Intersections of Buekenhout-Metz unitals

Angela Aguglia

On the occasion of the 60th birthday of Gábor Korchmáros

Abstract

Configurations arising as intersections of two Buekenhout-Metz unitals of a given family are studied and, in the case in which at most one of the unitals is classical, a new intersection size is found.

Keywords: projective plane, unital

MSC 2000: 05B25

1. Introduction

In [8] it has been shown that there are just seven configurations in which two classical unitals may intersect. There-within it has also been proved that the cardinality of the intersection of any two classical unitals in the desarguesian projective plane \( \text{PG}(2, q^2) \) is congruent to 1 modulo \( q \).

A family of non-classical Buekenhout-Metz unitals in \( \text{PG}(2, q^2) \), with \( q = p^h \) an odd prime power, has been constructed in [1]; the intersection of every unital of this family with a classical one contains a number of points congruent to 1 modulo \( p \). In the same paper, it is also conjectured that the size of the intersection of any classical unital with a non-classical one should be one of the following:

\[
q^2 \pm 2q + 1, \quad q^2 \pm q + 1, \quad q^2 + 1.
\]

Afterwards, in [3] it has been proved that an arbitrary unital in \( \text{PG}(2, q^2) \), with \( q = p^h \) any prime power, meets a classical unital in a number of points congruent

\(^\ast\)Research supported by the Italian Ministry MIUR, Strutture geometriche, combinatoria e loro applicazioni.
to 1 modulo \( p \). To classify intersections of two non-classical unitals seems to be a difficult question. Here we make some advances in this direction by looking at a suitable family \( \mathcal{F} \) of Buekenhout-Metz unitals in \( \text{PG}(2, q^2) \), \( q \) any prime power, containing both classical unitals as well as non-classical ones. We prove that any two classical unitals in \( \mathcal{F} \) intersect on \( q + 1 \) collinear points, whereas in all other cases the intersection number is one of the following:

\[
q + 1, \quad q^2 + 1, \quad 2q^2 - q + 1.
\]

This last size does not appear among those conjectured in [1].

2. Preliminaries

A set \( S \) of \( k \) points (or a \( k \)-set) in a projective plane of order \( q \) is of type \((k_1, k_2, \ldots, k_s)\), with \( k_1 < k_2 < \cdots < k_s \), if a line \( \ell \) may intersect \( S \) in only sets of \( k_1, k_2, \ldots \) or \( k_s \) points. A line \( \ell \) for which \(|\ell \cap S| = k_i\) is called a \( k_i \)-secant of \( S \) whereas the integers \( k_i \) are called characters of \( S \).

A unital in \( \text{PG}(2, q^2) \) is a \((q^3 + 1)\)-set of type \((1, q + 1)\). A class of unitals in \( \text{PG}(2, q^2) \) is given by the (non-degenerate) Hermitian curves, that is sets of absolute points with respect to (non-degenerate) unitary polarities; these are also called classical unitals.

Unitals which are not Hermitian curves are non-classical. A unital \( U \) in \( \text{PG}(2, q^2) \) is parabolic or hyperbolic according as the line at infinity contains 1 or \( q + 1 \) points of \( U \).

Every unital in \( \text{PG}(2, 2^2) \) is classical. The first non-classical unitals in \( \text{PG}(2, q^2) \) with \( q = 2^{2r+1}, \ r \geq 1 \) were found by Buekenhout in [4]. Using Buekenhout’s method, Metz extended this class of non-classical unitals in \( \text{PG}(2, q^2) \) to all values of \( q \geq 2 \); see [9]. A Buekenhout-Metz unital (BM unital for short) is a parabolic unital obtained with the construction given in [9] in which the ovoidal cone is an elliptic cone. This class also includes classical unitals. We refer the reader to [6] for a survey of results on these unitals.

Let \((X_0, X_1, X_2)\) denote homogeneous coordinates for points of \( \text{PG}(2, q^2) \). The line \( \ell_\infty : X_0 = 0 \) will be taken as the line at infinity, whereas \( P_\infty \) will denote the point \((0, 0, 1)\). For \( q = 2^h \), let \( C_0 \) be the additive subgroup of \( \text{GF}(q) \) defined by \( C_0 = \{ x \in \text{GF}(q) \mid \text{Tr}(x) = 0 \} \) where

\[
\text{Tr}: \text{GF}(q) \rightarrow \text{GF}(2): x \mapsto x + x^2 + \ldots + x^{2^{h-1}}
\]

is the trace map of \( \text{GF}(q) \) over \( \text{GF}(2) \). The following results come from [2] for \( q \) odd and from [5] for \( q \) even.
Lemma 2.1. Let $a, b \in \text{GF}(q^2)$. The point set
\[ U_{a,b} = \{(1, t, at^2 + bt^{q+1} + r) \mid t \in \text{GF}(q^2), r \in \text{GF}(q)\} \cup \{P_{\infty}\} \]
is a BM unital in $\text{PG}(2, q^2)$ if and only if either $q$ is odd and $4a^{q+1} + (b^q - b)^2$ is a non-square in $\text{GF}(q)$, or $q$ is even, $b \notin \text{GF}(q)$ and $a^{q+1}/(b^q + b)^2 \in C_0$.

The expression $4a^{q+1} + (b^q - b)^2$ for $q$ odd, and $a^{q+1}/(b^q + b)^2$ with $b \notin \text{GF}(q)$, for $q$ even, is the discriminant of the unital $U_{a,b}$.

Lemma 2.2. Every BM unital can be expressed as $U_{a,b}$, for some $a, b \in \text{GF}(q^2)$ which satisfy the discriminant condition of Lemma 2.1. Furthermore, a BM unital $U_{a,b}$ is classical if and only if $a = 0$.

3. Sets with few characters

In this section we are going to construct a family of $(q^2 + 1)$-sets with four characters in $\text{PG}(2, q^2)$, where $q = p^h$ and $p$ is any prime power. Some of them, as pointed out in Remark 4.2, may be obtained by intersecting two BM unitals, at least one of which is non-classical.

Let $\sigma$ denote the automorphism of $\text{GF}(q^2)$ defined by
\[ x^\sigma = x^{p^i}, \text{ with } i < h \text{ and } (i, h) = 1. \]

Write $T_0 = \{t \in \text{GF}(q^2) \mid T(t) = 0\}$, where
\[ T : x \in \text{GF}(q^2) \mapsto x^q + x \in \text{GF}(q) \]
is the trace function of $\text{GF}(q^2)$ over $\text{GF}(q)$.

Theorem 3.1. For each $a \in \text{GF}(q^2)^*$, the subset
\[ S = \{(1, t, at^\sigma + r) \mid t \in \text{GF}(q), r \in T_0\} \cup \{P_{\infty}\} \]
of $\text{PG}(2, q^2)$ is either of type $(0, 1, q, q+1)$ or of type $(0, 1, p, q+1)$ according as $a \in T_0$ or not.

Proof. By construction, $S$ consists of $q^2 + 1$ points not all on a same line. Observe that $S$ is not a blocking set with respect to the lines of $\text{PG}(2, q^2)$ since, otherwise, it would contain at least $q^2 + 3$ points; see [7, Lemma 13.4]. Therefore, there exist some 0-secants of $S$. We are going to show that for each $k$-secant of $S$ which is neither external nor tangent to it, $k \in \{q, q+1\}$ or $k \in \{p, q+1\}$ according as $a \in T_0$ or not.
We begin by considering the line \( P_\infty P_{t,r} \) joining the point \( P_\infty \) with another point \( P_{t,r} = (1, t, at^\sigma + r) \in S \). Such a line corresponds to the set

\[
\{(1, t, at^\sigma + r + \alpha) \mid \alpha \in \text{GF}(q^2)\} \cup \{P_\infty\};
\]
hence, the intersection of \( P_\infty P_{t,r} \) and \( S \) is

\[
\{(1, t, at^\sigma + r + \alpha) \mid \alpha \in T_0\} \cup \{P_\infty\};
\]
that is the line \( P_\infty P_{t,r} \) is a \((q+1)\)-secant of \( S \).

Now take the line \( P_{t_1,r_1}P_{t_2,r_2} \) through two distinct points

\[
P_{t_1,r_1} = (1, t_1, at_1^\sigma + r_1) \quad \text{and} \quad P_{t_2,r_2} = (1, t_2, at_2^\sigma + r_2)
\]
of \( S \). Such a line consists of all the points

\[
Q_\alpha = (\alpha + 1, \alpha t_1 + t_2, a(\alpha t_1^\sigma + t_2^\sigma) + \alpha r_1 + r_2)
\]
with \( \alpha \) ranging over \( \text{GF}(q^2) \), plus the point \( P_{t_1,r_1} \).

If \( t_1 = t_2 \), then the line \( P_{t_1,r_1}P_{t_2,r_2} \) passes through the point \( P_\infty \) and hence is a \((q+1)\)-secant of \( S \).

When \( t_1 \neq t_2 \), observe that the point at infinity \( Q_{-1} = (0, t_2 - t_1, a(t_2^\sigma - t_1^\sigma) + r_2 - r_1) \) of the line \( P_{t_1,r_1}P_{t_2,r_2} \) is not on \( S \). Thus, we restrict our attention to the affine points \( Q_\alpha \), where \( \alpha \neq -1 \). The normalized homogeneous coordinates for these points are

\[
\left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a \frac{(\alpha t_1^\sigma + t_2^\sigma)}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right).
\]

A point \( Q_\alpha \) is on \( S \) if and only if the following conditions hold:

(i) \( \frac{\alpha t_1 + t_2}{\alpha + 1} \in \text{GF}(q) \);

(ii) \( a(\alpha t_1^\sigma + t_2^\sigma) + \alpha r_1 + r_2 - a(\alpha t_1 + t_2)^\sigma \) \( \frac{\alpha + 1}{\alpha + 1} \in T_0 \).

Condition (i) implies \((\alpha^q - \alpha)(t_1 - t_2) = 0\), therefore, as \( t_1 \neq t_2 \), we have \( \alpha \in \text{GF}(q) \). Hence condition (ii) can be written as

\[
(a^q + a) \left[ \frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1} - \frac{(\alpha^q t_1^\sigma + t_2^\sigma)}{(\alpha^q + 1)} \right] = 0.
\]

If \( a^q + a = 0 \), then the intersection of \( P_{t_1,r_1}P_{t_2,r_2} \) and \( S \) is the set

\[
\left\{ \left(1, \frac{\alpha t_1 + t_2}{\alpha + 1}, a \frac{\alpha t_1^\sigma + t_2^\sigma}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right) \mid \alpha \in \text{GF}(q) \setminus \{-1\} \right\} \cup \{P_{t_1,r_1}\},
\]
that is the line $P_{t_1,r_1}P_{t_2,r_2}$ is a $q$-secant to $S$.

In the case where $a^q + a \neq 0$, (1) gives $(\alpha^q - \alpha)(t_1^q - t_2^q) = 0$; thus, $\alpha^q - \alpha = 0$ as $t_1 \neq t_2$. Whence $\alpha = 0$ or $\alpha^{(\sigma - 1)} = 1$.

As $(p^h - 1, p^i - 1) = (p^{(h,i)} - 1$ and $(i, h) = 1$ the equation $\alpha^{(\sigma - 1)} = 1$ has $p - 1$ solutions in $GF(q)$, one of them is $\alpha = -1$. Thus, there are $p - 2 + 2$ affine points $Q_\alpha$ on $P_{t_1,r_1}P_{t_2,r_2} \cap S$, that is the line $P_{t_1,r_1}P_{t_2,r_2}$ is a $p$-secant of $S$. \qed

Let $s$ be an element of $GF(q^2) \setminus \{1\}$ such that $s^{q+1} = 1$. Set

$$\mathcal{A} = \{(1, 0, r) \mid r \in GF(q)\}.$$

For each $a \in GF(q^2)^*$, write

$$\mathcal{B} = \{(1, t, at^2 + r) \mid t \in GF(q^2), t^{q-1} = s, r \in GF(q)\}.

**Theorem 3.2.** The subset

$$S = \mathcal{A} \cup \mathcal{B} \cup \{P_\infty\}$$

of $PG(2, q^2)$ is either of type $(0, 1, q, q + 1)$ or of type $(0, 1, 2, q + 1)$ according as $a^{q-1}s^2 = 1$ or not.

**Proof.** By definition, $S$ consists of $q^2 + 1$ points. As seen in the proof of Theorem 3.1 there exist some 0-secants of $S$. We are going to show that for each $k$-secant of $S$ which is neither external nor tangent to $S$, $k \in \{q, q + 1\}$ or $k \in \{2, q + 1\}$ according as $a^{q-1}s^2 = 1$ or not.

Arguing as in the proof of Theorem 3.1, it can be verified that a line trough the point $P_\infty$ which is not tangent to the set $S$, meets $S$ in $q + 1$ points.

Next, we consider the line $P_rP_{t,m}$ joining the point $P_r = (1, 0, r) \in \mathcal{A}$, with the point $P_{t,m} = (1, t, at^2 + m) \in \mathcal{B}$. Such a line corresponds to the set

$$\{(\alpha + 1, t, at^2 + m + \alpha r) \mid \alpha \in GF(q^2)\} \cup \{P_r\}.$$

Since $t \neq 0$, the point at infinity $(0, t, at^2 + m - r)$ of the line $P_rP_{t,m}$ is not on $S$. Thus, we restrict our attention to the affine points $Q_\alpha = (\alpha + 1, t, at^2 + m + \alpha r)$, with $\alpha \neq -1$, on the line $P_rP_{t,m}$. The normalized homogeneous coordinates for these points are

$$\left(\frac{1}{\alpha + 1}, \frac{t}{\alpha + 1}, \frac{at^2 + m + \alpha r}{\alpha + 1}\right).

A point $Q_\alpha$ is on $S$ if and only if the following conditions hold:

1. \[
\left(\frac{t}{\alpha + 1}\right)^{q-1} = s;
\]
(ii) \( \frac{at^2 + m + \alpha r}{\alpha + 1} - \frac{at^2}{(\alpha + 1)^2} \in \text{GF}(q) \).

Condition (i) implies \((\alpha + 1) \in \text{GF}(q)^*\), therefore, \(\alpha \in \text{GF}(q) \setminus \{-1\}\). Hence (ii) becomes \(at^2 \in \text{GF}(q)\), that is \(a^{q-1}s^2 = 1\). Thus, if \(a^{q-1}s^2 = 1\), then the intersection of \(P_rP_{t,m}\) and \(S\) is

\[
\left\{ \left(1, \frac{t}{\alpha + 1}, \frac{at^2 + m + \alpha r}{\alpha + 1} \right) \mid \alpha \in \text{GF}(q) \setminus \{-1\} \right\} \cup \{P_r\},
\]

that is the line \(P_rP_{t,m}\) is a \(q\)-secant to \(S\). In the case \(a^{q-1}s^2 \neq 1\) the line \(P_rP_{t,m}\) is a 2-secant.

Now take the line \(P_{t_1,r_1}P_{t_2,r_2}\) joining two distinct points

\[P_{t_1,r_1} = (1, t_1, at_1^2 + r_1) \text{ and } P_{t_2,r_2} = (1, t_2, at_2^2 + r_2)\]

of \(B\). Such a line consists of \(P_{t_1,r_1}\) plus the points

\[Q_\alpha = (\alpha + 1, at_1 + t_2, a(at_1^2 + t_2^2) + \alpha r_1 + r_2)\]

as \(\alpha\) ranges over \(\text{GF}(q^2)\).

If \(t_1 = t_2\), the line \(P_{t_1,r_1}P_{t_2,r_2}\) passes through the point \(P_{\infty};\) hence it is a \((q + 1)\)-secant of \(S\).

When \(t_1 \neq t_2\), observe that the point at infinity \((0, t_2 - t_1, a(t_2 - t_1)^2 + r_2 - r_1)\) of \(P_{t_1,r_1}P_{t_2,r_2}\) is not on \(S\). Thus, we restrict our attention to the affine points \(Q_\alpha\) with \(\alpha \neq -1\). Their normalized homogeneous coordinates are

\[
\left(1, \frac{at_1 + t_2}{\alpha + 1}, a \frac{(at_1^2 + t_2^2)}{\alpha + 1} + \frac{\alpha r_1 + r_2}{\alpha + 1}\right).
\]

A point \(Q_\alpha\) is on \(S\) if and only if the following conditions hold:

(i) \(\alpha t_1 + t_2 = 0\) or \(\left(\frac{at_1 + t_2}{\alpha + 1}\right)^{q-1} = s;\)

(ii) \(\frac{a(at_1^2 + t_2^2) + \alpha r_1 + r_2}{\alpha + 1} - \frac{a(at_1 + t_2)^2}{(\alpha + 1)^2} \in \text{GF}(q)\).

When \(\alpha = -\frac{t_2}{t_1} \in \text{GF}(q)\), condition (ii) becomes \(-at_1t_2 \in \text{GF}(q)\); hence we have \(a^{(q-1)s^2} = 1\). Therefore the point \((1, 0, \frac{at_1t_2(t_2 - t_1 + r_2t_1 - r_1t_2)}{t_1 - t_2})\) belongs to \(S\) if and only if \(a^{(q-1)s^2} = 1\).

In the case \(\left(\frac{at_1 + t_2}{\alpha + 1}\right)^{q-1} = s\) we get \(\alpha \in \text{GF}(q) \setminus \{-1, -\frac{t_2}{t_1}\}\). Hence condition (ii) can be written as \(a(t_1^2 - t_2^2)^{q-1} \in \text{GF}(q)\), that is \(a^{q-1}s^2 = 1\). Therefore, if \(a^{q-1}s^2 = 1\) the line \(P_{t_1,r_1}P_{t_2,r_2}\) is a \(q\)-secant to \(S\), otherwise it meets \(S\) only in \(P_{t_1,r_1}\) and \(P_{t_2,r_2}\), thus it is a 2-secant to \(S\). \(\square\)
4. Main result

In this section we study the cardinality of the intersection of two distinct BM unitals in the family
\[ \mathcal{F} = \{ U_{a,b} \}_{(a,b) \in \text{GF}(q^2) \times \text{GF}(q^2)}, \]
where
\[ U_{a,b} = \{ (1, t, at^2 + bt^{q+1} + r) \mid t \in \text{GF}(q^2), r \in \text{GF}(q) \} \cup \{ P_\infty \} \]
and the coefficients \( a \) and \( b \) satisfy the discriminant condition of Lemma 2.1.

**Theorem 4.1.** In \( \text{PG}(2, q^2) \), with \( q \) a prime power, the intersection size of two unitals of \( \mathcal{F} \) is one of the following:
\[ q + 1, q^2 + 1, 2q^2 - q + 1. \]
Furthermore, any two classical unitals of \( \mathcal{F} \) can only intersect in \( q + 1 \) collinear points.

**Proof.** Let \( U_{a_1,b_1} \) and \( U_{a_2,b_2} \) be two distinct unitals in \( \mathcal{F} \). Denote by \( \mathcal{I} \) their intersection and set \( \alpha = a_1 - a_2, \beta = b_1 - b_2 \). We distinguish the following cases:

(A) \( \alpha + \beta = 0 \) and \( \alpha \in \text{GF}(q)^* \);
(B) \( \alpha + \beta = 0 \) and \( \alpha \notin \text{GF}(q) \);
(C) \( \alpha + \beta \in \text{GF}(q)^* \);
(D) \( \alpha + \beta \notin \text{GF}(q) \).

**Case (A)**
Since \( a_1 + b_1 = a_2 + b_2 \), the points in
\[ S_1 = \{ (1, t, (a_1 + b_1)t^2 + r) \mid t, r \in \text{GF}(q) \} \cup \{ P_\infty \} \]
are on both unitals. Therefore, the cardinality of \( \mathcal{I} \) is at least \( q^2 + 1 \).

Let \( Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1} \), for a suitable \( t \in \text{GF}(q^2) \setminus \text{GF}(q) \). The point \( Q' \) lies also on \( U_{a_2,b_2} \) if and only if
\[ \alpha t^2 + \beta t^{q+1} \in \text{GF}(q), \quad (2) \]
or equivalently
\[ \alpha (t^2 - t^{q+1}) \in \text{GF}(q). \quad (3) \]
By the hypothesis $\alpha \in \mathbb{GF}(q)^*$, (3) may be rewritten as

$$t^{q-1} = \pm 1.$$  \hfill (4)

There are now two possibilities.

(A) $q$ is even.

Then (4) implies $t \in \mathbb{GF}(q)$; hence, by the assumption made on $t$, there are no points $Q'$ on $I$; thus

$$|I| = q^2 + 1.$$  

(B) $q$ is odd.

Since $t \not\in \mathbb{GF}(q)$, (4) necessarily gives $t^{q-1} = -1$. This condition is satisfied by $q - 1$ values of $t$ and to any such a value there correspond $q$ points $Q' \in I$ as $r$ ranges over $\mathbb{GF}(q)$. Therefore

$$|I| = q^2 + 1 + q(q - 1) = 2q^2 - q + 1.$$  

Case (B)

Arguing as in Case (A) we have that $S_1$ is a subset of $I$ and so $|I| \geq q^2 + 1$. Again a point $Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1}$, with $t \in \mathbb{GF}(q^2) \setminus \mathbb{GF}(q)$, lies on $U_{a_2,b_2}$ if and only if (3) holds. Setting

$$y = t^{q-1},$$  \hfill (5)

condition (3) can be rewritten as

$$\alpha^qy^2 - (\alpha^q - \alpha)y - \alpha = 0.$$  \hfill (6)

As $\alpha \neq 0$, equation (6) has solutions $y = 1$ or $y = -\alpha^{-1-q}$. We distinguish the following subcases.

(B1) $q$ is even.

Since $\alpha \notin \mathbb{GF}(q)$, $-\alpha^{-1-q}$ is different from 1. Because of (5), we necessarily have

$$t^{q-1} = -\alpha^{-1-q}$$  \hfill (7)

as $t \in \mathbb{GF}(q^2) \setminus \mathbb{GF}(q)$. Equation (7) gives $q - 1$ possible values for $t$; for any such a value, we get $q$ points $Q' \in I$ as $r$ varies in $\mathbb{GF}(q)$. Therefore we get again $|I| = 2q^2 - q + 1$.

(B2) $q$ is odd.
(B$_{21}$) If $\alpha \in T_0$, that is $\alpha^q + \alpha = 0$, then $-\alpha^{-1-q} = 1$. From (5) it follows that $t^{q-1} = 1$, which is not allowed. Thus, there are no points $Q'$ on $I$, and hence $|I| = q^2 + 1$.

(B$_{22}$) Assume $\alpha \notin T_0$. In this case 1 and $-\alpha^{-1-q}$ are two distinct solutions of (6). Arguing as in case (B$_1$) it follows that $I$ consists of $2q^2 - q + 1$ points.

Case (C)

Let
\[ S_i = \{(1, t, (a_i + b_i)t^2 + r) \mid t, r \in GF(q)\} \cup \{(0, 0, 1)\} \subset U_{a_i, b_i} \]
for $i \in \{1, 2\}$. We are going to show that $S_1 = S_2$. To this end, observe that a point $Q = (1, t, (a_i + b_i)t^2 + r) \in S_i$ lies also on $S_j$ for any distinct $i, j \in \{1, 2\}$, since
\[ (a_j + b_j)t^2 + (\alpha + \beta)t^2 = (a_i + b_i)t^2 \]
and $\alpha + \beta \in GF(q)^*$. Hence, $S_1 \subseteq I$ and thus $|I| \geq q^2 + 1$.

Now, consider a point $Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1, b_1}$ for a suitable $t \in GF(q^2) \setminus GF(q)$. The point $Q'$ is also a point on $U_{a_2, b_2}$ if and only if (2) holds, namely, in this case,
\[ \alpha q^2 t^{q-1} + (\beta^q - \beta)t^{q-1} - \alpha = 0. \] (8)
Setting $y$ as in (5), condition (8) becomes
\[ \alpha q^2 y^2 + (\beta^q - \beta)y - \alpha = 0. \] (9)
Observe that $\alpha \neq 0$, since, otherwise, $\beta^q - \beta = 0$ and (9) would be always true; therefore, the two unitals would be the same, contradicting our assumption.

As $\alpha \neq 0$, equation (9) has solutions $y = 1$ or $y = -\alpha^{-1-q}$. There are now several subcases to consider.

(C$_1$) $\alpha \in GF(q^2) \setminus GF(q)$.

(C$_{12}$) $q$ is even.

In this case the solutions $y = 1$ and $y = -\alpha^{-1-q}$ of (9) are distinct. Because of (5), we can only have $t^{q-1} = -\alpha^{-1-q}$ as $t \notin GF(q)$; again we find $q - 1$ values for $t$ satisfying (8), and for any such a value, we obtain $q$ points $Q' \in I$, as $r$ ranges over $GF(q)$. Therefore,
\[ |I| = 2q^2 - q + 1. \]
(C₁₃) \( q \) is odd and \( \alpha \in T₀ \).
As \( \alpha \in T₀ \) then \(-\alpha^{1-q} = 1 \). From (5) we have \( t^{q-1} = 1 \), which is impossible. Thus, \( |I| = q^2 + 1 \).

(C₁₄) \( q \) is odd and \( \alpha \notin T₀ \).
In this case \(-\alpha^{1-q} \neq 1 \), therefore, arguing as in case (C₁₂), we get that \( I \) consists of \( 2q^2 - q + 1 \) points.

(C₂) \( \alpha \in GF(q)^* \).
Equation (8) gives
\[
\frac{t^{q-1}}{t} = \pm 1. 
\] (10)

(C₂₁) If \( q \) is even, condition (10) implies \( t \in GF(q) \), which is not allowed; thus,
\[
|I| = q^2 + 1. 
\]

(C₂₂) Suppose \( q \) to be odd. As \( t \notin GF(q) \), we necessarily have from (10) that \( t^{q-1} = -1 \), a condition satisfied by \( q - 1 \) possible values for \( t \); to any such a value of \( t \) there correspond \( q \) points \( Q' \in I \) as \( r \) ranges over \( GF(q) \). Therefore, again
\[
|I| = 2q^2 - q + 1. 
\]

Case (D)

Let us again consider the point-sets
\[
S_i = \{(1, t, (a_i + b_i)t^2 + r) \mid t, r \in GF(q)\} \cup \{P_\infty\}
\]
where \( i = 1, 2 \). A point \( Q = (1, t, (a_i + b_i)t^2 + r) \in S_i \) lies also on \( S_j \) for \( i \neq j \), if and only if the element \((\alpha + \beta)t^2 \in GF(q)\); the hypothesis \( \alpha + \beta \notin GF(q) \) forces \( t \) to be zero. Thus, \( S_1 \cap S_2 = \{(1, 0, r) \mid r \in GF(q)\} \) and \( |I| \geq q + 1 \).

Next, take a point \( Q' = (1, t, a_1t^2 + b_1t^{q+1} + r) \in U_{a_1,b_1} \) with \( t \in GF(q^2) \setminus GF(q) \). The point \( Q' \) is on \( U_{a_2,b_2} \) if and only if (8) holds. We distinguish three possibilities.

(D₁) \( \alpha = 0 \).
In this case \( \beta^q - \beta \neq 0 \) and (8) gives \( t = 0 \) which is not allowed. Thus
\[
|I| = q + 1. 
\]

(D₂) \( q \) is even and \( \alpha \neq 0 \).

(D₂₁) \( \beta \in GF(q) \).
Condition (8) gives \( t^{q-1} = \sqrt{1/\alpha^{q-1}} \) with \( \alpha^{q-1} \neq 1 \). Once again, we get \( q - 1 \) possible values for \( t \); so for any such a value, we get \( q \) points \( Q' \in I \) as \( r \) ranges over \( \text{GF}(q) \). Hence,

\[
|I| = q^2 + 1.
\]

\((\text{D}_{22})\) \( \beta \notin \text{GF}(q) \).

Let \( y \) be as in (5); we get again (9). This equation has 2 solutions as \( \delta = \alpha^{q+1}/(\beta^q - \beta)^2 \) belongs to \( \text{GF}(q) \) and hence the absolute trace of \( \delta \) is zero. Furthermore, both solutions are different from 1 as \( \alpha + \beta \notin \text{GF}(q) \).

Therefore, by (5), we find \( 2(q - 1) \) possible values for \( t \) and thus, \( 2q(q - 1) \) points \( Q' \) on \( I \). Hence \( I \) consists of \( 2q(q - 1) + q + 1 = 2q^2 - q + 1 \) points.

\((\text{D}_3)\) \( q \) is odd and \( \alpha \neq 0 \).

We need to consider the discriminant of (9), that is

\[
\Delta = (\beta^q - \beta)^2 + 4\alpha^{q+1} \in \text{GF}(q) .
\]

\((\text{D}_{31})\) \( \Delta = 0 \).

Condition (9) has the unique solution \( y = \beta - \beta^q \neq 1 \) which gives \( q - 1 \) possible values for \( t \) because of (5); hence

\[
|I| = q^2 + 1.
\]

\((\text{D}_{32})\) \( \Delta \neq 0 \).

As \( \Delta \in \text{GF}(q)^* \) we get \( \Delta^{(q^2-1)/2} = 1 \), that is \( \Delta \) is a non-zero square in \( \text{GF}(q^2) \).

Therefore, (9) has two non-zero solutions different from 1. Each of them provides \( q - 1 \) possible values for \( t \); thus

\[
|I| = 2q^2 - q + 1 .
\]

Finally, assume both \( U_{a_1,b_1} \) and \( U_{a_2,b_2} \) to be classical. From Lemma 2.2 it follows that \( \alpha = 0 \); this only happens in case (D1) giving \( |I| = q + 1 \). \( \Box \)

**Remark 4.2.** The configurations for the intersection \( I \) of two BM unitals \( U_{a_1,b_1}, U_{a_2,b_2} \) in \( \mathfrak{F} \) are the following:

1. \( I \) consists of \( q + 1 \) collinear points;
2. \( I \) consists of \( q \) sets of \( q + 1 \) collinear points. The \( q \) lines all meet at \( P_{\infty} \);
(3) \( \mathcal{I} \) consists of \( 2q - 1 \) sets of \( q + 1 \) collinear points. The \( 2q - 1 \) lines all pass through the point \( P_\infty \).

Furthermore, it follows from the proof of Theorem 4.1 that

(a) in cases (A\(_1\)) and (C\(_{21}\)) the intersection \( \mathcal{I} \) is the set

\[
\mathcal{I} = \{(1, t, (a_1 + b_1)t^2 + r) \mid t, r \in \text{GF}(q)\} \cup \{P_\infty\}.
\]

Hence \( \mathcal{I} \) is one of the \( (q^2 + 1) \)-sets defined in Theorem 3.1 with \( a = a_1 + b_1 \) and \( \sigma \) the automorphism of \( \text{GF}(q^2) \) such that \( x^\sigma = x^2 \).

(b) In cases (D\(_{21}\)) and (D\(_{31}\)) the intersection \( \mathcal{I} \) turns out to be one of the \( (q^2 + 1) \)-sets defined in Theorem 3.2 with respectively \( s = \sqrt{1/\alpha^{q-1}} \) or \( s = \frac{\beta - \beta^q}{2\alpha^r} \), and \( a = a_1 + sb_1 \).

5. Examples

In this section we show that all the cases discussed in Theorem 4.1 effectively occur for \( q = 4 \) and 5. If \( q = 4 \), denote by \( \omega \) a primitive element of \( \text{GF}(16) \), such that \( \omega^2 + \omega + \delta = 0 \), with \( \delta \) any element of \( \text{GF}(4) \setminus \text{GF}(2) \). Furthermore, put \( a_1 = \omega^3 \) and \( b_1 = \omega \).

When \( q = 5 \), take \( \xi \) as a primitive element of \( \text{GF}(25) \) such that \( \xi^2 - \xi + 2 = 0 \) and set \( a_1 = \xi^7 \), \( b_1 = \xi^{12} \).

Under these assumptions, \( U_{a_1,b_1} \) turns out to be a non-classical BM unital respectively in \( \text{PG}(2, 16) \) or in \( \text{PG}(2, 25) \). Let \( a_2 \) and \( b_2 \) be two coefficients ranging over \( \text{GF}(16) \) or \( \text{GF}(25) \) in such a way that the discriminant condition of Lemma 2.1 is satisfied.

By different choices of \( a_2 \) and \( b_2 \) we get all the cases for \( U_{a_1,b_1} \cap U_{a_2,b_2} \) occurring in the proof of Theorem 4.1 but case (D\(_1\)); see Table 1.

References


Table 1: Intersection cases for $q$ small

<table>
<thead>
<tr>
<th>Case</th>
<th>$a_2$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A$_1$)</td>
<td>$\omega^{12}$</td>
<td>$\omega^8$</td>
</tr>
<tr>
<td>(B$_1$)</td>
<td>$\omega^4$</td>
<td>$\omega^{14}$</td>
</tr>
<tr>
<td>(C$_{12}$)</td>
<td>$\omega$</td>
<td>$\omega^{11}$</td>
</tr>
<tr>
<td>(C$_{21}$)</td>
<td>$\omega^{12}$</td>
<td>$\omega^4$</td>
</tr>
<tr>
<td>(D$_{21}$)</td>
<td>$\omega^9$</td>
<td>$\omega^2$</td>
</tr>
<tr>
<td>(D$_{22}$)</td>
<td>$\omega^{11}$</td>
<td>$\omega^6$</td>
</tr>
</tbody>
</table>

$q = 4$

<table>
<thead>
<tr>
<th>Case</th>
<th>$a_2$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A$_2$)</td>
<td>$\xi^{23}$</td>
<td>1</td>
</tr>
<tr>
<td>(B$_{21}$)</td>
<td>$\xi^{10}$</td>
<td>$\xi^{19}$</td>
</tr>
<tr>
<td>(B$_{22}$)</td>
<td>$\xi^{16}$</td>
<td>$\xi^5$</td>
</tr>
<tr>
<td>(C$_{13}$)</td>
<td>$\xi^{10}$</td>
<td>$\xi^{16}$</td>
</tr>
<tr>
<td>(C$_{14}$)</td>
<td>$\xi^{20}$</td>
<td>$\xi^3$</td>
</tr>
<tr>
<td>(C$_{22}$)</td>
<td>$\xi^{23}$</td>
<td>$\xi^{18}$</td>
</tr>
<tr>
<td>(D$_{31}$)</td>
<td>$\xi^{22}$</td>
<td>$\xi^7$</td>
</tr>
<tr>
<td>(D$_{32}$)</td>
<td>$\xi^{16}$</td>
<td>$\xi^{15}$</td>
</tr>
</tbody>
</table>

$q = 5$


Angela Aguglia

**Dipartimento di Matematica, Politecnico di Bari, Via G. Amendola 126/B, 70126 Bari, Italy**

e-mail: a.aguglia@poliba.it

website: http://www.dm.uniba.it/~aguglia