Intersections of Buekenhout-Metz unitals

Angela Aguglia*

On the occasion of the 60th birthday of Gàbor Korchmàros

Abstract

Configurations arising as intersections of two Buekenhout-Metz unitals of a given family are studied and, in the case in which at most one of the unitals is classical, a new intersection size is found.

Keywords: projective plane, unital

MSC 2000: 05B25

1. Introduction

In [8] it has been shown that there are just seven configurations in which two classical unitals may intersect. There-within it has also been proved that the cardinality of the intersection of any two classical unitals in the desarguesian projective plane $\mathrm{PG}(2, q^2)$ is congruent to 1 modulo q.

A family of non-classical Buekenhout-Metz unitals in $\mathrm{PG}(2, q^2)$, with $q = p^h$ an odd prime power, has been constructed in [1]; the intersection of every unital of this family with a classical one contains a number of points congruent to 1 modulo p. In the same paper, it is also conjectured that the size of the intersection of any classical unital with a non-classical one should be one of the following:

$$q^2 \pm 2q + 1,\ q^2 \pm q + 1,\ q^2 + 1\,.$$

Afterwards, in [3] it has been proved that an arbitrary unital in $\mathrm{PG}(2, q^2)$, with $q = p^h$ any prime power, meets a classical unital in a number of points congruent

*Research supported by the Italian Ministry MIUR, Strutture geometriche, combinatoria e loro applicazioni.

to 1 modulo p. To classify intersections of two non-classical unitals seems to be a difficult question. Here we make some advances in this direction by looking at a suitable family $\mathfrak{F}$ of Buekenhout-Metz unitals in $\mathsf{PG}(2, q^2)$, q any prime power, containing both classical unitals as well as non-classical ones. We prove that any two classical unitals in $\mathfrak{F}$ intersect on $q+1$ collinear points, whereas in all other cases the intersection number is one of the following:

$$q+1,\ q^2+1, 2q^2-q+1\,.$$

This last size does not appear among those conjectured in [1].

2. Preliminaries

A set $\mathcal{S}$ of k points (or a k-set) in a projective plane of order q is *of type* $(k_1, k_2, \ldots, k_s)$, with $k_1 < k_2 < \cdots < k_s$, if a line ℓ may intersect $\mathcal{S}$ in only sets of $k_1, k_2, \ldots$ or k_s points. A line ℓ for which $|\ell \cap \mathcal{S}| = k_i$ is called a k_i-secant of $\mathcal{S}$ whereas the integers k_i are called *characters* of $\mathcal{S}$.

A *unital* in $\mathsf{PG}(2, q^2)$ is a (q^3+1)-set of type $(1, q+1)$. A class of unitals in $\mathsf{PG}(2, q^2)$ is given by the (non-degenerate) Hermitian curves, that is sets of absolute points with respect to (non-degenerate) unitary polarities; these are also called *classical unitals*.

Unitals which are not Hermitian curves are *non-classical*. A unital U in $\mathsf{PG}(2, q^2)$ is *parabolic* or *hyperbolic* according as the line at infinity contains 1 or $q+1$ points of U.

Every unital in $\mathsf{PG}(2, 2^2)$ is classical. The first non-classical unitals in $\mathsf{PG}(2, q^2)$ with $q = 2^{2r+1}$, $r \geq 1$ were found by Buekenhout in [4]. Using Buekenhout's method, Metz extended this class of non-classical unitals in $\mathsf{PG}(2, q^2)$ to all values of $q \geq 2$; see [9]. A *Buekenhout-Metz unital* (BM unital for short) is a parabolic unital obtained with the construction given in [9] in which the ovoidal cone is an elliptic cone. This class also includes classical unitals. We refer the reader to [6] for a survey of results on these unitals.

Let (X_0, X_1, X_2) denote homogeneous coordinates for points of $\mathsf{PG}(2, q^2)$. The line $\ell_\infty : X_0 = 0$ will be taken as the line at infinity, whereas P_∞ will denote the point $(0,0,1)$. For $q = 2^h$, let C_0 be the additive subgroup of $\mathsf{GF}(q)$ defined by $C_0 = \{x \in \mathsf{GF}(q) \mid \mathrm{Tr}(x) = 0\}$ where

$$\mathrm{Tr}\colon \mathsf{GF}(q) \to \mathsf{GF}(2)\colon x \mapsto x + x^2 + \ldots + x^{2^{h-1}}$$

is the trace map of $\mathsf{GF}(q)$ over $\mathsf{GF}(2)$. The following results come from [2] for q odd and from [5] for q even.

Lemma 2.1. *Let $a, b \in \mathsf{GF}(q^2)$. The point set*

$$U_{a,b} = \big\{(1, t, at^2 + bt^{q+1} + r) \mid t \in \mathsf{GF}(q^2), r \in \mathsf{GF}(q)\big\} \cup \{P_\infty\}$$

is a BM unital in $\mathsf{PG}(2, q^2)$ if and only if either q is odd and $4a^{q+1} + (b^q - b)^2$ is a non-square in $\mathsf{GF}(q)$, or q is even, $b \notin \mathsf{GF}(q)$ and $a^{q+1}/(b^q + b)^2 \in C_0$.

The expression $4a^{q+1} + (b^q - b)^2$ for q odd, and $a^{q+1}/(b^q + b)^2$ with $b \notin \mathsf{GF}(q)$, for q even, is the *discriminant* of the unital $U_{a,b}$.

Lemma 2.2. *Every BM unital can be expressed as $U_{a,b}$, for some $a, b \in \mathsf{GF}(q^2)$ which satisfy the discriminant condition of Lemma 2.1. Furthermore, a BM unital $U_{a,b}$ is classical if and only if $a = 0$.*

3. Sets with few characters

In this section we are going to construct a family of $(q^2 + 1)$-sets with four characters in $\mathsf{PG}(2, q^2)$, where $q = p^h$ and p is any prime power. Some of them, as pointed out in Remark 4.2, may be obtained by intersecting two BM unitals, at least one of which is non-classical.

Let σ denote the automorphism of $\mathsf{GF}(q^2)$ defined by

$$x^\sigma = x^{p^i}, \text{with } i < h \text{ and } (i, h) = 1\,.$$

Write $T_0 = \{t \in \mathsf{GF}(q^2) \mid \mathrm{T}(t) = 0\}$, where

$$\mathrm{T} : x \in \mathsf{GF}(q^2) \mapsto x^q + x \in \mathsf{GF}(q)$$

is the trace function of $\mathsf{GF}(q^2)$ over $\mathsf{GF}(q)$.

Theorem 3.1. *For each $a \in \mathsf{GF}(q^2)^*$, the subset*

$$\mathcal{S} = \{(1, t, at^\sigma + r) \mid t \in \mathsf{GF}(q), r \in T_0\} \cup \{P_\infty\}$$

of $\mathsf{PG}(2, q^2)$ is either of type $(0, 1, q, q + 1)$ or of type $(0, 1, p, q + 1)$ according as $a \in T_0$ or not.

Proof. By construction, $\mathcal{S}$ consists of q^2+1 points not all on a same line. Observe that $\mathcal{S}$ is not a blocking set with respect to the lines of $\mathsf{PG}(2, q^2)$ since, otherwise, it would contain at least $q^2 + 3$ points; see [7, Lemma 13.4]. Therefore, there exist some 0-secants of $\mathcal{S}$. We are going to show that for each k-secant of $\mathcal{S}$ which is neither external nor tangent to it, $k \in \{q, q + 1\}$ or $k \in \{p, q + 1\}$ according as $a \in T_0$ or not.

We begin by considering the line $P_\infty P_{t,r}$ joining the point P_∞ with another point $P_{t,r} = (1, t, at^\sigma + r) \in \mathcal{S}$. Such a line corresponds to the set

$$\left\{(1, t, at^\sigma + r + \alpha) \mid \alpha \in \mathsf{GF}(q^2)\right\} \cup \{P_\infty\};$$

hence, the intersection of $P_\infty P_{t,r}$ and $\mathcal{S}$ is

$$\{(1, t, at^\sigma + r + \alpha) \mid \alpha \in T_0)\} \cup \{P_\infty\};$$

that is the line $P_\infty P_{t,r}$ is a $(q+1)$-secant of $\mathcal{S}$.

Now take the line $P_{t_1,r_1}P_{t_2,r_2}$ through two distinct points

$$P_{t_1,r_1} = (1, t_1, at_1^\sigma + r_1) \text{ and } P_{t_2,r_2} = (1, t_2, at_2^\sigma + r_2)$$

of $\mathcal{S}$. Such a line consists of all the points

$$Q_\alpha = \left(\alpha + 1, \alpha t_1 + t_2, a(\alpha t_1^\sigma + t_2^\sigma) + \alpha r_1 + r_2\right)$$

with α ranging over $\mathsf{GF}(q^2)$, plus the point P_{t_1,r_1}.

If $t_1 = t_2$, then the line $P_{t_1,r_1}P_{t_2,r_2}$ passes through the point P_∞ and hence is a $(q+1)$-secant of $\mathcal{S}$.

When $t_1 \neq t_2$, observe that the point at infinity $Q_{-1} = (0, t_2 - t_1, a(t_2^\sigma - t_1^\sigma) + r_2 - r_1)$ of the line $P_{t_1,r_1}P_{t_2,r_2}$ is not on $\mathcal{S}$. Thus, we restrict our attention to the affine points Q_α where $\alpha \neq -1$. The normalized homogeneous coordinates for these points are

$$\left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a\,\frac{(\alpha t_1^\sigma + t_2^\sigma)}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right).$$

A point Q_α is on $\mathcal{S}$ if and only if the following conditions hold:

(i) $\dfrac{\alpha t_1 + t_2}{\alpha + 1} \in \mathsf{GF}(q)$;

(ii) $\dfrac{a(\alpha t_1^\sigma + t_2^\sigma) + \alpha r_1 + r_2}{\alpha + 1} - \dfrac{a(\alpha t_1 + t_2)^\sigma}{\alpha^\sigma + 1} \in T_0$.

Condition (i) implies $(\alpha^q - \alpha)(t_1 - t_2) = 0$, therefore, as $t_1 \neq t_2$, we have $\alpha \in \mathsf{GF}(q)$. Hence condition (ii) can be written as

$$(a^q + a)\left[\frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1} - \frac{(\alpha^\sigma t_1^\sigma + t_2^\sigma)}{(\alpha^\sigma + 1)}\right] = 0. \tag{1}$$

If $a^q + a = 0$, then the intersection of $P_{t_1,r_1}P_{t_2,r_2}$ and $\mathcal{S}$ is the set

$$\left\{\left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a\,\frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right) \,\middle|\, \alpha \in \mathsf{GF}(q) \setminus \{-1\}\right\} \cup \{P_{t_1,r_1}\},$$

that is the line $P_{t_1,r_1}P_{t_2,r_2}$ is a q-secant to $\mathcal{S}$.

In the case where $a^q + a \neq 0$, (1) gives $(\alpha^\sigma - \alpha)(t_1^\sigma - t_2^\sigma) = 0$; thus, $\alpha^\sigma - \alpha = 0$ as $t_1 \neq t_2$. Whence $\alpha = 0$ or $\alpha^{(\sigma-1)} = 1$.

As $(p^h - 1, p^i - 1) = p^{(h,i)} - 1$ and $(i, h) = 1$ the equation $\alpha^{(\sigma-1)} = 1$ has $p - 1$ solutions in $\mathsf{GF}(q)$, one of them is $\alpha = -1$. Thus, there are $p - 2 + 2$ affine points Q_α on $P_{t_1,r_1}P_{t_2,r_2} \cap \mathcal{S}$, that is the line $P_{t_1,r_1}P_{t_2,r_2}$ is a p-secant of $\mathcal{S}$. □

Let s be an element of $\mathsf{GF}(q^2) \setminus \{1\}$ such that $s^{q+1} = 1$. Set

$$\mathcal{A} = \{(1, 0, r) \mid r \in \mathsf{GF}(q)\}.$$

For each $a \in \mathsf{GF}(q^2)^*$, write

$$\mathcal{B} = \{(1, t, at^2 + r) \mid t \in \mathsf{GF}(q^2), t^{q-1} = s, r \in \mathsf{GF}(q)\}.$$

Theorem 3.2. *The subset*

$$\mathcal{S} = \mathcal{A} \cup \mathcal{B} \cup \{P_\infty\}$$

of $\mathsf{PG}(2, q^2)$ *is either of type* $(0, 1, q, q + 1)$ *or of type* $(0, 1, 2, q + 1)$ *according as* $a^{q-1}s^2 = 1$ *or not.*

Proof. By definition, $\mathcal{S}$ consists of $q^2 + 1$ points. As seen in the proof of Theorem 3.1 there exist some 0-secants of $\mathcal{S}$. We are going to show that for each k-secant of $\mathcal{S}$ which is neither external nor tangent to $\mathcal{S}$, $k \in \{q, q + 1\}$ or $k \in \{2, q + 1\}$ according as $a^{q-1}s^2 = 1$ or not.

Arguing as in the proof of Theorem 3.1, it can be verified that a line trough the point P_∞ which is not tangent to the set $\mathcal{S}$, meets $\mathcal{S}$ in $q + 1$ points.

Next, we consider the line $P_r P_{t,m}$ joining the point $P_r = (1, 0, r) \in \mathcal{A}$, with the point $P_{t,m} = (1, t, at^2 + m) \in \mathcal{B}$. Such a line corresponds to the set

$$\left\{(\alpha + 1, t, at^2 + m + \alpha r) \mid \alpha \in \mathsf{GF}(q^2)\right\} \cup \{P_r\}.$$

Since $t \neq 0$, the point at infinity $(0, t, at^2 + m - r)$ of the line $P_r P_{t,m}$ is not on $\mathcal{S}$. Thus, we restrict our attention to the affine points $Q_\alpha = (\alpha + 1, t, at^2 + m + \alpha r)$, with $\alpha \neq -1$, on the line $P_r P_{t,m}$. The normalized homogeneous coordinates for these points are

$$\left(1, \frac{t}{\alpha + 1}, \frac{at^2 + m + \alpha r}{\alpha + 1}\right).$$

A point Q_α is on $\mathcal{S}$ if and only if the following conditions hold:

(i) $\left(\dfrac{t}{\alpha + 1}\right)^{q-1} = s;$

(ii) $\dfrac{at^2+m+\alpha r}{\alpha+1}-\dfrac{at^2}{(\alpha+1)^2}\in \mathsf{GF}(q)\,.$

Condition (i) implies $(\alpha+1)\in \mathsf{GF}(q)^*$, therefore, $\alpha\in \mathsf{GF}(q)\setminus\{-1\}$. Hence (ii) becomes $at^2\in \mathsf{GF}(q)$, that is $a^{q-1}s^2=1$. Thus, if $a^{q-1}s^2=1$, then the intersection of $P_rP_{t,m}$ and $\mathcal{S}$ is

$$\left\{\left(1,\frac{t}{\alpha+1},\frac{at^2+m+\alpha r}{\alpha+1}\right)\,\middle|\,\alpha\in \mathsf{GF}(q)\setminus\{-1\}\right\}\cup\{P_r\}\,,$$

that is the line $P_rP_{t,m}$ is a q-secant to $\mathcal{S}$. In the case $a^{q-1}s^2\neq 1$ the line $P_rP_{t,m}$ is a 2-secant.

Now take the line $P_{t_1,r_1}P_{t_2,r_2}$ joining two distinct points

$$P_{t_1,r_1}=(1,t_1,at_1^2+r_1)\ \text{ and }\ P_{t_2,r_2}=(1,t_2,at_2^2+r_2)$$

of $\mathcal{B}$. Such a line consists of P_{t_1,r_1} plus the points

$$Q_\alpha=\big(\alpha+1,\alpha t_1+t_2,a(\alpha t_1^2+t_2^2)+\alpha r_1+r_2\big)$$

as α ranges over $\mathsf{GF}(q^2)$.

If $t_1=t_2$, the line $P_{t_1,r_1}P_{t_2,r_2}$ passes through the point P_∞; hence it is a $(q+1)$-secant of $\mathcal{S}$.

When $t_1\neq t_2$, observe that the point at infinity $(0,t_2-t_1,a(t_2-t_1^2)+r_2-r_1)$ of $P_{t_1,r_1}P_{t_2,r_2}$ is not on $\mathcal{S}$. Thus, we restrict our attention to the affine points Q_α with $\alpha\neq -1$. Their normalized homogeneous coordinates are

$$\left(1,\frac{\alpha t_1+t_2}{\alpha+1},\,a\,\frac{(\alpha t_1^2+t_2^2)}{\alpha+1}+\frac{\alpha r_1+r_2}{\alpha+1}\right).$$

A point Q_α is on $\mathcal{S}$ if and only if the following conditions hold:

(i) $\alpha t_1+t_2=0$ or $\left(\dfrac{\alpha t_1+t_2}{\alpha+1}\right)^{q-1}=s\,;$

(ii) $\dfrac{a(\alpha t_1^2+t_2^2)+\alpha r_1+r_2}{\alpha+1}-\dfrac{a(\alpha t_1+t_2)^2}{(\alpha+1)^2}\in \mathsf{GF}(q)\,.$

When $\alpha=-\frac{t_2}{t_1}\in \mathsf{GF}(q)$, condition (ii) becomes $-at_1t_2\in \mathsf{GF}(q)$; hence we have $a^{(q-1)}s^2=1$. Therefore the point $(1,0,\frac{at_1t_2(t_2-t_1)+r_2t_1-r_1t_2}{t_1-t_2})$ belongs to $\mathcal{S}$ if and only if $a^{(q-1)}s^2=1$.

In the case $\left(\frac{\alpha t_1+t_2}{\alpha+1}\right)^{q-1}=s$ we get $\alpha\in \mathsf{GF}(q)\setminus\{-1,-\frac{t_1}{t_2}\}$. Hence condition (ii) can be written as $a(t_1^2-t_2^2)^{q-1}\in \mathsf{GF}(q)$, that is $a^{q-1}s^2=1$. Therefore, if $a^{q-1}s^2=1$ the line $P_{t_1,r_1}P_{t_2,r_2}$ is a q-secant to $\mathcal{S}$, otherwise it meets $\mathcal{S}$ only in P_{t_1,r_1} and P_{t_2,r_2}, thus it is a 2-secant to $\mathcal{S}$. □

ACADEMIA PRESS
UNIVERSITEIT GENT

4. Main result

In this section we study the cardinality of the intersection of two distinct BM unitals in the family

$$\mathfrak{F} = \{U_{a,b}\}_{(a,b)\in \mathsf{GF}(q^2)\times \mathsf{GF}(q^2)}\,,$$

where

$$U_{a,b} = \left\{(1, t, at^2 + bt^{q+1} + r) \mid t \in \mathsf{GF}(q^2), r \in \mathsf{GF}(q)\right\} \cup \{P_\infty\}$$

and the coefficients a and b satisfy the discriminant condition of Lemma 2.1.

Theorem 4.1. *In* $\mathsf{PG}(2, q^2)$*, with* q *a prime power, the intersection size of two unitals of* $\mathfrak{F}$ *is one of the following:*

$$q + 1, q^2 + 1, 2q^2 - q + 1\,.$$

Furthermore, any two classical unitals of $\mathfrak{F}$ *can only intersect in* $q + 1$ *collinear points.*

Proof. Let U_{a_1,b_1} and U_{a_2,b_2} be two distinct unitals in $\mathfrak{F}$. Denote by $\mathcal{I}$ their intersection and set $\alpha = a_1 - a_2$, $\beta = b_1 - b_2$. We distinguish the following cases:

(A) $\alpha + \beta = 0$ and $\alpha \in \mathsf{GF}(q)^*$;

(B) $\alpha + \beta = 0$ and $\alpha \notin \mathsf{GF}(q)$;

(C) $\alpha + \beta \in \mathsf{GF}(q)^*$;

(D) $\alpha + \beta \notin \mathsf{GF}(q)$.

Case (A)

Since $a_1 + b_1 = a_2 + b_2$, the points in

$$\mathcal{S}_1 = \left\{(1, t, (a_1 + b_1)t^2 + r) \mid t, r \in \mathsf{GF}(q)\right\} \cup \{P_\infty\}$$

are on both unitals. Therefore, the cardinality of $\mathcal{I}$ is at least $q^2 + 1$.

Let $Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1}$, for a suitable $t \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$. The point Q' lies also on U_{a_2,b_2} if and only if

$$\alpha t^2 + \beta t^{q+1} \in \mathsf{GF}(q)\,, \tag{2}$$

or equivalently

$$\alpha(t^2 - t^{q+1}) \in \mathsf{GF}(q)\,. \tag{3}$$

By the hypothesis $\alpha \in \mathsf{GF}(q)^*$, (3) may be rewritten as

$$t^{q-1} = \pm 1\,. \tag{4}$$

There are now two possibilities.

(A$_1$) q is even.

Then (4) implies $t \in \mathsf{GF}(q)$; hence, by the assumption made on t, there are no points Q' on $\mathcal{I}$; thus

$$|\mathcal{I}| = q^2 + 1\,.$$

(A$_2$) q is odd.

Since $t \notin \mathsf{GF}(q)$, (4) necessarily gives $t^{q-1} = -1$. This condition is satisfied by $q-1$ values of t and to any such a value there correspond q points $Q' \in \mathcal{I}$ as r ranges over $\mathsf{GF}(q)$. Therefore

$$|\mathcal{I}| = q^2 + 1 + q(q-1) = 2q^2 - q + 1\,.$$

Case (B)

Arguing as in Case (A) we have that $\mathcal{S}_1$ is a subset of $\mathcal{I}$ and so $|\mathcal{I}| \geq q^2 + 1$. Again a point $Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1}$, with $t \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$, lies on U_{a_2,b_2} if and only if (3) holds. Setting

$$y = t^{q-1}, \tag{5}$$

condition (3) can be rewritten as

$$\alpha^q y^2 - (\alpha^q - \alpha)y - \alpha = 0\,. \tag{6}$$

As $\alpha \neq 0$, equation (6) has solutions $y = 1$ or $y = -\alpha^{1-q}$. We distinguish the following subcases.

(B$_1$) q is even.

Since $\alpha \notin \mathsf{GF}(q)$, $-\alpha^{1-q}$ is different from 1. Because of (5), we necessarily have

$$t^{q-1} = -\alpha^{1-q} \tag{7}$$

as $t \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$. Equation (7) gives $q-1$ possible values for t; for any such a value, we get q points $Q' \in \mathcal{I}$ as r varies in $\mathsf{GF}(q)$. Therefore we get again $|\mathcal{I}| = 2q^2 - q + 1$.

(B$_2$) q is odd.

(B_{21}) If $\alpha \in T_0$, that is $\alpha^q + \alpha = 0$, then $-\alpha^{1-q} = 1$. From (5) it follows that $t^{q-1} = 1$, which is not allowed. Thus, there are no points Q' on $\mathcal{I}$, and hence $|\mathcal{I}| = q^2 + 1$.

(B_{22}) Assume $\alpha \notin T_0$. In this case 1 and $-\alpha^{1-q}$ are two distinct solutions of (6). Arguing as in case (B_1) it follows that $\mathcal{I}$ consists of $2q^2 - q + 1$ points.

Case (C)

Let

$$\mathcal{S}_i = \{(1, t, (a_i + b_i)t^2 + r) \mid t, r \in \mathsf{GF}(q)\} \cup \{(0, 0, 1)\} \subset U_{a_i, b_i}$$

for $i \in \{1, 2\}$. We are going to show that $\mathcal{S}_1 = \mathcal{S}_2$. To this end, observe that a point $Q = (1, t, (a_i + b_i)t^2 + r) \in \mathcal{S}_i$ lies also on $\mathcal{S}_j$ for any distinct $i, j \in \{1, 2\}$, since

$$(a_j + b_j)t^2 + (\alpha + \beta)t^2 = (a_i + b_i)t^2$$

and $\alpha + \beta \in \mathsf{GF}(q)^*$. Hence, $\mathcal{S}_1 \subseteq \mathcal{I}$ and thus $|\mathcal{I}| \geq q^2 + 1$.

Now, consider a point $Q' = (1, t, a_1 t^2 + b_1 t^{q+1} + r) \in U_{a_1, b_1}$ for a suitable $t \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$. The point Q' is also a point on U_{a_2, b_2} if and only if (2) holds, namely, in this case,

$$\alpha^q t^{2(q-1)} + (\beta^q - \beta)t^{q-1} - \alpha = 0\,. \tag{8}$$

Setting y as in (5), condition (8) becomes

$$\alpha^q y^2 + (\beta^q - \beta)y - \alpha = 0\,. \tag{9}$$

Observe that $\alpha \neq 0$, since, otherwise, $\beta^q - \beta = 0$ and (9) would be always true; therefore, the two unitals would be the same, contradicting our assumption.

As $\alpha \neq 0$, equation (9) has solutions $y = 1$ or $y = -\alpha^{1-q}$. There are now several subcases to consider.

(C_1) $\alpha \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$.

(C_{12}) q is even.

In this case the solutions $y = 1$ and $y = -\alpha^{1-q}$ of (9) are distinct. Because of (5), we can only have $t^{q-1} = -\alpha^{1-q}$ as $t \notin \mathsf{GF}(q)$; again we find $q - 1$ values for t satisfying (8), and for any such a value, we obtain q points $Q' \in \mathcal{I}$, as r ranges over $\mathsf{GF}(q)$. Therefore,

$$|\mathcal{I}| = 2q^2 - q + 1\,.$$

(C_{13}) q is odd and $\alpha \in T_0$.
As $\alpha \in T_0$ then $-\alpha^{1-q} = 1$. From (5) we have $t^{q-1} = 1$, which is impossible. Thus, $|\mathcal{I}| = q^2 + 1$.

(C_{14}) q is odd and $\alpha \notin T_0$.
In this case $-\alpha^{1-q} \neq 1$, therefore, arguing as in case (C_{12}), we get that $\mathcal{I}$ consists of $2q^2 - q + 1$ points.

(C_2) $\alpha \in \mathsf{GF}(q)^*$.
Equation (8) gives

$$t^{q-1} = \pm 1\,. \tag{10}$$

(C_{21}) If q is even, condition (10) implies $t \in \mathsf{GF}(q)$, which is not allowed; thus,

$$|\mathcal{I}| = q^2 + 1\,.$$

(C_{22}) Suppose q to be odd. As $t \notin \mathsf{GF}(q)$, we necessarily have from (10) that $t^{q-1} = -1$, a condition satisfied by $q-1$ possible values for t; to any such a value of t there correspond q points $Q' \in \mathcal{I}$ as r ranges over $\mathsf{GF}(q)$. Therefore, again

$$|\mathcal{I}| = 2q^2 - q + 1\,.$$

Case (D)

Let us again consider the point-sets

$$\mathcal{S}_i = \left\{(1, t, (a_i + b_i)t^2 + r) \mid t, r \in \mathsf{GF}(q)\right\} \cup \{P_\infty\}$$

where $i = 1, 2$. A point $Q = (1, t, (a_i + b_i)t^2 + r) \in \mathcal{S}_i$ lies also on $\mathcal{S}_j$ for $i \neq j$, if and only if the element $(\alpha + \beta)t^2 \in \mathsf{GF}(q)$; the hypothesis $\alpha + \beta \notin \mathsf{GF}(q)$ forces t to be zero. Thus, $\mathcal{S}_1 \cap \mathcal{S}_2 = \{(1, 0, r) \mid r \in \mathsf{GF}(q)\}$ and $|\mathcal{I}| \geq q + 1$.

Next, take a point $Q' = (1, t, a_1 t^2 + b_1 t^{q+1} + r) \in U_{a_1,b_1}$ with $t \in \mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$. The point Q' is on U_{a_2,b_2} if and only if (8) holds. We distinguish three possibilities.

(D_1) $\alpha = 0$.
In this case $\beta^q - \beta \neq 0$ and (8) gives $t = 0$ which is not allowed. Thus

$$|\mathcal{I}| = q + 1\,.$$

(D_2) q is even and $\alpha \neq 0$.

(D_{21}) $\beta \in \mathsf{GF}(q)$.

Condition (8) gives $t^{q-1} = \sqrt{1/\alpha^{q-1}}$ with $\alpha^{q-1} \neq 1$. Once again, we get $q-1$ possible values for t; so for any such a value, we get q points $Q' \in \mathcal{I}$ as r ranges over $\mathsf{GF}(q)$. Hence,

$$|\mathcal{I}| = q^2 + 1\,.$$

(D$_{22}$) $\beta \notin \mathsf{GF}(q)$.

Let y be as in (5); we get again (9). This equation has 2 solutions as $\delta = \alpha^{q+1}/(\beta^q - \beta)^2$ belongs to $\mathsf{GF}(q)$ and hence the absolute trace of δ is zero. Furthermore, both solutions are different from 1 as $\alpha + \beta \notin \mathsf{GF}(q)$.

Therefore, by (5), we find $2(q-1)$ possible values for t and thus, $2q(q-1)$ points Q' on $\mathcal{I}$. Hence $\mathcal{I}$ consists of $2q(q-1) + q + 1 = 2q^2 - q + 1$ points.

(D$_3$) q is odd and $\alpha \neq 0$.

We need to consider the discriminant of (9), that is

$$\Delta = (\beta^q - \beta)^2 + 4\alpha^{q+1} \in \mathsf{GF}(q)\,.$$

(D$_{31}$) $\Delta = 0$.

Condition (9) has the unique solution $y = \frac{\beta - \beta^q}{2\alpha^q} \neq 1$ which gives $q-1$ possible values for t because of (5); hence

$$|\mathcal{I}| = q^2 + 1\,.$$

(D$_{32}$) $\Delta \neq 0$.

As $\Delta \in \mathsf{GF}(q)^*$ we get $\Delta^{(q^2-1)/2} = 1$, that is Δ is a non-zero square in $\mathsf{GF}(q^2)$.

Therefore, (9) has two non-zero solutions different from 1. Each of them provides $q-1$ possible values for t; thus

$$|\mathcal{I}| = 2q^2 - q + 1\,.$$

Finally, assume both $U_{a_1 b_1}$ and $U_{a_2 b_2}$ to be classical. From Lemma 2.2 it follows that $\alpha = 0$; this only happens in case (D$_1$) giving $|\mathcal{I}| = q + 1$. □

Remark 4.2. The configurations for the intersection $\mathcal{I}$ of two BM unitals U_{a_1,b_1}, U_{a_2,b_2} in $\mathfrak{F}$ are the following:

(1) $\mathcal{I}$ consists of $q+1$ collinear points;

(2) $\mathcal{I}$ consists of q sets of $q+1$ collinear points. The q lines all meet at P_∞;

(3) $\mathcal{I}$ consists of $2q-1$ sets of $q+1$ collinear points. The $2q-1$ lines all pass through the point P_∞.

Furthermore, it follows from the proof of Theorem 4.1 that

(a) in cases (A_1) and (C_{21}) the intersection $\mathcal{I}$ is the set

$$\mathcal{I} = \left\{(1, t, (a_1+b_1)t^2 + r) \mid t, r \in \mathsf{GF}(q)\right\} \cup \{P_\infty\}.$$

Hence $\mathcal{I}$ is one of the (q^2+1)-sets defined in Theorem 3.1 with $a = a_1 + b_1$ and σ the automorphism of $\mathsf{GF}(q^2)$ such that $x^\sigma = x^2$.

(b) In cases (D_{21}) and (D_{31}) the intersection $\mathcal{I}$ turns out to be one of the (q^2+1)-sets defined in Theorem 3.2 with respectively $s = \sqrt{1/\alpha^{q-1}}$ or $s = \frac{\beta-\beta^q}{2\alpha^q}$, and $a = a_1 + sb_1$.

5. Examples

In this section we show that all the cases discussed in Theorem 4.1 effectively occur for $q = 4$ and 5. If $q = 4$, denote by ω a primitive element of $\mathsf{GF}(16)$, such that $\omega^2 + \omega + \delta = 0$, with δ any element of $\mathsf{GF}(4) \setminus \mathsf{GF}(2)$. Furthermore, put $a_1 = \omega^3$ and $b_1 = \omega$.

When $q = 5$, take ξ as a primitive element of $\mathsf{GF}(25)$ such that $\xi^2 - \xi + 2 = 0$ and set $a_1 = \xi^7$, $b_1 = \xi^{12}$.

Under these assumptions, U_{a_1,b_1} turns out to be a non-classical BM unital respectively in $\mathsf{PG}(2,16)$ or in $\mathsf{PG}(2,25)$. Let a_2 and b_2 be two coefficients ranging over $\mathsf{GF}(16)$ or $\mathsf{GF}(25)$ in such a way that the discriminant condition of Lemma 2.1 is satisfied.

By different choices of a_2 and b_2 we get all the cases for $U_{a_1,b_1} \cap U_{a_2,b_2}$ occurring in the proof of Theorem 4.1 but case (D_1); see Table 1.

References

[1] **R. D. Baker** and **G. L. Ebert**, Intersection of unitals in the Desarguesian plane, Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). *Congr. Numer.* **70** (1990), 87–94.

[2] ______, On Buekenhout-Metz unitals of odd order, *J. Combin. Theory Ser. A* **60** (1992), no. 1, 67–84.

$q = 4$		
Case	a_2	b_2
(A_1)	ω^{12}	ω^{8}
(B_1)	ω^{4}	ω^{14}
(C_{12})	ω	ω^{11}
(C_{21})	ω^{12}	ω^{4}
(D_{21})	ω^{9}	ω^{2}
(D_{22})	ω^{11}	ω^{6}

$q = 5$		
Case	a_2	b_2
(A_2)	ξ^{23}	1
(B_{21})	ξ^{10}	ξ^{19}
(B_{22})	ξ^{16}	ξ^{5}
(C_{13})	ξ^{10}	ξ^{16}
(C_{14})	ξ^{20}	ξ^{3}
(C_{22})	ξ^{23}	ξ^{18}
(D_{31})	ξ^{22}	ξ^{7}
(D_{32})	ξ^{16}	ξ^{15}

Table 1: Intersection cases for q small

[3] **A. Blokhuis**, **A. Brouwer** and **H. Wilbrink**, Hermitian unitals are code words, *Discrete Math.* **97** (1991), no. 1–3, 63–68.

[4] **F. Buekenhout**, Existence of unitals in finite translation planes of order q^2 with a kernel of order q, *Geom. Dedicata* **5** (1976), 189–194.

[5] **G. L. Ebert**, On Buekenhout-Metz unitals of even order, *European J. Combin.* **13** (1992), no. 2, 109–117.

[6] ______, Buekenhout unitals, Combinatorics (Assisi, 1996). *Discrete Math.* **208/209** (1999), 247–260.

[7] **J. W. P. Hirschfeld**, *Projective Geometries over Finite Fields*, Second edition, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.

[8] **B. C. Kestenband**, Unital intersections in finite projective planes, *Geom. Dedicata* **11** (1981), no. 1, 107–117.

[9] **R. Metz**, On a class of unitals, *Geom. Dedicata* **8** (1979), 125–126.

Angela Aguglia

Dipartimento di Matematica, Politecnico di Bari, Via G. Amendola 126/B, 70126 Bari, Italy

e-mail: a.aguglia@poliba.it

website: http://www.dm.uniba.it/~aguglia