Innovations in Incidence Geometry
Volume 6-7 (2007-2008), Pages 73–94
ISSN 1781-6475

On the finite projective planes of order up to q^4, q odd, admitting $\mathsf{PSL}(3,q)$ as a collineation group

Mauro Biliotti Alessandro Montinaro

Dedicated to Prof. Gábor Korchmáros on occasion of his 60^th birthday

Abstract

In this paper, it is shown that any projective plane Π of order $n \leq q^4$, q odd, that admits a group $G \cong \mathsf{PSL}(3,q)$ as a collineation group contains a G-invariant Desarguesian subplane of order q. Moreover, the involutions and suitable p-elements in G are homologies and elations of Π, respectively. In particular, if $n \leq q^3$, actually, $n = q$, q^2 or q^3.

Keywords: projective plane, collineation group, orbit

MSC 2000: 51E15, 20B25

1 Introduction and result

The problem of determining a projective plane Π of order n admitting $G \cong \mathsf{PSL}(3,q)$ as a collineation group has been largely investigated in the last decades. The first significant result related to this problem is the celebrated theorem of Ostrom and Wagner [21], dating back to 1959, which asserts that the projective plane Π is Desarguesian when $n = q$. In 1976, Lüneburg [15] proves that either Π is a Desarguesian plane or a Generalized Hughes plane when $n = q^2$. In 1985, Dempwolff [4] proves that any projective plane Π of order $n = q^3$ that admits $G \cong \mathsf{PSL}(3,q)$ as a collineation group contains a Desarguesian subplane Π_0 of order q on which G acts faithfully in its natural permutation representation. Despite the fact that Dempwolff provides a complete description of the G-orbits on the points and on the lines of Π, he emphasizes the difficulty in obtaining a characterization of Π. In 1989, Moorhouse obtains for projective planes of order $n = q^4$, q odd, the analogue of Dempwolff's result. Recently, Montinaro investigated the projective planes of order $n \leq q^3$ admitting a group inducing a 2-transitive group (namely, $\mathsf{PSL}(3, q)$) on a subplane of Π, showing that $n = q$, q^2 or q^3 and the results of Ostrom and Wagner, Lüneburg, Dempwolff, occur, respectively. This paper represents a further contribution to the study of the projective planes of order $n \leq q^4$, q odd, that admit $\mathsf{PSL}(3, q)$ as a collineation group. In particular, it represents a conclusive result when the plane has order $n \leq q^3$.

Theorem 1.1. *Let Π be a finite projective plane of order n that admits $G \cong \mathsf{PSL}(3, q)$, q odd, as a collineation group. If $n \leq q^4$, then the following occurs:*

(I) *There exists a subplane $\Pi_0 \cong \mathsf{PG}(2, q)$ of Π on which G acts in the natural way;*

(II) *The involutions in G are homologies of Π;*

(III) *The p-elements of G inducing elations on Π_0 are elations of Π.*

Moreover, one of the following occurs:

(i) $n = q$ *and* $\Pi = \Pi_0 \cong \mathsf{PG}(2, q)$*;*

(ii) $n = q^2$*,* Π *is a Desarguesian plane or a Generalized Hughes plane and* Π_0 *is a Baer subplane of* Π*;*

(iii) $n = q^3$*;*

(iv) $n = q^2(\lambda(q - 1) + 1)$*, where* $1 < \lambda \leq q + 1$ *and* $q + 1 \mid \lambda(\lambda - 1)$.

The cases (i) and (ii) clearly occur. The only known occurrences of the case (iii) are in the Desarguesian planes and in the Figueroa planes [5], [7]. The only known occurrences of the case (iv) are in the Desarguesian planes and in the Generalized Hughes planes when $\lambda = q + 1$, i.e. $n = q^4$.

The strategy of the proof is the following. Firstly, we prove that G is irreducible on Π. Hence, Π consists of nontrivial G-orbits. If ψ is a Baer collineation of Π, we determine the general structure of the action of the group induced by $C_G(\psi)$ on $\mathrm{Fix}(\psi)$ by Theorem 2.1. This forces any admissible G-orbit on the points of Π to be divisible by either q^2 or $q\sqrt{q}$ for q square. So, $n^2 + n + 1$, i.e. the number of points of Π, is divisible by either q^2 or $q\sqrt{q}$ for q square, as Π consists of nontrivial G-orbits. This yields a Diophantine equation involving $n^2 + n + 1$ and either q^2 or $q\sqrt{q}$ for q square. However, such an equation has no admissible solutions by [13, Lemma 6.2]. Therefore, the involutions in G are homologies of Π. At this point, the proof of our result easily follows.

2 Background

The notation used in this paper is standard. For what concerns finite groups, the reader is referred to [11] and to [3]. The necessary background about finite projective planes may be found in [10].

Now, we collect some information about the structure of the groups $\mathsf{PSL}(2, q)$ and $\mathsf{PSL}(3, q)$ and some results on the projective planes admitting one of these as a collineation group. Based on the results of Lüneburg [14], Yaqub [22] and Moorhouse [19], the following theorem, due to Montinaro, determines the general structure of the projective planes of order up to q^2 admitting $\mathsf{PSL}(2, q)$, $q > 3$, as a collineation group. Recall that a collineation group of a projective plane Π is said to be *irreducible on* Π if the group does not fix any point, line, triangle of Π. An irreducible collineation group of Π which does not fix any proper subplane of Π is said to be *strongly irreducible on* Π.

Theorem 2.1. *Let Π be a projective plane of order n admitting a collineation group $H \cong \mathsf{PSL}(2, q)$, $q > 3$. If $n \leq q^2$, then one of the following occurs:*

(1) *$n < q$ and one of the following occurs:*

  (a) *$n = 4$, $\Pi \cong \mathsf{PG}(2, 4)$ and $H \cong \mathsf{PSL}(2, 5)$;*

  (b) *$n = 2$ or 4, $\Pi \cong \mathsf{PG}(2, 2)$ or $\mathsf{PG}(2, 4)$, respectively, and $H \cong \mathsf{PSL}(2, 7)$;*

  (c) *$n = 4$, $\Pi \cong \mathsf{PG}(2, 4)$ and $H \cong \mathsf{PSL}(2, 9)$.*

(2) *$n = q$, $\Pi \cong \mathsf{PG}(2, q)$ and one of the following occurs:*

  (a) *H fixes a line or a point and q is even;*

  (b) *H is strongly irreducible and q is odd.*

(3) *$q < n < q^2$ and one of the following occurs:*

  (a) *H fixes a point or a line, and one of the following occurs:*

    (i) *$n = 16$ and $H \cong \mathsf{PSL}(2, 5)$;*

    (ii) *$n = 16$, Π is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and $H \cong \mathsf{PSL}(2, 7)$.*

  (b) *H fixes a subplane Π_0 of Π, q is odd and one of the following occurs:*

    (i) *$n = 16$, $\Pi_0 \cong \mathsf{PG}(2, 4)$ and $H \cong \mathsf{PSL}(2, 5)$;*

    (ii) *$\Pi_0 \cong \mathsf{PG}(2, 2)$ or $\mathsf{PG}(2, 4)$, and $H \cong \mathsf{PSL}(2, 7)$;*

    (iii) *$\Pi_0 \cong \mathsf{PG}(2, 4)$ and $G \cong \mathsf{PSL}(2, 9)$.*

(c) *H is strongly irreducible and q is odd.*

(4) *$n = q^2$ and one of the following occurs:*

(a) *H fixes a point or a line, and one of the following occurs:*

(i) *$n = 25$ and $H \cong \mathsf{PSL}(2,5)$;*
(ii) *$n = 81$ and $H \cong \mathsf{PSL}(2,9)$;*
(iii) *$n = q^2$, q even, and $G \cong \mathsf{PSL}(2,q)$.*

(b) *H fixes a subplane Π_0 of Π, q is odd and one of the following occurs:*

(i) *$n = q^2$, $\Pi_0 \cong \mathsf{PG}(2,q)$ and $H \cong \mathsf{PSL}(2,q)$;*
(ii) *$n = 25$, $\Pi_0 \cong \mathsf{PG}(2,4)$ and $H \cong \mathsf{PSL}(2,5)$;*
(iii) *$n = 81$, $\Pi_0 \cong \mathsf{PG}(2,4)$ and $H \cong \mathsf{PSL}(2,9)$;*
(iv) *$n = 81$, Π_0 is a Hughes plane of order 9 and $H \cong \mathsf{PSL}(2,9)$.*

(c) *H is strongly irreducible.*

Proof. See [18, Theorem 1]. □

As we shall see, such a theorem will play a central role in our investigation due to the fact that the centralizer of an involution involves a group isomorphic to $\mathsf{PSL}(2,q)$.

Now, we recall some basic facts about the structure of the group $G \cong \mathsf{PSL}(3,q)$ (the reader is referred to [16]).

1. Let ψ and β be the involutions in G represented by $\mathrm{diag}(1,-1,-1)$ and $\mathrm{diag}(-1,1,-1)$, respectively. Then $\langle \psi, \beta \rangle \cong E_4$.

2. Let U be the Sylow p-subgroup of G represented by all the matrices

$$\begin{bmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

where $x_1, x_2, x_3 \in \mathsf{GF}(q)$. Clearly, $|U| = q^3$. Let U_0 be the subgroup of U represented by the matrices in (1) having $x_1 = x_3 = 0$. Then U_0 has order q and $U_0 = Z(U) = U'$. Thus, U is a special p-group. Finally, let U^* be the subgroup of U represented by the matrices in (1) having $x_3 = 0$. Then U^* is elementary abelian of order q^2 which is normalized by ψ.

3. Let S be the Sylow p-subgroup of G represented by all the matrices

$$\begin{bmatrix} 1 & 0 & y_2 \\ y_3 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where $y_1, y_2, y_3 \in \mathsf{GF}(q)$, and let S_0 be the subgroup of S represented by those having $y_2 = y_3 = 0$. Then $S_0 = Z(S) = S'$. In particular, $U \cap S$ is an elementary abelian group of order q^2 containing S_0. Namely, $U \cap S$ consists of all the matrices in (2) having $y_3 = 0$.

4. The group $S_0.\langle\psi, \beta\rangle$ has order $4q$. In particular, ψ centralizes S_0, while β inverts S_0.

5. The group $C_G(\psi)$ consists of the matrices

$$\begin{bmatrix} e^{-1} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix},$$

where $a, b, c, d, e \in \mathsf{GF}(q)$, $e = ad - bc \neq 0$. Denote by Z_ψ the subgroup of $C_G(\psi)$ represented by all the matrices $\mathrm{diag}(d^{-2}, d, d)$, where $d \in \mathsf{GF}(q)^*$. Then $Z_\psi = Z(C_G(\psi))$. In particular, Z_ψ is a cyclic group of order $\frac{q-1}{\mu}$, where $\mu = (3, q-1)$ and $C_G(\psi) \cong Z_\psi.\mathsf{PGL}(2,q)$.

6. The group $U^* : C_G(\psi)$ is a maximal parabolic subgroup of G. Furthermore, $U^*\langle\psi\rangle \lhd U^* : C_G(\psi)$ and $C'_G(\psi) \cong \mathsf{SL}(2,q)$.

7. Let W^* be the subgroup of G represented by all the matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z_1 & z_2 & 1 \end{bmatrix},$$

where $z_1, z_2 \in \mathsf{GF}(q)$. Then W^* is an elementary abelian group of order q^2 which is normalized by $C_G(\psi)$. Moreover, the groups $U^* : C_G(\psi)$ and $W^* : C_G(\psi)$ are the representatives of the two distinct conjugate classes of maximal parabolic subgroups of G. The groups U^* and W^* are conjugate by the inverse-transpose automorphism.

We shall use the facts stated above without recalling them, unless it is explicitly required. In particular, since the Sylow p-subgroups of G are conjugate, we shall mainly refer either to U or to S. Furthermore, despite the fact that there are two distinct conjugate classes of maximal parabolic subgroups in G by (7), what we prove to be true for $U^* : C_G(\psi)$ can always be proven to be true for $W^* : C_G(\psi)$. Hence, for our purposes we may always refer to $U^* : C_G(\psi)$, without loss of generality.

Lemma 2.2. *The group* $G \cong \mathsf{PSL}(3,q)$ *contains two distinct involutions* ψ_1 *and* ψ_2 *such that* $C'_G(\psi_1) \cap C'_G(\psi_2) \neq \langle 1 \rangle$ *and* $\langle C'_G(\psi_1), C'_G(\psi_2) \rangle = G$.

Proof. See [20, Lemma 4.1.vi]. □

Some geometrical results involving the group $G \cong \mathsf{PSL}(3,q)$ are in order. By using the results of Ostrom-Wagner [21], Lüneburg [15] and Dempwolff [4], Montinaro proved the following.

Theorem 2.3. *Let* Π *be a finite projective plane of order* n *and let* G *be a collineation group of* Π *inducing a group containing* $\mathsf{PSL}(3,q)$ *on a subplane* Π_0 *of order* q. *If* $n \leq q^3$, *then one of the following occurs:*

(1) $\Pi_0 \cong \mathsf{PG}(2,q)$, $\mathsf{PSL}(3,q) \leq G$ *and one of the following occurs:*

  (a) $n = q$ *and* $\Pi = \Pi_0$ *;*

  (b) $n = q^2$, Π *is a Desarguesian plane or a Generalized Hughes plane and* Π_0 *is a Baer subplane of* Π *;*

  (c) $n = q^3$.

(2) $\Pi_0 \cong \mathsf{PG}(2,7)$, Π *is the generalized Hughes plane over the exceptional nearfield of order* 7^2 *and* $\mathsf{SL}(3,7) \leq G$.

Proof. See [17]. □

Finally, we quote this useful final result, due to Moorhouse [20], which inspired the present paper, and that allows to reduce our investigation to $n < q^4$ (when $q > 3$).

Theorem 2.4 (Moorhouse)**.** *Let* Π *be a projective plane of order* q^4 *admitting* $G \cong \mathsf{PSL}(3,q)$, q *odd. If* $q > 3$, *then the following must hold.*

(i) G *leaves invariant a Desarguesian subplane* Π_0 *of order* q, *on which* G *acts 2-transitively ;*

(ii) *The involutions in* G *are homologies of* Π, *and those* p*-elements of* G *which induce elations of* Π_0 *are elations of* Π.

If $q = 3$ *then the same two conclusions must hold, under the additional hypothesis that* G *acts irreducibly on* Π.

Proof. See [20, Theorem 1.3]. □

3 Preliminary reductions

The aim of this section is to show that G is irreducible on Π and that the involutions in G are perspectivities of Π, in order to apply Hering-Walker theory on the strong irreducibility (e.g. see [6], [8] and [9]).

In view of Theorem 2.1, we treat the cases $q = 3$ and $q > 3$ separately.

Lemma 3.1. *Let Π be a finite projective plane of order n that admits $G \cong$ PSL$(3,3)$ as a collineation group. If $n \leq 3^4$, then each involution in G is a perspectivity of Π.*

Proof. Assume that the involutions in G are Baer collineations of Π. Hence, $\sqrt{n} \leq 9$. Let J be a Sylow 2-subgroup of G. As $n^2 + n + 1$ is odd, then J fixes a secant s of $\mathrm{Fix}(\psi)$. Let $J_0 = J \cap C_G(\psi)$. Then $J_0 \cong Q_8$. Thus, J_0 is semiregular on $s - \mathrm{Fix}(\psi)$. So, $8 \mid \sqrt{n}(\sqrt{n} - 1)$, since $|s - \mathrm{Fix}(\psi)| = \sqrt{n}(\sqrt{n} - 1)$. Consequently, either $\sqrt{n} = 8$ or 9, as $\sqrt{n} \leq 9$. Note that $J = J_0.\langle\beta\rangle$ is known to be semidihedral of order 16. As J_0 is semiregular on $s - \mathrm{Fix}(\psi)$, then each J-orbit on $s - \mathrm{Fix}(\psi)$ has length either 8 or 16. Therefore, let x and y be the number of J-orbits on $s - \mathrm{Fix}(\psi)$ of length 8 and 16, respectively. It follows that

$$8x + 16y = \sqrt{n}(\sqrt{n} - 1), \tag{3}$$

where $\sqrt{n} = 8$ or 9. As J is semidihedral of order 16, then J contains two distinct conjugate classes of involutions, one consisting of ψ and the other consisting of the four conjugates of β (including β). Furthermore, $C_J(\beta) \cong \langle\psi,\beta\rangle \cong E_4$. Thus, by [19, Relation (8)], the involution β fixes 2 and 0 points on the J-orbits on $s - \mathrm{Fix}(\psi)$ of length 8 and 16, respectively, since $\psi \in J_0$ and since J_0 is semiregular on $s - \mathrm{Fix}(\psi)$. Hence, β fixes exactly $2x$ points on $s - \mathrm{Fix}(\psi)$. If x is even, then $16 \mid \sqrt{n}(\sqrt{n} - 1)$ by (3), which is impossible as $\sqrt{n} = 8$ or 9. Therefore, x is odd. Hence, β cannot induce either the identity or a perspectivity of axis s on $\mathrm{Fix}(\psi)$, otherwise $x = 0$, since β is a Baer collineation on Π (recall that $G \cong$ PSL$(3,3)$ has a unique conjugate class of involutions). Suppose that β induces a perspectivity on $\mathrm{Fix}(\psi)$ of axis distinct from s. Clearly, β induces on $\mathrm{Fix}(\psi)$ either an elation when $\sqrt{n} = 8$ or a homology when $\sqrt{n} = 9$. Then $x = \sqrt{n}$ or $\sqrt{n} - 1$, respectively, again since β is a Baer collineation on Π . So, x is even in any case, which is a contradiction. Finally, assume that β induces a Baer collineation on $\mathrm{Fix}(\psi)$ when $\sqrt{n} = 9$. Arguing as above, we have that $x = \sqrt{n} - \sqrt[4]{n}$ which is even and we again obtain a contradiction. Thus, the involutions in G are perspectivities of Π. □

Proposition 3.2. *Let Π be a finite projective plane of order n that admits $G \cong$ PSL$(3,3)$ as a collineation group. If $n \leq 3^4$, then the following occurs:*

(1) *There exists a subplane* $\Pi_0 \cong \mathsf{PG}(2,3)$ *of* Π *on which* G *acts in the natural way;*

(2) *The group* G *is irreducible on* Π*;*

(3) *The involutions in* G *are homologies of* Π*;*

(4) *The* 3*-elements that induce elations on* Π_0 *are elations of* Π.

Moreover, one of the following occurs:

(i) $n = 3$ *and* $\Pi = \Pi_0 \cong \mathsf{PG}(2,q)$*;*

(ii) $n = 3^2$, Π *is a Desarguesian plane or a Generalized Hughes plane and* Π_0 *is a Baer subplane of* Π*;*

(iii) $n = 3^3$*;*

(iv) $n = 3^4$.

Proof. Assume that $G \cong \mathsf{PSL}(3,3)$ fixes a line l of Π. As $n \leq 3^4$, then each nontrivial G-orbit on l has length divisible by 13 by [2]. Actually, G contains such orbits, since G acts faithfully, G being nonabelian simple. Let $X \in l$ such that $13 \mid \left|X^G\right|$. So, $G_X \leq E_9.\mathsf{GL}(2,3)$. Let $\mathcal{B}_X$ be the block of imprimitivity in X^G containing X. Clearly $\left|X^G\right| = 13\,|\mathcal{B}_X|$ ($|\mathcal{B}_X|$ might be 1). Furthermore, $E_9.\mathsf{GL}(2,3)$ acts transitively on $\mathcal{B}_X$. As the socle of $E_9.\mathsf{GL}(2,3)$ is E_9, then either $E_9 \trianglelefteq G_X$ or $13 \cdot 9 \mid \left|X^G\right|$ by [3, Theorem 4.1A]. Actually, the latter cannot occur, since $\left|X^G\right| \leq n+1$ and $n \leq 3^4$. Hence, each nontrivial G-orbit on l has length $\left|X^G\right| = 13\,|\mathcal{B}_X|$, where $|\mathcal{B}_X| \mid |\mathsf{GL}(2,3)|$. Actually, $|\mathcal{B}_X| = 1,2,3,4$ or 6, since $|\mathcal{B}_{X_i}| \leq 6$, as $n \leq 3^4$. Since the blocks of imprimitivity are 13, then there exists a point P, lying in a nontrivial G-orbit on l, such that J_0 fixes $\mathcal{B}_P$, where J_0 is the 2-group isomorphic to Q_8 containing the involution ψ. Thus, ψ fixes $\mathcal{B}_P$ pointwise, since $|\mathcal{B}_P| \leq 6$. Then $|\mathcal{B}_P| \leq 2$, since ψ is a perspectivity of Π having axis distinct from l. If $|\mathcal{B}_P| = 1$, then P^G is a 2-transitive G-orbit. Hence, ψ fixes exactly 5 points on X^G. So, we arrive at a contradiction, since ψ is a perspectivity of Π having axis distinct from l. Thus, $|\mathcal{B}_P| = 2$ and hence $l = P^G$, since ψ fixes $\mathcal{B}_P$ pointwise. In particular, $n = 25$, since $\left|P^G\right| = 26$. Since G acts faithfully on l, there are no involutory homologies of axis l. Therefore, no involutions lie in a triangular configuration. In particular, since ψ is the unique central involution in J (recall that J is semidihedral of order 16), each involution in J has center and axis C_ψ and a_ψ, where C_ψ and a_ψ denote the center and the axis of ψ, respectively. So, each involution in J fixes exactly two points on l, namely C_ψ and $a_\psi \cap l$. Hence, J is semiregular on $l - \{C_\psi, W\}$, where $\{W\} = a_\psi \cap l$. Then $16 \mid n-1$, which is a contradiction, since $n = 25$

while $|J| = 16$. Therefore, G does not fix lines. By the dual of the previous proof, we obtain that G does not fix points. Finally, these two facts, combined with the fact that G is nonabelian simple, yield that G does not fix triangles of Π. Thus, G is irreducible on Π and hence the assertion (2).

Since $G \cong \mathsf{PSL}(3,3)$ is irreducible on Π, and since each involution in G is a perspectivity by Lemma 3.1, then G leaves invariant a subplane Π_0 on which it acts strongly irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong \mathsf{PG}(2,3)$ by [8, Theorem 1.1], and we obtain the assertion (1). Therefore, the involutions in G are homologies of Π and hence the assertion (3). For $n \leq 3^3$, the assertions (4) and (i)–(iii) follow by Theorem 2.3. Furthermore, for $n = 3^4$ the assertions (4) and (iv) follow by Theorem 2.4, since we proved the irreducibility of G on Π. Hence, assume that $3^3 < n < 3^4$. Note that G contains an elementary abelian group H of order 3^2 consisting of elations with the same axis r and distinct centres lying in $\Pi_0 \cap r$ by [10, Theorem 4.25]. As H is semiregular on $[Q] - \{l\}$, for any $Q \in r - \Pi_0$, then $3^2 \mid n$. So, $3^3 < n < 3^4$, n odd, and $3^2 \mid n$ yield that $n = 3^2 5$ or $3^2 7$. Let $\mathcal{E}$ be the set of external lines to Π_0. Easy computations yield $|\mathcal{E}| = 1512$ or 3240, respectively. Let R be any Sylow 2-subgroup of G. Then $|R| = 16$. Since each involution in G, and hence in R, is a homology of axis a secant to Π_0, then R is semiregular on $\mathcal{E}$. So, $16 \mid |\mathcal{E}|$, which is impossible as $|\mathcal{E}| = 1512$ or 3240. This completes the proof. □

It should be pointed out that the previous theorem extends the Theorem 2.4 also for $n = 3^4$. Indeed, Theorem 2.4 works for $q = 3$ under the additional assumption that G is irreducible on Π. In particular, Moorhouse shows that the irreducibility of G on Π implies that the involutions in G are homologies of Π. We, instead, prove that the involutions are perspectivities of Π and then we use this fact to prove that G is irreducible on Π.

From now on, we assume that $q > 3$.

Lemma 3.3. *The group G is irreducible on Π.*

Proof. Assume that G fixes a line l of Π. Then $\sqrt{n} < q^2$, since for $n = q^4$ the assertion follows by [20] (e.g. see the proof of Theorem 1.3). Let ψ be the involution in G defined in Section 2. Then, by Theorem 2.1 and by bearing in mind that q is odd and $\sqrt{n} < q^2$, one of the following occurs:

(1) $\sqrt{n} = 4$, $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,4)$ and $C_G(\psi)'/\langle\psi\rangle \cong \mathsf{PSL}(2,5)$;

(2) $\sqrt{n} = 16$ and $C_G(\psi)'/\langle\psi\rangle \cong \mathsf{PSL}(2,5)$;

(3) $\sqrt{n} = 16$, $\mathrm{Fix}(\psi)$ is either the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C_G(\psi)'/\langle\psi\rangle \cong \mathsf{PSL}(2,7)$.

Assume that the case (1) occurs. Since $n+1 = 17$ and since these primitive permutation representations of G have a degree greater than 17 by [2], then G fixes l pointwise. That is, G is a group of perspectivities of axis l. So, G should be a Frobenius group by [10, Theorem 4.25], which is impossible as G is nonabelian simple.

We treat the cases (2)–(3) simultaneously. By a direct inspection of [2], it is plain that the unique nontrivial orbits on l under $G \cong \mathsf{PSL}(3,q)$, $q = 5$ or 7, are those of length a multiple of d_0, the minimal primitive permutation representation degree of G. By [2], such a d_0 is equal to 31 or 57, respectively. Let r be the minimal nonnegative integer such that $n+1 \equiv r_0 \mod d_0$. Easy computations yield that $r_0 = 9$, 29 or 6 in the cases (1)–(3), respectively. So, $6 \leq r_0 < n+1$ and $\sqrt{n}+1 \not\equiv r_0 \mod d_0$ in any case. Therefore, G fixes at least 6 points on l in any case. Let P be any of these points. Now, by repeating the above argument with $[P]$ in the role of l, we obtain that G fixes at least 6 lines of $[P]$ (clearly, the line l is included). Again, by repeating the above argument for any for each of these 6 lines, we obtain that G fixes a subplane Σ of Π pointwise. Let r be the order of Σ. Then $r = r_0 + hd_0 - 1$, where $h \geq 0$. Note that $r_0 + hd_0 - 1 \leq \sqrt{n}$ by [10, Theorem 3.7]. Hence, the case (3) is ruled out. Actually, $r_0 + hd_0 - 1 < \sqrt{n}$, since $\sqrt{n}+1 \not\equiv r_0 \mod d_0$. Thus, $\Sigma \subset \mathrm{Fix}(\psi)$, since $\Sigma \subseteq \mathrm{Fix}(\psi)$. Therefore, $(r_0 + hd_0 - 1)^2 \leq \sqrt{n}$ by [10, Theorem 3.7]. This forces $h = 0$ in any admissible case. In particular, the case (2) is ruled out. Consequently, G is irreducible on Π. □

Throughout this section, we assume that ψ is a Baer collineation of Π. *Then $n < q^4$ by Theorem 2.4, as $q > 3$.*

The following lemma determines the structure of the kernel K_ψ of the action of $C_G(\psi)$ on $\mathrm{Fix}(\psi)$.

Lemma 3.4. $\langle\psi\rangle \leq K_\psi \leq Z_\psi$.

Proof. Clearly, $\langle\psi\rangle \trianglelefteq K_\psi \trianglelefteq C_G(\psi)$. Recall that $C_G(\psi) \cong Z_\psi.\mathsf{PGL}(2,q)$. Since $K_\psi Z_\psi/Z_\psi \trianglelefteq \mathsf{PGL}(2,q)$, then either $K_\psi Z_\psi/Z_\psi = \langle 1\rangle$ or $\mathsf{PSL}(2,q) \leq \bar{K}\bar{Z}_\psi/\bar{Z}_\psi$. Assume that the latter occurs. Then $C_G'(\psi) \leq K_\psi$, since $C_G'(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2,q)$ and since $\langle\psi\rangle \trianglelefteq K_\psi \trianglelefteq C_G(\psi)$. Since for each involution $\beta \in G$ there exists $g \in G$ such that $\psi^g = \beta$, then $C_G'(\psi)^g = C_G'(\beta)$. Hence $C_G'(\beta)$ fixes $\mathrm{Fix}(\beta)$ pointwise for each involution β in G. By Lemma 2.2, there exist two involutions ψ_1 and ψ_2 such that $C_G'(\psi_1) \cap C_G'(\psi_2) \neq \langle 1\rangle$ and $\langle C_G'(\psi_1), C_G'(\psi_2)\rangle = G$. Since

$C'_G(\psi_i)$ fixes the Baer subplane $\mathrm{Fix}(\psi_i)$ pointwise for each $i = 1, 2$, and since $C'_G(\psi_1) \cap C'_G(\psi_2) \neq \langle 1 \rangle$, then $\mathrm{Fix}(\psi_1) = \mathrm{Fix}(\psi_2)$. Thus, $G = \langle C'_G(\psi_1), C'_G(\psi_2) \rangle$ fixes $\mathrm{Fix}(\psi_1)$ pointwise, which is impossible by Lemma 3.3. Consequently, $K_\psi Z_\psi / Z_\psi = \langle 1 \rangle$. That is, $K_\psi \leq Z_\psi$ and hence we obtain the assertion. □

For each subgroup X of $C_G(\psi)$, we denote by $\bar{X}$ the group XK_ψ / K_ψ.

Lemma 3.5. *For each point $X \in \Pi$ such that G_X lies in a maximal parabolic subgroup of G, one of the following occurs:*

(1) *X^G is a 2-transitive orbit;*

(2) *$\mathrm{Fix}(U^* \langle \psi \rangle)$ is either a flag, or an antiflag or a proper subplane of $\mathrm{Fix}(\psi)$. Furthermore, $\overline{C_G(\psi)}$ leaves $\mathrm{Fix}(U^* \langle \psi \rangle)$ invariant;*

(3) $q^2 \mid \left| X^G \right|$.

Proof. Let $X \in \Pi$ and assume that G_X lies in a maximal parabolic subgroup of G. As mentioned in Section 2, for our purposes we may reduce to study the case when $G_X \leq U^* : C_G(\psi)$, where $C_G(\psi) \cong Z_\psi.\mathrm{PGL}(2, q)$ and $Z_\psi \cong Z_{\frac{q-1}{\mu}}$, $\mu = (3, q-1)$. If $G_X = U^* : C_G(\psi)$, then X^G is a 2-transitive orbit and we obtain the assertion (1). If $G_X < U^* : C_G(\psi)$, denoted by $\mathcal{B}_X$ the block of imprimitivity in X^G containing X, we have $|\mathcal{B}_X| > 1$. Clearly, $U^* : C_G(\psi)$ acts on $\mathcal{B}_X$.

Assume that $U^* : C_G(\psi)$ does not act faithfully on $\mathcal{B}_X$, then U^* lies in the kernel of the action, since U^* is the socle of $U^* : C_G(\psi)$ by [3, Theorem 4.3B]. Thus, $\mathrm{Fix}(U^*) \neq \varnothing$. Since $U^* \lhd U^* : C_G(\psi)$, and since $\mathrm{Fix}(U^* : C_G(\psi)) = \varnothing$, being $G_X < U^* : C_G(\psi)$, either $\mathrm{Fix}(U^*) = \Delta$, where Δ is a triangle of Π, or $\mathrm{Fix}(U^*)$ is a subplane of Π by [6, Corollary 3.6]. This yields that $\mathrm{Fix}(U^* \langle \psi \rangle)$ consists of either a flag, or an antiflag or a plane. Clearly, $\mathrm{Fix}(U^* \langle \psi \rangle) \subseteq \mathrm{Fix}(\psi)$. Furthermore, $\overline{C_G(\psi)}$ acts on $\mathrm{Fix}(\psi)$ leaving $\mathrm{Fix}(U^* \langle \psi \rangle)$ invariant, since $U^* \langle \psi \rangle \lhd U^* : C'_G(\psi)$. If $\mathrm{Fix}(U^* \langle \psi \rangle) = \mathrm{Fix}(\psi)$, then $\mathrm{Fix}(U^*) = \mathrm{Fix}(\psi)$, since $\mathrm{Fix}(\psi)$ is a Baer subplane of Π. So, U^* is semiregular on $s - \mathrm{Fix}(U^*)$, where s is a secant of $\mathrm{Fix}(U^*)$. Therefore, $q^2 \mid n - \sqrt{n}$, since $|U^*| = q^2$. That is, either $q^2 \mid \sqrt{n} - 1$ or $q^2 \mid \sqrt{n}$, and we have a contradiction in any case since $n < q^4$ and $q > 3$. Thus, we obtain the assertion (2)

Assume that $U^* : C_G(\psi)$ acts faithfully on $\mathcal{B}_X$. Then $q^2 \mid |\mathcal{B}_X|$ by [3, Theorem 4.1A], since U^* is the socle of $U^* : C_G(\psi)$. Thus, $q^2 \mid \left| X^G \right|$ and we obtain the assertion (3). □

Lemma 3.6. *One of the following occurs:*

(I) *The groups $\overline{C'_G(\psi)}$ and $\overline{C_G(\psi)}$ are strongly irreducible on $\mathrm{Fix}(\psi)$;*

(II) $q = 5$ *and* $n = 4$*;*

(III) $q = 9$ *and* $9^2 < n < 9^4$.

Proof. Assume that the cases (II) and (III) do not occur. Note that $\overline{C'_G(\psi)} \cong \mathsf{PSL}(2,q)$, since $C'_G(\psi) \cap K_\psi = \langle\psi\rangle$ by Lemma 3.4. Suppose that the $\overline{C'_G(\psi)}$ is not strongly irreducible on $\mathrm{Fix}(\psi)$. The case $\sqrt{n} = q$ is ruled out by Theorem 2.1. As $\sqrt{n} < q^2$, then either $\sqrt{n} < q$ or $q < \sqrt{n} < q^2$. Then, again by Theorem 2.1 and bearing in mind that the cases (II) and (III) do not occur by our assumptions, one of the following occurs:

(1) $n = 4$ or 16, $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,2)$ or $\mathsf{PG}(2,4)$, respectively, and $\overline{C'_G(\psi)} \cong \mathsf{PSL}(2,7)$;

(2) $n = 16$, $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,4)$ and $\overline{C'_G(\psi)} \cong \mathsf{PSL}(2,9)$;

(3) $n = 16^2$, $\overline{C'_G(\psi)} \cong \mathsf{PSL}(2,5)$ fixes a subplane of $\mathrm{Fix}(\psi)$ isomorphic to $\mathsf{PG}(2,4)$;

(4) $7^2 < n < 49^2$, $\overline{C'_G(\psi)} \cong \mathsf{PSL}(2,7)$ fixes a subplane of $\mathrm{Fix}(\psi)$ isomorphic either to $\mathsf{PG}(2,2)$ or to $\mathsf{PG}(2,4)$.

Actually, in the cases (1)–(4), the group $\overline{C_G(\psi)} \cong \bar{Z}_\psi.\mathsf{PGL}(2,q)$ acts on $\mathrm{Fix}(\psi)$. The group $\overline{C_G(\psi)}$ fixes a subplane Π_0 of $\mathrm{Fix}(\psi)$ isomorphic either to $\mathsf{PG}(2,2)$ or to $\mathsf{PG}(2,4)$ for $q = 7$, or to $\mathsf{PG}(2,4)$ for $q \neq 7$ (note that it might be $\Pi_0 = \mathrm{Fix}(\psi)$).

Assume that $q = 7$. Then $\bar{Z}_\psi = \langle 1\rangle$, since $Z_\psi = \langle\psi\rangle$. Therefore $\overline{C_G(\psi)} = \mathsf{PGL}(2,7)$ acts on Π_0. Then the case $\Pi_0 \cong \mathsf{PG}(2,2)$ is ruled out, since the full automorphism group of $\mathsf{PG}(2,2)$ is isomorphic to $\mathsf{PSL}(2,7)$. Hence, assume that $\Pi_0 \cong \mathsf{PG}(2,4)$ and $\overline{C'_G(\psi)} \cong \mathsf{PSL}(2,7)$. It is easy to see that $\mathsf{PSL}(2,7)$ fixes a subplane Π_1 of Π_0 which is isomorphic to $\mathsf{PG}(2,2)$. In particular, $\mathsf{PGL}(2,7)$ leaves Π_1 invariant. So, we arrive at a contradiction by the above argument with Π_1 in the role of Π_0. Therefore, $q \neq 7$ and hence the cases (1) and (4) are ruled out.

Assume that $q = 5$ or 9. Then $\Pi_0 \cong \mathsf{PG}(2,4)$ and hence $\overline{C_G(\psi)} \leq \mathsf{P\Gamma L}(3,4)$. Furthermore, $\bar{Z}_\psi = \langle 1\rangle$ by [2]. Consequently, Z_ψ fixes $\mathrm{Fix}(\psi)$ pointwise and $\overline{C_G(\psi)} \cong \mathsf{PGL}(2,q)$ in any case. Since Z_ψ is semiregular $s - \mathrm{Fix}(\psi)$, then $\frac{q-1}{\mu} \mid n - \sqrt{n}$, where $\frac{q-1}{\mu} = |Z_\psi|$ and $\mu = (3, q-1)$. That is, $\frac{q-1}{\mu} \mid \sqrt{n}$ or $\frac{q-1}{\mu} \mid \sqrt{n} - 1$, since $q = 5$ or 9. Thus, the case (2) is ruled out, since $\sqrt{n} = 4$, while $\frac{q-1}{\mu} = 8$.

It remains to investigate the case (3). In this case, any subgroup Z_{31} of G fixes a subplane of Π of order $7 + 31k$, $k \geq 0$. Actually, $k = 0$ by [10, Theorem 3.7], since $n = 16^2$. Therefore, Z_{31} fixes exactly 57 points of Π. Note that $Z_{31} \leq$

$G_X \leq Z_{31}.Z_3$ for any point X of Π fixed by Z_{31} by [2]. Moreover, $Z_{31}.Z_3$ is maximal in G. So, either $G_X = Z_{31}$, $\left|X^G\right| = 12000$ and Z_{31} fixes 3 points on X^G, or $G_X = Z_{31}.Z_3$, $\left|X^G\right| = 4000$ and Z_{31} fixes 1 point on X^G. Let x and y be the number of G-orbits on Π of length 12000 and 4000, respectively. Then $12000x + 4000y \leq 65793$, since $n^2+n+1 = 65793$. Furthermore, $3x+y = 57$, since Z_{31} fixes exactly 57 points of Π. By combining the previous relations involving x and y, we obtain a contradiction. Thus, $\overline{C'_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$. Then $\overline{C_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$, since $\overline{C'_G(\psi)} \leq \overline{C_G(\psi)}$. That is, the assertion (I) occurs. □

Lemma 3.7. *The group G does not admit 2-transitive point-orbits on Π.*

Proof. Let $\mathcal{O}$ be a 2-transitive G-orbit on Π. Then $|\mathcal{O}| = q^2+q+1$. Clearly, $\mathcal{O}$ cannot be contained in a line by lemma 3.3. Then, it is a plain that, either $\mathcal{O}$ is an arc or $\mathcal{O} \cong$ PG$(2,q)$. Assume that the former occurs. Let U^* be the elementary abelian p-group defined in Section 2. Then $U^*\langle\psi\rangle$ fixes exactly $q+1$ points on $\mathcal{O}$. So $U^*\langle\psi\rangle$ is planar, since $\mathcal{O}$ is an arc. Since $U^*\langle\psi\rangle \lhd U^*C'_G(\psi)$ and since $C'_G(\psi)$ acts 2-transitively on $\mathrm{Fix}(U^*\langle\psi\rangle)\cap\mathcal{O}$, then $C'_G(\psi)$ acts as PSL$(2,q)$ on $\mathrm{Fix}(U^*\langle\psi\rangle)$. Note that

$$|\mathrm{Fix}(U^*\langle\psi\rangle)\cap\mathcal{O}| = q+1 \text{ and } |\mathrm{Fix}(\langle\psi\rangle)\cap\mathcal{O}| = q+2\,,$$

as q is odd. Thus $\mathrm{Fix}(U^*\langle\psi\rangle) \subsetneq \mathrm{Fix}(\psi) \subsetneq \Pi$, with $o(\mathrm{Fix}(U^*\langle\psi\rangle)) \geq q-1$. Assume that $o(\mathrm{Fix}(U^*\langle\psi\rangle)) = q$. Then $\sqrt{n} \geq q^2$ by [10, Theorem 3.7], since $\mathrm{Fix}(U^*\langle\psi\rangle) \subset \mathrm{Fix}(\psi)$, which is contrary to the assumption $\sqrt{n} < q^2$. So, $o(\mathrm{Fix}(U^*\langle\psi\rangle)) = q-1$. Then $\mathrm{Fix}(U^*\langle\psi\rangle)\cap\mathcal{O}$ is a hyperoval of $\mathrm{Fix}(U^*\langle\psi\rangle)$, as $|\mathrm{Fix}(U^*\langle\psi\rangle)\cap\mathcal{O}| = q+1$. Furthermore, $C_G(\psi)/\langle\psi\rangle \cong Z_\psi/\langle\psi\rangle$.PGL$(2,q)$, where $|Z_\psi| = \frac{q-1}{\mu}$ and $\mu = (3,q-1)$, acts 2-transitively on $\mathrm{Fix}(U^*\langle\psi\rangle)\cap\mathcal{O}$. Then $q-1=4$ and $C_G(\psi)/\langle\psi\rangle \leq S_6$ by [1], as $q>3$. This implies that $\mathrm{Fix}(Z_\psi) = \mathrm{Fix}(\psi)$. So, $C_G(\psi)$ acts on $\mathrm{Fix}(\psi)$ as PGL$(2,5)$ leaving invariant a subplane $\mathrm{Fix}(U^*\langle\psi\rangle) \cong$ PG$(2,4)$, which is impossible by Lemma 3.6, as $n>4$.

Assume that $\mathcal{O} \cong$ PG$(2,q)$. As ψ is Baer collineation of Π and ψ induces a homology on $\mathcal{O}$, then $C'_G(\psi)$ acts on $\mathrm{Fix}(\psi)$ as PSL$(2,q)$ and it also fixes an antiflag. Note that $q^3 < n < q^4$ by [17, Proposition 11], and since $n \neq q^4$ by our assumption. Then, by Theorem 2.1 (3a), either $\mathrm{Fix}(\psi)$ has order 16 and $C'_G(\psi)/\langle\psi\rangle \cong$ PSL$(2,5)$, or $\mathrm{Fix}(\psi)$ is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C'_G(\psi)/\langle\psi\rangle \cong$ PSL$(2,7)$. However, the same argument as in Lemma 3.6 rules out both these cases, since $C'_G(\psi)/\langle\psi\rangle$ fixes an antiflag. This completes the proof. □

Lemma 3.8. *The groups $\overline{C'_G(\psi)}$ and $\overline{C_G(\psi)}$ are strongly irreducible on $\mathrm{Fix}(\psi)$.*

Proof. In order to prove the assertion, by Lemma 3.6, we need to analyze only the case $(q, n) = (5, 4)$ and $q = 9$ when $9^2 < n < 9^4$. Recall that $G \cong \mathsf{PSL}(3, q)$ is irreducible on Π by Lemma 3.3. Then Π consists of nontrivial G-orbits. Since each G-orbit has length $\lambda_j d_j(G)$, where $\lambda_j \geq 0$ and $d_j(G)$ is the degree of some primitive permutation representation of G, then

$$n^2 + n + 1 = \sum_{j \geq 0} \lambda_j d_j(G) . \tag{4}$$

That is, $n^2 + n + 1$ must admit a partition restricted to

$$D(G) = [d_0(G), d_1(G), ..d_k(G)] \ ,$$

the spectrum of the degrees of the primitive permutation representations of G. So, the case $(q, n) = (5, 4)$ is ruled out, since $n^2 + n + 1 = 21$, while $D(G) = [31, 3100, 3875, 4000]$ by [2].

Assume that $q = 9$ and $9^2 < n < 9^4$. As above, by [2], $n^2 + n + 1$ must admit a partition restricted to

$$D(G) = [91, 7020, 7560, 58968, 110565, 155520] \ .$$

Note that $9 \mid d_j(G)$ for each $j > 0$. If $\lambda_0 = 0$, then $9 \mid n^2 + n + 1$ by (4), while it is known that either $n^2 + n + 1 \equiv 1 \mod 3$ or $n^2 + n + 1 \equiv 3 \mod 9$. Hence, $\lambda_0 > 0$. So, there exists a point $X \in \Pi$ such that $G_X \leq U^* : C_G(\psi)$, where $C_G(\psi) \cong \mathsf{GL}(2, 9)$, by [2]. Since the group G does not admit 2-transitive point-orbits on Π for $n < 9^4$ by Lemma 3.7, then $G_X < U^* : C_G(\psi)$. Hence, by Lemma 3.5, either $9 \mid \left|X^G\right|$ or $\mathrm{Fix}(U^* \langle\psi\rangle)$ is either a flag, or an antiflag or a proper subplane of $\mathrm{Fix}(\psi)$. Furthermore, $\overline{C_G(\psi)}$ leaves $\mathrm{Fix}(U^* \langle\psi\rangle)$ invariant.

Assume that the latter occurs. If $\mathrm{Fix}(U^* \langle\psi\rangle)$ consists of a flag or an antiflag, again by Theorem 2.1, the case (3) inside the proof of Lemma 3.6 occurs, which leads to a contradiction, as we have seen. So, $\mathrm{Fix}(U^* \langle\psi\rangle)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Then $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ by Theorem 2.1. Note that either $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ is a Baer subplane of $\mathrm{Fix}(U^*)$ or $\mathrm{Fix}(U^* \langle\psi\rangle) = \mathrm{Fix}(U^*) \cong \mathsf{PG}(2, 4)$. Suppose that $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ is a Baer subplane of $\mathrm{Fix}(U^*)$. Note that $\bar{Z}_\psi = \langle 1 \rangle$ by [2]. Consequently, Z_ψ fixes $\mathrm{Fix}(\psi)$ pointwise and $\overline{C_G(\psi)} \cong \mathsf{PGL}(2, 9)$. Hence, $\mathrm{Fix}(\langle\psi\rangle) = \mathrm{Fix}(Z_\psi)$. Thus, $\mathrm{Fix}(U^* \langle\psi\rangle) = \mathrm{Fix}(U^*.Z_\psi)$, as Z_ψ normalizes U^*. That is, $\mathrm{Fix}(U^*.Z_\psi) \cong \mathsf{PG}(2, 4)$ is a Baer subplane $\mathrm{Fix}(U^*)$. Then Z_ψ is semiregular on $s \cap (\mathrm{Fix}(U^*) - \mathrm{Fix}(U^*.Z_\psi))$, where s is a secant of $\mathrm{Fix}(U^*.Z_\psi)$. So $8 \mid 16 - 4$, since $o(\mathrm{Fix}(U^*)) = 16$, $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ and since $|Z_\psi| = \frac{q-1}{\mu} = 8$; this is a contradiction. Thus, $\mathrm{Fix}(U^* \langle\psi\rangle) = \mathrm{Fix}(U^*) \cong \mathsf{PG}(2, 4)$. If there exists a nontrivial element ρ in U^* fixing a point in $\Pi - \mathrm{Fix}(U^*)$, then $\mathrm{Fix}(\rho)$ is a Baer subplane of Π, since $\mathrm{Fix}(U^*) \cong \mathsf{PG}(2, 4)$ and $n = 16^2$. Then each non trivial element in

U^* fixes a subplane of order 16 of Π, since the nontrivial elements in U^* are conjugate under $C_G(\psi) \cong \mathsf{GL}(2,9)$. Hence, if Q is a point fixed by U^*, then $9^2 \mid (9^2-1)(\sqrt{n}+1)+(n+1)$ by Cauchy-Frobenius Lemma, since $|U^*| = 9^2$. So, $9^2 \mid n-\sqrt{n}$, which is a contradiction, since $n = 16^2$. Therefore, U^* is semiregular on $r - \mathrm{Fix}(U^*)$, where r is a secant to U^*. Hence, $9^2 \mid n-4$ and we again obtain a contradiction, as $n = 16^2$. Thus, $9 \mid \left|X^G\right|$. Actually the previous argument can be repeated for each point $Y \in \Pi$ such that G_Y lies in a maximal parabolic subgroup of G. Consequently, any orbit divisible by $d_0(G)$ is actually divisible by $9d_0(G)$. Therefore, bearing in mind that $9 \mid d_j(G)$ for each $j > 0$, any admissible G-orbit has length divisible by 9. So, $9 \mid n^2+n+1$ by (4), and we obtain a contradiction as above. This completes the proof. □

Lemma 3.9. *The group $\overline{C_G(\psi)}$ contains Baer involutions of* $\mathrm{Fix}(\psi)$. *In particular, $\sqrt[4]{n}$ is an integer.*

Proof. Assume that all the involutions in $\overline{C_G(\psi)}$ are perspectivities of $\mathrm{Fix}(\psi)$. If $\sqrt{n}$ is even, then either $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,2)$ and $C_G^{'}(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2,7)$ or $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,4)$ and $C_G^{'}(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2,9)$ by [9]. However, both these cases cannot occur by the same argument as in Lemma 3.6. Hence, $\sqrt{n}$ is odd and the involutions in $\overline{C_G(\psi)}$ are homologies of $\mathrm{Fix}(\psi)$.

If $K = Z_\psi$, then $\overline{C_G(\psi)} \cong \mathsf{PGL}(2,q)$. Then $q \mid \sqrt{n}$ and $q-1 \mid \sqrt{n}-1$ by [12, Theorem C.ii]. As $q \mid \sqrt{n}$, then $\sqrt{n} = \lambda_1 q$ for some $\lambda_1 \geq 0$. Furthermore, $\lambda_1 = (q-1)\lambda_2+1$ for some $\lambda_2 \geq 0$, since $q-1 \mid \sqrt{n}-1$. Hence, $\sqrt{n} = q(q-1)\lambda_2+q$. However, this is impossible, since $n < q^4$ by our assumption.

If $K < Z_\psi$. Then $\bar{Z}_\psi \neq \langle 1\rangle$. Since $\overline{C_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$ by Lemma 3.8, and since each nontrivial subgroup of $\bar{Z}_\psi$ is normal in $\overline{C_G(\psi)}$, then $\bar{Z}_\psi$ is semiregular on $\mathrm{Fix}(\psi)$. Let $\bar{\sigma}$ be any involutory $(C_{\bar{\sigma}}, a_{\bar{\sigma}})$-homology of $\overline{C'_G(\psi)}$. Note that $\overline{C'_G(\psi)} \times \bar{Z}_\psi \lhd \overline{C_G(\psi)}$. That is, $\bar{Z}_\psi$ centralizes $\bar{\sigma}$ and hence $\bar{Z}_\psi$ fixes $(C_{\bar{\sigma}}, a_{\bar{\sigma}})$. This is impossible, since $\bar{Z}_\psi$ is semiregular on $\mathrm{Fix}(\psi)$. Therefore, $\overline{C_G(\psi)}$ contains Baer collineation of $\mathrm{Fix}(\psi)$ and hence $\sqrt[4]{n}$ is an integer. □

Proposition 3.10. *For each $X \in \Pi$ such that G_X lies in a maximal parabolic subgroup of G, then $q^2 \mid \left|X^G\right|$.*

Proof. Since $\overline{C_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$ by Lemma 3.8 and since the group G does not admit 2-transitive point-orbits on Π by Lemma 3.7, the assertion follows by Lemma 3.5. □

Lemma 3.11. *One of the following occurs:*

(1) $q^2 \mid n^2+n+1$;

(2) *q is a square, $q\sqrt{q} \mid n^2+n+1$, and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3,\sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3,\sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$.*

Proof. Since G is irreducible on Π, then Π consists of nontrivial G-orbits. By a direct inspection of the list of maximal subgroups of $\mathsf{PSL}(3,q)$ given in [16], we have that $q^2 \mid \left|X^G\right|$ for each point $X \in \Pi$, unless q is a square and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3,\sqrt{q})$, or $G_Y \leq \mathsf{PSU}(3,\sqrt{q})$, with $(|G_Y|, q\sqrt{q}) > q$, or $G_Y \leq E_{q^2} : C_G(\gamma)$ for some involution γ of G. Actually, if $G_Y \leq E_{q^2} : C_G(\gamma)$, then $q^2 \mid \left|Y^G\right|$ by Proposition 3.10.

If either there are no Z in Π such that $G_Z \leq \mathsf{PSL}(3,\sqrt{q})$ or $G_Z \leq \mathsf{PSU}(3,\sqrt{q})$ and $(|G_Z|, q\sqrt{q}) > q$, each admissible G-orbit on Π is divisible by q^2. Therefore, $q^2 \mid n^2+n+1$, since Π consists of nontrivial G-orbits. That is, the assertion (1).

If q is square and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3,\sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3,\sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$, each G-orbits is divisible by $q\sqrt{q}$ and hence $q\sqrt{q} \mid n^2+n+1$ by the above argument. That is, the assertion (2). □

Corollary 3.12. *$p \neq 3$.*

Proof. Assume that $p=3$. As $q>3$, then $9 \mid q$. Hence, $9 \mid n^2+n+1$ by Lemma 3.11. However, this is impossible, since it is known that either $n^2+n+1 \equiv 1 \mod 3$ or $n^2+n+1 \equiv 3 \mod 9$. □

Lemma 3.13. *Let S_0 be the p-group and let ψ and β be the involutions defined in Section 2. If q is a square and $q\sqrt{q} \mid n^2+n+1$, then one of the following occurs:*

(1) *The group S_0 is semiregular on Π and hence on $\mathrm{Fix}(\psi)$;*

(2) *$\mathrm{Fix}(S_0)$ is a subplane of Π. Furthermore, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$;*

(3) *There exists a nontrivial proper subgroup S^* of S_0 such that $\mathrm{Fix}(S^*)$ is a subplane of Π of order m and one of the following occurs:*

   (a) *$\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$ and the involution β induces a Baer collineation on it. In particular, $\sqrt[4]{m}$ is an integer.*

   (b) *$\mathrm{Fix}(S^*)$ is a proper subplane of $\mathrm{Fix}(\psi)$ and hence $m \leq \sqrt[4]{n}$.*

   Furthermore, in the cases (3a)–(3b), the group S_0/S^* acts on $\mathrm{Fix}(S^*)$ and on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly.

Proof. Let S_0 be the p-group defined in Section 2. Recall that $p \neq 3$ by Corollary 3.12. Also, recall that ψ centralizes S_0, that β inverts S_0 and that $\langle \psi, \beta \rangle \cong$

E_4. Since $q\sqrt{q} \mid n^2 + n + 1$ and $n^2 + n + 1 = (n - \sqrt{n} + 1)(n + \sqrt{n} + 1)$, with $(n - \sqrt{n} + 1, n + \sqrt{n} + 1) = 1$, either $q\sqrt{q} \mid n - \sqrt{n} + 1$ or $q\sqrt{q} \mid n + \sqrt{n} + 1$.

Assume that $q\sqrt{q} \mid n - \sqrt{n} + 1$. Thus $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi) \neq \varnothing$. In particular, $\mathrm{Fix}(S_0) \neq \varnothing$. As $p \neq 3$ and that $q\sqrt{q} \mid n^2 + n + 1$, we have that $(q, n) = (q, n \pm 1) = 1$. Therefore, $\mathrm{Fix}(S_0)$ is a subplane of Π. As ψ centralizes S_0, then ψ acts on $\mathrm{Fix}(S_0)$. Hence, $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi) \neq \varnothing$. Actually, $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a subplane of $\mathrm{Fix}(\psi)$, again since ψ centralizes S_0, $q\sqrt{q} \mid n + \sqrt{n} + 1$ and $p \neq 3$. Moreover, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0) \subseteq \mathrm{Fix}(\psi)$. Assume that $\mathrm{Fix}(S_0) = \mathrm{Fix}(\psi)$. Then S_0 is semiregular on $s - \mathrm{Fix}(S_0)$, where s is a secant of $\mathrm{Fix}(S_0)$, since $\mathrm{Fix}(S_0)$ is a Baer subplane of Π. Therefore, $q \mid n - \sqrt{n}$, since $|S_0| = q$. That is, $q \mid \sqrt{n}(\sqrt{n} - 1)$. So, we obtain a contradiction, since $(\sqrt{n}(\sqrt{n} - 1), n - \sqrt{n} - 1) = 1$. Thus, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$, and we obtain the assertion (2).

Assume that $q\sqrt{q} \mid n + \sqrt{n} + 1$. If $\mathrm{Fix}(S_0) \neq \varnothing$, we still obtain the assertion (2) by the previous argument, by bearing in mind that $(\sqrt{n}(\sqrt{n} - 1), n + \sqrt{n} + 1) \mid 3$ and that $q > 3$. Hence, assume that $\mathrm{Fix}(S_0) = \varnothing$. At this point, either S_0 is semiregular on Π and we obtain the assertion (1), or there exists a nontrivial subgroup S_1 of S_0 such that $\mathrm{Fix}(S_1) \neq \varnothing$. By bearing in mind that ψ centralizes S_0 and hence S_1, that $(\sqrt{n}(\sqrt{n} - 1), n + \sqrt{n} + 1) \mid 3$ and that $p \neq 3$ by Corollary 3.12, the previous argument, with S_1 in the role of S_0, yields that either $\mathrm{Fix}(S_1) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_1)$ or $\mathrm{Fix}(S_1)$ is a proper subplane of $\mathrm{Fix}(\psi)$.

Let $\mathcal{S}$ be the set of the nontrivial subgroups of S_0 fixing a subplane of Π whose intersection with $\mathrm{Fix}(\psi)$ is in turn a subplane of this one. Clearly, $\mathcal{S} \neq \varnothing$, since $S_1 \in \mathcal{S}$. Let S^* be an element of $\mathcal{S}$ of maximal order. Hence, $\mathrm{Fix}(S^*)$ is a subplane of Π and $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a subplane of $\mathrm{Fix}(\psi)$. Moreover, either $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$ or $\mathrm{Fix}(S^*)$ is a proper subplane of $\mathrm{Fix}(\psi)$, again by the above argument with S^* in the role of S_1. Let m be the order of $\mathrm{Fix}(S^*)$. If $\mathrm{Fix}(S^*)$ is a proper subplane of $\mathrm{Fix}(\psi)$, then $m \leq \sqrt[4]{n}$ by [10, Theorem 3.7], and we obtain the assertion (3a). Hence, assume that $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$. Note that S_0/S^* is nontrivial and acts semiregularly on $\mathrm{Fix}(S^*)$ and on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$, since $\mathrm{Fix}(S_0) = \varnothing$, the group S^* is an element of $\mathcal{S}$ of maximal order, the group S_0 is abelian and since ψ centralizes S_0. Denote by m_ψ the order of $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Then $m = m_\psi^2$ by [10, Theorem 3.7], since $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$. As β inverts S_0 and as $\langle \psi, \beta \rangle \cong E_4$, then β normalizes $S^* \langle \psi \rangle$ and acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Denote by $S_0^+ = S_0/S^*$. Hence, S_0^+ is nontrivial and acts semiregularly on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$, as we have seen above. Furthermore, $S_0^+ \langle \beta \rangle$ acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Assume that β induces a perspectivity on $\mathrm{Fix}(S^*) \cap$

$\mathrm{Fix}(\psi)$. Let $\rho \in S_0^+$, $\rho \neq 1$. Then β^ρ is also a perspectivity of $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$, and $\mathrm{Fix}([\beta^\rho, \beta]) \cap \mathrm{Fix}(\psi) \neq \varnothing$ by [6, Lemma 5.1]. This is a contradiction, since $[\beta^\rho, \beta] \in S_0^+$, the group S_0^+ is nontrivial and acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly. Therefore, β induces a Baer collineation on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Then m_ψ is a square by [10, Theorem 3.7]. Consequently, $\sqrt[4]{m}$ is an integer, since we proved that $m = m_\psi^2$, and we obtain the assertion (3b). □

4 The proof of Theorem 1.1

Proposition 4.1. *The involutions in G are perspectivities of* Π.

Proof. We proceed with a series of steps to show that no one of the cases of Lemma 3.13 occurs, obtaining the assertion in this way.

Step I: The case (1) of Lemma 3.13 does not occur.

Assume that S_0 is semiregular on Π and on $\mathrm{Fix}(\psi)$. So, $q \mid n + \sqrt{n} + 1$. Recall that either $q^2 \mid n^2 + n + 1$ or, q is a square, $q\sqrt{q} \mid n^2 + n + 1$, and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3, \sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3, \sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$, by Lemma 3.11. In particular, either $q^2 \mid n + \sqrt{n} + 1$ or $q\sqrt{q} \mid n + \sqrt{n} + 1$, respectively, since $n^2 + n + 1 = (n + \sqrt{n} + 1)(n - \sqrt{n} + 1)$, $(n + \sqrt{n} + 1, n - \sqrt{n} + 1) = 1$, and since $q \mid n + \sqrt{n} + 1$. If $q^2 \mid n + \sqrt{n} + 1$, then we obtain a contradiction by [13, Lemma 6.2], since $\sqrt{n}$ is a square by Lemma 3.9. Thus, q is a square, $q\sqrt{q} \mid n + \sqrt{n} + 1$, and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3, \sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3, \sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$.

Assume that there exists a point $Y \in \Pi$, such that either $G_Y \leq M$, where M is either $\mathsf{PSL}(3\sqrt{q})$ or $\mathsf{PSU}(3, \sqrt{q})$, and such that $(|G_Y|, q\sqrt{q}) > q$. Without loss of generality, we may assume that a Sylow p-subgroup of G_Y is contained in U, the group defined in Section 2. Set $U_Y = G_Y \cap U$ and $U(M) = M \cap U$. Clearly, $U_Y \leq U(M)$, with $(|U_Y|, q\sqrt{q}) > q$ and $|U(M)| = q\sqrt{q}$. In particular, $U(M)$ consists of matrices of type (1) given in Section 2 whose entries are all the elements of $\mathsf{GF}(\sqrt{q})$, while U_Y consists of some of these matrices. Let W be the subgroup of S_0, represented by the matrices type (2) given in Section 2, with $y_2 = y_3 = 0$ and with $y_1 \in \mathsf{GF}(\sqrt{q})$. Hence, $|W| = \sqrt{q}$ and $W \leq U(M)$. Therefore, $\langle U_Y, W \rangle \leq U(M)$, as $U_Y \leq U(M)$. Hence, $|\langle U_Y, W \rangle| \leq q\sqrt{q}$. On the other hand, $|\langle U_Y, W \rangle| \geq \frac{|U_Y||W|}{|U_Y \cap W|}$. Thus, $\frac{|U_Y||W|}{|U_Y \cap W|} \leq q\sqrt{q}$. Therefore, $p \mid |U_Y \cap W|$ since $(|U_Y|, q\sqrt{q}) > q$ and $|W| = \sqrt{q}$. So, $p \mid |U_Y \cap S_0|$, since $W \leq S_0$. Hence, we arrive at a contradiction, since S_0 is semiregular on Π.

Step II: The case (2) of Lemma 3.13 does not occur.

Recall that S, S_0 and ψ are defined as in Section 2. Hence, $Z(S) = S' = S_0$. Furthermore, ψ normalizes S and S_0. Assume that the case (2) of Lemma 3.13 occurs. Hence, $\mathrm{Fix}(S_0)$ is a subplane of Π. Moreover, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Clearly, S acts on $\mathrm{Fix}(S_0)$ (the action is unfaithful). Assume $S_0 < S_Q$ for some point $Q \in \mathrm{Fix}(S_0)$. Then S_Q lies in G_Q which, in turn, lies in a maximal parabolic subgroup of G by a direct inspection of the list of maximal subgroups of $G \cong$ PSL$(3, q)$, q odd, given in [16]. Then $q^2 \mid \left|Q^G\right|$ by Proposition 3.10. However, this is impossible, since $S_0 < S_Q \leq S$, while $|S_0| = q$ and $|S| = q^3$. Hence S induces the group S/S_0 on $\mathrm{Fix}(S_0)$ acting semiregularly. Thus $q^2 \mid h^2 + h + 1$, where h is the order of $\mathrm{Fix}(S_0)$.

Assume that $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$. Then h is a square. Moreover $h \leq \sqrt{n}$ by [10, Theorem 3.7]. Hence $h^2+h+1 \leq q^4+q+1$, since $n \leq q^4$ by our assumption. However, this yields a contradiction, by [13, Lemma 6.2], since $q^2 \mid h^2 + h + 1$ and h is a square.

Assume that $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Then $h \leq \sqrt[4]{n}$ and hence $h^2+h+1 \leq q^2+q+1$. Thus, $q^2 = h^2+h+1$, since $q^2 \mid h^2+h+1$, and we still obtain a contradiction by [13, Lemma 6.2].

Step III: The final contradiction.

By (I) and (II) it follows that only case (3) of Lemma 3.13 might occur. Hence, assume there exists a nontrivial proper subgroup S^* of S_0 such that S_0/S^* acts on $\mathrm{Fix}(S^*)$ and on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly. In particular, if m is the order of $\mathrm{Fix}(S^*)$, then $\sqrt[4]{m}$ is an integer. Since $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$ an since S_0/S^* is nontrivial and acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly, then $p \mid m + \sqrt{m} + 1$. Furthermore, as $S^* < S_0$, the group S centralizes S^* and acts on $\mathrm{Fix}(S^*)$. From the proof of Lemma 3.11, we actually obtain that any G-orbit has length divisible by either q^2 or, when q is a square, by $q\sqrt{q}$. This implies that each orbit under the group induced by S on $\mathrm{Fix}(S^*)$ has length divisible by either q^2 or, when q is a square, by $q\sqrt{q}$. So, we obtain that either $q^2 \mid m^2 + m + 1$, or $q\sqrt{q} \mid m^2 + m + 1$ when q is a square. Actually, either $q^2 \mid m + \sqrt{m} + 1$ or $q\sqrt{q} \mid m + \sqrt{m} + 1$, respectively, since $m^2 + m + 1 = (m + \sqrt{m} + 1)(m - \sqrt{m} + 1)$, $(m + \sqrt{m} + 1, m - \sqrt{m} + 1) = 1$, and since $p \mid m + \sqrt{m} + 1$.

Assume that $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Then $\sqrt{m} \leq \sqrt[4]{n}$ by [10, Theorem 3.7]. So, $m + \sqrt{m} + 1 \leq q^2 + q + 1$. This fact, in conjunction with either $q^2 \mid m+\sqrt{m}+1$ or $q\sqrt{q} \mid m+\sqrt{m}+1$, yields that $q\sqrt{q} = m+\sqrt{m}+1$

with $q = 7$ and $\sqrt{m} = 18$ by [13, Lemma 6.2]. However, this is a contradiction, since $\sqrt{m}$ is a square.

Since none of the cases of Lemma 3.13 occurs, then ψ cannot be a Baer collineation of Π. Therefore, any involution of G is a perspectivity of Π, since $G \cong \mathsf{PSL}(3, q)$ contains a unique conjugate class of involutions. □

Now, using Proposition 4.1, we prove our main result.

Proof of Theorem 1.1. The assertion follows by Proposition 3.2 for $q = 3$. Hence, assume that $q > 3$. Since $G \cong \mathsf{PSL}(3, q)$ is irreducible on Π by Lemma 3.3 and since each involution in G is a perspectivity of Π by Proposition 4.1, then G leaves invariant a subplane Π_0 on which it acts strong irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong \mathsf{PG}(2, q)$ by [8, Theorem 1.1]. If $n \leq q^3$, the assertion follows from Theorem 2.3. Hence, assume that $q^3 < n \leq q^4$. As the involutions in G are homologies of Π_0, they are also homologies of Π. Furthermore, each p-element inducing an elation on Π_0 is also an elation of Π by [10, Theorem 4.25]. Finally, by [12, Theorem C.ii], we have that $q^2 \mid n$, that $q - 1 \mid n - 1$ and that $q + 1 \mid n^2 - 1$. It is a straightforward computation to show that this numerical information yield that Π has order $n = \lambda q^3 + (1 - \lambda) q^2$, where $1 < \lambda \leq q + 1$ and $q + 1 \mid \lambda(\lambda - 1)$, since $q^3 < n \leq q^4$. This completes the proof. □

Remark 4.2. It seems to be tough proving that there are no planes of order $q^3 < n < q^4$ admitting $G \cong \mathsf{PSL}(3, q)$ as a collineation group. Indeed, although it is easy to show that a nontrivial stabilizer of a point has order odd and coprime to p, it is difficult to determine the exact orbital decomposition of the set of external lines to Π_0, especially when the stabilizer of a line of such a set is a subgroup of a Singer cycle of G.

References

[1] **V. Abatangelo**, Doubly transitive $(n + 2)$-arcs in projective planes of even order, J. Combin. Theory. Ser. A **42** (1986), 1–8.

[2] **J. H. Conway**, **R. T. Curtis**, **R. A. Parker** and **R. A. Wilson**, *Atlas of Finite Groups. Maximal subgroups and ordinary characters for simple groups*, Oxford University Press, 1985.

[3] **J. D. Dixon** and **B. Mortimer**, *Permutation groups*, Springer Verlag, New York, 1966.

[4] **U. Dempwolff**, PSL$(3, q)$ on projective planes of order q^3, *Geom. Dedicata* **18** (1985), 101–112.

[5] **R. Figueroa**, A family of not (V, l)-transitive projective planes of order q^3, $q \not\equiv 1 \mod 3$ and $q > 2$, *Math. Z.* **181** (1982), 471–479.

[6] **C. Hering**, On the structure of finite collineation groups of projective planes, *Abh. Math. Sem. Univ. Hamburg* **49** (1979), 155–182.

[7] **C. Hering** and **H. J. Schaeffer**, On the new projective planes of R. Figueroa, in: Combinatorial Theory, Proc. Schloss Rauischholzhausen 1982, D. Jungnickel and K. Wedder (eds.), Springer, Berlin, (1982), 187–190.

[8] **C. Hering** and **M. Walker**, Perspectivities in irreducible collineation groups of projective planes I, *Math. Z.* **155** (1977), 95–101.

[9] ______, Perspectivities in irreducible collineation groups of projective planes II, *J. Statist. Plann. Inference* **3** (1979), 151–177.

[10] **D. R. Hughes** and **F. C. Piper**, *Projective Planes*, Springer Verlag, New York, Berlin, 1973.

[11] **B. Huppert**, *Endliche Gruppen I*, Springer Verlag, New York - Berlin, 1967.

[12] **W. M. Kantor**, On the structure of collineations group of finite projective planes, *Proc. London Math. Soc.* **32** (1976), 385–420.

[13] ______, Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1987), 15–45.

[14] **H. Lüneburg**, Charakterisierungen der endlichen Desarguesschen projectiven Ebenen, *Math. Z.* **85** (1964), 419–450.

[15] ______, Characterizations of the generalized Hughes planes, *Canad. J. Math.* **28** (1976), 376–402.

[16] **H. H. Mitchell**, Determination of ordinary and modular ternary linear groups, *Trans. Amer. Math. Soc.* **12** (1911), 207–242.

[17] **A. Montinaro**, Projective Planes with a Doubly Transitive Projective Subplane, *Bull. Belgian Math. Soc. Simon Stevin* **114** (2007), 117–134.

[18] ______, The general structure of the projective planes admitting PSL$(2, q)$ as a collineation group, *Innov. Incidence Geom.* **5**, 35–116.

[19] **G. E. Moorhouse**, PSL$(2, q)$ as a collineation group of projective planes of small order, *Geom. Dedicata* **31** (1989), 63–88.

[20] ______, PSL$(3, q)$ and PSU$(3, q)$ on projective planes of order q^4, Ph.D. Thesis, University of Toronto, Canada, 1987.

[21] **T. G. Ostrom** and **A. Wagner**, On projective and affine planes with transitive collineation groups, *Math. Z.* **71** (1959), 186–199.

[22] **J. C. D. S. Yaqub**, On two theorems of Lüneburg, *Arch. Math.* **17** (1966), 485–488.

Mauro Biliotti
Dipartimento di Matematica, Università del Salento, Via per Arnesano, 73100 Lecce, Italy
e-mail: mauro.biliotti@unile.it

Alessandro Montinaro
Dipartimento di Matematica, Università del Salento, Via per Arnesano, 73100 Lecce, Italy
e-mail: alessandro.montinaro@unile.it