On the finite projective planes of order \( q^4 \), \( q \) odd, admitting \( \text{PSL}(3, q) \) as a collineation group

Mauro Biliotti    Alessandro Montinaro

Dedicated to Prof. Gábor Korchmáros on occasion of his 60th birthday

Abstract

In this paper, it is shown that any projective plane \( \Pi \) of order \( n \leq q^4 \), \( q \) odd, that admits a group \( G \cong \text{PSL}(3, q) \) as a collineation group contains a \( G \)-invariant Desarguesian subplane of order \( q \). Moreover, the involutions and suitable \( p \)-elements in \( G \) are homologies and elations of \( \Pi \), respectively. In particular, if \( n \leq q^3 \), actually, \( n = q, q^2 \) or \( q^3 \).

Keywords: projective plane, collineation group, orbit

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1 Introduction and result

The problem of determining a projective plane \( \Pi \) of order \( n \) admitting \( G \cong \text{PSL}(3, q) \) as a collineation group has been largely investigated in the last decades. The first significant result related to this problem is the celebrated theorem of Ostrom and Wagner [21], dating back to 1959, which asserts that the projective plane \( \Pi \) is Desarguesian when \( n = q \). In 1976, Lüneburg [15] proves that either \( \Pi \) is a Desarguesian plane or a Generalized Hughes plane when \( n = q^2 \). In 1985, Dempwolff [4] proves that any projective plane \( \Pi \) of order \( n = q^3 \) that admits \( G \cong \text{PSL}(3, q) \) as a collineation group contains a Desarguesian subplane \( \Pi_0 \) of order \( q \) on which \( G \) acts faithfully in its natural permutation representation. Despite the fact that Dempwolff provides a complete description of the \( G \)-orbits on the points and on the lines of \( \Pi \), he emphasizes the difficulty in obtaining
a characterization of \( \Pi \). In 1989, Moorhouse obtains for projective planes of order \( n = q^4 \), \( q \) odd, the analogue of Dempwolff’s result. Recently, Montinaro investigated the projective planes of order \( n \leq q^3 \) admitting a group inducing a 2-transitive group (namely, \( \text{PSL}(3, q) \)) on a subplane of \( \Pi \), showing that \( n = q \), \( q^2 \) or \( q^3 \) and the results of Ostrom and Wagner, Lüneburg, Dempwolff, occur, respectively. This paper represents a further contribution to the study of the projective planes of order \( n \leq q^4 \), \( q \) odd, that admit \( \text{PSL}(3, q) \) as a collineation group. In particular, it represents a conclusive result when the plane has order \( n \leq q^3 \).

**Theorem 1.1.** Let \( \Pi \) be a finite projective plane of order \( n \) that admits \( G \cong \text{PSL}(3, q) \), \( q \) odd, as a collineation group. If \( n \leq q^4 \), then the following occurs:

(I) There exists a subplane \( \Pi_0 \cong \text{PG}(2, q) \) of \( \Pi \) on which \( G \) acts in the natural way;

(II) The involutions in \( G \) are homologies of \( \Pi \);

(III) The \( p \)-elements of \( G \) inducing elations on \( \Pi_0 \) are elations of \( \Pi \).

Moreover, one of the following occurs:

(i) \( n = q \) and \( \Pi = \Pi_0 \cong \text{PG}(2, q) \);

(ii) \( n = q^2 \), \( \Pi \) is a Desarguesian plane or a Generalized Hughes plane and \( \Pi_0 \) is a Baer subplane of \( \Pi \);

(iii) \( n = q^3 \);

(iv) \( n = q^2(\lambda(q - 1) + 1) \), where \( 1 < \lambda \leq q + 1 \) and \( q + 1 \mid \lambda(\lambda - 1) \).

The cases (i) and (ii) clearly occur. The only known occurrences of the case (iii) are in the Desarguesian planes and in the Figueroa planes [5], [7]. The only known occurrences of the case (iv) are in the Desarguesian planes and in the Generalized Hughes planes when \( \lambda = q + 1 \), i.e. \( n = q^4 \).

The strategy of the proof is the following. Firstly, we prove that \( G \) is irreducible on \( \Pi \). Hence, \( \Pi \) consists of nontrivial \( G \)-orbits. If \( \psi \) is a Baer collineation of \( \Pi \), we determine the general structure of the action of the group induced by \( C_G(\psi) \) on \( \text{Fix}(\psi) \) by Theorem 2.1. This forces any admissible \( G \)-orbit on the points of \( \Pi \) to be divisible by either \( q^2 \) or \( q\sqrt{q} \) for \( q \) square. So, \( n^2 + n + 1 \), i.e. the number of points of \( \Pi \), is divisible by either \( q^2 \) or \( q\sqrt{q} \) for \( q \) square, as \( \Pi \) consists of nontrivial \( G \)-orbits. This yields a Diophantine equation involving \( n^2 + n + 1 \) and either \( q^2 \) or \( q\sqrt{q} \) for \( q \) square. However, such an equation has no admissible solutions by [13, Lemma 6.2]. Therefore, the involutions in \( G \) are homologies of \( \Pi \). At this point, the proof of our result easily follows.
2 Background

The notation used in this paper is standard. For what concerns finite groups, the reader is referred to [11] and to [3]. The necessary background about finite projective planes may be found in [10].

Now, we collect some information about the structure of the groups $\text{PSL}(2,q)$ and $\text{PSL}(3,q)$ and some results on the projective planes admitting one of these as a collineation group. Based on the results of Lüneburg [14], Yaqub [22] and Moorhouse [19], the following theorem, due to Montinaro, determines the general structure of the projective planes of order up to $q^2$ admitting $\text{PSL}(2,q)$, $q > 3$, as a collineation group. Recall that a collineation group of a projective plane $\Pi$ is said to be irreducible on $\Pi$ if the group does not fix any point, line, triangle of $\Pi$. An irreducible collineation group of $\Pi$ which does not fix any proper subplane of $\Pi$ is said to be strongly irreducible on $\Pi$.

**Theorem 2.1.** Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $H \cong \text{PSL}(2,q)$, $q > 3$. If $n \leq q^2$, then one of the following occurs:

1. $n < q$ and one of the following occurs:
   
   (a) $n = 4$, $\Pi \cong \text{PG}(2,4)$ and $H \cong \text{PSL}(2,5)$;
   
   (b) $n = 2$ or $4$, $\Pi \cong \text{PG}(2,2)$ or $\text{PG}(2,4)$, respectively, and $H \cong \text{PSL}(2,7)$;
   
   (c) $n = 4$, $\Pi \cong \text{PG}(2,4)$ and $H \cong \text{PSL}(2,9)$.

2. $n = q$, $\Pi \cong \text{PG}(2,q)$ and one of the following occurs:
   
   (a) $H$ fixes a line or a point and $q$ is even;
   
   (b) $H$ is strongly irreducible and $q$ is odd.

3. $q < n < q^2$ and one of the following occurs:
   
   (a) $H$ fixes a point or a line, and one of the following occurs:
      
      (i) $n = 16$ and $H \cong \text{PSL}(2,5)$;
      
      (ii) $n = 16$, $\Pi$ is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and $H \cong \text{PSL}(2,7)$.
   
   (b) $H$ fixes a subplane $\Pi_0$ of $\Pi$, $q$ is odd and one of the following occurs:
      
      (i) $n = 16$, $\Pi_0 \cong \text{PG}(2,4)$ and $H \cong \text{PSL}(2,5)$;
      
      (ii) $\Pi_0 \cong \text{PG}(2,2)$ or $\text{PG}(2,4)$, and $H \cong \text{PSL}(2,7)$;
      
      (iii) $\Pi_0 \cong \text{PG}(2,4)$ and $G \cong \text{PSL}(2,9)$. 
(c) $H$ is strongly irreducible and $q$ is odd.

(4) $n = q^2$ and one of the following occurs:

(a) $H$ fixes a point or a line, and one of the following occurs:
   (i) $n = 25$ and $H \cong \operatorname{PSL}(2, 5)$;
   (ii) $n = 81$ and $H \cong \operatorname{PSL}(2, 9)$;
   (iii) $n = q^2$, $q$ even, and $G \cong \operatorname{PSL}(2, q)$.

(b) $H$ fixes a subplane $\Pi_0$ of $\Pi$, $q$ is odd and one of the following occurs:
   (i) $n = q^2$, $\Pi_0 \cong \operatorname{PG}(2, q)$ and $H \cong \operatorname{PSL}(2, q)$;
   (ii) $n = 25$, $\Pi_0 \cong \operatorname{PG}(2, 4)$ and $H \cong \operatorname{PSL}(2, 5)$;
   (iii) $n = 81$, $\Pi_0 \cong \operatorname{PG}(2, 4)$ and $H \cong \operatorname{PSL}(2, 9)$;
   (iv) $n = 81$, $\Pi_0$ is a Hughes plane of order $9$ and $H \cong \operatorname{PSL}(2, 9)$.

(c) $H$ is strongly irreducible.

Proof. See [18, Theorem 1].

As we shall see, such a theorem will play a central role in our investigation due to the fact that the centralizer of an involution involves a group isomorphic to $\operatorname{PSL}(2, q)$.

Now, we recall some basic facts about the structure of the group $G \cong \operatorname{PSL}(3, q)$ (the reader is referred to [16]).

1. Let $\psi$ and $\beta$ be the involutions in $G$ represented by $\text{diag}(1, -1, -1)$ and $\text{diag}(-1, 1, -1)$, respectively. Then $\langle \psi, \beta \rangle \cong E_4$.

2. Let $U$ be the Sylow $p$-subgroup of $G$ represented by all the matrices
   \[
   \begin{bmatrix}
   1 & x_1 & x_2 \\
   0 & 1 & x_3 \\
   0 & 0 & 1
   \end{bmatrix},
   \]  
   where $x_1, x_2, x_3 \in \text{GF}(q)$. Clearly, $|U| = q^3$. Let $U_0$ be the subgroup of $U$ represented by the matrices in (1) having $x_1 = x_3 = 0$. Then $U_0$ has order $q$ and $U_0 = Z(U) = U'$. Thus, $U$ is a special $p$-group. Finally, let $U^*$ be the subgroup of $U$ represented by the matrices in (1) having $x_3 = 0$. Then $U^*$ is elementary abelian of order $q^2$ which is normalized by $\psi$.

3. Let $S$ be the Sylow $p$-subgroup of $G$ represented by all the matrices
   \[
   \begin{bmatrix}
   1 & 0 & y_2 \\
   y_1 & 1 & y_1 \\
   0 & 0 & 1
   \end{bmatrix},
   \]
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where $y_1, y_2, y_3 \in \text{GF}(q)$, and let $S_0$ be the subgroup of $S$ represented by those having $y_2 = y_3 = 0$. Then $S_0 = Z(S) = S'$. In particular, $U \cap S$ is an elementary abelian group of order $q^2$ containing $S_0$. Namely, $U \cap S$ consists of all the matrices in (2) having $y_3 = 0$.

4. The group $S_0 \langle \psi, \beta \rangle$ has order $4q$. In particular, $\psi$ centralizes $S_0$, while $\beta$ inverts $S_0$.

5. The group $C_G(\psi)$ consists of the matrices

$$
\begin{bmatrix}
e^{-1} & 0 & 0 \\
0 & a & b \\
0 & c & d
\end{bmatrix}
$$

where $a, b, c, d, e \in \text{GF}(q)$, $e = ad - bc \neq 0$. Denote by $Z_\psi$ the subgroup of $C_G(\psi)$ represented by all the matrices $\text{diag}(d^{-2}, d, d)$, where $d \in \text{GF}(q)^*$. Then $Z_\psi = Z(C_G(\psi))$. In particular, $Z_\psi$ is a cyclic group of order $\frac{q^2 - 1}{\mu}$, where $\mu = (3, q - 1)$ and $C_G(\psi) \cong Z_\psi \cdot \text{PGL}(2, q)$.

6. The group $U^* : C_G(\psi)$ is a maximal parabolic subgroup of $G$. Furthermore, $U^* \langle \psi \rangle \triangleleft U^* : C_G(\psi)$ and $C'_G(\psi) \cong SL(2, q)$.

7. Let $W^*$ be the subgroup of $G$ represented by all the matrices of the form

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
z_1 & z_2 & 1
\end{bmatrix}
$$

where $z_1, z_2 \in \text{GF}(q)$. Then $W^*$ is an elementary abelian group of order $q^2$ which is normalized by $C_G(\psi)$. Moreover, the groups $U^* : C_G(\psi)$ and $W^* : C_G(\psi)$ are the representatives of the two distinct conjugate classes of maximal parabolic subgroups of $G$. The groups $U^*$ and $W^*$ are conjugate by the inverse-transpose automorphism.

We shall use the facts stated above without recalling them, unless it is explicitly required. In particular, since the Sylow $p$-subgroups of $G$ are conjugate, we shall mainly refer either to $U$ or to $S$. Furthermore, despite the fact that there are two distinct conjugate classes of maximal parabolic subgroups in $G$ by (7), what we prove to be true for $U^* : C_G(\psi)$ can always be proven to be true for $W^* : C_G(\psi)$. Hence, for our purposes we may always refer to $U^* : C_G(\psi)$, without loss of generality.
Lemma 2.2. The group $G \cong \text{PSL}(3, q)$ contains two distinct involutions $\psi_1$ and $\psi_2$ such that $C'_G(\psi_1) \cap C'_G(\psi_2) \neq \langle 1 \rangle$ and $(C'_G(\psi_1), C'_G(\psi_2)) = G$.

Proof. See [20, Lemma 4.1.vi].

Some geometrical results involving the group $G \cong \text{PSL}(3, q)$ are in order. By using the results of Ostrom-Wagner [21], Lüneburg [15] and Dempwolff [4], Montinaro proved the following.

Theorem 2.3. Let $\Pi$ be a finite projective plane of order $n$ and let $G$ be a collineation group of $\Pi$ inducing a group containing $\text{PSL}(3, q)$ on a subplane $\Pi_0$ of order $q$. If $n \leq q^3$, then one of the following occurs:

1. $\Pi_0 \cong \text{PG}(2, q)$, $\text{PSL}(3, q) \leq G$ and one of the following occurs:
   
   (a) $n = q$ and $\Pi = \Pi_0$;
   
   (b) $n = q^2$, $\Pi$ is a Desarguesian plane or a Generalized Hughes plane and $\Pi_0$ is a Baer subplane of $\Pi$;
   
   (c) $n = q^3$.

2. $\Pi_0 \cong \text{PG}(2, 7)$, $\Pi$ is the generalized Hughes plane over the exceptional nearfield of order $7^2$ and $\text{SL}(3, 7) \leq G$.

Proof. See [17].

Finally, we quote this useful final result, due to Moorhouse [20], which inspired the present paper, and that allows to reduce our investigation to $n < q^4$ (when $q > 3$).

Theorem 2.4 (Moorhouse). Let $\Pi$ be a projective plane of order $q^4$ admitting $G \cong \text{PSL}(3, q)$, $q$ odd. If $q > 3$, then the following must hold.

(i) $G$ leaves invariant a Desarguesian subplane $\Pi_0$ of order $q$, on which $G$ acts 2-transitively;

(ii) The involutions in $G$ are homologies of $\Pi$, and those $p$-elements of $G$ which induce elations of $\Pi_0$ are elations of $\Pi$.

If $q = 3$ then the same two conclusions must hold, under the additional hypothesis that $G$ acts irreducibly on $\Pi$.

Proof. See [20, Theorem 1.3].
3 Preliminary reductions

The aim of this section is to show that $G$ is irreducible on II and that the involutions in $G$ are perspectivities of II, in order to apply Hering-Walker theory on the strong irreducibility (e.g. see [6], [8] and [9]).

In view of Theorem 2.1, we treat the cases $q = 3$ and $q > 3$ separately.

Lemma 3.1. Let II be a finite projective plane of order $n$ that admits $G \cong PSL(3, q)$ as a collineation group. If $n \leq 3^4$, then each involution in $G$ is a perspectivity of II.

Proof. Assume that the involutions in $G$ are Baer collineations of II. Hence, $\sqrt{n} \leq 9$. Let $J$ be a Sylow 2-subgroup of $G$. As $n^2 + n + 1$ is odd, then $J$ fixes a secant $s$ of $\text{Fix}(\psi)$. Let $J_0 = J \cap C_G(\psi)$. Then $J_0 \cong Q_8$. Thus, $J_0$ is semiregular on $s = \text{Fix}(\psi)$. So, $8 \mid \sqrt{n}(\sqrt{n} - 1)$, since $|s - \text{Fix}(\psi)| = \sqrt{n}(\sqrt{n} - 1)$. Consequently, either $\sqrt{n} = 8$ or $9$, as $\sqrt{n} \leq 9$. Note that $J = J_0$. ($\beta$) is known to be semidihedral of order 16. As $J_0$ is semiregular on $s - \text{Fix}(\psi)$, then each $J$-orbit on $s - \text{Fix}(\psi)$ has length either 8 or 16. Therefore, let $x$ and $y$ be the number of $J$-orbits on $s - \text{Fix}(\psi)$ of length 8 and 16, respectively. It follows that

$$8x + 16y = \sqrt{n}(\sqrt{n} - 1),$$

where $\sqrt{n} = 8$ or 9. As $J$ is semidihedral of order 16, then $J$ contains two distinct conjugate classes of involutions, one consisting of $\psi$ and the other consisting of the four conjugates of $\beta$ (including $\beta$). Furthermore, $C_J(\beta) \cong \langle \psi, \beta \rangle \cong E_4$. Thus, by [19, Relation (8)], the involution $\beta$ fixes 2 and 0 points on the $J$-orbits on $s - \text{Fix}(\psi)$ of length 8 and 16, respectively, since $\psi \in J_0$ and since $J_0$ is semiregular on $s - \text{Fix}(\psi)$. Hence, $\beta$ fixes exactly $2x$ points on $s - \text{Fix}(\psi)$. If $x$ is even, then $16 \mid \sqrt{n}(\sqrt{n} - 1)$ by (3), which is impossible as $\sqrt{n} = 8$ or 9. Therefore, $x$ is odd. Hence, $\beta$ cannot induce either the identity or a perspectivity of axis $s$ on $\text{Fix}(\psi)$, otherwise $x = 0$, since $\beta$ is a Baer collineation on II (recall that $G \cong PSL(3, 3)$ has a unique conjugate class of involutions). Suppose that $\beta$ induces a perspectivity on $\text{Fix}(\psi)$ of axis distinct from $s$. Clearly, $\beta$ induces on $\text{Fix}(\psi)$ either an elation when $\sqrt{n} = 8$ or a homology when $\sqrt{n} = 9$. Then $x = \sqrt{n}$ or $\sqrt{n} - 1$, respectively, again since $\beta$ is a Baer collineation on II. So, $x$ is even in any case, which is a contradiction. Finally, assume that $\beta$ induces a Baer collineation on $\text{Fix}(\psi)$ when $\sqrt{n} = 9$. Arguing as above, we have that $x = \sqrt{n} - \sqrt{n}$ which is even and we again obtain a contradiction. Thus, the involutions in $G$ are perspectivities of II.

Proposition 3.2. Let II be a finite projective plane of order $n$ that admits $G \cong PSL(3, q)$ as a collineation group. If $n \leq 3^4$, then the following occurs:
There exists a subplane \( \Pi_0 \cong \text{PG}(2,3) \) of \( \Pi \) on which \( G \) acts in the natural way;

The group \( G \) is irreducible on \( \Pi \);

The involutions in \( G \) are homologies of \( \Pi \);

The 3-elements that induce elations on \( \Pi_0 \) are elations of \( \Pi \).

Moreover, one of the following occurs:

(i) \( n = 3 \) and \( \Pi = \Pi_0 \cong \text{PG}(2, q) \);

(ii) \( n = 3^2 \), \( \Pi \) is a Desarguesian plane or a Generalized Hughes plane and \( \Pi_0 \) is a Baer subplane of \( \Pi \);

(iii) \( n = 3^3 \);

(iv) \( n = 3^4 \).

Proof. Assume that \( G \cong \text{PSL}(3,3) \) fixes a line \( l \) of \( \Pi \). As \( n \leq 3^4 \), then each nontrivial \( G \)-orbit on \( l \) has length divisible by 13 by [2]. Actually, \( G \) contains such orbits, since \( G \) acts faithfully, \( G \) being nonabelian simple. Let \( X \in \Pi \) such that \( 13 \mid |X^G| \). So, \( G_X \leq E_9 \cdot \text{GL}(2,3) \). Let \( B_X \) be the block of imprimitivity in \( X^G \) containing \( X \). Clearly \( |X^G| = 13 |B_X| \) (\( |B_X| \) might be 1). Furthermore, \( E_9 \cdot \text{GL}(2,3) \) acts transitively on \( B_X \). As the socle of \( E_9 \cdot \text{GL}(2,3) \) is \( E_9 \), then either \( E_9 \leq G_X \) or \( 13 \cdot 9 \mid |X^G| \) by [3, Theorem 4.1A]. Actually, the latter cannot occur, since \( |X^G| \leq n + 1 \) and \( n \leq 3^4 \). Hence, each nontrivial \( G \)-orbit on \( l \) has length \( |X^G| = 13 |B_X| \), where \( |B_X| \mid | \text{GL}(2,3)| \). Actually, \( |B_X| = 1, 2, 3, 4 \) or 6, since \( |B_X| \leq 6 \), as \( n \leq 3^4 \). Since the blocks of imprimitivity are 13, then there exists a point \( P \), lying in a nontrivial \( G \)-orbit on \( l \), such that \( J_0 \) fixes \( B_P \), where \( J_0 \) is the 2-group isomorphic to \( Q_8 \) containing the involution \( \psi \). Thus, \( \psi \) fixes \( B_P \) pointwise, since \( |B_P| \leq 6 \). Then \( |B_P| \leq 2 \), since \( \psi \) is a perspectivity of \( \Pi \) having axis distinct from \( l \). If \( |B_P| = 1 \), then \( P^G \) is a 2-transitive \( G \)-orbit. Hence, \( \psi \) fixes exactly 5 points on \( X^G \). So, we arrive at a contradiction, since \( \psi \) is a perspectivity of \( \Pi \) having axis distinct from \( l \). Thus, \( |B_P| = 2 \) and hence \( l = P^G \), since \( \psi \) fixes \( B_P \) pointwise. In particular, \( n = 25 \), since \( |P^G| = 26 \). Since \( G \) acts faithfully on \( l \), there are no involutory homologies of axis \( l \). Therefore, no involutions lie in a triangular configuration. In particular, since \( \psi \) is the unique central involution in \( J \) (recall that \( J \) is semidihedral of order 16), each involution in \( J \) has center and axis \( C_\psi \) and \( a_\psi \), where \( C_\psi \) and \( a_\psi \) denote the center and the axis of \( \psi \), respectively. So, each involution in \( J \) fixes exactly two points on \( l \), namely \( C_\psi \) and \( a_\psi \cap l \). Hence, \( J \) is semiregular on \( l - \{C_\psi, W\} \), where \( \{W\} = a_\psi \cap l \). Then 16 \( \mid n - 1 \), which is a contradiction, since \( n = 25 \).
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while $|J| = 16$. Therefore, $G$ does not fix lines. By the dual of the previous proof, we obtain that $G$ does not fix points. Finally, these two facts, combined with the fact that $G$ is nonabelian simple, yield that $G$ does not fix triangles of $\Pi$. Thus, $G$ is irreducible on $\Pi$ and hence the assertion (2).

Since $G \cong \text{PSL}(3, 3)$ is irreducible on $\Pi$, and since each involution in $G$ is a perspectivity by Lemma 3.1, then $G$ leaves invariant a subplane $\Pi_0$ on which it acts strongly irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong \text{PG}(2, 3)$ by [8, Theorem 1.1], and we obtain the assertion (1). Therefore, the involutions in $G$ are homologies of $\Pi$ and hence the assertion (3). For $n \leq 3^3$, the assertions (4) and (i)-(iii) follow by Theorem 2.3. Furthermore, for $n = 3^4$ the assertions (4) and (iv) follow by Theorem 2.4, since we proved the irreducibility of $G$ on $\Pi$. Hence, assume that $3^3 < n < 3^4$. Note that $G$ contains an elementary abelian group $H$ of order $3^2$ consisting of elations with the same axis $r$ and distinct centres lying in $\Pi_0 \cap r$ by [10, Theorem 4.25]. As $H$ is semiregular on $[Q] - \{l\}$, for any $Q \in r - \Pi_0$, then $3^2 \mid n$. So, $3^3 < n < 3^4$, $n$ odd, and $3^2 \mid n$ yield that $n = 3^2 5$ or $3^2 7$. Let $E$ be the set of external lines to $\Pi_0$. Easy computations yield $|E| = 1512$ or $3240$, respectively. Let $R$ be any Sylow 2-subgroup of $G$. Then $|R| = 16$. Since each involution in $G$, and hence in $R$, is a homology of axis a secant to $\Pi_0$, then $R$ is semiregular on $E$. So, $16 \mid |E|$, which is impossible as $|E| = 1512$ or $3240$. This completes the proof. □

It should be pointed out that the previous theorem extends the Theorem 2.4 also for $n = 3^4$. Indeed, Theorem 2.4 works for $q = 3$ under the additional assumption that $G$ is irreducible on $\Pi$. In particular, Moorhouse shows that the irreducibility of $G$ on $\Pi$ implies that the involutions in $G$ are homologies of $\Pi$. We, instead, prove that the involutions are perspectivities of $\Pi$ and then we use this fact to prove that $G$ is irreducible on $\Pi$.

From now on, we assume that $q > 3$.

**Lemma 3.3.** The group $G$ is irreducible on $\Pi$.

**Proof.** Assume that $G$ fixes a line $l$ of $\Pi$. Then $\sqrt{n} < q^2$, since for $n = q^4$ the assertion follows by [20] (e.g. see the proof of Theorem 1.3). Let $\psi$ be the involution in $G$ defined in Section 2. Then, by Theorem 2.1 and by bearing in mind that $q$ is odd and $\sqrt{n} < q^2$, one of the following occurs:

1. $\sqrt{n} = 4$, $\text{Fix}(\psi) \cong \text{PG}(2, 4)$ and $C_G(\psi)' / \langle \psi \rangle \cong \text{PSL}(2, 5)$;
2. $\sqrt{n} = 16$ and $C_G(\psi)' / \langle \psi \rangle \cong \text{PSL}(2, 5)$;
(3) \( \sqrt{n} = 16 \), \( \text{Fix}(\psi) \) is either the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and \( C_G(\psi)/\langle \psi \rangle \cong \text{PSL}(2, 7) \).

Assume that the case (1) occurs. Since \( n + 1 = 17 \) and since these primitive permutation representations of \( G \) have a degree greater than 17 by [2], then \( G \) fixes \( l \) pointwise. That is, \( G \) is a group of perspectivities of axis \( l \). So, \( G \) should be a Frobenius group by [10, Theorem 4.25], which is impossible as \( G \) is nonabelian simple.

We treat the cases (2)–(3) simultaneously. By a direct inspection of [2], it is plain that the unique nontrivial orbits on \( l \) under \( G \cong \text{PSL}(3, q) \), \( q = 5 \) or 7, are those of length a multiple of \( d_0 \), the minimal primitive permutation representation degree of \( G \). By [2], such a \( d_0 \) is equal to 31 or 57, respectively. Let \( r \) be the minimal nonnegative integer such that \( n + 1 \equiv r_0 \mod d_0 \). Easy computations yield that \( r_0 = 9, 29 \) or 6 in the cases (1)–(3), respectively. So, \( 6 \leq r_0 < n + 1 \) and \( \sqrt{n} + 1 \not\equiv r_0 \mod d_0 \) in any case. Therefore, \( G \) fixes at least 6 points on \( l \) in any case. Let \( P \) be any of these points. Now, by repeating the above argument with \([P]\) in the role of \( l \), we obtain that \( G \) fixes at least 6 lines of \([P]\) (clearly, the line \( l \) is included). Again, by repeating the above argument for any for each of these 6 lines, we obtain that \( G \) fixes a subplane \( \Sigma \) of \( \Pi \) pointwise. Let \( r \) be the order of \( \Sigma \). Then \( r = r_0 + h d_0 - 1 \), where \( h \geq 0 \). Note that \( r_0 + h d_0 - 1 \leq \sqrt{n} \) by [10, Theorem 3.7]. Hence, the case (3) is ruled out. Actually, \( r_0 + h d_0 - 1 < \sqrt{n} \), since \( \sqrt{n} + 1 \not\equiv r_0 \mod d_0 \). Thus, \( \Sigma \subseteq \text{Fix}(\psi) \), since \( \Sigma \subseteq \text{Fix}(\psi) \). Therefore, \( (r_0 + h d_0 - 1)^2 \leq n \) by [10, Theorem 3.7]. This forces \( h = 0 \) in any admissible case. In particular, the case (2) is ruled out. Consequently, \( G \) is irreducible on \( \Pi \).

Throughout this section, we assume that \( \psi \) is a Baer collineation of \( \Pi \).

Then \( n < q^4 \) by Theorem 2.4, as \( q > 3 \).

The following lemma determines the structure of the kernel \( K_\psi \) of the action of \( C_G(\psi) \) on \( \text{Fix}(\psi) \).

**Lemma 3.4.** \( \langle \psi \rangle \leq K_\psi \leq Z_\psi \).

**Proof.** Clearly, \( \langle \psi \rangle \subseteq K_\psi \leq C_G(\psi) \). Recall that \( C_G(\psi) \cong Z_\psi \cdot \text{PGL}(2, q) \). Since \( K_\psi Z_\psi / Z_\psi \cong \text{PGL}(2, q) \) then either \( K_\psi Z_\psi / Z_\psi = \{1\} \) or \( \text{PSL}(2, q) \cong K_\psi Z_\psi / Z_\psi \). Assume that the latter occurs. Then \( C'_G(\psi) \leq K_\psi \), since \( C'_G(\psi)/\langle \psi \rangle \cong \text{PSL}(2, q) \) and since \( \langle \psi \rangle \leq K_\psi \leq C_G(\psi) \). Since for each involution \( \beta \in G \) there exists \( g \in G \) such that \( \psi^g = \beta \), then \( C'_G(\psi) = C'_G(\beta) \). Hence \( C'_G(\beta) \) fixes \( \text{Fix}(\beta) \) pointwise for each involution \( \beta \) in \( G \). By Lemma 2.2, there exist two involutions \( \psi_1 \) and \( \psi_2 \) such that \( C'_G(\psi_1) \cap C'_G(\psi_2) \neq \{1\} \) and \( (C'_G(\psi_1), C'_G(\psi_2)) = G \). Since
$C'_{\Pi}(\psi)$ fixes the Baer subplane $\text{Fix}(\psi_i)$ pointwise for each $i = 1, 2$, and since $C'_{\Pi}(\psi_1) \cap C'_{\Pi}(\psi_2) \neq \{1\}$, then $\text{Fix}(\psi_1) = \text{Fix}(\psi_2)$. Thus, $G = (C'_{\Pi}(\psi_1), C'_{\Pi}(\psi_2))$ fixes $\text{Fix}(\psi_1)$ pointwise, which is impossible by Lemma 3.3. Consequently, $K_\psi Z_\psi / Z_\psi = \{1\}$. That is, $K_\psi \leq Z_\psi$ and hence we obtain the assertion. □

For each subgroup $X$ of $C_\Pi(\psi)$, we denote by $X$ the group $XK_\psi / K_\psi$.

**Lemma 3.5.** For each point $X \in \Pi$ such that $G_X$ lies in a maximal parabolic subgroup of $G$, one of the following occurs:

1. $X^G$ is a 2-transitive orbit;
2. $\text{Fix}(U^*(\psi))$ is either a flag, or an antiflag or a proper subplane of $\text{Fix}(\psi)$. Furthermore, $\overline{C}(\psi)$ leaves $\text{Fix}(U^*(\psi))$ invariant;
3. $q^2 \mid |X^G|$.

**Proof.** Let $X \in \Pi$ and assume that $G_X$ lies in a maximal parabolic subgroup of $G$. As mentioned in Section 2, for our purposes we may reduce to study the case when $G_X \leq U^*: C_\Pi(\psi)$, where $C_\Pi(\psi) \cong Z_\psi$. PGL(2, $q^2$) and $Z_\psi \cong Z_{q^2}$, $\mu = (3, q - 1)$. If $G_X = U^* : C_\Pi(\psi)$, then $X^G$ is a 2-transitive orbit and we obtain the assertion (1). If $G_X < U^* : C_\Pi(\psi)$, denoted by $B_X$ the block of imprimitivity in $X^G$ containing $X$, we have $|B_X| > 1$. Clearly, $U^* : C_\Pi(\psi)$ acts on $B_X$.

Assume that $U^* : C_\Pi(\psi)$ does not act faithfully on $B_X$, then $U^*$ lies in the kernel of the action, since $U^*$ is the socle of $U^* : C_\Pi(\psi)$ by [3, Theorem 4.3B]. Thus, $\text{Fix}(U^*) \neq \emptyset$. Since $U^* \triangleleft U^* : C_\Pi(\psi)$, and since $\text{Fix}(U^* : C_\Pi(\psi)) = \emptyset$, being $G_X < U^* : C_\Pi(\psi)$, either $\text{Fix}(U^*) = \Delta$, where $\Delta$ is a triangle of $\Pi$, or $\text{Fix}(U^*)$ is a subplane of $\Pi$ by [6, Corollary 3.6]. This yields that $\text{Fix}(U^*(\psi))$ consists of either a flag, or an antiflag or a plane. Clearly, $\text{Fix}(U^*(\psi)) \subseteq \text{Fix}(\psi)$. Furthermore, $\overline{C}(\psi)$ acts on $\text{Fix}(\psi)$ leaving $\text{Fix}(U^*(\psi))$ invariant, since $U^*(\psi) \triangleleft U^* : C_\Pi(\psi)$. If $\text{Fix}(U^*(\psi)) = \text{Fix}(\psi)$, then $\text{Fix}(U^*) = \text{Fix}(\psi)$, since $\text{Fix}(\psi)$ is a Baer subplane of $\Pi$. So, $U^*$ is semiregular on $s - \text{Fix}(U^*)$, where $s$ is a secant of $\text{Fix}(U^*)$. Therefore, $q^2 \mid n - \sqrt{n}$, since $|U^*| = q^2$. That is, either $q^2 \mid \sqrt{n} - 1$ or $q^2 \mid \sqrt{n}$, and we have a contradiction in any case since $n < q^4$ and $q > 3$. Thus, we obtain the assertion (2)

Assume that $U^* : C_\Pi(\psi)$ acts faithfully on $B_X$. Then $q^2 \mid |B_X|$ by [3, Theorem 4.1A], since $U^*$ is the socle of $U^* : C_\Pi(\psi)$. Thus, $q^2 \mid |X^G|$ and we obtain the assertion (3). □

**Lemma 3.6.** One of the following occurs:

1. The groups $\overline{C}_G(\psi)$ and $\overline{C}_\Pi(\psi)$ are strongly irreducible on $\text{Fix}(\psi)$;
(II) $q = 5$ and $n = 4$;

(III) $q = 9$ and $9^2 < n < 9^4$.

Proof. Assume that the cases (II) and (III) do not occur. Note that $C_G(\psi) \cong PSL(2, q)$, since $C_G(\psi) \cap K_\psi = \langle \psi \rangle$ by Lemma 3.4. Suppose that the $C_G(\psi)$ is not strongly irreducible on $Fix(\psi)$. The case $\sqrt{n} = q$ is ruled out by Theorem 2.1. Since $\sqrt{n} < q$, then either $\sqrt{n} < q$ or $q < \sqrt{n} < q^2$. Then, again by Theorem 2.1 and bearing in mind that the cases (II) and (III) do not occur by our assumptions, one of the following occurs:

1. $n = 4$ or 16, $Fix(\psi) \cong PG(2, 2)$ or $PG(2, 4)$, respectively, and $C_G(\psi) \cong PSL(2, 7)$;

2. $n = 16$, $Fix(\psi) \cong PG(2, 4)$ and $C_G(\psi) \cong PSL(2, 9)$;

3. $n = 16^2$, $C_G(\psi) \cong PSL(2, 5)$ fixes a subplane of $Fix(\psi)$ isomorphic to $PG(2, 4)$;

4. $7^2 < n < 49^2$, $C_G(\psi) \cong PSL(2, 7)$ fixes a subplane of $Fix(\psi)$ isomorphic either to $PG(2, 2)$ or to $PG(2, 4)$.

Actually, in the cases (1)–(4), the group $C_G(\psi) \cong Z_\psi \cdot PGL(2, q)$ acts on $Fix(\psi)$. The group $C_G(\psi)$ fixes a subplane $\Pi_0$ of $Fix(\psi)$ isomorphic either to $PG(2, 2)$ or to $PG(2, 4)$ for $q = 7$, or to $PG(2, 4)$ for $q \neq 7$ (note that it might be $\Pi_0$ or $Fix(\psi)$).

Assume that $q = 7$. Then $Z_\psi = \langle 1 \rangle$, since $Z_\psi = \langle \psi \rangle$. Therefore $C_G(\psi) = PGL(2, 7)$ acts on $\Pi_0$. Then the case $\Pi_0 \cong PG(2, 2)$ is ruled out, since the full automorphism group of $PG(2, 2)$ is isomorphic to $PSL(2, 7)$. Hence, assume that $\Pi_0 \cong PG(2, 4)$ and $C_G(\psi) \cong PSL(2, 7)$. It is easy to see that $PSL(2, 7)$ fixes a subplane $\Pi_1$ of $\Pi_0$ which is isomorphic to $PG(2, 2)$. In particular, $PGL(2, 7)$ leaves $\Pi_1$ invariant. So, we arrive at a contradiction by the above argument with $\Pi_1$ in the role of $\Pi_0$. Therefore, $q \neq 7$ and hence the cases (1) and (4) are ruled out.

Assume that $q = 5$ or 9. Then $\Pi_0 \cong PG(2, 4)$ and hence $C_G(\psi) \leq PGL(3, 4)$. Furthermore, $Z_\psi = \langle 1 \rangle$ by [2]. Consequently, $Z_\psi$ fixes $Fix(\psi)$ pointwise and $C_G(\psi) \cong PGL(2, 4)$ in any case. Since $Z_\psi$ is semiregular $s = Fix(\psi)$, then $\frac{n - 1}{\mu} \mid n - \sqrt{n}$, where $\frac{n - 1}{\mu} = |Z_\psi|$ and $\mu = (3, q - 1)$. That is, $\frac{2^{\frac{q - 1}{\mu}}}{\mu} \mid \sqrt{n}$ or $\frac{2^{\frac{q - 1}{\mu}}}{\mu} \mid \sqrt{n} - 1$, since $q = 5$ or 9. Thus, the case (2) is ruled out, since $\sqrt{n} = 4$, while $\frac{2^{\frac{q - 1}{\mu}}}{\mu} = 8$.

It remains to investigate the case (3). In this case, any subgroup $Z_{31}$ of $G$ fixes a subplane of $\Pi$ of order $7 + 31k$, $k \geq 0$. Actually, $k = 0$ by [10, Theorem 3.7], since $n = 16^2$. Therefore, $Z_{31}$ fixes exactly 57 points of $\Pi$. Note that $Z_{31} \leq
$G_X \leq Z_{31}Z_3$ for any point $X$ of $\Pi$ fixed by $Z_{31}$ by [2]. Moreover, $Z_{31}Z_3$ is maximal in $G$. So, either $G_X = Z_{31}$, $|X^G| = 12000$ and $Z_{31}$ fixes 3 points on $X^G$, or $G_X = Z_{31}Z_3$, $|X^G| = 4000$ and $Z_{31}$ fixes 1 point on $X^G$. Let $x$ and $y$ be the number of $G$-orbits on $\Pi$ of length 12000 and 4000, respectively. Then $12000x + 4000y \leq 65793$, since $n^2 + n + 1 = 65793$. Furthermore, $3x + y = 57$, since $Z_{31}$ fixes exactly 57 points of $\Pi$. By combining the previous relations involving $x$ and $y$, we obtain a contradiction. Thus, $C_G(\psi)$ is strongly irreducible on $\Fix(\psi)$. Then $C_G(\psi)$ is strongly irreducible on $\Fix(\psi)$, since $C_G(\psi) \leq C_G(\psi)$. That is, the assertion (I) occurs. □

**Lemma 3.7.** The group $G$ does not admit 2-transitive point-orbits on $\Pi$.

**Proof.** Let $O$ be a 2-transitive $G$-orbit on $\Pi$. Then $|O| = q^2 + q + 1$. Clearly, $O$ cannot be contained in a line by lemma 3.3. Then, it is a plain that, either $O$ is an arc or $O \equiv \PG(2, q)$. Assume that the former occurs. Let $U^*$ be the elementary abelian $p$-group defined in Section 2. Then $U^*(\psi)$ fixes exactly $q + 1$ points on $O$. So $U^*(\psi)$ is planar, since $O$ is an arc. Since $U^*(\psi) \triangleleft U^*C_G(\psi)$ and $C_G(\psi)$ acts 2-transitively on $\Fix(U^*(\psi)) \cap O$, then $C_G(\psi)$ acts as $\PSL(2, q)$ on $\Fix(U^*(\psi))$. Note that

$$|\Fix(U^*(\psi)) \cap O| = q + 1 \quad \text{and} \quad |\Fix(\psi) \cap O| = q + 2,$$

as $q$ is odd. Thus $\Fix(U^*(\psi)) \subseteq \Fix(\psi) \subseteq \Pi$, with $o(\Fix(U^*(\psi))) \geq q - 1$. Assume that $o(\Fix(U^*(\psi))) = q$. Then $\sqrt{n} \geq q^2$ by [10, Theorem 3.7], since $\Fix(U^*(\psi)) \subset \Fix(\psi)$, which is contrary to the assumption $\sqrt{n} < q^2$. So, $o(\Fix(U^*(\psi))) = q - 1$. Then $\Fix(U^*(\psi)) \cap O$ is a hyperoval of $\Fix(U^*(\psi))$, as $|\Fix(U^*(\psi)) \cap O| = q + 1$. Furthermore, $C_G(\psi)/\langle \psi \rangle \equiv Z_3$. Since $\PSL(2, q)$, where $|Z_3| = \frac{q - 1}{2}$ and $\mu = (3, q - 1)$, acts 2-transitively on $\Fix(U^*(\psi)) \cap O$. Then $q - 1 = \frac{q - 1}{2}$ and $C_G(\psi)/\langle \psi \rangle \leq S_6$ by [1], as $q > 3$. This implies that $\Fix(Z_3) = \Fix(\psi)$. So, $C_G(\psi)$ acts on $\Fix(\psi)$ as $\PG(2, 5)$ leaving invariant a subplane $\Fix(U^*(\psi)) \cong \PG(2, 4)$, which is impossible by Lemma 3.6, as $n > 4$.

Assume that $O \cong \PG(2, q)$. As $\psi$ is Baer collineation of $\Pi$ and $\psi$ induces a homology on $O$, then $C_G(\psi)$ acts on $\Fix(\psi)$ as $\PSL(2, q)$ and it also fixes an antiflag. Note that $q^3 < n < q^4$ by [17, Proposition 11], and since $n \neq q^4$ by our assumption. Then, by Theorem 2.1 (3a), either $\Fix(\psi)$ has order 16 and $C_G(\psi)/\langle \psi \rangle \cong \PSL(2, 5)$, or $\Fix(\psi)$ is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C_G(\psi)/\langle \psi \rangle \cong \PSL(2, 7)$. However, the same argument as in Lemma 3.6 rules out both these cases, since $C_G(\psi)/\langle \psi \rangle$ fixes an antiflag. This completes the proof. □

**Lemma 3.8.** The groups $\overline{C_G(\psi)}$ and $\overline{C_G(\psi)}$ are strongly irreducible on $\Fix(\psi)$. 

Finite projective planes of order up to $q^4$ admitting $\PSL(3, q)$
Proof. In order to prove the assertion, by Lemma 3.6, we need to analyze only the case \((q, n) = (5, 4)\) and \(q = 9\) when \(9^2 < n < 9^4\). Recall that \(G \cong \text{PSL}(3, q)\) is irreducible on II by Lemma 3.3. Then II consists of nontrivial G-orbits. Since each \(G\)-orbits have length \(\lambda_j d_j(G)\), where \(\lambda_j \geq 0\) and \(d_j(G)\) is the degree of some primitive permutation representation of \(G\), then

\[
n^2 + n + 1 = \sum_{j \geq 0} \lambda_j d_j(G).
\]

That is, \(n^2 + n + 1\) must admit a partition restricted to

\[D(G) = [d_0(G), d_1(G), \ldots, d_k(G)],\]

the spectrum of the degrees of the primitive permutation representations of \(G\). So, the case \((q, n) = (5, 4)\) is ruled out, since \(n^2 + n + 1 = 21\), while \(D(G) = [31, 3100, 3875, 4000]\) by [2].

Assume that \(q = 9\) and \(9^2 < n < 9^4\). As above, by [2], \(n^2 + n + 1\) must admit a partition restricted to

\[D(G) = [91, 7020, 7560, 58968, 110565, 155520].\]

Note that \(9 \mid d_j(G)\) for each \(j > 0\). If \(\lambda_0 = 0\), then \(9 \mid n^2 + n + 1\) by (4), while it is known that either \(n^2 + n + 1 \equiv 1 \mod 3\) or \(n^2 + n + 1 \equiv 3 \mod 9\). Hence, \(\lambda_0 > 0\). So, there exists a point \(X \in \Pi\) such that \(G_X \leq U^* : C_G(\psi)\), where \(C_G(\psi) \cong \text{GL}(2, 9)\), by [2]. Since the group \(G\) does not admit 2-transitive point-orbits on II for \(n < 9^4\) by Lemma 3.7, then \(G_X < U^* : C_G(\psi)\). Hence, by Lemma 3.5, either \(9 \mid |X^G|\) or \(\text{Fix}(U^* \langle \psi \rangle)\) is either a flag, or an antiflag or a proper subplane of \(\text{Fix}(\psi)\). Furthermore, \(\text{C}_G(\psi)\) leaves \(\text{Fix}(U^* \langle \psi \rangle)\) invariant.

Assume that the latter occurs. If \(\text{Fix}(U^* \langle \psi \rangle)\) consists of a flag or an antiflag, again by Theorem 2.1, the case (3) inside the proof of Lemma 3.6 occurs, which leads to a contradiction, as we have seen. So, \(\text{Fix}(U^* \langle \psi \rangle)\) is a proper subplane of \(\text{Fix}(\psi)\). Then \(\text{Fix}(U^* \langle \psi \rangle) \cong \text{PG}(2, 4)\) by Theorem 2.1. Note that either \(\text{Fix}(U^* \langle \psi \rangle) \cong \text{PG}(2, 4)\) is a Baer subplane of \(\text{Fix}(U^*)\) or \(\text{Fix}(U^* \langle \psi \rangle) = \text{Fix}(U^* \langle \psi \rangle) \cong \text{PG}(2, 4)\). Suppose that \(\text{Fix}(U^* \langle \psi \rangle) \cong \text{PG}(2, 4)\) is a Baer subplane of \(\text{Fix}(U^*)\). Note that \(Z_\psi = \langle 1 \rangle\) by [2]. Consequently, \(Z_\psi\) fixes \(\text{Fix}(\psi)\), \(\text{C}_G(\psi) \cong \text{PGL}(2, 9)\). Hence, \(\text{Fix}(\langle \psi \rangle) = \text{Fix}(Z_\psi)\). Thus, \(\text{Fix}(U^* \langle \psi \rangle) = \text{Fix}(U^* \langle \psi \rangle) = \text{Fix}(U^*) \cong \text{PG}(2, 4)\). As \(Z_\psi\) normalizes \(U^*\), that is, \(\text{Fix}(U^* \langle Z_\psi \rangle) \cong \text{PG}(2, 4)\) is a Baer subplane of \(\text{Fix}(U^*)\). Then \(Z_\psi\) is semiregular on \(s \cap (\text{Fix}(U^*) - \text{Fix}(U^* \langle Z_\psi \rangle))\), where \(s\) is a secant of \(\text{Fix}(U^* \langle Z_\psi \rangle)\). So \(8 \mid 16 - 4\), since \(o(\text{Fix}(U^*)) = 16\), \(\text{Fix}(U^* \langle \psi \rangle) \cong \text{PG}(2, 4)\) and since \(|Z_\psi| = \frac{q^2 - 1}{q - 1} = 8\); this is a contradiction. Thus, \(\text{Fix}(U^* \langle \psi \rangle) = \text{Fix}(U^*) \cong \text{PG}(2, 4)\). If there exists a nontrivial element \(\rho\) in \(U^*\) fixing a point in \(\Pi - \text{Fix}(U^*)\), then \(\text{Fix}(\rho)\) is a Baer subplane of \(\Pi\), since \(\text{Fix}(U^*) \cong \text{PG}(2, 4)\) and \(n = 16^2\). Then each non trivial element in
$U^*$ fixes a subplane of order 16 of II, since the nontrivial elements in $U^*$ are conjugate under $C_G(\psi) \cong \text{GL}(2, 9)$. Hence, if $Q$ is a point fixed by $U^*$, then $9^2 \mid (9^2 - 1)(\sqrt{n} + 1) + (n + 1)$ by Cauchy-Frobenius Lemma, since $|U^*| = 9^2$. So, $9^2 \mid n - \sqrt{n}$, which is a contradiction, since $n = 16^2$. Therefore, $U^*$ is semiregular on $r - \text{Fix}(U^*)$, where $r$ is a secant to $U^*$. Hence, $9^2 \mid n - 4$ and we again obtain a contradiction, as $n = 16^2$. Thus, $9 \mid |X^G|$. Actually the previous argument can be repeated for each point $Y \in \Pi$ such that $G_Y$ lies in a maximal parabolic subgroup of $G$. Consequently, any orbit divisible by $d_0(G)$ is actually divisible by $9d_0(G)$. Therefore, bearing in mind that $9 \mid d_j(G)$ for each $j > 0$, any admissible $G$-orbit has length divisible by 9. So, $9 \mid n^2 + n + 1$ by (4), and we obtain a contradiction as above. This completes the proof. □

**Lemma 3.9.** The group $C_G(\psi)$ contains Baer involutions of $\text{Fix}(\psi)$. In particular, $\sqrt{n}$ is an integer.

**Proof.** Assume that all the involutions in $C_G(\psi)$ are perspectivities of $\text{Fix}(\psi)$. If $\sqrt{n}$ is even, then either $\text{Fix}(\psi) \cong \text{PG}(2, 2)$ and $C_G(\psi)/\langle \psi \rangle \cong \text{PSL}(2, 7)$ or $\text{Fix}(\psi) \cong \text{PG}(2, 4)$ and $C_G(\psi)/\langle \psi \rangle \cong \text{PSL}(2, 9)$ by [9]. However, both these cases cannot occur by the same argument as in Lemma 3.6. Hence, $\sqrt{n}$ is odd and the involutions in $C_G(\psi)$ are homologies of $\text{Fix}(\psi)$.

If $K = Z_\psi$, then $C_G(\psi) \cong \text{PGL}(2, q)$. Then $q \mid \sqrt{n}$ and $q - 1 \mid \sqrt{n} - 1$ by [12, Theorem C.ii]. As $q \mid \sqrt{n}$, then $\sqrt{n} = \lambda_1 q$ for some $\lambda_1 \geq 0$. Furthermore, $\lambda_1 = (q - 1)\lambda_2 + 1$ for some $\lambda_2 \geq 0$, since $q - 1 \mid \sqrt{n} - 1$. Hence, $\sqrt{n} = q(q - 1)\lambda_2 + q$. However, this is impossible, since $n < q^4$ by our assumption.

If $K < Z_\psi$. Then $\tilde{Z}_\psi \neq \langle 1 \rangle$. Since $C_G(\psi)$ is strongly irreducible on $\text{Fix}(\psi)$ by Lemma 3.8, and since each nontrivial subgroup of $Z_\psi$ is normal in $C_G(\psi)$, then $\tilde{Z}_\psi$ is semiregular on $\text{Fix}(\psi)$. Let $\sigma$ be any involutory $(C_\sigma, a_\sigma)$-homology of $C_G(\psi)$. Note that $C_G(\psi) \times Z_\psi \lhd C_G(\psi)$. That is, $\tilde{Z}_\psi$ centralizes $\sigma$ and hence $\tilde{Z}_\psi$ fixes $(C_\sigma, a_\sigma)$. This is impossible, since $\tilde{Z}_\psi$ is semiregular on $\text{Fix}(\psi)$. Therefore, $C_G(\psi)$ contains Baer collineation of $\text{Fix}(\psi)$ and hence $\sqrt{n}$ is an integer. □

**Proposition 3.10.** For each $X \in \Pi$ such that $G_X$ lies in a maximal parabolic subgroup of $G$, then $q^2 \mid |X^G|$. □

**Proof.** Since $C_G(\psi)$ is strongly irreducible on $\text{Fix}(\psi)$ by Lemma 3.8 and since the group $G$ does not admit 2-transitive point-orbits on II by Lemma 3.7, the assertion follows by Lemma 3.5.

**Lemma 3.11.** One of the following occurs:

(1) $q^2 \mid n^2 + n + 1$;
(2) $q$ is a square, $q\sqrt{q} | n^2 + n + 1$, and there exists a point $Y \in II$ such that either $G_Y \leq \text{PSL}(3, \sqrt{q})$ or $G_Y \leq \text{PSU}(3, \sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$.

Proof. Since $G$ is irreducible on II, then II consists of nontrivial $G$-orbits. By a direct inspection of the list of maximal subgroups of $\text{PSL}(3, q)$ given in [16], we have that $q^2 | |X^G|$ for each point $X \in II$, unless $q$ is a square and there exists a point $Y \in II$ such that either $G_Y \leq \text{PSL}(3, \sqrt{q})$, or $G_Y \leq \text{PSU}(3, \sqrt{q})$, with $(|G_Y|, q\sqrt{q}) > q$, or $G_Y \leq E_{q^2} : C_{G}(\gamma)$ for some involution $\gamma$ of $G$. Actually, if $G_Y \leq E_{q^2} : C_{G}(\gamma)$, then $q^2 | |Y^G|$ by Proposition 3.10.

If either there are no $Z$ in II such that $G_Z \leq \text{PSL}(3, \sqrt{q})$ or $G_Z \leq \text{PSU}(3, \sqrt{q})$ and $(|G_Z|, q\sqrt{q}) > q$, each admissible $G$-orbit on II is divisible by $q^2$. Therefore, $q^2 | n^2 + n + 1$, since II consists of nontrivial $G$-orbits. That is, the assertion (1).

If $q$ is square and there exists a point $Y \in II$ such that either $G_Y \leq \text{PSL}(3, \sqrt{q})$ or $G_Y \leq \text{PSU}(3, \sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$, each $G$-orbits is divisible by $q\sqrt{q}$ and hence $q\sqrt{q} | n^2 + n + 1$ by the above argument. That is, the assertion (2). □

Corollary 3.12. $p \neq 3$.

Proof. Assume that $p = 3$. As $q > 3$, then $9 \mid q$. Hence, $9 \mid n^2 + n + 1$ by Lemma 3.11. However, this is impossible, since it is known that either $n^2 + n + 1 \equiv 1 \mod 3$ or $n^2 + n + 1 \equiv 3 \mod 9$. □

Lemma 3.13. Let $S_0$ be the $p$-group and let $\psi$ and $\beta$ be the involutions defined in Section 2. If $q$ is a square and $q\sqrt{q} | n^2 + n + 1$, then one of the following occurs:

(1) The group $S_0$ is semiregular on II and hence on $\text{Fix}(\psi)$;

(2) $\text{Fix}(S_0)$ is a subplane of II. Furthermore, either $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$ or $\text{Fix}(S_0)$ is a proper subplane of $\text{Fix}(\psi)$;

(3) There exists a nontrivial proper subgroup $S^*$ of $S_0$ such that $\text{Fix}(S^*)$ is a subplane of II of order $m$ and one of the following occurs:

(a) $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$ and the involution $\beta$ induces a Baer collineation on it. In particular, $\sqrt{m}$ is an integer.

(b) $\text{Fix}(S^*)$ is a proper subplane of $\text{Fix}(\psi)$ and hence $m \leq \sqrt{n}$.

Furthermore, in the cases (3a)–(3b), the group $S_0/S^*$ acts on $\text{Fix}(S^*)$ and on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ semiregularly.

Proof. Let $S_0$ be the $p$-group defined in Section 2. Recall that $p \neq 3$ by Corollary 3.12. Also, recall that $\psi$ centralizes $S_0$, that $\beta$ inverts $S_0$ and that $(\psi, \beta) \cong$
Assume that $q\sqrt{n} | n + \sqrt{n} + 1$. Thus $\text{Fix}(S_0) \cap \text{Fix}(\psi) \neq \emptyset$. In particular, $\text{Fix}(S_0) \neq \emptyset$. As $p \neq 3$ and that $q\sqrt{n} | n^2 + n + 1$, we have that $(q,n) = (q,n \pm 1) = 1$. Therefore, $\text{Fix}(S_0)$ is a subplane of II. As $\psi$ centralizes $S_0$, then $\psi$ acts on $\text{Fix}(S_0)$. Hence, $\text{Fix}(S_0) \cap \text{Fix}(\psi) \neq \emptyset$. Actually, $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a subplane of $\text{Fix}(\psi)$, again since $\psi$ centralizes $S_0$, $q\sqrt{n} | n + \sqrt{n} + 1$ and $p \neq 3$. Moreover, either $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$ or $\text{Fix}(S_0) \subseteq \text{Fix}(\psi)$. Assume that $\text{Fix}(S_0) = \text{Fix}(\psi)$. Then $S_0$ is semiregular on $s - \text{Fix}(S_0)$, where $s$ is a secant of $\text{Fix}(S_0)$, since $\text{Fix}(S_0)$ is a Baer subplane of II. Therefore, $q | n - \sqrt{n}$, since $|S_0| = q$. That is, $q | \sqrt{n}(\sqrt{n} - 1)$. So, we obtain a contradiction, since $(\sqrt{n}(\sqrt{n} - 1), n - \sqrt{n} - 1) = 1$. Thus, either $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$ or $\text{Fix}(S_0)$ is a proper subplane of $\text{Fix}(\psi)$, and we obtain the assertion (2).

Assume that $q\sqrt{n} | n + \sqrt{n} + 1$. If $\text{Fix}(S_0) \neq \emptyset$, we still obtain the assertion (2) by the previous argument, by bearing in mind that $(\sqrt{n}(\sqrt{n} - 1), n + \sqrt{n} + 1) = 3$ and that $q > 3$. Hence, assume that $\text{Fix}(S_0) = \emptyset$. At this point, either $S_0$ is semiregular on II and we obtain the assertion (1), or there exists a nontrivial subgroup $S_1$ of $S_0$ such that $\text{Fix}(S_1) \neq \emptyset$. By bearing in mind that $\psi$ centralizes $S_0$ and hence $S_1$, that $(\sqrt{n}(\sqrt{n} - 1), n + \sqrt{n} + 1) = 3$ and that $p \neq 3$ by Corollary 3.12, the previous argument, with $S_1$ in the role of $S_0$, yields that either $\text{Fix}(S_1) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_1)$ or $\text{Fix}(S_1)$ is a proper subplane of $\text{Fix}(\psi)$.

Let $S$ be the set of the nontrivial subgroups of $S_0$ fixing a subplane of II whose intersection with $\text{Fix}(\psi)$ is in turn a subplane of this one. Clearly, $S \neq \emptyset$, since $S_1 \in S$. Let $S^*$ be an element of $S$ of maximal order. Hence, $\text{Fix}(S^*)$ is a subplane of II and $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a subplane of $\text{Fix}(\psi)$. Moreover, either $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$ or $\text{Fix}(S^*)$ is a proper subplane of $\text{Fix}(\psi)$, again by the above argument with $S^*$ in the role of $S_1$. Let $m$ be the order of $\text{Fix}(S^*)$. If $\text{Fix}(S^*)$ is a proper subplane of $\text{Fix}(\psi)$, then $m \leq \sqrt{n}$ by [10, Theorem 3.7], and we obtain the assertion (3a). Hence, assume that $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$. Note that $S_0/S^*$ is nontrivial and acts semiregularly on $\text{Fix}(S^*)$ and on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$, since $\text{Fix}(S_0) = \emptyset$, the group $S^*$ is an element of $S$ of maximal order, the group $S_0$ is abelian and since $\psi$ centralizes $S_0$. Denote by $m_\psi$ the order of $\text{Fix}(S^*) \cap \text{Fix}(\psi)$. Then $m = m_\psi^2$ by [10, Theorem 3.7], since $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$. As $\beta$ inverts $S_0$ and as $\langle \psi, \beta \rangle \cong E_4$, then $\beta$ normalizes $S^* \langle \psi \rangle$ and acts on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$. Denote by $S_0^+ = S_0/S^*$. Hence, $S_0^+$ is nontrivial and acts semiregularly on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$, as we have seen above. Furthermore, $S_0^+ \langle \beta \rangle$ acts on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$. Assume that $\beta$ induces a perspectivity on $\text{Fix}(S^*) \cap$
Fix(\psi). Let \( \rho \in S_0^+ \), \( \rho \neq 1 \). Then \( \beta^\rho \) is also a perspectivity of \( \text{Fix}(S^*) \cap \text{Fix}(\psi) \), and \( \text{Fix}([\beta^\rho, \beta]) \cap \text{Fix}(\psi) \neq \emptyset \) by [6, Lemma 5.1]. This is a contradiction, since \([\beta^\rho, \beta] \in S_0^+\), the group \( S_0^+ \) is nontrivial and acts on \( \text{Fix}(S^*) \cap \text{Fix}(\psi) \) semiregularly. Therefore, \( \beta \) induces a Baer collineation on \( \text{Fix}(S^*) \cap \text{Fix}(\psi) \).

Then \( m_\psi \) is a square by [10, Theorem 3.7]. Consequently, \( \sqrt{m} \) is an integer, since we proved that \( m = m_\psi^2 \), and we obtain the assertion (3b). \( \square \)

## 4 The proof of Theorem 1.1

**Proposition 4.1.** The involutions in \( G \) are perspectivities of \( \Pi \).

**Proof.** We proceed with a series of steps to show that no one of the cases of Lemma 3.13 occurs, obtaining the assertion in this way.

### Step I: The case (1) of Lemma 3.13 does not occur.

Assume that \( S_0 \) is semiregular on \( \Pi \) and on \( \text{Fix}(\psi) \). So, \( q | n + \sqrt{n} + 1 \). Recall that either \( q^2 | n^2 + n + 1 \) or \( q \) is a square, \( q \sqrt{q} | n^2 + n + 1 \), and there exists a point \( Y \in \Pi \) such that either \( G_Y \leq \text{PSL}(3, \sqrt{q}) \) or \( G_Y \leq \text{PSU}(3, \sqrt{q}) \), where \( (|G_Y|, q \sqrt{q}) > q \), by Lemma 3.11. In particular, either \( q^2 | n + \sqrt{n} + 1 \) or \( q \sqrt{q} | n + \sqrt{n} + 1 \), respectively, since \( n^2 + n + 1 = (n + \sqrt{n} + 1)(n - \sqrt{n} + 1) \), \( (n + \sqrt{n} + 1, n - \sqrt{n} + 1) = 1 \), and since \( q | n + \sqrt{n} + 1 \). If \( q^2 | n + \sqrt{n} + 1 \), then we obtain a contradiction by [13, Lemma 6.2], since \( \sqrt{n} \) is a square by Lemma 3.9. Thus, \( q \) is a square, \( q \sqrt{q} | n + \sqrt{n} + 1 \), and there exists a point \( Y \in \Pi \) such that either \( G_Y \leq \text{PSL}(3, \sqrt{q}) \) or \( G_Y \leq \text{PSU}(3, \sqrt{q}) \), where \( (|G_Y|, q \sqrt{q}) > q \).

Assume that there exists a point \( Y \in \Pi \), such that either \( G_Y \leq M \), where \( M \) is either \( \text{PSL}(3, \sqrt{q}) \) or \( \text{PSU}(3, \sqrt{q}) \), and such that \( (|G_Y|, q \sqrt{q}) > q \). Without loss of generality, we may assume that a Sylow \( p \)-subgroup of \( G_Y \) is contained in \( U \), the group defined in Section 2. Set \( U_Y = G_Y \cap U \) and \( U(M) = M \cap U \). Clearly, \( U_Y \leq U(M) \), with \( (|U_Y|, q \sqrt{q}) > q \) and \( |U(M)| = q \sqrt{q} \). In particular, \( U(M) \) consists of matrices of type (1) given in Section 2 whose entries are all the elements of \( \text{GF}(\sqrt{q}) \), while \( U_Y \) consists of some of these matrices. Let \( W \) be the subgroup of \( S_0 \), represented by the matrices type (2) given in Section 2, with \( y_2 = y_3 = 0 \) and with \( y_1 \in \text{GF}(\sqrt{q}) \). Hence, \( |W| = \sqrt{q} \) and \( W \leq U(M) \). Therefore, \( \langle U_Y, W \rangle \leq U(M) \), as \( U_Y \leq U(M) \). Hence, \( |\langle U_Y, W \rangle| \leq q \sqrt{q} \). On the other hand, \( |\langle U_Y, W \rangle| \geq |U_Y||W| \). Thus, \( |\langle U_Y, W \rangle| \leq q \sqrt{q} \). Therefore, \( p | |U_Y \cap W| \) since \( (|U_Y|, q \sqrt{q}) > q \) and \( |W| = \sqrt{q} \). So, \( p | |U_Y \cap S_0| \), since \( W \leq S_0 \). Hence, we arrive at a contradiction, since \( S_0 \) is semiregular on \( \Pi \).
Step II: The case (2) of Lemma 3.13 does not occur.

Recall that $S$, $S_0$ and $\psi$ are defined as in Section 2. Hence, $Z(S) = S' = S_0$. Furthermore, $\psi$ normalizes $S$ and $S_0$. Assume that the case (2) of Lemma 3.13 occurs. Hence, $\text{Fix}(S_0)$ is a subplane of II. Moreover, either $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$ or $\text{Fix}(S_0)$ is a proper subplane of $\text{Fix}(\psi)$. Clearly, $S$ acts on $\text{Fix}(S_0)$ (the action is unfaithful). Assume $S_0 < S_Q$ for some point $Q \in \text{Fix}(S_0)$. Then $S_Q$ lies in $G_Q$ which, in turn, lies in a maximal parabolic subgroup of $G$ by a direct inspection of the list of maximal subgroups of $G \cong \text{PSL}(3, q)$, $q$ odd, given in [16]. Then $q^2 \mid |Q^G|$ by Proposition 3.10. However, this is impossible, since $S_0 < S_Q \leq S$, while $|S_0| = q$ and $|S| = q^3$. Hence $S$ induces the group $S/S_0$ on $\text{Fix}(S_0)$ acting semiregularly. Thus $q^2 \mid h^2 + h + 1$, where $h$ is the order of $\text{Fix}(S_0)$.

Assume that $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$. Then $h$ is a square. Moreover $h \leq \sqrt{m}$ by [10, Theorem 3.7]. Hence $h^2 + h + 1 \leq q^4 + q + 1$, since $n \leq q^4$ by our assumption. However, this yields a contradiction, by [13, Lemma 6.2], since $q^2 \mid h^2 + h + 1$ and $h$ is a square.

Assume that $\text{Fix}(S_0)$ is a proper subplane of $\text{Fix}(\psi)$. Then $h \leq \sqrt{m}$ and hence $h^2 + h + 1 \leq q^2 + q + 1$. Thus, $q^2 = h^2 + h + 1$, since $q^2 \mid h^2 + h + 1$, and we still obtain a contradiction by [13, Lemma 6.2].

Step III: The final contradiction.

By (I) and (II) it follows that only case (3) of Lemma 3.13 might occur. Hence, assume there exists a nontrivial proper subgroup $S^*$ of $S_0$ such that $S_0/S^*$ acts on $\text{Fix}(S^*)$ and on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ semiregularly. In particular, if $m$ is the order of $\text{Fix}(S^*)$, then $\sqrt{m}$ is an integer. Since $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$ an since $S_0/S^*$ is nontrivial and acts on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ semiregularly, then $p \mid m + \sqrt{m} + 1$. Furthermore, as $S^* < S_Q$, the group $S$ centralizes $S^*$ and acts on $\text{Fix}(S^*)$. From the proof of Lemma 3.11, we actually obtain that any $G$-orbit has length divisible by either $q^2$ or, when $q$ is a square, by $q\sqrt{q}$. This implies that each orbit under the group induced by $S$ on $\text{Fix}(S^*)$ has length divisible by either $q^2$ or, when $q$ is a square, by $q\sqrt{q}$. So, we obtain that either $q^2 \mid m^2 + m + 1$, or $q\sqrt{q} \mid m^2 + m + 1$ when $q$ is a square. Actually, either $q^2 \mid m + \sqrt{m} + 1$ or $q\sqrt{q} \mid m + \sqrt{m} + 1$, respectively, since $m^2 + m + 1 = (m + \sqrt{m} + 1)(m - \sqrt{m} + 1)$, $(m + \sqrt{m} + 1, m - \sqrt{m} + 1) = 1$, and since $p \mid m + \sqrt{m} + 1$.

Assume that $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a proper subplane of $\text{Fix}(\psi)$. Then $\sqrt{m} \leq \sqrt{q}$ by [10, Theorem 3.7]. So, $m + \sqrt{m} + 1 \leq q^2 + q + 1$. This fact, in conjunction with either $q^2 \mid m + \sqrt{m} + 1$ or $q\sqrt{q} \mid m + \sqrt{m} + 1$, yields that $q\sqrt{q} = m + \sqrt{m} + 1$. 

with \( q = 7 \) and \( \sqrt{m} = 18 \) by [13, Lemma 6.2]. However, this is a contradiction, since \( \sqrt{m} \) is a square.

Since none of the cases of Lemma 3.13 occurs, then \( \psi \) cannot be a Baer collineation of \( \Pi \). Therefore, any involution of \( G \) is a perspectivity of \( \Pi \), since \( G \cong \text{PSL}(3, q) \) contains a unique conjugate class of involutions. \( \square \)

Now, using Proposition 4.1, we prove our main result.

**Proof of Theorem 1.1.** The assertion follows by Proposition 3.2 for \( q = 3 \). Hence, assume that \( q > 3 \). Since \( G \cong \text{PSL}(3, q) \) is irreducible on \( \Pi \) by Lemma 3.3 and since each involution in \( G \) is a perspectivity of \( \Pi \) by Proposition 4.1, then \( G \) leaves invariant a subplane \( \Pi_0 \) on which it acts strongly irreducibly by [6, Lemmas 5.2 and 5.3]. Then \( \Pi_0 \cong \text{PG}(2, q) \) by [8, Theorem 1.1]. If \( n \leq q^3 \), the assertion follows from Theorem 2.3. Hence, assume that \( q^3 < n \leq q^4 \).

As the involutions in \( G \) are homologies of \( \Pi_0 \), they are also homologies of \( \Pi \). Furthermore, each \( p \)-element inducing an elation on \( \Pi_0 \) is also an elation of \( \Pi \) by [10, Theorem 4.25]. Finally, by [12, Theorem C.ii], we have that \( q^2 \mid n \), that \( q - 1 \mid n - 1 \) and that \( q + 1 \mid n^2 - 1 \). It is a straightforward computation to show that this numerical information yield that \( \Pi \) has order \( n = \lambda q^3 + (1 - \lambda)q^2 \), where \( 1 < \lambda \leq q + 1 \) and \( q + 1 \mid \lambda(\lambda - 1) \), since \( q^3 < n \leq q^4 \). This completes the proof. \( \square \)

**Remark 4.2.** It seems to be tough proving that there are no planes of order \( q^3 < n < q^4 \) admitting \( G \cong \text{PSL}(3, q) \) as a collineation group. Indeed, although it is easy to show that a nontrivial stabilizer of a point has order odd and coprime to \( p \), it is difficult to determine the exact orbital decomposition of the set of external lines to \( \Pi_0 \), especially when the stabilizer of a line of such a set is a subgroup of a Singer cycle of \( G \).

**References**


Finite projective planes of order up to $q^4$ admitting $\text{PSL}(3, q)$


Mauro Biliotti
Dipartimento di Matematica, Università del Salento, Via per Arnesano, 73100 Lecce, Italy
email: mauro.biliotti@unile.it

Alessandro Montinaro
Dipartimento di Matematica, Università del Salento, Via per Arnesano, 73100 Lecce, Italy
e-mail: alessandro.montinaro@unile.it