On the finite projective planes of order up to $q^4$, $q$ odd, admitting $\text{PSL}(3, q)$ as a collineation group

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Dedicated to Prof. Gábor Korchmáros on occasion of his 60th birthday

Abstract

In this paper, it is shown that any projective plane $\Pi$ of order $n \leq q^4$, $q$ odd, that admits a group $G \cong \text{PSL}(3, q)$ as a collineation group contains a $G$-invariant Desarguesian subplane of order $q$. Moreover, the involutions and suitable $p$-elements in $G$ are homologies and elations of $\Pi$, respectively. In particular, if $n \leq q^3$, actually, $n = q$, $q^2$ or $q^3$.

Keywords: projective plane, collineation group, orbit

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1. Introduction and result

The problem of determining a projective plane $\Pi$ of order $n$ admitting $G \cong \text{PSL}(3, q)$ as a collineation group has been largely investigated in the last decades. The first significant result related to this problem is the celebrated theorem of Ostrom and Wagner [21], dating back to 1959, which asserts that the projective plane $\Pi$ is Desarguesian when $n = q$. In 1976, Lüneburg [15] proves that either $\Pi$ is a Desarguesian plane or a Generalized Hughes plane when $n = q^2$. In 1985, Dempwolff [4] proves that any projective plane $\Pi$ of order $n = q^3$ that admits $G \cong \text{PSL}(3, q)$ as a collineation group contains a Desarguesian subplane $\Pi_0$ of order $q$ on which $G$ acts faithfully in its natural permutation representation. Despite the fact that Dempwolff provides a complete description of the $G$-orbits on the points and on the lines of $\Pi$, he emphasizes the difficulty in obtaining
a characterization of $\Pi$. In 1989, Moorhouse obtains for projective planes of order $n = q^4$, $q$ odd, the analogue of Dempwolff’s result. Recently, Montinaro investigated the projective planes of order $n \leq q^3$ admitting a group inducing a 2-transitive group (namely, $\text{PSL}(3, q)$) on a subplane of $\Pi$, showing that $n = q$, $q^2$ or $q^3$ and the results of Ostrom and Wagner, Lüneburg, Dempwolff, occur, respectively. This paper represents a further contribution to the study of the projective planes of order $n \leq q^4$, $q$ odd, that admit $\text{PSL}(3, q)$ as a collineation group. In particular, it represents a conclusive result when the plane has order $n \leq q^3$.

**Theorem 1.1.** Let $\Pi$ be a finite projective plane of order $n$ that admits $G \cong \text{PSL}(3, q)$, $q$ odd, as a collineation group. If $n \leq q^4$, then the following occurs:

(I) There exists a subplane $\Pi_0 \cong \text{PG}(2, q)$ of $\Pi$ on which $G$ acts in the natural way;

(II) The involutions in $G$ are homologies of $\Pi$;

(III) The $p$-elements of $G$ inducing elations on $\Pi_0$ are elations of $\Pi$.

Moreover, one of the following occurs:

(i) $n = q$ and $\Pi = \Pi_0 \cong \text{PG}(2, q)$;

(ii) $n = q^2$, $\Pi$ is a Desarguesian plane or a Generalized Hughes plane and $\Pi_0$ is a Baer subplane of $\Pi$;

(iii) $n = q^3$;

(iv) $n = q^2(\lambda(q - 1) + 1)$, where $1 \leq \lambda \leq q + 1$ and $q + 1 \mid \lambda(\lambda - 1)$.

The cases (i) and (ii) clearly occur. The only known occurrences of the case (iii) are in the Desarguesian planes and in the Figueroa planes [5], [7]. The only known occurrences of the case (iv) are in the Desarguesian planes and in the Generalized Hughes planes when $\lambda = q + 1$, i.e. $n = q^4$.

The strategy of the proof is the following. Firstly, we prove that $G$ is irreducible on $\Pi$. Hence, $\Pi$ consists of nontrivial $G$-orbits. If $\psi$ is a Baer collineation of $\Pi$, we determine the general structure of the action of the group induced by $C_G(\psi)$ on $\text{Fix}(\psi)$ by Theorem 2.1. This forces any admissible $G$-orbit on the points of $\Pi$ to be divisible by either $q^2$ or $q\sqrt{q}$ for $q$ square. So, $n^2 + n + 1$, i.e. the number of points of $\Pi$, is divisible by either $q^2$ or $q\sqrt{q}$ for $q$ square, as $\Pi$ consists of nontrivial $G$-orbits. This yields a Diophantine equation involving $n^2 + n + 1$ and either $q^2$ or $q\sqrt{q}$ for $q$ square. However, such an equation has no admissible solutions by [13, Lemma 6.2]. Therefore, the involutions in $G$ are homologies of $\Pi$. At this point, the proof of our result easily follows.
2. Background

The notation used in this paper is standard. For what concerns finite groups, the reader is referred to [11] and to [3]. The necessary background about finite projective planes may be found in [10].

Now, we collect some information about the structure of the groups $\text{PSL}(2, q)$ and $\text{PSL}(3, q)$ and some results on the projective planes admitting one of these as a collineation group. Based on the results of Lüneburg [14], Yaqub [22] and Moorhouse [19], the following theorem, due to Montinaro, determines the general structure of the projective planes of order up to $q^2$ admitting $\text{PSL}(2, q)$, $q > 3$, as a collineation group. Recall that a collineation group of a projective plane $\Pi$ is said to be irreducible on $\Pi$ if the group does not fix any point, line, triangle of $\Pi$. An irreducible collineation group of $\Pi$ which does not fix any proper subplane of $\Pi$ is said to be strongly irreducible on $\Pi$.

**Theorem 2.1.** Let $\Pi$ be a projective plane of order $n$ admitting a collineation group $H \cong \text{PSL}(2, q)$, $q > 3$. If $n \leq q^2$, then one of the following occurs:

1. $n < q$ and one of the following occurs:
   
   (a) $n = 4$, $\Pi \cong \text{PG}(2, 4)$ and $H \cong \text{PSL}(2, 5)$;
   
   (b) $n = 2$ or $4$, $\Pi \cong \text{PG}(2, 2)$ or $\text{PG}(2, 4)$, respectively, and $H \cong \text{PSL}(2, 7)$;
   
   (c) $n = 4$, $\Pi \cong \text{PG}(2, 4)$ and $H \cong \text{PSL}(2, 9)$.

2. $n = q$, $\Pi \cong \text{PG}(2, q)$ and one of the following occurs:
   
   (a) $H$ fixes a line or a point and $q$ is even;
   
   (b) $H$ is strongly irreducible and $q$ is odd.

3. $q < n < q^2$ and one of the following occurs:
   
   (a) $H$ fixes a point or a line, and one of the following occurs:
      
      (i) $n = 16$ and $H \cong \text{PSL}(2, 5)$;
      
      (ii) $n = 16$, $\Pi$ is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and $H \cong \text{PSL}(2, 7)$.
   
   (b) $H$ fixes a subplane $\Pi_0$ of $\Pi$, $q$ is odd and one of the following occurs:
      
      (i) $n = 16$, $\Pi_0 \cong \text{PG}(2, 4)$ and $H \cong \text{PSL}(2, 5)$;
      
      (ii) $\Pi_0 \cong \text{PG}(2, 2)$ or $\text{PG}(2, 4)$, and $H \cong \text{PSL}(2, 7)$;
      
      (iii) $\Pi_0 \cong \text{PG}(2, 4)$ and $G \cong \text{PSL}(2, 9)$. 

(c) \( H \) is strongly irreducible and \( q \) is odd.

(4) \( n = q^2 \) and one of the following occurs:

(a) \( H \) fixes a point or a line, and one of the following occurs:

(i) \( n = 25 \) and \( H \cong \text{PSL}(2,5) \);

(ii) \( n = 81 \) and \( H \cong \text{PSL}(2,9) \);

(iii) \( n = q^2, q \) even, and \( G \cong \text{PSL}(2,q) \).

(b) \( H \) fixes a subplane \( \Pi_0 \) of \( \Pi \), \( q \) is odd and one of the following occurs:

(i) \( n = q^2 \), \( \Pi_0 \cong \text{PG}(2,q) \) and \( H \cong \text{PSL}(2,q) \);

(ii) \( n = 25 \), \( \Pi_0 \cong \text{PG}(2,4) \) and \( H \cong \text{PSL}(2,5) \);

(iii) \( n = 81 \), \( \Pi_0 \cong \text{PG}(2,4) \) and \( H \cong \text{PSL}(2,9) \);

(iv) \( n = 81 \), \( \Pi_0 \) is a Hughes plane of order \( 9 \) and \( H \cong \text{PSL}(2,9) \).

(c) \( H \) is strongly irreducible.

**Proof.** See [18, Theorem 1]. \( \square \)

As we shall see, such a theorem will play a central role in our investigation due to the fact that the centralizer of an involution involves a group isomorphic to \( \text{PSL}(2,q) \).

Now, we recall some basic facts about the structure of the group \( G \cong \text{PSL}(3,q) \) (the reader is referred to [16]).

1. Let \( \psi \) and \( \beta \) be the involutions in \( G \) represented by \( \text{diag}(1,-1,-1) \) and \( \text{diag}(-1,1,-1) \), respectively. Then \( \langle \psi, \beta \rangle \cong E_4 \).

2. Let \( U \) be the Sylow \( p \)-subgroup of \( G \) represented by all the matrices

\[
\begin{bmatrix}
1 & x_1 & x_2 \\
0 & 1 & x_3 \\
0 & 0 & 1
\end{bmatrix}, \tag{1}
\]

where \( x_1, x_2, x_3 \in \text{GF}(q) \). Clearly, \( |U| = q^3 \). Let \( U_0 \) be the subgroup of \( U \) represented by the matrices in (1) having \( x_1 = x_3 = 0 \). Then \( U_0 \) has order \( q \) and \( U_0 = Z(U) = U' \). Thus, \( U \) is a special \( p \)-group. Finally, let \( U^* \) be the subgroup of \( U \) represented by the matrices in (1) having \( x_3 = 0 \). Then \( U^* \) is elementary abelian of order \( q^2 \) which is normalized by \( \psi \).

3. Let \( S \) be the Sylow \( p \)-subgroup of \( G \) represented by all the matrices

\[
\begin{bmatrix}
1 & 0 & y_2 \\
y_3 & 1 & y_1 \\
0 & 0 & 1
\end{bmatrix}, \tag{2}
\]
where \( y_1, y_2, y_3 \in \text{GF}(q) \), and let \( S_0 \) be the subgroup of \( S \) represented by those having \( y_2 = y_3 = 0 \). Then \( S_0 = Z(S) = S' \). In particular, \( U \cap S \) is an elementary abelian group of order \( q^2 \) containing \( S_0 \). Namely, \( U \cap S \) consists of all the matrices in (2) having \( y_3 = 0 \).

4. The group \( S_0, \langle \psi, \beta \rangle \) has order \( 4q \). In particular, \( \psi \) centralizes \( S_0 \), while \( \beta \) inverts \( S_0 \).

5. The group \( C_G(\psi) \) consists of the matrices

\[
\begin{bmatrix}
e^{-1} & 0 & 0 \\
0 & a & b \\
0 & c & d
\end{bmatrix},
\]

where \( a, b, c, d, e \in \text{GF}(q) \), \( e = ad - bc \neq 0 \). Denote by \( Z_\psi \) the subgroup of \( C_G(\psi) \) represented by all the matrices \( \text{diag}(d^{-2}, d, d) \), where \( d \in \text{GF}(q)^* \). Then \( Z_\psi = Z(C_G(\psi)) \). In particular, \( Z_\psi \) is a cyclic group of order \( \frac{q-1}{\mu} \), where \( \mu = (3, q - 1) \) and \( C_G(\psi) \cong Z_\psi \cdot \text{PGL}(2, q) \).

6. The group \( U^* : C_G(\psi) \) is a maximal parabolic subgroup of \( G \). Furthermore, \( U^* (\psi) \triangleleft U^* : C_G(\psi) \) and \( C'_G(\psi) \cong \text{SL}(2, q) \).

7. Let \( W^* \) be the subgroup of \( G \) represented by all the matrices of the form

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
z_1 & z_2 & 1
\end{bmatrix},
\]

where \( z_1, z_2 \in \text{GF}(q) \). Then \( W^* \) is an elementary abelian group of order \( q^2 \) which is normalized by \( C_G(\psi) \). Moreover, the groups \( U^* : C_G(\psi) \) and \( W^* : C_G(\psi) \) are the representatives of the two distinct conjugate classes of maximal parabolic subgroups of \( G \). The groups \( U^* \) and \( W^* \) are conjugate by the inverse-transpose automorphism.

We shall use the facts stated above without recalling them, unless it is explicitly required. In particular, since the Sylow \( p \)-subgroups of \( G \) are conjugate, we shall mainly refer either to \( U \) or to \( S \). Furthermore, despite the fact that there are two distinct conjugate classes of maximal parabolic subgroups in \( G \) by (7), what we prove to be true for \( U^* : C_G(\psi) \) can always be proven to be true for \( W^* : C_G(\psi) \). Hence, for our purposes we may always refer to \( U^* : C_G(\psi) \), without loss of generality.
Lemma 2.2. The group $G \cong \text{PSL}(3,q)$ contains two distinct involutions $\psi_1$ and $\psi_2$ such that $C_G'(\psi_1) \cap C_G'(\psi_2) \neq \langle 1 \rangle$ and $\langle C_G'(\psi_1), C_G'(\psi_2) \rangle = G$.

Proof. See [20, Lemma 4.1.vi].

Some geometrical results involving the group $G \cong \text{PSL}(3,q)$ are in order. By using the results of Ostrom-Wagner [21], Lüneburg [15] and Dempwolff [4], Montinaro proved the following.

Theorem 2.3. Let $\Pi$ be a finite projective plane of order $n$ and let $G$ be a collineation group of $\Pi$ inducing a group containing $\text{PSL}(3,q)$ on a subplane $\Pi_0$ of order $q$. If $n \leq q^3$, then one of the following occurs:

1. $\Pi_0 \cong \text{PG}(2,q)$, $\text{PSL}(3,q) \leq G$ and one of the following occurs:
   
   - (a) $n = q$ and $\Pi = \Pi_0$;
   - (b) $n = q^2$, $\Pi$ is a Desarguesian plane or a Generalized Hughes plane and $\Pi_0$ is a Baer subplane of $\Pi$;
   - (c) $n = q^3$.

2. $\Pi_0 \cong \text{PG}(2,7)$, $\Pi$ is the generalized Hughes plane over the exceptional nearfield of order $7^2$ and $\text{SL}(3,7) \leq G$.

Proof. See [17].

Finally, we quote this useful final result, due to Moorhouse [20], which inspired the present paper, and that allows to reduce our investigation to $n < q^4$ (when $q > 3$).

Theorem 2.4 (Moorhouse). Let $\Pi$ be a projective plane of order $q^4$ admitting $G \cong \text{PSL}(3,q)$, $q$ odd. If $q > 3$, then the following must hold.

(i) $G$ leaves invariant a Desarguesian subplane $\Pi_0$ of order $q$, on which $G$ acts 2-transitively;

(ii) The involutions in $G$ are homologies of $\Pi$, and those $p$-elements of $G$ which induce elations of $\Pi_0$ are elations of $\Pi$.

If $q = 3$ then the same two conclusions must hold, under the additional hypothesis that $G$ acts irreducibly on $\Pi$.

Proof. See [20, Theorem 1.3].
3. Preliminary reductions

The aim of this section is to show that $G$ is irreducible on $\Pi$ and that the involutions in $G$ are perspectivities of $\Pi$, in order to apply Hering-Walker theory on the strong irreducibility (e.g. see [6, 8] and [9]).

In view of Theorem 2.1, we treat the cases $q = 3$ and $q > 3$ separately.

**Lemma 3.1.** Let $\Pi$ be a finite projective plane of order $n$ that admits $G \cong \text{PSL}(3, 3)$ as a collineation group. If $n \leq 3^4$, then each involution in $G$ is a perspectivity of $\Pi$.

**Proof.** Assume that the involutions in $G$ are Baer collineations of $\Pi$. Hence, $\sqrt{n} \leq 9$. Let $J$ be a Sylow 2-subgroup of $G$. As $n^2 + n + 1$ is odd, then $J$ fixes a secant $s$ of $\text{Fix}(\psi)$. Let $J_0 = J \cap C_G(\psi)$. Then $J_0 \cong Q_8$. Thus, $J_0$ is semiregular on $s - \text{Fix}(\psi)$. So, $8 \mid \sqrt{n}(\sqrt{n} - 1)$, since $|s - \text{Fix}(\psi)| = \sqrt{n}(\sqrt{n} - 1)$. Consequently, either $\sqrt{n} = 8$ or 9, as $\sqrt{n} \leq 9$. Note that $J = J_0$. $\langle \beta \rangle$ is known to be semidihedral of order 16. As $J_0$ is semiregular on $s - \text{Fix}(\psi)$, then each $J$-orbit on $s - \text{Fix}(\psi)$ has length either 8 or 16. Therefore, let $x$ and $y$ be the number of $J$-orbits on $s - \text{Fix}(\psi)$ of length 8 and 16, respectively. It follows that

$$8x + 16y = \sqrt{n}(\sqrt{n} - 1),$$

where $\sqrt{n} = 8$ or 9. As $J$ is semidihedral of order 16, then $J$ contains two distinct conjugate classes of involutions, one consisting of $\psi$ and the other consisting of the four conjugates of $\beta$ (including $\beta$). Furthermore, $C_J(\beta) \cong \langle \psi, \beta \rangle \cong E_4$. Thus, by [19, Relation (8)], the involution $\beta$ fixes 2 and 0 points on the $J$-orbits on $s - \text{Fix}(\psi)$ of length 8 and 16, respectively, since $\psi \in J_0$ and since $J_0$ is semiregular on $s - \text{Fix}(\psi)$. Hence, $\beta$ fixes exactly $2x$ points on $s - \text{Fix}(\psi)$. If $x$ is even, then $16 \mid \sqrt{n}(\sqrt{n} - 1)$ by (3), which is impossible as $\sqrt{n} = 8$ or 9. Therefore, $x$ is odd. Hence, $\beta$ cannot induce either the identity or a perspectivity of axis $s$ on $\text{Fix}(\psi)$, otherwise $x = 0$, since $\beta$ is a Baer collineation on $\Pi$ (recall that $G \cong \text{PSL}(3, 3)$ has a unique conjugate class of involutions). Suppose that $\beta$ induces a perspectivity on $\text{Fix}(\psi)$ of axis distinct from $s$. Clearly, $\beta$ induces on $\text{Fix}(\psi)$ either an elation when $\sqrt{n} = 8$ or a homology when $\sqrt{n} = 9$. Then $x = \sqrt{n}$ or $\sqrt{n} - 1$, respectively, again since $\beta$ is a Baer collineation on $\Pi$. So, $x$ is even in any case, which is a contradiction. Finally, assume that $\beta$ induces a Baer collineation on $\text{Fix}(\psi)$ when $\sqrt{n} = 9$. Arguing as above, we have that $x = \sqrt{n} - \sqrt{n}$ which is even and we again obtain a contradiction. Thus, the involutions in $G$ are perspectivities of $\Pi$. □

**Proposition 3.2.** Let $\Pi$ be a finite projective plane of order $n$ that admits $G \cong \text{PSL}(3, 3)$ as a collineation group. If $n \leq 3^4$, then the following occurs:
(1) There exists a subplane \( \Pi_0 \cong \text{PG}(2, 3) \) of \( \Pi \) on which \( G \) acts in the natural way;

(2) The group \( G \) is irreducible on \( \Pi \);

(3) The involutions in \( G \) are homologies of \( \Pi \);

(4) The 3-elements that induce elations on \( \Pi_0 \) are elations of \( \Pi \).

Moreover, one of the following occurs:

(i) \( n = 3 \) and \( \Pi = \Pi_0 \cong \text{PG}(2, q) \);

(ii) \( n = 3^2 \), \( \Pi \) is a Desarguesian plane or a Generalized Hughes plane and \( \Pi_0 \) is a Baer subplane of \( \Pi \);

(iii) \( n = 3^3 \);

(iv) \( n = 3^4 \).

**Proof.** Assume that \( G \cong \text{PSL}(3, 3) \) fixes a line \( l \) of \( \Pi \). As \( n \leq 3^4 \), then each nontrivial \( G \)-orbit on \( l \) has length divisible by 13 by [2]. Actually, \( G \) contains such orbits, since \( G \) acts faithfully, \( G \) being nonabelian simple. Let \( X \in l \) such that \( 13 \mid |X^G| \). So, \( G_X \leq E_9.\text{GL}(2, 3) \). Let \( B_X \) be the block of imprimitivity in \( X^G \) containing \( X \). Clearly \( |X^G| = 13 |B_X| \) (\(|B_X| \) might be 1). Furthermore, \( E_9.\text{GL}(2, 3) \) acts transitively on \( B_X \). As the socle of \( E_9.\text{GL}(2, 3) \) is \( E_9 \), then either \( E_9 \leq G_X \) or \( 13 \cdot 9 \mid |X^G| \) by [3, Theorem 4.1A]. Actually, the latter cannot occur, since \( |X^G| \leq n + 1 \) and \( n \leq 3^4 \). Hence, each nontrivial \( G \)-orbit on \( l \) has length \( |X^G| = 13 |B_X| \), where \( |B_X| \mid |\text{GL}(2, 3)| \). Actually, \( |B_X| = 1, 2, 3, 4 \) or 6, since \( |B_X| \leq 6 \), as \( n \leq 3^4 \). Since the blocks of imprimitivity are 13, then there exists a point \( P \), lying in a nontrivial \( G \)-orbit on \( l \), such that \( J_0 \) fixes \( B_P \), where \( J_0 \) is the 2-group isomorphic to \( Q_8 \) containing the involution \( \psi \). Thus, \( \psi \) fixes \( B_P \) pointwise, since \( |B_P| \leq 6 \). Then \( |B_P| \leq 2 \), since \( \psi \) is a perspectivity of \( \Pi \) having axis distinct from \( l \). If \( |B_P| = 1 \), then \( P^G \) is a 2-transitive \( G \)-orbit. Hence, \( \psi \) fixes exactly 5 points on \( X^G \). So, we arrive at a contradiction, since \( \psi \) is a perspectivity of \( \Pi \) having axis distinct from \( l \). Thus, \( |B_P| = 2 \) and hence \( l = P^G \), since \( \psi \) fixes \( B_P \) pointwise. In particular, \( n = 25 \), since \( |P^G| = 26 \). Since \( G \) acts faithfully on \( l \), there are no involutional homologies of axis \( l \). Therefore, no involutions lie in a triangular configuration. In particular, since \( \psi \) is the unique central involution in \( J \) (recall that \( J \) is semidihedral of order 16), each involution in \( J \) has center and axis \( C_\psi \) and \( a_\psi \), where \( C_\psi \) and \( a_\psi \) denote the center and the axis of \( \psi \), respectively. So, each involution in \( J \) fixes exactly two points on \( l \), namely \( C_\psi \) and \( a_\psi \cap l \). Hence, \( J \) is semiregular on \( l - \{C_\psi, W\} \), where \( \{W\} = a_\psi \cap l \). Then \( 16 \mid n - 1 \), which is a contradiction, since \( n = 25 \).
while $|J| = 16$. Therefore, $G$ does not fix lines. By the dual of the previous proof, we obtain that $G$ does not fix points. Finally, these two facts, combined with the fact that $G$ is nonabelian simple, yield that $G$ does not fix triangles of $\Pi$. Thus, $G$ is irreducible on $\Pi$ and hence the assertion (2).

Since $G \cong PSL(3, 3)$ is irreducible on $\Pi$, and since each involution in $G$ is a perspectivity by Lemma 3.1, then $G$ leaves invariant a subplane $\Pi_0$ on which it acts strongly irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong PG(2, 3)$ by [8, Theorem 1.1], and we obtain the assertion (1). Therefore, the involutions in $G$ are homologies of $\Pi$ and hence the assertion (3). For $n \leq 3^3$, the assertions (4) and (i)–(iii) follow by Theorem 2.3. Furthermore, for $n = 3^4$ the assertions (4) and (iv) follow by Theorem 2.4, since we proved the irreducibility of $G$ on $\Pi$.

Hence, assume that $3^3 < n < 3^4$. Note that $G$ contains an elementary abelian group $H$ of order $3^2$ consisting of elations with the same axis $r$ and distinct centres lying in $\Pi_0 \cap r$ by [10, Theorem 4.25]. As $H$ is semiregular on $[Q] - \{l\}$, for any $Q \in r - \Pi_0$, then $3^2 | n$. So, $3^3 < n < 3^4$, $n$ odd, and $3^2 | n$ yield that $n = 3^25$ or $3^27$. Let $E$ be the set of external lines to $\Pi_0$. Easy computations yield $|E| = 1512$ or $3240$, respectively. Let $R$ be any Sylow 2-subgroup of $G$. Then $|R| = 16$. Since each involution in $G$, and hence in $R$, is a homology of axis a secant to $\Pi_0$, then $R$ is semiregular on $E$. So, $16 \mid |E|$, which is impossible as $|E| = 1512$ or $3240$. This completes the proof. □

It should be pointed out that the previous theorem extends the Theorem 2.4 also for $n = 3^4$. Indeed, Theorem 2.4 works for $q = 3$ under the additional assumption that $G$ is irreducible on $\Pi$. In particular, Moorhouse shows that the irreducibility of $G$ on $\Pi$ implies that the involutions in $G$ are homologies of $\Pi$. We, instead, prove that the involutions are perspectivities of $\Pi$ and then we use this fact to prove that $G$ is irreducible on $\Pi$.

*From now on, we assume that $q > 3$.

**Lemma 3.3.** The group $G$ is irreducible on $\Pi$.

**Proof.** Assume that $G$ fixes a line $l$ of $\Pi$. Then $\sqrt{n} < q^2$, since for $n = q^4$ the assertion follows by [20] (e.g. see the proof of Theorem 1.3). Let $\psi$ be the involution in $G$ defined in Section 2. Then, by Theorem 2.1 and by bearing in mind that $q$ is odd and $\sqrt{n} < q^2$, one of the following occurs:

1. $\sqrt{n} = 4$, $\text{Fix}(\psi) \cong PG(2, 4)$ and $C_G(\psi)'/\langle \psi \rangle \cong PSL(2, 5)$;
2. $\sqrt{n} = 16$ and $C_G(\psi)'/\langle \psi \rangle \cong PSL(2, 5)$;
(3) $\sqrt{n} = 16$, Fix$(\psi)$ is either the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C_G(\psi)/(\psi) \cong PSL(2, 7)$.

Assume that the case (1) occurs. Since $n + 1 = 17$ and since these primitive permutation representations of $G$ have a degree greater than 17 by [2], then $G$ fixes $l$ pointwise. That is, $G$ is a group of perspectivities of axis $l$. So, $G$ should be a Frobenius group by [10, Theorem 4.25], which is impossible as $G$ is nonabelian simple.

We treat the cases (2)–(3) simultaneously. By a direct inspection of [2], it is plain that the unique nontrivial orbits on $l$ under $G \cong PSL(3, q)$, $q = 5$ or 7, are those of length a multiple of $d_0$, the minimal primitive permutation representation degree of $G$. By [2], such a $d_0$ is equal to 31 or 57, respectively. Let $r$ be the minimal nonnegative integer such that $n + 1 \equiv r_0 \mod d_0$. Easy computations yield that $r_0 = 9, 29$ or 6 in the cases (1)–(3), respectively. So, $6 \leq r_0 < n+1$ and $\sqrt{n}+1 \neq r_0 \mod d_0$ in any case. Therefore, $G$ fixes at least 6 points on $l$ in any case. Let $P$ be any of these points. Now, by repeating the above argument with $[P]$ in the role of $l$, we obtain that $G$ fixes at least 6 lines of $[P]$ (clearly, the line $l$ is included). Again, by repeating the above argument for any for each of these 6 lines, we obtain that $G$ fixes a subplane $\Sigma$ of $\Pi$ pointwise. Let $r$ be the order of $\Sigma$. Then $r = r_0 + h d_0 - 1$, where $h \geq 0$. Note that $r_0 + h d_0 - 1 \leq \sqrt{n}$ by [10, Theorem 3.7]. Hence, the case (3) is ruled out. Actually, $r_0 + h d_0 - 1 < \sqrt{n}$, since $\sqrt{n} + 1 \neq r_0 \mod d_0$. Thus, $\Sigma \subset \text{Fix}(\psi)$, since $\Sigma \subseteq \text{Fix}(\psi)$. Therefore, $(r_0 + h d_0 - 1)^2 \leq \sqrt{n}$ by [10, Theorem 3.7]. This forces $h = 0$ in any admissible case. In particular, the case (2) is ruled out. Consequently, $G$ is irreducible on $\Pi$. 

\[ \text{Throughout this section, we assume that } \psi \text{ is a Baer collineation of } \Pi. \]
\[ \text{Then } n < q^4 \text{ by Theorem 2.4, as } q > 3. \]

The following lemma determines the structure of the kernel $K_\psi$ of the action of $C_G(\psi)$ on Fix$(\psi)$.

**Lemma 3.4.** \( \langle \psi \rangle \leq K_\psi \leq Z_\psi. \)

**Proof.** Clearly, \( \langle \psi \rangle \leq K_\psi \leq C_G(\psi). \) Recall that $C_G(\psi) \cong Z_\psi PGL(2, q)$. Since $K_\psi Z_\psi / Z_\psi \leq PGL(2, q)$, then either $K_\psi Z_\psi / Z_\psi = \langle 1 \rangle$ or $PSL(2, q) \leq K \bar{Z}_\psi / \bar{Z}_\psi$. Assume that the latter occurs. Then $C_G'(\psi) \leq K_\psi$, since $C_G'(\psi) / \langle \psi \rangle \cong PSL(2, q)$ and since $\langle \psi \rangle \leq K_\psi \leq C_G(\psi)$. Since for each involution $\beta \in G$ there exists $g \in G$ such that $\psi^g = \beta$, then $C_G'(\psi)^g = C_G'(\beta)$. Hence $C_G'(\beta)$ fixes Fix$(\beta)$ pointwise for each involution $\beta$ in $G$. By Lemma 2.2, there exist two involutions $\psi_1$ and $\psi_2$ such that $C_G'(\psi_1) \cap C_G'(\psi_2) \neq \langle 1 \rangle$ and $\langle C_G'(\psi_1), C_G'(\psi_2) \rangle = G$. Since
For each subgroup $X$ of $C_G(\psi)$, we denote by $\tilde{X}$ the group $X K_\psi / K_\psi$.

**Lemma 3.5.** For each point $X \in \Pi$ such that $G_X$ lies in a maximal parabolic subgroup of $G$, one of the following occurs:

1. $X^G$ is a 2-transitive orbit;
2. $\text{Fix}(U^* \langle \psi \rangle)$ is either a flag, or an antiflag or a proper subplane of $\text{Fix}(\psi)$. Furthermore, $C_G(\psi)$ leaves $\text{Fix}(U^* \langle \psi \rangle)$ invariant;
3. $q^2 \mid |X^G|$.

**Proof.** Let $X \in \Pi$ and assume that $G_X$ lies in a maximal parabolic subgroup of $G$. As mentioned in Section 2, for our purposes we may reduce to study the case when $G_X \leq U^* : C_G(\psi)$, where $C_G(\psi) \cong Z_\psi . \text{PGL}(2,q)$ and $Z_\psi \cong Z_{\frac{q-1}{2}}$, $\mu = (3,q - 1)$. If $G_X = U^* : C_G(\psi)$, then $X^G$ is a 2-transitive orbit and we obtain the assertion (1). If $G_X < U^* : C_G(\psi)$, denoted by $B_X$ the block of imprimitivity in $X^G$ containing $X$, we have $|B_X| > 1$. Clearly, $U^* : C_G(\psi)$ acts on $B_X$.

Assume that $U^* : C_G(\psi)$ does not act faithfully on $B_X$, then $U^*$ lies in the kernel of the action, since $U^*$ is the socle of $U^* : C_G(\psi)$ by [3, Theorem 4.3B]. Thus, $\text{Fix}(U^*) \neq \emptyset$. Since $U^* \not< U^* : C_G(\psi)$, and since $\text{Fix}(U^* : C_G(\psi)) = \emptyset$, being $G_X < U^* : C_G(\psi)$, either $\text{Fix}(U^*) = \Delta$, where $\Delta$ is a triangle of $\Pi$, or $\text{Fix}(U^*)$ is a subplane of $\Pi$ by [6, Corollary 3.6]. This yields that $\text{Fix}(U^* \langle \psi \rangle)$ consists of either a flag, or an antiflag or a plane. Clearly, $\text{Fix}(U^* \langle \psi \rangle) \subseteq \text{Fix}(\psi)$. Furthermore, $C_G(\psi)$ acts on $\text{Fix}(\psi)$ leaving $\text{Fix}(U^* \langle \psi \rangle)$ invariant, since $U^* \langle \psi \rangle < U^* : C_G(\psi)$. If $\text{Fix}(U^* \langle \psi \rangle) = \text{Fix}(\psi)$, then $\text{Fix}(U^*) = \text{Fix}(\psi)$, since $\text{Fix}(\psi)$ is a Baer subplane of $\Pi$. So, $U^*$ is semiregular on $s - \text{Fix}(U^*)$, where $s$ is a secant of $\text{Fix}(U^*)$. Therefore, $q^2 \mid n - \sqrt{n}$, since $|U^*| = q^2$. That is, either $q^2 \mid \sqrt{n} - 1$ or $q^2 \mid \sqrt{n}$, and we have a contradiction in any case since $n < q^4$ and $q > 3$. Thus, we obtain the assertion (2).

Assume that $U^* : C_G(\psi)$ acts faithfully on $B_X$. Then $q^2 \mid |B_X|$ by [3, Theorem 4.1A], since $U^*$ is the socle of $U^* : C_G(\psi)$. Thus, $q^2 \mid |X^G|$ and we obtain the assertion (3). \qed

**Lemma 3.6.** One of the following occurs:

1. The groups $C_G'(\psi)$ and $C_G(\psi)$ are strongly irreducible on $\text{Fix}(\psi)$;
(II) $q = 5$ and $n = 4$;

(III) $q = 9$ and $9^2 < n < 9^4$.

Proof. Assume that the cases (II) and (III) do not occur. Note that $\overline{C_G(P)} \cong \text{PSL}(2, q)$, since $C_G(P) \cap K = \langle \psi \rangle$ by Lemma 3.4. Suppose that the $\overline{C_G(P)}$ is not strongly irreducible on $\text{Fix}(\psi)$. The case $\sqrt{n} = q$ is ruled out by Theorem 2.1. As $\sqrt{n} < q^2$, then either $\sqrt{n} < q$ or $q < \sqrt{n} < q^2$. Then, again by Theorem 2.1 and bearing in mind that the cases (II) and (III) do not occur by our assumptions, one of the following occurs:

1. $n = 4$ or $16$, $\text{Fix}(\psi) \cong \text{PG}(2, 2)$ or $\text{PG}(2, 4)$, respectively, and $\overline{C_G(P)} \cong \text{PSL}(2, 7)$;

2. $n = 16$, $\text{Fix}(\psi) \cong \text{PG}(2, 4)$ and $\overline{C_G(P)} \cong \text{PSL}(2, 9)$;

3. $n = 16^2$, $\overline{C_G(P)} \cong \text{PSL}(2, 5)$ fixes a subplane of $\text{Fix}(\psi)$ isomorphic to $\text{PG}(2, 4)$;

4. $7^2 < n < 49^2$, $\overline{C_G(P)} \cong \text{PSL}(2, 7)$ fixes a subplane of $\text{Fix}(\psi)$ isomorphic either to $\text{PG}(2, 2)$ or to $\text{PG}(2, 4)$.

Actually, in the cases (1)–(4), the group $\overline{C_G(P)} \cong \bar{Z}_\psi \cdot \text{PGL}(2, q)$ acts on $\text{Fix}(\psi)$. The group $\overline{C_G(P)}$ fixes a subplane $\Pi_0$ of $\text{Fix}(\psi)$ isomorphic either to $\text{PG}(2, 2)$ or to $\text{PG}(2, 4)$ for $q = 7$, or to $\text{PG}(2, 4)$ for $q \neq 7$ (note that it might be $\Pi_0 = \text{Fix}(\psi)$).

Assume that $q = 7$. Then $\bar{Z}_\psi = \langle 1 \rangle$, since $Z_\psi = \langle \psi \rangle$. Therefore $\overline{C_G(P)} \cong \text{PGL}(2, 7)$ acts on $\Pi_0$. Then the case $\Pi_0 \cong \text{PG}(2, 2)$ is ruled out, since the full automorphism group of $\text{PG}(2, 2)$ is isomorphic to $\text{PSL}(2, 7)$. Hence, assume that $\Pi_0 \cong \text{PG}(2, 4)$ and $\overline{C_G(P)} \cong \text{PSL}(2, 7)$. It is easy to see that $\text{PSL}(2, 7)$ fixes a subplane $\Pi_1$ of $\Pi_0$ which is isomorphic to $\text{PG}(2, 2)$. In particular, $\text{PGL}(2, 7)$ leaves $\Pi_1$ invariant. So, we arrive at a contradiction by the above argument with $\Pi_1$ in the role of $\Pi_0$. Therefore, $q \neq 7$ and hence the cases (1) and (4) are ruled out.

Assume that $q = 5$ or 9. Then $\Pi_0 \cong \text{PG}(2, 4)$ and hence $\overline{C_G(P)} \leq \text{PGL}(3, 4)$. Furthermore, $\bar{Z}_\psi = \langle 1 \rangle$ by [2]. Consequently, $\bar{Z}_\psi$ fixes $\text{Fix}(\psi)$ pointwise and $\overline{C_G(P)} \cong \text{PGL}(2, q)$ in any case. Since $Z_\psi$ is semiregular on $\text{Fix}(\psi)$, then $\frac{q-1}{\mu} | n - \sqrt{n}$, where $\frac{q-1}{\mu} = |Z_\psi|$ and $\mu = (3, q - 1)$. That is, $\frac{q-1}{\mu} | \sqrt{n}$ or $\frac{q-1}{\mu} | \sqrt{n} - 1$, since $q = 5$ or 9. Thus, the case (2) is ruled out, since $\sqrt{n} = 4$, while $\frac{q-1}{\mu} = 8$.

It remains to investigate the case (3). In this case, any subgroup $Z_{31}$ of $G$ fixes a subplane of $\Pi$ of order $7 + 31k$, $k \geq 0$. Actually, $k = 0$ by [10, Theorem 3.7], since $n = 16^2$. Therefore, $Z_{31}$ fixes exactly 57 points of $\Pi$. Note that $Z_{31} \leq \text{PG}(2, 4)$.
$G_X \leq Z_{31}.Z_3$ for any point $X$ of $\Pi$ fixed by $Z_{31}$ by [2]. Moreover, $Z_{31}.Z_3$ is maximal in $G$. So, either $G_X = Z_{31}, \mid X^G \mid = 12000$ and $Z_{31}$ fixes 3 points on $X^G$, or $G_X = Z_{31}.Z_3, \mid X^G \mid = 4000$ and $Z_{31}$ fixes 1 point on $X^G$. Let $x$ and $y$ be the number of $G$-orbits on $\Pi$ of length 12000 and 4000, respectively. Then $12000x + 4000y \leq 65793$, since $n^2 + n + 1 = 65793$. Furthermore, $3x + y = 57$, since $Z_{31}$ fixes exactly 57 points of $\Pi$. By combining the previous relations involving $x$ and $y$, we obtain a contradiction. Thus, $C'_G(\psi)$ is strongly irreducible on $Fix(\psi)$. Then $\overline{C}_G(\psi)$ is strongly irreducible on $Fix(\psi)$, since $\overline{C}_G(\psi) \leq C_G(\psi)$. That is, the assertion (I) occurs.

**Lemma 3.7.** The group $G$ does not admit 2-transitive point-orbits on $\Pi$.

**Proof.** Let $\mathcal{O}$ be a 2-transitive $G$-orbit on $\Pi$. Then $|\mathcal{O}| = q^2 + q + 1$. Clearly, $\mathcal{O}$ cannot be contained in a line by lemma 3.3. Then, it is a plain that, either $\mathcal{O}$ is an arc or $\mathcal{O} \cong PG(2,q)$. Assume that the former occurs. Let $U^*$ be the elementary abelian $p$-group defined in Section 2. Then $U^*(\langle \psi \rangle)$ fixes exactly $q + 1$ points on $\mathcal{O}$. So $U^*(\langle \psi \rangle)$ is planar, since $\mathcal{O}$ is an arc. Since $U^*(\langle \psi \rangle) \leq U^*C'_G(\psi)$ and since $C'_G(\psi)$ acts 2-transitively on $Fix(U^*(\langle \psi \rangle)) \cap \mathcal{O}$, then $C'_G(\psi)$ acts as $PSL(2,q)$ on $Fix(U^*(\langle \psi \rangle))$. Note that

$$|Fix(U^*(\langle \psi \rangle)) \cap \mathcal{O}| = q + 1 \quad \text{and} \quad |Fix(\langle \psi \rangle) \cap \mathcal{O}| = q + 2,$$

as $q$ is odd. Thus $Fix(U^*(\langle \psi \rangle)) \subset Fix(\langle \psi \rangle) \subset \mathcal{O}$, with $o(Fix(U^*(\langle \psi \rangle))) \geq q - 1$. Assume that $o(Fix(U^*(\langle \psi \rangle))) = q$. Then $\sqrt{n} \geq q^2$ by [10, Theorem 3.7], since $Fix(U^*(\langle \psi \rangle)) \subset Fix(\langle \psi \rangle)$, which is contrary to the assumption $\sqrt{n} < q^2$. So, $o(Fix(U^*(\langle \psi \rangle))) = q - 1$. Then $Fix(U^*(\langle \psi \rangle)) \cap \mathcal{O}$ is a hyperoval of $Fix(U^*(\langle \psi \rangle))$, as $|Fix(U^*(\langle \psi \rangle)) \cap \mathcal{O}| = q + 1$. Furthermore, $C_G(\langle \psi \rangle) \cong Z_3/\langle \psi \rangle . PGL(2,q)$, where $|Z_\psi| = 2^{q-1}$ and $\mu = (3,q-1)$, acts 2-transitively on $Fix(U^*(\langle \psi \rangle)) \cap \mathcal{O}$. Then $q - 1 = 4$ and $C_G(\langle \psi \rangle) \cong S_6$ by [1], as $q > 3$. This implies that $Fix(Z_\psi) = Fix(\langle \psi \rangle)$. So, $C'_G(\langle \psi \rangle)$ acts on $Fix(\langle \psi \rangle)$ as $PGL(2,5)$ leaving invariant a subplane $Fix(U^*(\langle \psi \rangle)) \cong PG(2,4)$, which is impossible by Lemma 3.6, as $n > 4$.

Assume that $\mathcal{O} \cong PG(2,q)$. As $\psi$ is Baer collineation of $\Pi$ and $\psi$ induces a homology on $\mathcal{O}$, then $C'_G(\langle \psi \rangle)$ acts on $Fix(\langle \psi \rangle)$ as $PSL(2,q)$ and it also fixes an antiflag. Note that $q^3 < n < q^4$ by [17, Proposition 11], and since $n \neq q^4$ by our assumption. Then, by Theorem 2.1 (3a), either $Fix(\langle \psi \rangle)$ has order 16 and $C'_G(\langle \psi \rangle)/\langle \psi \rangle \cong PSL(2,5)$, or $Fix(\langle \psi \rangle)$ is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C'_G(\langle \psi \rangle)/\langle \psi \rangle \cong PSL(2,7)$. However, the same argument as in Lemma 3.6 rules out both these cases, since $C'_G(\langle \psi \rangle)/\langle \psi \rangle$ fixes an antiflag. This completes the proof.

**Lemma 3.8.** The groups $\overline{C}_G'(\psi)$ and $\overline{C}_G(\psi)$ are strongly irreducible on $Fix(\psi)$. 

Proof. In order to prove the assertion, by Lemma 3.6, we need to analyze only the case \((q, n) = (5, 4)\) and \(q = 9\) when \(9^2 < n < 9^4\). Recall that \(G \cong \text{PSL}(3, q)\) is irreducible on \(\Pi\) by Lemma 3.3. Then \(\Pi\) consists of nontrivial \(G\)-orbits. Since each \(G\)-orbit has length \(\lambda_jd_j(G)\), where \(\lambda_j \geq 0\) and \(d_j(G)\) is the degree of some primitive permutation representation of \(G\), then

\[
n^2 + n + 1 = \sum_{j \geq 0} \lambda_jd_j(G).
\]

That is, \(n^2 + n + 1\) must admit a partition restricted to

\[D(G) = [d_0(G), d_1(G), \ldots, d_k(G)]\]

the spectrum of the degrees of the primitive permutation representations of \(G\). So, the case \((q, n) = (5, 4)\) is ruled out, since \(n^2 + n + 1 = 21\), while \(D(G) = [31, 3100, 3875, 4000]\) by [2].

Assume that \(q = 9\) and \(9^2 < n < 9^4\). As above, by [2], \(n^2 + n + 1\) must admit a partition restricted to

\[D(G) = [91, 7020, 7560, 58968, 110565, 155520]\]

Note that \(9 \mid d_j(G)\) for each \(j > 0\). If \(\lambda_0 = 0\), then \(9 \mid n^2 + n + 1\) by (4), while it is known that either \(n^2 + n + 1 \equiv 1 \mod 3\) or \(n^2 + n + 1 \equiv 3 \mod 9\). Hence, \(\lambda_0 > 0\). So, there exists a point \(X \in \Pi\) such that \(G_X \leq U^* : C_G(\psi)\), where \(C_G(\psi) \cong \text{GL}(2, 9)\), by [2]. Since the group \(G\) does not admit 2-transitive point-orbits on \(\Pi\) for \(n < 9^4\) by Lemma 3.7, then \(G_X < U^* : C_G(\psi)\). Hence, by Lemma 3.5, either \(9 \mid |X^G|\) or \(\text{Fix}(U^*(\psi))\) is either a flag, or an antiflag or a proper subplane of \(\text{Fix}(\psi)\). Furthermore, \(C_G(\psi)\) leaves \(\text{Fix}(U^*(\psi))\) invariant.

Assume that the latter occurs. If \(\text{Fix}(U^*(\psi))\) consists of a flag or an antiflag, again by Theorem 2.1, the case (3) inside the proof of Lemma 3.6 occurs, which leads to a contradiction, as we have seen. So, \(\text{Fix}(U^*(\psi))\) is a proper subplane of \(\text{Fix}(\psi)\). Then \(\text{Fix}(U^*(\psi)) \cong \text{PG}(2, 4)\) by Theorem 2.1. Note that either \(\text{Fix}(U^*(\psi)) \cong \text{PG}(2, 4)\) is a Baer subplane of \(\text{Fix}(U^*)\) or \(\text{Fix}(U^*(\psi)) = \text{Fix}(U^* \cong \text{PG}(2, 4))\). Suppose that \(\text{Fix}(U^*(\psi)) \cong \text{PG}(2, 4)\) is a Baer subplane of \(\text{Fix}(U^*)\). Note that \(Z_\psi = \langle 1 \rangle\) by [2]. Consequently, \(Z_\psi\) fixes \(\text{Fix}(\psi)\) pointwise and \(C_G(\psi) \cong \text{PGL}(2, 9)\). Hence, \(\text{Fix}(\psi) = \text{Fix}(Z_\psi)\). Thus, \(\text{Fix}(U^*(\psi)) = \text{Fix}(U^*.Z_\psi)\), as \(Z_\psi\) normalizes \(U^*\). That is, \(\text{Fix}(U^*.Z_\psi) \cong \text{PG}(2, 4)\) is a Baer subplane \(\text{Fix}(U^*)\). Then \(Z_\psi\) is semiregular on \(s \cap (\text{Fix}(U^*) - \text{Fix}(U^*.Z_\psi))\), where \(s\) is a secant of \(\text{Fix}(U^*.Z_\psi)\). So \(8 \mid 16 - 4\), since \(o(\text{Fix}(U^*)) = 16\), \(\text{Fix}(U^*(\psi)) \cong \text{PG}(2, 4)\) and since \(|Z_\psi| = \frac{q^3 - 1}{m} = 8\); this is a contradiction. Thus, \(\text{Fix}(U^*(\psi)) = \text{Fix}(U^*) \cong \text{PG}(2, 4)\). If there exists a nontrivial element \(\rho\) in \(U^*\) fixing a point in \(\Pi - \text{Fix}(U^*)\), then \(\text{Fix}(\rho)\) is a Baer subplane of \(\Pi\), since \(\text{Fix}(U^*) \cong \text{PG}(2, 4)\) and \(n = 16^2\). Then each nontrivial element in
$U^*$ fixes a subplane of order 16 of $\Pi$, since the nontrivial elements in $U^*$ are conjugate under $C_G(\psi) \cong \text{GL}(2,9)$. Hence, if $Q$ is a point fixed by $U^*$, then $9^2 | (9^2 - 1)(\sqrt{n} + 1) + (n + 1)$ by Cauchy-Frobenius Lemma, since $|U^*| = 9^2$. So, $9^2 | n - \sqrt{n}$, which is a contradiction, since $n = 16^2$. Therefore, $U^*$ is semiregular on $r - \text{Fix}(U^*)$, where $r$ is a secant to $U^*$. Hence, $9^2 | n - 4$ and we again obtain a contradiction, as $n = 16^2$. Thus, $9 | \langle |X^G\rangle$. Actually the previous argument can be repeated for each point $Y \in \Pi$ such that $G_Y$ lies in a maximal parabolic subgroup of $G$. Consequently, any orbit divisible by $d_0(G)$ is actually divisible by $9d_0(G)$. Therefore, bearing in mind that $9 | d_j(G)$ for each $j > 0$, any admissible $G$-orbit has length divisible by 9. So, $9 | n^2 + n + 1$ by (4), and we obtain a contradiction as above. This completes the proof. \hfill \Box

**Lemma 3.9.** The group $\overline{C_G(\psi)}$ contains Baer involutions of $\text{Fix}(\psi)$. In particular, $\sqrt{n}$ is an integer.

**Proof.** Assume that all the involutions in $\overline{C_G(\psi)}$ are perspectivities of $\text{Fix}(\psi)$. If $\sqrt{n}$ is even, then either $\text{Fix}(\psi) \cong \text{PG}(2,2)$ and $C_G(\psi) / \langle \psi \rangle \cong \text{PSL}(2,7)$ or $\text{Fix}(\psi) \cong \text{PG}(2,4)$ and $C_G(\psi) / \langle \psi \rangle \cong \text{PSL}(2,9)$ by [9]. However, both these cases cannot occur by the same argument as in Lemma 3.6. Hence, $\sqrt{n}$ is odd and the involutions in $\overline{C_G(\psi)}$ are homologies of $\text{Fix}(\psi)$.

If $K = Z_\psi$, then $\overline{C_G(\psi)} \cong \text{PGL}(2,q)$. Then $q | \sqrt{n}$ and $q - 1 | \sqrt{n} - 1$ by [12, Theorem C.ii]. As $q | \sqrt{n}$, then $\sqrt{n} = \lambda_1 q$ for some $\lambda_1 \geq 0$. Furthermore, $\lambda_1 = (q - 1)\lambda_2 + 1$ for some $\lambda_2 \geq 0$, since $q - 1 | \sqrt{n} - 1$. Hence, $\sqrt{n} = q(q - 1)\lambda_2 + q$. However, this is impossible, since $n < q^4$ by our assumption.

If $K < Z_\psi$. Then $\overline{Z_\psi} \neq \langle 1 \rangle$. Since $\overline{C_G(\psi)}$ is strongly irreducible on $\text{Fix}(\psi)$ by Lemma 3.8, and since each nontrivial subgroup of $\overline{Z_\psi}$ is normal in $\overline{C_G(\psi)}$, then $\overline{Z_\psi}$ is semiregular on $\text{Fix}(\psi)$. Let $\bar{\sigma}$ be any involutory $(C_\sigma, a_\sigma)$-homology of $\overline{C_G(\psi)}$. Note that $\overline{C_G(\psi)} \times \overline{Z_\psi} \leq \overline{C_G(\psi)}$. That is, $\overline{Z_\psi}$ centralizes $\bar{\sigma}$ and hence $\overline{Z_\psi}$ fixes $(C_\sigma, a_\sigma)$. This is impossible, since $\overline{Z_\psi}$ is semiregular on $\text{Fix}(\psi)$. Therefore, $\overline{C_G(\psi)}$ contains Baer collineation of $\text{Fix}(\psi)$ and hence $\sqrt{n}$ is an integer. \hfill \Box

**Proposition 3.10.** For each $X \in \Pi$ such that $G_X$ lies in a maximal parabolic subgroup of $G$, then $q^2 | \langle |X^G\rangle$.

**Proof.** Since $\overline{C_G(\psi)}$ is strongly irreducible on $\text{Fix}(\psi)$ by Lemma 3.8 and since the group $G$ does not admit 2-transitive point-orbits on $\Pi$ by Lemma 3.7, the assertion follows by Lemma 3.5. \hfill \Box

**Lemma 3.11.** One of the following occurs:

(1) $q^2 | n^2 + n + 1$;
(2) \( q \) is a square, \( q\sqrt{q} \mid n^2 + n + 1 \), and there exists a point \( Y \in \Pi \) such that either \( G_Y \leq \text{PSL}(3, \sqrt{q}) \) or \( G_Y \leq \text{PSU}(3, \sqrt{q}) \), where \( |G_Y|, q\sqrt{q} > q \).

**Proof.** Since \( G \) is irreducible on \( \Pi \), then \( \Pi \) consists of nontrivial \( G \)-orbits. By a direct inspection of the list of maximal subgroups of \( \text{PSL}(3, q) \) given in [16], we have that \( q^2 \mid |X^G| \) for each point \( X \in \Pi \), unless \( q \) is a square and there exists a point \( Y \in \Pi \) such that either \( G_Y \leq \text{PSL}(3, \sqrt{q}) \), or \( G_Y \leq \text{PSU}(3, \sqrt{q}) \), with \(|G_Y|, q\sqrt{q} > q\), or \( G_Y \leq E_{q^2} \cdot C_G(\gamma) \) for some involution \( \gamma \) of \( G \). Actually, if \( G_Y \leq E_{q^2} \cdot C_G(\gamma) \), then \( q^2 \mid |Y^G| \) by Proposition 3.10.

If either there are no \( Z \) in \( \Pi \) such that \( G_Z \leq \text{PSL}(3, \sqrt{q}) \) or \( G_Z \leq \text{PSU}(3, \sqrt{q}) \) and \(|G_Z|, q\sqrt{q} > q\), each admissible \( G \)-orbit on \( \Pi \) is divisible by \( q^2 \). Therefore, \( q^2 \mid n^2 + n + 1 \), since \( \Pi \) consists of nontrivial \( G \)-orbits. That is, the assertion (1).

If \( q \) is square and there exists a point \( Y \in \Pi \) such that either \( G_Y \leq \text{PSL}(3, \sqrt{q}) \) or \( G_Y \leq \text{PSU}(3, \sqrt{q}) \), where \(|G_Y|, q\sqrt{q} > q\), each \( G \)-orbits is divisible by \( q\sqrt{q} \) and hence \( q\sqrt{q} \mid n^2 + n + 1 \) by the above argument. That is, the assertion (2). \( \square \)

**Corollary 3.12.** \( p \neq 3 \).

**Proof.** Assume that \( p = 3 \). As \( q > 3 \), then \( 9 \mid q \). Hence, \( 9 \mid n^2 + n + 1 \) by Lemma 3.11. However, this is impossible, since it is known that either \( n^2 + n + 1 \equiv 1 \mod 3 \) or \( n^2 + n + 1 \equiv 3 \mod 9 \). \( \square \)

**Lemma 3.13.** Let \( S_0 \) be the \( p \)-group and let \( \psi \) and \( \beta \) be the involutions defined in Section 2. If \( q \) is a square and \( q\sqrt{q} \mid n^2 + n + 1 \), then one of the following occurs:

1. The group \( S_0 \) is semiregular on \( \Pi \) and hence on \( \text{Fix}(\psi) \);
2. \( \text{Fix}(S_0) \) is a subplane of \( \Pi \). Furthermore, either \( \text{Fix}(S_0) \cap \text{Fix}(\psi) \) is a Baer subplane of \( \text{Fix}(S_0) \) or \( \text{Fix}(S_0) \) is a proper subplane of \( \text{Fix}(\psi) \);
3. There exists a nontrivial proper subgroup \( S^* \) of \( S_0 \) such that \( \text{Fix}(S^*) \) is a subplane of \( \Pi \) of order \( m \) and one of the following occurs:
   - (a) \( \text{Fix}(S^*) \cap \text{Fix}(\psi) \) is a Baer subplane of \( \text{Fix}(S^*) \) and the involution \( \beta \) induces a Baer collineation on it. In particular, \( \sqrt{m} \) is an integer;
   - (b) \( \text{Fix}(S^*) \) is a proper subplane of \( \text{Fix}(\psi) \) and hence \( m \leq \sqrt{n} \).

Furthermore, in the cases (3a)–(3b), the group \( S_0/S^* \) acts on \( \text{Fix}(S^*) \) and on \( \text{Fix}(S^*) \cap \text{Fix}(\psi) \) semiregularly.

**Proof.** Let \( S_0 \) be the \( p \)-group defined in Section 2. Recall that \( p \neq 3 \) by Corollary 3.12. Also, recall that \( \psi \) centralizes \( S_0 \), that \( \beta \) inverts \( S_0 \) and that \( \langle \psi, \beta \rangle \cong \)
$E_4$. Since $q\sqrt{q} | n^2 + n + 1$ and $n^2 + n + 1 = (n - \sqrt{n} + 1)(n + \sqrt{n} + 1)$, with $(n - \sqrt{n} + 1, n + \sqrt{n} + 1) = 1$, either $q\sqrt{q} | n - \sqrt{n} + 1$ or $q\sqrt{q} | n + \sqrt{n} + 1$.

Assume that $q\sqrt{q} | n - \sqrt{n} + 1$. Thus $\text{Fix}(S_0) \cap \text{Fix}(\psi) \neq \emptyset$. In particular, $\text{Fix}(S_0) \neq \emptyset$. As $p \neq 3$ and that $q\sqrt{q} | n^2 + n + 1$, we have that $(q, n) = (q, n \pm 1) = 1$. Therefore, $\text{Fix}(S_0)$ is a subplane of II. As $\psi$ centralizes $S_0$, then $\psi$ acts semiregularly on $S_0$. Hence, $\text{Fix}(S_0) \cap \text{Fix}(\psi) \neq \emptyset$. Actually, $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a subplane of $\text{Fix}(\psi)$, again since $\psi$ centralizes $S_0$, $q\sqrt{q} | n + \sqrt{n} + 1$ and $p \neq 3$. Moreover, either $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$ or $\text{Fix}(S_0) \subseteq \text{Fix}(\psi)$. Assume that $\text{Fix}(S_0) = \text{Fix}(\psi)$. Then $S_0$ is semiregular on $s - \text{Fix}(S_0)$, where $s$ is a secant of $\text{Fix}(S_0)$, since $\text{Fix}(S_0)$ is a Baer subplane of II. Therefore, $q \mid n - \sqrt{n}$, since $|S_0| = q$. That is, $q \mid \sqrt{n} - 1$. So, we obtain a contradiction, since $(\sqrt{n}(\sqrt{n} - 1), n - \sqrt{n} - 1) = 1$. Thus, either $\text{Fix}(S_0) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_0)$ or $\text{Fix}(S_0)$ is a proper subplane of $\text{Fix}(\psi)$, and we obtain the assertion (2).

Assume that $q\sqrt{q} | n + \sqrt{n} + 1$. If $\text{Fix}(S_0) \neq \emptyset$, we still obtain the assertion (2) by the previous argument, by bearing in mind that $(\sqrt{n}(\sqrt{n} - 1), n + \sqrt{n} + 1) \mid 3$ and that $q > 3$. Hence, assume that $\text{Fix}(S_0) = \emptyset$. At this point, either $S_0$ is semiregular on II and we obtain the assertion (1), or there exists a nontrivial subgroup $S_1$ of $S_0$ such that $\text{Fix}(S_1) \neq \emptyset$. By bearing in mind that $\psi$ centralizes $S_0$ and hence $S_1$, that $(\sqrt{n}(\sqrt{n} - 1), n + \sqrt{n} + 1) \mid 3$ and that $p \neq 3$ by Corollary 3.12, the previous argument, with $S_1$ in the role of $S_0$, yields that either $\text{Fix}(S_1) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S_1)$ or $\text{Fix}(S_1)$ is a proper subplane of $\text{Fix}(\psi)$.

Let $S$ be the set of the nontrivial subgroups of $S_0$ fixing a subplane of II whose intersection with $\text{Fix}(\psi)$ is in turn a subplane of this one. Clearly, $S \neq \emptyset$, since $S_1 \in S$. Let $S^*$ be an element of $S$ of maximal order. Hence, $\text{Fix}(S^*)$ is a subplane of II and $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a subplane of $\text{Fix}(\psi)$. Moreover, either $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$ or $\text{Fix}(S^*)$ is a proper subplane of $\text{Fix}(\psi)$, again by the above argument with $S^*$ in the role of $S_1$. Let $m$ be the order of $\text{Fix}(S^*)$. If $\text{Fix}(S^*)$ is a proper subplane of $\text{Fix}(\psi)$, then $m \leq \sqrt[n]{m}$ by [10, Theorem 3.7], and we obtain the assertion (3a). Hence, assume that $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$. Note that $S_0/S^*$ is nontrivial and acts semiregularly on $\text{Fix}(S^*)$ and on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$, since $\text{Fix}(S_0) = \emptyset$, the group $S^*$ is an element of $S$ of maximal order, the group $S_0$ is abelian and since $\psi$ centralizes $S_0$. Denote by $m_\psi$ the order of $\text{Fix}(S^*) \cap \text{Fix}(\psi)$. Then $m = m_\psi^2$ by [10, Theorem 3.7], since $\text{Fix}(S^*) \cap \text{Fix}(\psi)$ is a Baer subplane of $\text{Fix}(S^*)$. As $\beta$ inverts $S_0$ and as $\langle \psi, \beta \rangle \cong E_4$, then $\beta$ normalizes $S^* \langle \psi \rangle$ and acts on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$. Denote by $S_0^+ = S_0/S^*$. Hence, $S_0^+$ is nontrivial and acts semiregularly on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$, as we have seen above. Furthermore, $S_0^+ \langle \beta \rangle$ acts on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$. Assume that $\beta$ induces a perspectivity on $\text{Fix}(S^*) \cap \text{Fix}(\psi)$.
Fix(ψ). Let ρ ∈ S0+, ρ ≠ 1. Then βρ is also a perspectivity of Fix(S∗) ∩ Fix(ψ), and Fix([βρ, β]) ∩ Fix(ψ) ≠ ∅ by [6, Lemma 5.1]. This is a contradiction, since [βρ, β] ∈ S0+, the group S0+ is nontrivial and acts on Fix(S∗) ∩ Fix(ψ) semiregularly. Therefore, β induces a Baer collineation on Fix(S∗) ∩ Fix(ψ). Then mψ is a square by [10, Theorem 3.7]. Consequently, √m is an integer, since we proved that m = mψ2, and we obtain the assertion (3b).

4. The proof of Theorem 1.1

Proposition 4.1. The involutions in G are perspectivities of Π.

Proof. We proceed with a series of steps to show that no one of the cases of Lemma 3.13 occurs, obtaining the assertion in this way.

Step I: The case (1) of Lemma 3.13 does not occur.

Assume that S0 is semiregular on Π and on Fix(ψ). So, q | n + √n + 1. Recall that either q2 | n2 + n + 1 or, q is a square, q√q | n2 + n + 1, and there exists a point Y ∈ Π such that either GY ≤ PSL(3, √q) or GY ≤ PSU(3, √q), where (|GY|, q√q) > q, by Lemma 3.11. In particular, either q2 | n + √n + 1 or q√q | n + √n + 1, respectively, since n2 + n + 1 = (n + √n + 1)(n − √n + 1), (n + √n + 1, n − √n + 1) = 1, and since q | n + √n + 1. If q2 | n + √n + 1, then we obtain a contradiction by [13, Lemma 6.2], since √n is a square by Lemma 3.9. Thus, q is a square, q√q | n + √n + 1, and there exists a point Y ∈ Π such that either GY ≤ PSL(3, √q) or GY ≤ PSU(3, √q), where (|GY|, q√q) > q.

Assume that there exists a point Y ∈ Π, such that either GY ≤ M, where M is either PSL(3 √q) or PSU(3, √q), and such that (|GY|, q√q) > q. Without loss of generality, we may assume that a Sylow p-subgroup of GY is contained in U, the group defined in Section 2. Set U = GY ∩ U and U(M) = M ∩ U. Clearly, UY ≤ U(M), with (|UY|, q√q) > q and |U(M)| = q√q. In particular, U(M) consists of matrices of type (1) given in Section 2 whose entries are all the elements of GF(√q), while UY consists of some of these matrices. Let W be the subgroup of S0, represented by the matrices type (2) given in Section 2, with y2 = y3 = 0 and with y1 ∈ GF(√q). Hence, |W| = √q and W ≤ U(M). Therefore, ⟨UY, W⟩ ≤ U(M), as UY ≤ U(M). Hence, |⟨UY, W⟩| ≤ q√q. On the other hand, |⟨UY, W⟩| ≥ |UY||W|/|UY ∩ W|. Thus, |⟨UY, W⟩|/|UY ∩ W| ≤ q√q. Therefore, p | |UY ∩ W| since (|UY|, q√q) > q and |W| = √q. So, p | |UY ∩ S0|, since W ≤ S0. Hence, we arrive at a contradiction, since S0 is semiregular on Π.
Step II: The case (2) of Lemma 3.13 does not occur.

Recall that $S$, $S_0$ and $\psi$ are defined as in Section 2. Hence, $Z(S) = S' = S_0$. Furthermore, $\psi$ normalizes $S$ and $S_0$. Assume that the case (2) of Lemma 3.13 occurs. Hence, Fix($S_0$) is a subplane of $\Pi$. Moreover, either Fix($S_0$) $\cap$ Fix($\psi$) is a Baer subplane of Fix($S_0$) or Fix($S_0$) is a proper subplane of Fix($\psi$). Clearly, $S$ acts on Fix($S_0$) (the action is unfaithful). Assume $S_0 < S_Q$ for some point $Q \in$ Fix($S_0$). Then $S_Q$ lies in $G_Q$ which, in turn, lies in a maximal parabolic subgroup of $G$ by a direct inspection of the list of maximal subgroups of $G \cong PSL(3,q)$, $q$ odd, given in [16]. Then $q^2 | |Q^G|$ by Proposition 3.10. However, this is impossible, since $S_0 < S_Q < S$, while $|S_0| = q$ and $|S| = q^3$. Hence $S$ induces the group $S/S_0$ on Fix($S_0$) acting semiregularly. Thus $q^2 | h^2 + h + 1$, where $h$ is the order of Fix($S_0$).

Assume that Fix($S_0$) $\cap$ Fix($\psi$) is a Baer subplane of Fix($S_0$). Then $h$ is a square. Moreover $h \leq \sqrt{n}$ by [10, Theorem 3.7]. Hence $h^2 + h + 1 \leq q^4 + q + 1$, since $n \leq q^4$ by our assumption. However, this yields a contradiction, by [13, Lemma 6.2], since $q^2 | h^2 + h + 1$ and $h$ is a square.

Assume that Fix($S_0$) is a proper subplane of Fix($\psi$). Then $h \leq \sqrt[4]{n}$ and hence $h^2 + h + 1 \leq q^2 + q + 1$. Thus, $q^2 = h^2 + h + 1$, since $q^2 | h^2 + h + 1$, and we still obtain a contradiction by [13, Lemma 6.2].

Step III: The final contradiction.

By (I) and (II) it follows that only case (3) of Lemma 3.13 might occur. Hence, assume there exists a nontrivial proper subgroup $S^*$ of $S_0$ such that $S_0/S^*$ acts on Fix($S^*$) and on Fix($S^*$) $\cap$ Fix($\psi$) semiregularly. In particular, if $m$ is the order of Fix($S^*$), then $\sqrt{m}$ is an integer. Since Fix($S^*$) $\cap$ Fix($\psi$) is a Baer subplane of Fix($S^*$) and since $S_0/S^*$ is nontrivial and acts on Fix($S^*$) $\cap$ Fix($\psi$) semiregularly, then $p | m + \sqrt{m} + 1$. Furthermore, as $S^* < S_0$, the group $S$ centralizes $S^*$ and acts on Fix($S^*$). From the proof of Lemma 3.11, we actually obtain that any $G$-orbit has length divisible by either $q^2$ or, when $q$ is a square, by $q\sqrt{q}$. This implies that each orbit under the group induced by $S$ on Fix($S^*$) has length divisible by either $q^2$ or, when $q$ is a square, by $q\sqrt{q}$. So, we obtain that either $q^2 | m^2 + m + 1$, or $q\sqrt{q} | m^2 + m + 1$ when $q$ is a square. Actually, either $q^2 | m + \sqrt{m} + 1$ or $q\sqrt{q} | m + \sqrt{m} + 1$, respectively, since $m^2 + m + 1 = (m + \sqrt{m} + 1)(m - \sqrt{m} + 1)$, $(m + \sqrt{m} + 1, m - \sqrt{m} + 1) = 1$, and since $p | m + \sqrt{m} + 1$.

Assume that Fix($S^*$) $\cap$ Fix($\psi$) is a proper subplane of Fix($\psi$). Then $\sqrt{m} \leq \sqrt[4]{n}$ by [10, Theorem 3.7]. So, $m + \sqrt{m} + 1 \leq q^2 + q + 1$. This fact, in conjunction with either $q^2 | m + \sqrt{m} + 1$ or $q\sqrt{q} | m + \sqrt{m} + 1$, yields that $q\sqrt{q} = m + \sqrt{m} + 1$.
with $q = 7$ and $\sqrt{m} = 18$ by [13, Lemma 6.2]. However, this is a contradiction, since $\sqrt{m}$ is a square.

Since none of the cases of Lemma 3.13 occurs, then $\psi$ cannot be a Baer collineation of $\Pi$. Therefore, any involution of $G$ is a perspectivity of $\Pi$, since $G \cong PSL(3, q)$ contains a unique conjugate class of involutions. □

Now, using Proposition 4.1, we prove our main result.

**Proof of Theorem 1.1.** The assertion follows by Proposition 3.2 for $q = 3$. Hence, assume that $q > 3$. Since $G \cong PSL(3, q)$ is irreducible on $\Pi$ by Lemma 3.3 and since each involution in $G$ is a perspectivity of $\Pi$ by Proposition 4.1, then $G$ leaves invariant a subplane $\Pi_0$ on which it acts strongly irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong PG(2, q)$ by [8, Theorem 1.1]. If $n \leq q^3$, the assertion follows from Theorem 2.3. Hence, assume that $q^3 < n \leq q^4$. As the involutions in $G$ are homologies of $\Pi_0$, they are also homologies of $\Pi$. Furthermore, each $p$-element inducing an elation on $\Pi_0$ is also an elation of $\Pi$ by [10, Theorem 4.25]. Finally, by [12, Theorem C.ii], we have that $q^2 | n$, that $q - 1 | n - 1$ and that $q + 1 | n^2 - 1$. It is a straightforward computation to show that this numerical information yield that $\Pi$ has order $n = \lambda q^3 + (1 - \lambda)q^2$, where $1 < \lambda \leq q + 1$ and $q + 1 | \lambda(\lambda - 1)$, since $q^3 < n \leq q^4$. This completes the proof. □

**Remark 4.2.** It seems to be tough proving that there are no planes of order $q^3 < n < q^4$ admitting $G \cong PSL(3, q)$ as a collineation group. Indeed, although it is easy to show that a nontrivial stabilizer of a point has order odd and coprime to $p$, it is difficult to determine the exact orbital decomposition of the set of external lines to $\Pi_0$, especially when the stabilizer of a line of such a set is a subgroup of a Singer cycle of $G$.

**References**


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