Intersections of Buekenhout-Metz unitals

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On the occasion of the 60th birthday of Gábor Korchmáros

Abstract

Configurations arising as intersections of two Buekenhout-Metz unitals of a given family are studied and, in the case in which at most one of the unitals is classical, a new intersection size is found.

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1 Introduction

In [8] it has been shown that there are just seven configurations in which two classical unitals may intersect. There-within it has also been proved that the cardinality of the intersection of any two classical unitals in the desarguesian projective plane \( PG(2, q^2) \) is congruent to 1 modulo \( q \).

A family of non-classical Buekenhout-Metz unitals in \( PG(2, q^2) \), with \( q = p^h \) an odd prime power, has been constructed in [1]; the intersection of every unital of this family with a classical one contains a number of points congruent to 1 modulo \( p \). In the same paper, it is also conjectured that the size of the intersection of any classical unital with a non-classical one should be one of the following:

\[
q^2 \pm 2q + 1, \quad q^2 \pm q + 1, \quad q^2 + 1.
\]

Afterwards, in [3] it has been proved that an arbitrary unital in \( PG(2, q^2) \), with \( q = p^h \) any prime power, meets a classical unital in a number of points congruent

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to 1 modulo $p$. To classify intersections of two non-classical unitals seems to be a difficult question. Here we make some advances in this direction by looking at a suitable family $\mathcal{F}$ of Buekenhout-Metz unitals in $\text{PG}(2, q^2)$, $q$ any prime power, containing both classical unitals as well as non-classical ones. We prove that any two classical unitals in $\mathcal{F}$ intersect on $q + 1$ collinear points, whereas in all other cases the intersection number is one of the following:

$$q + 1, q^2 + 1, 2q^2 - q + 1.$$ 

This last size does not appear among those conjectured in [1].

2 Preliminaries

A set $S$ of $k$ points (or a $k$-set) in a projective plane of order $q$ is of type $(k_1, k_2, \ldots, k_s)$, with $k_1 < k_2 < \cdots < k_s$, if a line $\ell$ may intersect $S$ in only sets of $k_1, k_2, \ldots$ or $k_s$ points. A line $\ell$ for which $|\ell \cap S| = k_i$ is called a $k_i$-secant of $S$ whereas the integers $k_i$ are called characters of $S$.

A unital in $\text{PG}(2, q^2)$ is a $(q^3 + 1)$-set of type $(1, q + 1)$. A class of unitals in $\text{PG}(2, q^2)$ is given by the (non-degenerate) Hermitian curves, that is sets of absolute points with respect to (non-degenerate) unitary polarities; these are also called classical unitals.

Unitals which are not Hermitian curves are non-classical. A unital $U$ in $\text{PG}(2, q^2)$ is parabolic or hyperbolic according as the line at infinity contains 1 or $q + 1$ points of $U$.

Every unital in $\text{PG}(2, 2^2)$ is classical. The first non-classical unitals in $\text{PG}(2, q^2)$ with $q = 2^{2r+1}$, $r \geq 1$ were found by Buekenhout in [4]. Using Buekenhout’s method, Metz extended this class of non-classical unitals in $\text{PG}(2, q^2)$ to all values of $q \geq 2$; see [9]. A Buekenhout-Metz unital (BM unital for short) is a parabolic unital obtained with the construction given in [9] in which the ovoidal cone is an elliptic cone. This class also includes classical unitals. We refer the reader to [6] for a survey of results on these unitals.

Let $(X_0, X_1, X_2)$ denote homogeneous coordinates for points of $\text{PG}(2, q^2)$. The line $\ell_\infty : X_0 = 0$ will be taken as the line at infinity, whereas $P_\infty$ will denote the point $(0, 0, 1)$. For $q = 2^h$, let $C_0$ be the additive subgroup of $\text{GF}(q)$ defined by $C_0 = \{x \in \text{GF}(q) \mid \text{Tr}(x) = 0\}$ where

$$\text{Tr} : \text{GF}(q) \to \text{GF}(2) : x \mapsto x + x^2 + \cdots + x^{2^{h-1}}$$

is the trace map of $\text{GF}(q)$ over $\text{GF}(2)$. The following results come from [2] for $q$ odd and from [5] for $q$ even.
Lemma 2.1. Let \( a, b \in \text{GF}(q^2) \). The point set 
\[
U_{a,b} = \{(1, t, at^2 + bt^{q+1} + r) \mid t \in \text{GF}(q^2), r \in \text{GF}(q)\} \cup \{P_\infty\}
\]
is a BM unital in \( \text{PG}(2, q^2) \) if and only if either \( q \) is odd and \( 4a^{q+1} + (b^q - b)^2 \) is a non-square in \( \text{GF}(q) \), or \( q \) is even, \( b \notin \text{GF}(q) \) and \( a^{q+1}/(b^q + b)^2 \in C_0 \).

The expression \( 4a^{q+1} + (b^q - b)^2 \) for \( q \) odd, and \( a^{q+1}/(b^q + b)^2 \) with \( b \notin \text{GF}(q) \), for \( q \) even, is the discriminant of the unital \( U_{a,b} \).

Lemma 2.2. Every BM unital can be expressed as \( U_{a,b} \), for some \( a, b \in \text{GF}(q^2) \) which satisfy the discriminant condition of Lemma 2.1. Furthermore, a BM unital \( U_{a,b} \) is classical if and only if \( a = 0 \).

3 Sets with few characters

In this section we are going to construct a family of \((q^2 + 1)\)-sets with four characters in \( \text{PG}(2, q^2) \), where \( q = p^h \) and \( p \) is any prime power. Some of them, as pointed out in Remark 4.2, may be obtained by intersecting two BM unitals, at least one of which is non-classical.

Let \( \sigma \) denote the automorphism of \( \text{GF}(q^2) \) defined by 
\[
x^\sigma = x^{p^i}, \text{ with } i < h \text{ and } (i, h) = 1.
\]
Write \( T_0 = \{t \in \text{GF}(q^2) \mid T(t) = 0\} \), where 
\[
T : x \in \text{GF}(q^2) \mapsto x^q + x \in \text{GF}(q)
\]
is the trace function of \( \text{GF}(q^2) \) over \( \text{GF}(q) \).

Theorem 3.1. For each \( a \in \text{GF}(q^2)^* \), the subset 
\[
S = \{(1, t, at^\sigma + r) \mid t \in \text{GF}(q), r \in T_0\} \cup \{P_\infty\}
\]
of \( \text{PG}(2, q^2) \) is either of type \((0, 1, q, q + 1)\) or of type \((0, 1, p, q + 1)\) according as \( a \in T_0 \) or not.

Proof. By construction, \( S \) consists of \( q^2 + 1 \) points not all on a same line. Observe that \( S \) is not a blocking set with respect to the lines of \( \text{PG}(2, q^2) \) since, otherwise, it would contain at least \( q^2 + 3 \) points; see [7, Lemma 13.4]. Therefore, there exist some \( 0\)-secants of \( S \). We are going to show that for each \( k\)-secant of \( S \) which is neither external nor tangent to it, \( k \in \{q, q + 1\} \) or \( k \in \{p, q + 1\} \) according as \( a \in T_0 \) or not.
We begin by considering the line $P_{\infty}P_{t,r}$ joining the point $P_{\infty}$ with another point $P_{t,r} = (1, t, at^\sigma + r) \in S$. Such a line corresponds to the set

$$\{(1, t, at^\sigma + r + \alpha) \mid \alpha \in GF(q^2)\} \cup \{P_{\infty}\};$$

hence, the intersection of $P_{\infty}P_{t,r}$ and $S$ is

$$\{(1, t, at^\sigma + r + \alpha) \mid \alpha \in T_0\} \cup \{P_{\infty}\};$$

that is the line $P_{\infty}P_{t,r}$ is a $(q + 1)$-secant of $S$.

Now take the line $P_{t_1, r_1}P_{t_2, r_2}$ through two distinct points

$P_{t_1, r_1} = (1, t_1, at_1^\sigma + r_1)$ and $P_{t_2, r_2} = (1, t_2, at_2^\sigma + r_2)$

of $S$. Such a line consists of all the points

$$Q_\alpha = (\alpha + 1, \alpha t_1 + t_2, a(\alpha t_1^\sigma + t_2^\sigma) + \alpha r_1 + r_2)$$

with $\alpha$ ranging over $GF(q^2)$, plus the point $P_{t_1, r_1}$.

If $t_1 = t_2$, then the line $P_{t_1, r_1}P_{t_2, r_2}$ passes through the point $P_{\infty}$ and hence is a $(q + 1)$-secant of $S$.

When $t_1 \neq t_2$, observe that the point at infinity $Q_{-1} = (0, t_2 - t_1, a(t_2^\sigma - t_1^\sigma) + r_2 - r_1)$ of the line $P_{t_1, r_1}P_{t_2, r_2}$ is not on $S$. Thus, we restrict our attention to the affine points $Q_\alpha$ where $\alpha \neq -1$. The normalized homogeneous coordinates for these points are

$$\left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a \left(\frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1}\right) + \frac{\alpha r_1 + r_2}{\alpha + 1}\right).$$

A point $Q_\alpha$ is on $S$ if and only if the following conditions hold:

(i) $\frac{\alpha t_1 + t_2}{\alpha + 1} \in GF(q);$  

(ii) $\frac{a(\alpha t_1^\sigma + t_2^\sigma) + \alpha r_1 + r_2}{\alpha + 1} - \frac{a(\alpha t_1 + t_2)^\sigma}{\alpha^\sigma + 1} \in T_0.$

Condition (i) implies $(\alpha^q - \alpha)(t_1 - t_2) = 0$, therefore, as $t_1 \neq t_2$, we have $\alpha \in GF(q)$. Hence condition (ii) can be written as

$$\left(a^q + a \right) \left[\frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1} - \frac{(\alpha^q t_1^\sigma + t_2^\sigma)}{(\alpha^\sigma + 1)}\right] = 0. \quad (1)$$

If $a^q + a = 0$, then the intersection of $P_{t_1, r_1}P_{t_2, r_2}$ and $S$ is the set

$$\left\{\left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a \frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right) \mid \alpha \in GF(q) \setminus \{-1\}\right\} \cup \{P_{t_1, r_1}\},$$
that is the line \( P_{t_1, r_1}P_{t_2, r_2} \) is a \( q \)-secant to \( S \).

In the case where \( a^q + a \neq 0 \), (1) gives \((\alpha^q - \alpha)(t_1^q - t_2^q) = 0\); thus, \( \alpha^q - \alpha = 0 \) as \( t_1 \neq t_2 \). Whence \( \alpha = 0 \) or \( \alpha^{(q-1)} = 1 \).

As \((p^h - 1, p^i - 1) = p^{(h,i)} - 1 \) and \((i, h) = 1 \) the equation \( \alpha^{(q-1)} = 1 \) has \( p - 1 \) solutions in \( \text{GF}(q) \), one of them is \( \alpha = -1 \). Thus, there are \( p - 2 + 2 \) affine points \( Q_\alpha \) on \( P_{t_1, r_1}P_{t_2, r_2} \cap S \), that is the line \( P_{t_1, r_1}P_{t_2, r_2} \) is a \( p \)-secant of \( S \). \( \square \)

Let \( s \) be an element of \( \text{GF}(q^2) \setminus \{1\} \) such that \( s^{q+1} = 1 \). Set

\[ A = \{ (1, 0, r) \mid r \in \text{GF}(q) \}. \]

For each \( a \in \text{GF}(q^2)^* \), write

\[ B = \{ (1, t, at^2 + r) \mid t \in \text{GF}(q^2), t^{q-1} = s, r \in \text{GF}(q) \}. \]

**Theorem 3.2.** The subset

\[ S = A \cup B \cup \{ P_\infty \} \]

of \( \text{PG}(2, q^2) \) is either of type \((0, 1, q, q + 1)\) or of type \((0, 1, 2, q + 1)\) according as \( a^{q-1}s^2 = 1 \) or not.

**Proof.** By definition, \( S \) consists of \( q^2 + 1 \) points. As seen in the proof of Theorem 3.1 there exist some 0-secants of \( S \). We are going to show that for each \( k \)-secant of \( S \) which is neither external nor tangent to \( S \), \( k \in \{q, q + 1\} \) or \( k \in \{2, q + 1\} \) according as \( a^{q-1}s^2 = 1 \) or not.

Arguing as in the proof of Theorem 3.1, it can be verified that a line through the point \( P_\infty \) which is not tangent to the set \( S \), meets \( S \) in \( q + 1 \) points.

Next, we consider the line \( P_rP_{t, m} \) joining the point \( P_r = (1, 0, r) \in A \), with the point \( P_{t, m} = (1, t, at^2 + m) \in B \). Such a line corresponds to the set

\[ \{ (\alpha + 1, t, at^2 + m + \alpha r) \mid \alpha \in \text{GF}(q^2) \} \cup \{ P_r \}. \]

Since \( t \neq 0 \), the point at infinity \((0, t, at^2 + m - r)\) of the line \( P_rP_{t, m} \) is not on \( S \). Thus, we restrict our attention to the affine points \( Q_\alpha = (\alpha + 1, t, at^2 + m + \alpha r) \), with \( \alpha \neq -1 \), on the line \( P_rP_{t, m} \). The normalized homogeneous coordinates for these points are

\[ \left( 1, \frac{t}{\alpha + 1}, \frac{at^2 + m + \alpha r}{\alpha + 1} \right). \]

A point \( Q_\alpha \) is on \( S \) if and only if the following conditions hold:

(i) \( \left( \frac{t}{\alpha + 1} \right)^{q-1} = s; \)
(ii) \( \frac{at^2 + m + \alpha r}{\alpha + 1} - \frac{at^2}{(\alpha + 1)^2} \in GF(q). \)

Condition (i) implies \((\alpha + 1) \in GF(q)^*, \) therefore, \(\alpha \in GF(q) \setminus \{-1\}.\) Hence (ii) becomes \(at^2 \in GF(q),\) that is \(a^{q-1}t^2 = 1.\) Thus, if \(a^{q-1}t^2 = 1,\) then the intersection of \(P_t P_{t,m}\) with \(S\) is

\[
\left\{ \left(1, \frac{t}{\alpha + 1}, \frac{at^2 + m + \alpha r}{\alpha + 1} \right) \mid \alpha \in GF(q) \setminus \{-1\} \right\} \cup \{P_t\},
\]

that is the line \(P_r P_{t,m}\) is a \(q\)-secant to \(S.\) In the case \(a^{q-1}t^2 \neq 1\) the line \(P_r P_{t,m}\) is a 2-secant.

Now take the line \(P_{t_1, r_1} P_{t_2, r_2}\) joining two distinct points

\[
P_{t_1, r_1} = (1, t_1, at_1^2 + r_1) \quad \text{and} \quad P_{t_2, r_2} = (1, t_2, at_2^2 + r_2)
\]

of \(B.\) Such a line consists of \(P_{t_1, r_1}\) plus the points

\[
Q_\alpha = (\alpha + 1, \alpha t_1 + t_2, a(\alpha t_1^2 + t_2^2) + \alpha r_1 + r_2)
\]

as \(\alpha\) ranges over \(GF(q^2).\)

If \(t_1 = t_2,\) the line \(P_{t_1, r_1} P_{t_2, r_2}\) passes through the point \(P_\infty,\) hence it is a \((q+1)\)-secant of \(S.\)

When \(t_1 \neq t_2,\) observe that the point at infinity \((0, t_2 - t_1, a(t_2 - t_1^2) + r_2 - r_1)\) of \(P_{t_1, r_1} P_{t_2, r_2}\) is not on \(S.\) Thus, we restrict our attention to the affine points \(Q_\alpha\) with \(\alpha \neq -1.\) Their normalized homogeneous coordinates are

\[
\left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a \frac{(\alpha t_1^2 + t_2^2)}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right).
\]

A point \(Q_\alpha\) is on \(S\) if and only if the following conditions hold:

(i) \(\alpha t_1 + t_2 = 0\) or \(\left(\frac{\alpha t_1 + t_2}{\alpha + 1}\right)^{q-1} = s;\)

(ii) \(\frac{a(\alpha t_1^2 + t_2^2) + \alpha r_1 + r_2}{\alpha + 1} = \frac{a(\alpha t_1 + t_2)^2}{(\alpha + 1)^2} \in GF(q).\)

When \(\alpha = -\frac{t_2}{t_1} \in GF(q),\) condition (ii) becomes \(-\alpha t_1 t_2 \in GF(q);\) hence we have \(a^{(q-1)}t^2 = 1.\) Therefore the point \((1, 0, \frac{a(t(t_2-t_1)^2)}{t_1-t_2})\) belongs to \(S\) if and only if \(a^{(q-1)}t^2 = 1.\)

In the case \(\left(\frac{\alpha t_1 + t_2}{\alpha + 1}\right)^{q-1} = s\) we get \(\alpha \in GF(q) \setminus \{-1, -\frac{t_1}{t_2}\}.\) Hence condition (ii) can be written as \(a(t_1^2 - t_2^2)^{q-1} \in GF(q),\) that is \(a^{q-1}s^2 = 1.\) Therefore, if \(a^{(q-1)}s^2 = 1\) the line \(P_{t_1, r_1} P_{t_2, r_2}\) is a \(q\)-secant to \(S,\) otherwise it meets \(S\) only in \(P_{t_1, r_1}\) and \(P_{t_2, r_2},\) thus it is a 2-secant to \(S.\) \(\square\)
4 Main result

In this section we study the cardinality of the intersection of two distinct BM unitals in the family
\[ \mathcal{F} = \{ U_{a,b} \}_{(a,b) \in \text{GF}(q^2) \times \text{GF}(q^2)}, \]
where
\[ U_{a,b} = \{ (1, t, at^2 + btq^{q+1} + r) \mid t \in \text{GF}(q^2), r \in \text{GF}(q) \} \cup \{ P_{\infty} \} \]
and the coefficients \( a \) and \( b \) satisfy the discriminant condition of Lemma 2.1.

**Theorem 4.1.** In PG(2, \( q^2 \)), with \( q \) a prime power, the intersection size of two unitals of \( \mathcal{F} \) is one of the following:
\[ q + 1, q^2 + 1, 2q^2 - q + 1. \]
Furthermore, any two classical unitals of \( \mathcal{F} \) can only intersect in \( q + 1 \) collinear points.

**Proof.** Let \( U_{a_1,b_1} \) and \( U_{a_2,b_2} \) be two distinct unitals in \( \mathcal{F} \). Denote by \( I \) their intersection and set \( \alpha = a_1 - a_2, \beta = b_1 - b_2 \). We distinguish the following cases:

(A) \( \alpha + \beta = 0 \) and \( \alpha \in \text{GF}(q)^* \);
(B) \( \alpha + \beta = 0 \) and \( \alpha \notin \text{GF}(q) \);
(C) \( \alpha + \beta \in \text{GF}(q)^* \);
(D) \( \alpha + \beta \notin \text{GF}(q) \).

**Case (A)**

Since \( a_1 + b_1 = a_2 + b_2 \), the points in
\[ S_1 = \{ (1, t, (a_1 + b_1)t^2 + r) \mid t, r \in \text{GF}(q) \} \cup \{ P_{\infty} \} \]
are on both unitals. Therefore, the cardinality of \( I \) is at least \( q^2 + 1 \).

Let \( Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1}, \) for a suitable \( t \in \text{GF}(q^2) \setminus \text{GF}(q) \). The point \( Q' \) lies also on \( U_{a_2,b_2} \) if and only if
\[ \alpha t^2 + \beta t^{q+1} \in \text{GF}(q), \] (2)
or equivalently
\[ \alpha (t^2 - t^{q+1}) \in \text{GF}(q). \] (3)
By the hypothesis $\alpha \in \text{GF}(q)^*$, (3) may be rewritten as

$$t^{q-1} = \pm 1.$$  \hfill (4)

There are now two possibilities.

(A$_1$) $q$ is even.

Then (4) implies $t \in \text{GF}(q)$; hence, by the assumption made on $t$, there are no points $Q'$ on $I$; thus

$$|I| = q^2 + 1.$$  \hfill (A$_1$)

(A$_2$) $q$ is odd.

Since $t \notin \text{GF}(q)$, (4) necessarily gives $t^{q-1} = -1$. This condition is satisfied by $q - 1$ values of $t$ and to any such a value there correspond $q$ points $Q' \in I$ as $r$ ranges over $\text{GF}(q)$. Therefore

$$|I| = q^2 + 1 + q(q - 1) = 2q^2 - q + 1.$$  \hfill (A$_2$)

Case (B)

Arguing as in Case (A) we have that $S_1$ is a subset of $I$ and so $|I| \geq q^2 + 1$.

Again a point $Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1}$, with $t \in \text{GF}(q^2) \setminus \text{GF}(q)$, lies on $U_{a_2,b_2}$ if and only if (3) holds. Setting

$$y = t^{q-1},$$  \hfill (5)

condition (3) can be rewritten as

$$\alpha^q y^2 - (\alpha^q - \alpha)y - \alpha = 0.$$  \hfill (6)

As $\alpha \neq 0$, equation (6) has solutions $y = 1$ or $y = -\alpha^{-1-q}$. We distinguish the following subcases.

(B$_1$) $q$ is even.

Since $\alpha \notin \text{GF}(q)$, $-\alpha^{-1-q}$ is different from 1. Because of (5), we necessarily have

$$t^{q-1} = -\alpha^{-1-q}$$  \hfill (7)

as $t \in \text{GF}(q^2) \setminus \text{GF}(q)$. Equation (7) gives $q - 1$ possible values for $t$; for any such a value, we get $q$ points $Q' \in I$ as $r$ varies in $\text{GF}(q)$. Therefore we get again $|I| = 2q^2 - q + 1$.

(B$_2$) $q$ is odd.


(B21) If \( \alpha \in T_0 \), that is \( \alpha^q + \alpha = 0 \), then \( -\alpha^{1-q} = 1 \). From (5) it follows that \( t^{q-1} = 1 \), which is not allowed. Thus, there are no points \( Q' \) on \( \mathcal{I} \), and hence \(|\mathcal{I}| = q^2 + 1 \).

(B22) Assume \( \alpha \notin T_0 \). In this case 1 and \( -\alpha^{1-q} \) are two distinct solutions of (6). Arguing as in case (B1) it follows that \( \mathcal{I} \) consists of \( 2q^2 - q + 1 \) points.

**Case (C)**

Let

\[
S_i = \{(1, t, (a_i + b_i)t^2 + r) \mid t, r \in \text{GF}(q)\} \cup \{(0, 0, 1)\} \subset U_{a_i, b_i}
\]

for \( i \in \{1, 2\} \). We are going to show that \( S_1 = S_2 \). To this end, observe that a point \( Q = (1, t, (a_i + b_i)t^2 + r) \in S_i \) lies also on \( S_j \) for any distinct \( i, j \in \{1, 2\} \), since

\[
(a_j + b_j)t^2 + (\alpha + \beta)t^2 = (a_i + b_i)t^2
\]

and \( \alpha + \beta \in \text{GF}(q)^* \). Hence, \( S_1 \subseteq \mathcal{I} \) and thus \( |\mathcal{I}| \geq q^2 + 1 \).

Now, consider a point \( Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1, b_1} \) for a suitable \( t \in \text{GF}(q^2) \setminus \text{GF}(q) \). The point \( Q' \) is also a point on \( U_{a_2, b_2} \) if and only if (2) holds, namely, in this case,

\[
\alpha^q t^{2(q-1)} + (\beta^q - \beta)t^{q-1} - \alpha = 0 . \tag{8}
\]

Setting \( y \) as in (5), condition (8) becomes

\[
\alpha^q y^2 + (\beta^q - \beta)y - \alpha = 0 . \tag{9}
\]

Observe that \( \alpha \neq 0 \), since, otherwise, \( \beta^q - \beta = 0 \) and (9) would be always true; therefore, the two unitals would be the same, contradicting our assumption.

As \( \alpha \neq 0 \), equation (9) has solutions \( y = 1 \) or \( y = -\alpha^{-1-q} \). There are now several subcases to consider.

(C1) \( \alpha \in \text{GF}(q^2) \setminus \text{GF}(q) \).

(C1.1) \( q \) is even.

In this case the solutions \( y = 1 \) and \( y = -\alpha^{-1-q} \) of (9) are distinct. Because of (5), we can only have \( t^{q-1} = -\alpha^{1-q} \) as \( t \notin \text{GF}(q) \); again we find \( q-1 \) values for \( t \) satisfying (8), and for any such a value, we obtain \( q \) points \( Q' \in \mathcal{I} \), as \( r \) ranges over \( \text{GF}(q) \). Therefore,

\[
|\mathcal{I}| = 2q^2 - q + 1 .
\]
(C_{13}) \( q \) is odd and \( \alpha \in T_0 \).
As \( \alpha \in T_0 \) then \(-\alpha^{1-q} = 1 \). From (5) we have \( t^{q-1} = 1 \), which is impossible. Thus, \(|I| = q^2 + 1\).

(C_{14}) \( q \) is odd and \( \alpha \not\in T_0 \).
In this case \(-\alpha^{1-q} \neq 1 \), therefore, arguing as in case (C_{12}), we get that \( I \) consists of \( 2q^2 - q + 1 \) points.

(C_2) \( \alpha \in \text{GF}(q)^* \).
Equation (8) gives \( t^{q-1} = \pm 1 \). (10)

(C_{21}) If \( q \) is even, condition (10) implies \( t \in \text{GF}(q) \), which is not allowed; thus,
\[ |I| = q^2 + 1 \]

(C_{22}) Suppose \( q \) to be odd. As \( t \not\in \text{GF}(q) \), we necessarily have from (10) that \( t^{q-1} = -1 \), a condition satisfied by \( q - 1 \) possible values for \( t \); to any such a value of \( t \) there correspond \( q \) points \( Q' \in I \) as \( r \) ranges over \( \text{GF}(q) \). Therefore, again
\[ |I| = 2q^2 - q + 1. \]

Case (D)

Let us again consider the point-sets
\[ S_i = \{(1, t, (a_i + b_i)t^2 + r) \mid t, r \in \text{GF}(q)\} \cup \{P_\infty\} \]
where \( i = 1, 2 \). A point \( Q = (1, t, (a_i + b_i)t^2 + r) \in S_i \) lies also on \( S_j \) for \( i \neq j \), if and only if the element \((\alpha + \beta)t^2 \in \text{GF}(q)\); the hypothesis \( \alpha + \beta \not\in \text{GF}(q) \) forces \( t \) to be zero. Thus, \( S_1 \cap S_2 = \{(1, 0, r) \mid r \in \text{GF}(q)\} \) and \( |I| \geq q + 1 \).

Next, take a point \( Q' = (1, t, a_1 t^2 + b_1 t^{q+1} + r) \in U_{a_1,b_1} \) with \( t \in \text{GF}(q^2) \setminus \text{GF}(q) \). The point \( Q' \) is on \( U_{a_2,b_2} \) if and only if (8) holds. We distinguish three possibilities.

(D_1) \( \alpha = 0 \).
In this case \( \beta^{q} - \beta \neq 0 \) and (8) gives \( t = 0 \) which is not allowed. Thus
\[ |I| = q + 1 \]

(D_2) \( q \) is even and \( \alpha \neq 0 \).

(D_{21}) \( \beta \in \text{GF}(q) \).
Condition (8) gives \( t^{q-1} = \sqrt{1/\alpha^{q-1}} \) with \( \alpha^{q-1} \neq 1 \). Once again, we get \( q-1 \) possible values for \( t \); so for any such a value, we get \( q \) points \( Q' \in I \) as \( r \) ranges over \( GF(q) \). Hence,

\[
|I| = q^2 + 1.
\]

\((D_22)\) \( \beta \notin GF(q) \).

Let \( y \) be as in (5); we get again (9). This equation has 2 solutions as \( \delta = \alpha^{q+1}/(\beta^q - \beta)^2 \) belongs to \( GF(q) \) and hence the absolute trace of \( \delta \) is zero. Furthermore, both solutions are different from 1 as \( \alpha + \beta \notin GF(q) \).

Therefore, by (5), we find \( 2(q-1) \) possible values for \( t \) and thus, \( 2q(q-1) \) points \( Q' \) on \( I \). Hence \( I \) consists of \( 2q(q-1) + q + 1 = 2q^2 - q + 1 \) points.

\((D_3)\) \( q \) is odd and \( \alpha \neq 0 \).

We need to consider the discriminant of (9), that is

\[
\Delta = (\beta^q - \beta)^2 + 4\alpha^{q+1} \in GF(q)
\]

\((D_31)\) \( \Delta = 0 \).

Condition (9) has the unique solution \( y = \frac{\beta - \beta^q}{2\alpha^{q+1}} \neq 1 \) which gives \( q-1 \) possible values for \( t \) because of (5); hence

\[
|I| = q^2 + 1.
\]

\((D_32)\) \( \Delta \neq 0 \).

As \( \Delta \in GF(q)^* \) we get \( \Delta^{(q^2-1)/2} = 1 \), that is \( \Delta \) is a non-zero square in \( GF(q^2) \).

Therefore, (9) has two non-zero solutions different from 1. Each of them provides \( q-1 \) possible values for \( t \); thus

\[
|I| = 2q^2 - q + 1.
\]

Finally, assume both \( U_{a_1b_1} \) and \( U_{a_2b_2} \) to be classical. From Lemma 2.2 it follows that \( \alpha = 0 \); this only happens in case \((D_1)\) giving \( |I| = q + 1 \). \qed

**Remark 4.2.** The configurations for the intersection \( I \) of two BM unitals \( U_{a_1b_1}, U_{a_2b_2} \) in \( \mathcal{F} \) are the following:

1. \( I \) consists of \( q + 1 \) collinear points;
2. \( I \) consists of \( q \) sets of \( q + 1 \) collinear points. The \( q \) lines all meet at \( P_\infty \);
(3) $\mathcal{I}$ consists of $2q - 1$ sets of $q + 1$ collinear points. The $2q - 1$ lines all pass through the point $P_\infty$.

Furthermore, it follows from the proof of Theorem 4.1 that

(a) in cases (A$_1$) and (C$_{21}$) the intersection $\mathcal{I}$ is the set

$$\mathcal{I} = \{(1, t, (a_1 + b_1) t^2 + r) \mid t, r \in \text{GF}(q)\} \cup \{P_\infty\}.$$  

Hence $\mathcal{I}$ is one of the $(q^2 + 1)$-sets defined in Theorem 3.1 with $a = a_1 + b_1$ and $\sigma$ the automorphism of $\text{GF}(q^2)$ such that $x^\sigma = x^2$.

(b) In cases (D$_{21}$) and (D$_{31}$) the intersection $\mathcal{I}$ turns out to be one of the $(q^2 + 1)$-sets defined in Theorem 3.2 with respectively $s = \sqrt{1/\alpha q - 1}$ or $s = \sqrt{1/\beta q - 1}$, and $a = a_1 + sb_1$.

5 Examples

In this section we show that all the cases discussed in Theorem 4.1 effectively occur for $q = 4$ and 5. If $q = 4$, denote by $\omega$ a primitive element of $\text{GF}(16)$, such that $\omega^2 + \omega + \delta = 0$, with $\delta$ any element of $\text{GF}(4) \setminus \text{GF}(2)$. Furthermore, put $a_1 = \omega^3$ and $b_1 = \omega$.

When $q = 5$, take $\xi$ as a primitive element of $\text{GF}(25)$ such that $\xi^2 - \xi + 2 = 0$ and set $a_1 = \xi$, $b_1 = \xi^{12}$.

Under these assumptions, $U_{a_1, b_1}$ turns out to be a non-classical BM unital respectively in $\text{PG}(2, 16)$ or in $\text{PG}(2, 25)$. Let $a_2$ and $b_2$ be two coefficients ranging over $\text{GF}(16)$ or $\text{GF}(25)$ in such a way that the discriminant condition of Lemma 2.1 is satisfied.

By different choices of $a_2$ and $b_2$ we get all the cases for $U_{a_1, b_1} \cap U_{a_2, b_2}$ occurring in the proof of Theorem 4.1 but case (D$_1$); see Table 1.

References


Intersections of Buekenhout-Metz unitals

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Table 1: Intersection cases for $q$ small


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