition given for a GQ can be dualized by interchanging the words points and lines. It will therefore be assumed that whenever a definition or theorem is given, its dual has also been given.

Two points incident with a common line are said to be *collinear* and two lines incident with a common point are *concurrent*. If x and y are collinear we use the notation $x \sim y$. Similarly, if L and M are concurrent we denote this $L \sim M$.

If X is a set of points (respectively, lines) of \mathcal{S} , then X^\perp denotes the set of all points collinear (resp., lines concurrent) with everything in X . If $X = \{x\}$ is a singleton set, it is common to write X^\perp as x^\perp . The set $X^{\perp\perp}$, is the set of all points collinear (resp., lines concurrent) with all of X^\perp . By convention $x \in x^\perp$.

Let x, y be two noncollinear points of a $\text{GQ}(s, t)$. We say that $\{x, y\}$ is a *regular pair* provided $|\{x, y\}^{\perp\perp}| = t + 1$; that is, if $|\{x, y\}^{\perp\perp}|$ is as large as possible. If x is a point such that for every y , with $x \not\sim y$, we have $|\{x, y\}^{\perp\perp}| = t + 1$, then we say x is a *regular point*. A set $\{x, y, z\}$ of pairwise non-collinear points is called a *triad* of points. If $\{x, y, z\}$ is a triad of points, then all points in $\{x, y, z\}^\perp$ are called *centers*.

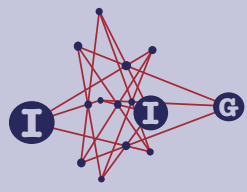
Recall that a GQ is *classical* if it is isomorphic to a GQ (or its dual) that can be embedded in a projective space.

2. Elation generalized quadrangles

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $\text{GQ}(s, t)$, $s \geq 1, t \geq 1$, and let $p \in \mathcal{P}$ be a point of \mathcal{S} . A *whorl* about p is a collineation of \mathcal{S} that leaves invariant each line incident with p . If there is a group of whorls acting transitively on the points not collinear with p we say that p is a *center of transitivity*. Let θ be a whorl about p . If $\theta = \text{id}$ or if θ fixes no point of $\mathcal{P} \setminus p^\perp$, then θ is an *elation* about p . If there is a group G of elations about p acting regularly on $\mathcal{P} \setminus p^\perp$, we say \mathcal{S} is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* p . We will often denote this quadrangle as $(\mathcal{S}^{(p)}, G)$, or simply $\mathcal{S}^{(p)}$. A *skew-translation GQ* (STGQ), is an EGQ $(\mathcal{S}^{(p)}, G)$ where G contains a full group of symmetries about p . All known GQ with parameters (q^2, q) are in fact STGQ. Moreover, a result

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obtained by X. Chen [2] and independently by D. Hachenberger [3] states that an STGQ must have both s and t powers of the same prime. If G is abelian we say $\mathcal{S}^{(p)}$ is a *translation generalized quadrangle*, denoted TGQ.

Let G be a group with order s^2t . Then let $F = \{A_0, A_1, \dots, A_t\}$ be a family of $t + 1$ subgroups of G , each with order s , and let $F^* = \{A_0^*, A_1^*, \dots, A_t^*\}$ be another family of $t + 1$ subgroups of G , each having order st where $A_i \leq A_i^*$ for each $0 \leq i \leq t$.

Using the group G we define a coset geometry, which we denote $\mathcal{S}^{(\infty)}$, as follows. There are three types of points; (i) elements $g \in G$, (ii) cosets A_i^*g , (iii) a symbol (∞) . There are two types of lines; (i) cosets $A_i g$, (ii) symbols $[A_i]$. Incidence is as follows; the symbol (∞) is incident with the $t + 1$ lines of type (ii), the s cosets of A_i^* are the other s points on a line $[A_i]$, each point A_i^*g is incident with lines corresponding to the cosets $A_i h$ that are completely contained in the coset A_i^*g , the remaining points on a line $A_i h$ are the group elements contained in the coset $A_i h$.

Theorem 2.1. *Let G be a group of order s^2t and let $F = \{A_0, A_1, \dots, A_t\}$ be a family of $t + 1$ subgroups, each with order s , and let $F^* = \{A_0^*, A_1^*, \dots, A_t^*\}$ be another family of $t + 1$ subgroups, each having order st where $A_i \leq A_i^*$ for each $0 \leq i \leq t$. Then if we build the coset geometry $\mathcal{S}^{(\infty)}$ as prescribed above, $\mathcal{S}^{(\infty)}$ is a GQ, having order (s, t) , if and only if properties K1 and K2 hold, where*

$$K1 : A_j A_i \cap A_k = \{\text{id}\} \text{ for all distinct } i, j, k ,$$

$$K2 : A_j^* \cap A_i = \{\text{id}\} \text{ for all } i \neq j .$$

In the previous theorem, we call F a 4-gonal family of G , and $\{G, F, F^*\}$ is called a Kantor family.

It is also well known that the set F^* is completely determined by the elements in F . Define $\Omega = \cup\{A_i : 0 \leq i \leq t\}$, then $A_i^* = A_i \cup \{A_i g : A_i g \cap \Omega = \emptyset\}$. The next theorem is also well known.

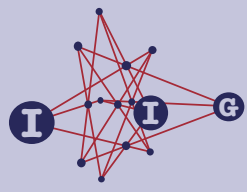
Theorem 2.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a GQ(s, t). If G is an elation group about a point p , q a point in $\mathcal{P} \setminus p^\perp$, and $\{p, q\}^\perp = \{x_0, \dots, x_t\}$, for $0 \leq i \leq t$, let A_i be the stabilizer of the line through q and x_i , and A_i^* be the stabilizer of the point x_i . Then $F = \{A_i : 0 \leq i \leq t\}$ is a 4-gonal family of G and the coset geometry $\mathcal{S}^{(\infty)}$ obtained from this Kantor family $\{G, F, F^*\}$ is a GQ isomorphic to \mathcal{S} .*

Theorem 2.3 (S.E. Payne and K. Thas). *Let \mathcal{S} be a GQ and let H be a group of whorls about the point x acting transitively on the set $X = P \setminus \{x\}^\perp$. The set of elations in H does not form a group if and only if (at least) one of the following conditions is satisfied:*

- (1) *There is a $j \geq 2$ for which $|\text{Fix}(\sigma)| = j$ for some $\sigma \in H$.*

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(2) *There is a proper thick sub-GQ of \mathcal{S} containing x (and all the lines through x) fixed pointwise by a non-identity element of H .*

3. Elation groups of $H(3, q^2)$

For this paper we will assume that $q = 2^e$, $\mathbb{F} = \text{GF}(q)$, and as usual the \mathbb{F} -trace function is defined as

$$\text{tr}(\alpha) = \sum_{i=0}^{e-1} \alpha^{2^i}.$$

We then choose $\delta \in \text{GF}(q)$ with $\text{tr}(\delta) = 1$, and let ζ be a root of the polynomial $x^2 + x + \delta$. Put $\mathbb{F}^2 = \{a + b\zeta : a, b \in \text{GF}(q)\}$; a quadratic extension of \mathbb{F} .

Lemma 3.1. *The element $\zeta^q = \zeta + 1$ is also a root of the polynomial, and if $\alpha = a + b\zeta$, then $\text{tr}(b) = \text{tr}(\alpha + \alpha^q)$.*

Let $\mathcal{S} = H(3, q^2)$ be the Hermitian surface in projective 3-space. Its construction is well known. Consider the projective space $\text{PG}(V)$, where V is a 4-dimensional vector space over \mathbb{F}^2 . Without loss of generality we choose the Hermitian form $H : V \times V \mapsto \mathbb{F}$ where

$$H(\bar{x}, \bar{y}) = x_1y_4^q + x_2y_3^q + x_3y_2^q + x_4y_1^q.$$

The set of all absolute points and totally isotropic lines of $\text{PG}(3, q^2)$ forms the Hermitian surface $H(3, q^2)$. This is a $\text{GQ}(q^2, q)$.

Theorem 3.2 ([4]). *Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a GQ of order (s, t) , $s, t > 1$, with s and t powers of the same prime p . Suppose (∞) is a regular point that is a center of transitivity, and let W_∞ be the full group of whorls about the point (∞) . Let S_p be a Sylow p subgroup of W_∞ . Then we have*

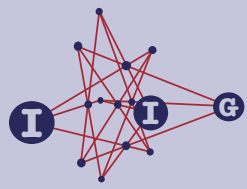
1. $|S_p| = s^2t$, or
2. $p = 2$, $|S_2| = 2s^2t$, and \mathcal{S} contains a proper thick sub-GQ of order t isomorphic to the symplectic GQ, denoted $W(t)$; consequently, $s = t^2$.

Corollary 3.3. *Let $q = 2^e$ and $\mathcal{S} = H(3, q^2)$. Every point of \mathcal{S} is a regular point, and for each point p of \mathcal{S} , if S_2 is a Sylow 2 subgroup of the entire group of whorls about p , then $|S_2| = 2q^5$.*

Consider the point $p = (0, 0, 0, 1)$. If \mathcal{P} is the set of all points of $H(3, q^2)$, then $\mathcal{P} \setminus p^\perp$ is the set of q^5 points $(1, \alpha, \beta, \mu) \in \text{PG}(3, q^2)$ satisfying $\mu + \mu^q + \alpha\beta^q + \alpha^q\beta =$

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0. The group of matrices

$$E_p = \left\{ \begin{pmatrix} 1 & \alpha & \beta & \mu \\ 0 & 1 & 0 & \beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{GL}(4, q^2) : \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0 \right\}$$

is an elation group about p , as this group acts regularly on the set of points in $\mathcal{P} \setminus \{p^\perp\}$ (the E_p -orbit of $(1, 0, 0, 0)$ is the set of points $(1, \alpha, \beta, \mu)$ where $\mu + \bar{\mu} + \alpha\bar{\beta} + \bar{\alpha}\beta = 0$) and fixes every line through p . It is often more convenient to represent this group as the set of triples

$$E_p = \{[\alpha, \beta, \mu] : \alpha, \beta, \mu \in \mathbb{F}^2 \text{ and } \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0\}$$

with group operation

$$[\alpha, \beta, \mu] * [\alpha', \beta', \mu'] = [\alpha + \alpha', \beta + \beta', \mu + \mu' + \alpha\beta'^q + \beta\alpha'^q].$$

Next define the Hermitian preserving involution $\phi : \text{PG}(3, q^2) \rightarrow \text{PG}(3, q^2) : (x, y, z, w) \mapsto (x^q, y^q, z^q, w^q)$. Then ϕ induces a collineation that is a whorl about the point p . If we adjoin ϕ to E_p we form the group

$$W_2 = \{[\alpha, \beta, \mu] \circ \phi^i : \alpha, \beta, \mu \in \mathbb{F}^2 \text{ and } \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0\}$$

with group operation being composition of maps. That is, for $g = [\alpha, \beta, \mu] \circ \phi^i$ and $g' = [\alpha', \beta', \mu'] \circ \phi^j$ we have

$$g * g' = [\alpha + \alpha'^{q^i}, \beta + \beta'^{q^i}, \mu + \mu'^{q^i} + \alpha\beta'^{q^{i+1}} + \beta\alpha'^{q^{i+1}}] \circ \phi^{i+j}.$$

Then $|W_2| = 2q^5$ and Theorem 3.2 guarantees W_2 is a Sylow₂ subgroup of the group of whorls about p .

Theorem 3.4. *The set of all elations about p does not form a group.*

Proof. Since $\text{Fix}(\phi) \supset \{(1, a, b, c) : a, b, c \in \mathbb{F}\}$, Theorem 2.3 guarantees that the set of all elations about p does not form a group. \square

Theorem 3.5. *The only elements in W_2 that fix any points not collinear with $p = (0, 0, 0, 1)$ are the conjugates of ϕ .*

Proof. Suppose that x is a point opposite p that is fixed by ϕ . As $E_p \leq W_2$ the size of the orbit of x under W_2 is exactly q^5 . By the orbit-stabilizer theorem the size of the stabilizer of x in W_2 is 2; i.e., $|W_{2_x}| = 2$. Therefore, $W_{2_x} = \{\text{id}, \phi\}$. Now choose any point y opposite p . Because E_p acts regularly on points not collinear with p , there is a unique $g \in M_p$ such that $y^g = x$. So $(y)^{g\phi g^{-1}} = y$. Thus $g\phi g^{-1} \in W_{2_y}$ and by the orbit-stabilizer theorem we get $W_{2_y} = \{\text{id}, g\phi g^{-1}\}$. \square



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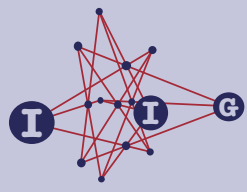
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Corollary 3.6. *A subgroup $E \leq W_2$, with $|E| = q^5$, is an elation group of $H(3, q^2)$ if and only if E contains no conjugates of ϕ .*

As usual, for $g, h \in W_2$, let $[g, h] = g^{-1}h^{-1}gh$; the commutator of g, h . Denote the commutator subgroup of W_2 as W'_2 . We observe that each conjugate of ϕ is in the coset of the commutator subgroup containing ϕ . That is,

$$g\phi g^{-1} = g\phi g^{-1}\phi^{-1}\phi = [g^{-1}, \phi] \cdot \phi.$$

Then since $W'_2 = \{[a, b, c] : a, b, c \in \mathbb{F}_q\}$ we see that $W'_2 \neq W'_2 \cdot \phi$.

We can choose the following $2q^2$ distinct coset representatives of the factor group W_2/W'_2 :

$$\{[\alpha, \beta, 0] \circ \phi^i : \alpha = 0 + ai, \beta = 0 + bi, i = 0, 1\}.$$

Put $\overline{W}_2 = W_2/W'_2$ and represent its elements as triples;

$$\overline{W}_2 = \{(a, b, i) : a, b \in \mathbb{F}_q, i = 0, 1\}.$$

Lemma 3.7 ([6]). *Let G be a finite p -group where $\Phi(G)$ is the Frattini subgroup. Then $\Phi(G) = G'G^p$ where G' is the commutator subgroup and G^p is the subgroup of G generated by all p^{th} powers. Moreover, $G/\Phi(G)$ is a vector space over $\text{GF}(p)$.*

Theorem 3.8. *The factor group \overline{W}_2 is a vector space over $\text{GF}(2)$.*

Proof. Every non-identity element in W_2 has order two or four. Moreover, if $|g| = 4$ it is easy to show that $g^2 \in S'_2$. It follows that $W_2^2 \leq W'_2$, and $W'_2 = \Phi(W_2)$. \square

We can treat \mathbb{F} as an e -dimensional vector space over $\text{GF}(2)$. Then for a fixed $\zeta \in \mathbb{F}^*$, the map

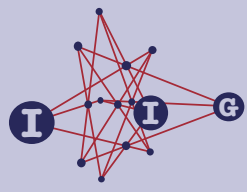
$$\text{tr}_\zeta(x) = \sum_{i=0}^{e-1} (\zeta x)^{2^i}$$

is a linear functional; i.e., $\text{tr}_\zeta : \mathbb{F} \mapsto \text{GF}(2)$. Letting ζ vary over \mathbb{F}^* gives us exactly $q - 1$ non-zero linear functionals on \mathbb{F} , and for each $\zeta \in \mathbb{F}^*$ there corresponds a unique hyperplane in \mathbb{F} giving $q - 1$ distinct hyperplanes. But every hyperplane of \mathbb{F} corresponds to a non-zero vector in \mathbb{F} . Hence $\{\text{tr}_\zeta : \zeta \in \mathbb{F}^*\}$ gives us all linear functionals from \mathbb{F} to $\text{GF}(2)$.

Next, for some $\zeta \in \mathbb{F}$ define the map $T_{\zeta, \sigma} : \overline{W}_2 \rightarrow \text{GF}(2)$ as $T_{\zeta, \sigma}(a, b, i) = \text{tr}_\zeta(a) + \text{tr}_\sigma(b) + i$. This is a linear functional from \overline{W}_2 onto $\text{GF}(2)$. The kernel is a hyperplane and it is easy to see that the vector $(0, 0, 1)$ is not in the kernel

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of $T_{\zeta, \sigma}$. This gives us q^2 hyperplanes of S_2/S'_2 without the vector $(0, 0, 1)$ and these correspond to subgroups of W_2 with order q^5 that do not contain any conjugates of ϕ . That is, these hyperplanes correspond to elation groups about the point p . If we then define the map $T_{\zeta, \sigma}^*(a, b, i) = \text{tr}_{\zeta}(a) + \text{tr}_{\sigma}(b)$ we get the other q^2 hyperplanes of $\overline{W_2}$, each one containing the vector $(0, 0, 1)$. Each of these hyperplanes corresponds to a subgroup of W_2 of order q^5 that contains conjugates of ϕ .

We have shown that there are q^2 elation groups about the point $p = (0, 0, 0, 1)$. Furthermore, $\text{P}\Gamma\text{U}(4, q^2)$ is transitive on the points of $H(3, q^2)$ and this holds for all points on the Hermitian surface. We summarize in the following theorem.

Theorem 3.9. *Given a point p of $H(3, q^2)$, there are exactly q^2 elation groups of $H(3, q^2)$ about p .*

We represent these groups in W_2 as follows. For some $\zeta \in \mathbb{F}$ and $\alpha \in \mathbb{F}^2$, define the map $T_{\zeta}(\alpha) = \text{tr}_{\zeta}(\alpha + \alpha^q) \in \text{GF}(2)$. Then using Observation 3.1 we see that for each pair $\zeta, \sigma \in \mathbb{F}$ we get an elation group

$$E_{\zeta, \sigma} = \left\{ [\alpha, \beta, \mu] \circ \phi^{T_{\zeta}(\alpha) + T_{\sigma}(\beta)} : \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0 \right\}.$$

We refer to $E_{0,0}$ as the *familiar* group and the other $q^2 - 1$ groups as *exotic*.

Theorem 3.10. *All of the $q^2 - 1$ exotic elation groups are pairwise isomorphic.*

Proof. We represent $E_{1,0}$ as a subgroup of $\text{P}\Gamma\text{U}(4, q^2)$, the entire stabilizer of $H(3, q^2)$. Recall that if $A, B \in \text{PGU}(4, q^2)$ and $\sigma, \sigma' \in \text{Aut}(\mathbb{F}^2)$, then in $\text{P}\Gamma\text{U}(4, q^2)$ we have the group product

$$(A \circ \sigma) * (B \circ \sigma') = (A \cdot B^{\sigma^{-1}}) \circ (\sigma \cdot \sigma'),$$

where $B^{\sigma^{-1}} = [b_{ij}^{\sigma^{-1}}]$.

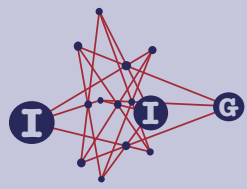
It is well known that $|\text{P}\Gamma\text{U}(4, q^2)| = 2e \cdot q^6(q+1)^2(q^3+1)(q^4-1)$. We first look at that particular Sylow₂ subgroup of $\text{PGU}(4, q^2)$ which is the subgroup of all upper-triangular matrices with ones on the diagonal. We denote this group as M_2 . For a matrix M to be in $\text{PGU}(4, q^2)$ it must satisfy $MB(M^q)^T = B$, where B is the bilinear form associated with $H(3, q^2)$. Therefore,

$$M_2 = \left\{ \left(\begin{array}{cccc} 1 & a & b & c \\ 0 & 1 & d & b^q + a^q d \\ 0 & 0 & 1 & a^q \\ 0 & 0 & 0 & 1 \end{array} \right) : \begin{array}{l} ab^q + a^q b + c + c^q = 0 \\ d = d^q \end{array} \right\}.$$

This subgroup has order q^6 and is the stabilizer of the flag (L, p) , where L is the line given by $x_1 = x_2 = 0$ and $p = (0, 0, 0, 1)$. If we adjoin each field

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automorphism whose order is a power of two we get a Sylow₂ subgroup of $\text{P}\Gamma\text{U}(4, q^2)$. The normalizer in $\text{P}\Gamma\text{U}(4, q^2)$ of M_2 , which we denote by N_{M_2} , is the group of all upper-triangular matrices.

$$N_{M_2} = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & w^{-q} & r & y^q w + r x^q \\ 0 & 0 & w & x^q w^{-q} \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} x y^q + x^q y + z + z^q = 0 \\ r w^q = w r^q \\ w \neq 0 \end{array} \right\}.$$

If we adjoin all the automorphisms of \mathbb{F}^2 this gives us the normalizer, which we denote as $N_{M_2}^*$, of the Sylow₂ subgroup of $\text{P}\Gamma\text{U}(4, q^2)$. Moreover, $|N_{M_2}^*| = 2e \cdot q^6 (q^2 - 1)$. We next show that the index of the normalizer of an exotic elation group in $N_{M_2}^*$ is $q^2 - 1$. That is, we show that the normalizer of an exotic elation group in $N_{M_2}^*$ has order $2e \cdot q^6$.

First note that all of the field automorphisms normalize the elation group $E_{1,0}$. So we consider the normalizer of $E_{1,0}$ in N_{M_2} . Choose the element

$$g = M \circ \phi^{T_1(\alpha)} = \begin{pmatrix} 1 & \alpha & \beta & \mu \\ 0 & 1 & 0 & \bar{\beta} \\ 0 & 0 & 1 & \bar{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \phi^{T_1(\alpha)}$$

in $E_{1,0}$ and choose an arbitrary matrix $A \in N_{M_2}$, so that $A \cdot g \cdot A^{-1} = B \circ \phi^{T_1(\alpha)}$, where $B = AM(A^{-1})^{q^{T_1(\alpha)}}$. First, when $T_1(\alpha) = 1$ we have $B_{21} = w^{q-q^2} = w^{q-1} = 1$ if and only if $w \in \mathbb{F}$. Then, with $w \in \mathbb{F}$ we have

$$B_{12} = w(x^{q^{T_1(\alpha)}} + x + \alpha).$$

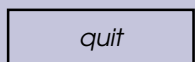
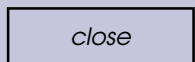
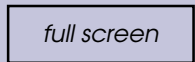
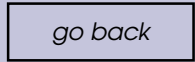
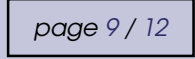
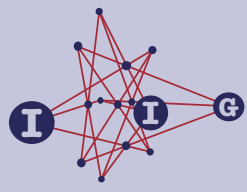
It follows that

$$\begin{aligned} T_1[w(x^{q^{T_1(\alpha)}} + x + \alpha)] &= T_1[w(x^{q^{T_1(\alpha)}} + x) + w\alpha] \\ &= T_1[w(x^{q^{T_1(\alpha)}} + x)] + T_1(w\alpha) = T_1(w\alpha) \end{aligned}$$

for all $\alpha \in \mathbb{F}^2$, and that $w = 1$. Then, given $w = 1$, we must have $B_{23} = r + r^{q^{T_1(\alpha)}} = 0$, and $r \in \mathbb{F}$. Given these conditions on w and r we have $AgA^{-1} =$

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$C \circ \phi^{T_1(\alpha)}$ where

$$C = \begin{pmatrix} 1 & x^{q^{T_1(\alpha)}} + x + \alpha & r(x^{q^{T_1(\alpha)}} + x + \alpha) + y + y^{q^{T_1(\alpha)}} + \beta & * \\ 0 & 1 & 0 & r(\bar{x}^{q^{T_1(\alpha)}} + \bar{x} + \bar{\alpha}) + \bar{y} + \bar{y}^{q^{T_1(\alpha)}} + \bar{\beta} \\ 0 & 0 & 1 & \bar{x}^{q^{T_1(\alpha)}} + \bar{x} + \bar{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$* = \mu + z + z^{q^{T_1(\alpha)}} + \bar{\beta}x + \beta\bar{x}^{q^{T_1(\alpha)}} + \bar{\alpha}y + \alpha\bar{y}^{q^{T_1(\alpha)}} + x\bar{y}^{q^{T_1(\alpha)}} + \bar{x}^{q^{T_1(\alpha)}}y^{q^{T_1(\alpha)}} + x^{q^{T_1(\alpha)}}\bar{y}^{q^{T_1(\alpha)}}.$$

So $AgA^{-1} \in E_{1,0}$ for all choices x, y, z satisfying $x\bar{y} + y\bar{x} + z + \bar{z} = 0$, and the normalizer of $E_{1,0}$ in $N_{M_2}^*$ has order $2e \cdot q^6$. Hence, there are $q^2 - 1$ conjugates of $E_{1,0}$ in $N_{M_2}^*$. It follows that all $q^2 - 1$ exotic elation groups are conjugate and hence isomorphic. \square

From now put $E = E_{1,0}$ and $T = T_1$. Further, we will denote an element $[\alpha, \beta, \mu] \circ \phi^{T(\alpha)}$ as simply $[\alpha, \beta, \mu]$. Let $\bar{x} = x^q$, and if we define the map $\hat{\alpha} : x \mapsto x^{\hat{\alpha}} = x^{q^{T(\alpha)}}$ we have the following binary operation in E .

$$[\alpha, \beta, \mu] * [\alpha', \beta', \mu'] = [\alpha + \alpha'^{\hat{\alpha}}, \beta + \beta'^{\hat{\alpha}}, \mu + \mu'^{\hat{\alpha}} + \alpha(\bar{\beta}')^{\hat{\alpha}} + \beta(\bar{\alpha}')^{\hat{\alpha}}].$$

Moreover, straightforward computations give us

$$[\alpha, \beta, \mu]^{-1} = [\alpha^{\hat{\alpha}}, \beta^{\hat{\alpha}}, \mu^{\hat{\alpha}} + (\bar{\alpha}\beta)^{\hat{\alpha}} + (\bar{\beta}\alpha)^{\hat{\alpha}}].$$

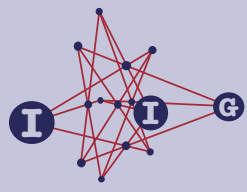
The following properties of the function $\hat{\alpha}$ will help with computations.

Lemma 3.11. *Suppose $a, b \in \mathbb{F}^2$ and $c \in \mathbb{F}$. Then for any $x \in \mathbb{F}^2$ we have*

- (i) $x^{\hat{0}} = x$.
- (ii) $x^{\hat{a}} = x^{\hat{a}}$.
- (iii) $x^{\hat{c}} = x$.
- (iv) $x^{\widehat{(a^b)}} = x^{\hat{a}}$.
- (v) $x^{\widehat{a+b}} = (x^{\hat{a}})^{\hat{b}} = (x^{\hat{b}})^{\hat{a}}$.

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Theorem 3.12. *Each of the exotic elation groups of $H(3, q^2)$ is not isomorphic to the familiar example.*

Proof. In the familiar example the center of the group is equal to the commutator subgroup and $E_{0,0}$ has nilpotency class 2. However, each of the exotic elation groups has nilpotency class 3. To see this choose $g = [\alpha, \beta, \mu]$ and $h = [\alpha', \beta', \mu']$ in E and compute $[g, h]$. Using the facts in Observation 3.11 we arrive at $[g, h] = [a, b, c]$, where

$$a = (\alpha + \alpha^{\widehat{\alpha}'})^{\widehat{\alpha}} + (\alpha' + \alpha'^{\widehat{\alpha}'})^{\widehat{\alpha}'},$$

$$b = (\beta + \beta^{\widehat{\alpha}'})^{\widehat{\alpha}} + (\beta' + \beta'^{\widehat{\alpha}'})^{\widehat{\alpha}'}$$

Now consider the following cases:

- (i) $\widehat{\alpha} = \widehat{\alpha}' = 1$: Then $[g, h] = [0, 0, c]$ with $c \in \mathbb{F}$.
- (ii) $\widehat{\alpha} = \widehat{\alpha}' = q$: Then $[g, h] = [\alpha + \bar{\alpha} + \alpha' + \bar{\alpha}', \beta + \bar{\beta} + \beta' + \bar{\beta}', c]$ with $c \in \mathbb{F}$.
- (iii) $\widehat{\alpha} = 1$ and $\widehat{\alpha}' = q$: Then $[g, h] = [\alpha + \bar{\alpha}, \beta + \bar{\beta}, c]$ with $c \in \mathbb{F}$.
- (iv) $\widehat{\alpha} = q$ and $\widehat{\alpha}' = 1$: Then $[g, h] = [\alpha' + \bar{\alpha}', \beta' + \bar{\beta}', c]$ with $c \in \mathbb{F}$.

Clearly, in case (ii) we have $\widehat{a} = 1$, where $a = (\alpha + \alpha') + \overline{(\alpha + \alpha')}$. So the possible values taken by the first coordinate in this case are a subset of the possible values of the first coordinate in cases (iii) and (iv). Then since the relative trace function $\text{tr}_q : x \mapsto x + x^q$ is a homomorphism from $\mathbb{F}^2 \rightarrow \mathbb{F}$, it follows that there are $q/2$ possible choices for the first coordinate of any commutator. It is then easy to see that for $[g, h] = [a, b, c]$ the coordinate b can take on all possible elements of \mathbb{F} . Next choose, $\alpha, \alpha', \beta' \in \mathbb{F}$, and we get $[g, h] = [0, 0, \alpha'(\beta + \bar{\beta})]$, and we have all of \mathbb{F} as possible entries in the third coordinate c . Then since the product of two commutators is again a commutator we get $E' = \{[\alpha + \bar{\alpha}, b, c] : b, c \in \mathbb{F}; \alpha \in \mathbb{F}^2 \text{ and } \widehat{\alpha} = 1\}$ and $|E'| = q^3/2$.

Finally, using case (iv) from above, we see that if $h \in E'$, then $[g, h]$ is in $Z(E)$ and E has nilpotency class 3. \square

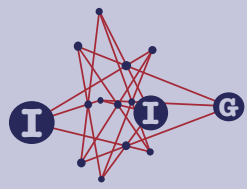
We summarize the above with the following main result.

Theorem 3.13. *Let $S = H(3, q^2)$ and p be any point of S . Then, up to isomorphism, there are exactly two elation groups about p .*

We also observe that $Z(E) = \{[0, 0, c] : c \in \text{GF}(q)\} = Z(E_{0,0})$.

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4. Looking forward

Consider the important theorem due to R.C. Bose.

Theorem 4.1 (Bose). *Let $S = (P, B, I)$ be a GQ of order (q^2, q) . Then every set of three pairwise non-concurrent lines has exactly $q + 1$ transversals.*

We can now give the following definition.

Definition 4.2. A GQ S with parameters (q^2, q) has Property (G) at the point p provided the following holds. Let L_1 and M_1 be distinct lines incident with the point p . Let M_1, M_2, M_3, M_4 be distinct lines and L_1, L_2, L_3, L_4 be distinct lines for which $L_i \sim M_j$ whenever $i + j \leq 7$. Then $L_4 \sim M_4$.

One of the most powerful recent results characterizing flock-GQ by Property (G) was given by J. A. Thas in [7], with one missing case in characteristic two, which was completed by M. Brown in [1].

Theorem 4.3 (J. A. Thas and M. Brown). *Let $S = (P, B, I)$ be a GQ (q, q^2) , $q > 1$, and assume that S satisfies Property (G) at some line l . Then S is the dual of a flock-GQ.*

Theorem 4.4 (S.E. Payne and K. Thas). *Let $S(\mathcal{F})$ be a non-classical flock generalized quadrangle of order (q^2, q) , $q > 1$, q even. Then the set of all elations about (∞) does form a group.*

The previous theorems should be helpful in characterizing any EGQ admitting an exotic elation group. This is the most obvious question. Another problem worth considering is the following.

Question 4.5. *Are there any non-classical EGQ admitting one of the exotic elation groups?*

Another problem worth considering is the following.

Question 4.6. *If G is an elation group, is there a bound on the nilpotency class of G ?*

We should remark that a very different and independent approach to the problem of whether an EGQ admits a non-classical elation group with base point p appears in a paper by K. Thas [8]. The author offers the following conjecture.

Conjecture 4.7. *Let $S = (S^{(p)}, G)$ be an EGQ with Kantor-family $\{G, F, F^*\}$ and suppose that, for some $A^* \in F^*$, A^* is not normal in G . Then S (which has order (s, t)) has non-isomorphic (full) elation groups, and S has a sub-GQ of order $(s/t, t)$ fixed pointwise by some non-trivial collineation (possibly under some mild extra assumption).*



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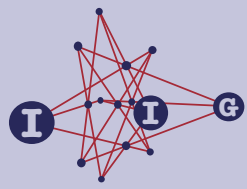
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5. Acknowledgments

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