On the finite projective planes of order up to q^4, q odd, admitting $\mathsf{PSL}(3, q)$ as a collineation group

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Dedicated to Prof. Gábor Korchmáros on occasion of his 60th birthday

Abstract

In this paper, it is shown that any projective plane Π of order $n \leq q^4$, q odd, that admits a group $G \cong \mathsf{PSL}(3, q)$ as a collineation group contains a G-invariant Desarguesian subplane of order q. Moreover, the involutions and suitable p-elements in G are homologies and elations of Π, respectively. In particular, if $n \leq q^3$, actually, $n = q$, q^2 or q^3.

Keywords: projective plane, collineation group, orbit

MSC 2000: 51E15, 20B25

1. Introduction and result

The problem of determining a projective plane Π of order n admitting $G \cong \mathsf{PSL}(3, q)$ as a collineation group has been largely investigated in the last decades. The first significant result related to this problem is the celebrated theorem of Ostrom and Wagner [21], dating back to 1959, which asserts that the projective plane Π is Desarguesian when $n = q$. In 1976, Lüneburg [15] proves that either Π is a Desarguesian plane or a Generalized Hughes plane when $n = q^2$. In 1985, Dempwolff [4] proves that any projective plane Π of order $n = q^3$ that admits $G \cong \mathsf{PSL}(3, q)$ as a collineation group contains a Desarguesian subplane Π_0 of order q on which G acts faithfully in its natural permutation representation. Despite the fact that Dempwolff provides a complete description of the G-orbits on the points and on the lines of Π, he emphasizes the difficulty in obtaining

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a characterization of Π. In 1989, Moorhouse obtains for projective planes of order $n = q^4$, q odd, the analogue of Dempwolff's result. Recently, Montinaro investigated the projective planes of order $n \leq q^3$ admitting a group inducing a 2-transitive group (namely, $\mathsf{PSL}(3,q)$) on a subplane of Π, showing that $n = q$, q^2 or q^3 and the results of Ostrom and Wagner, Lüneburg, Dempwolff, occur, respectively. This paper represents a further contribution to the study of the projective planes of order $n \leq q^4$, q odd, that admit $\mathsf{PSL}(3,q)$ as a collineation group. In particular, it represents a conclusive result when the plane has order $n \leq q^3$.

Theorem 1.1. *Let Π be a finite projective plane of order n that admits $G \cong \mathsf{PSL}(3,q)$, q odd, as a collineation group. If $n \leq q^4$, then the following occurs:*

(I) *There exists a subplane $\Pi_0 \cong \mathsf{PG}(2,q)$ of Π on which G acts in the natural way;*

(II) *The involutions in G are homologies of Π;*

(III) *The p-elements of G inducing elations on Π_0 are elations of Π.*

Moreover, one of the following occurs:

(i) *$n = q$ and $\Pi = \Pi_0 \cong \mathsf{PG}(2,q)$;*

(ii) *$n = q^2$, Π is a Desarguesian plane or a Generalized Hughes plane and Π_0 is a Baer subplane of Π;*

(iii) *$n = q^3$;*

(iv) *$n = q^2(\lambda(q-1)+1)$, where $1 < \lambda \leq q+1$ and $q+1 \mid \lambda(\lambda-1)$.*

The cases (i) and (ii) clearly occur. The only known occurrences of the case (iii) are in the Desarguesian planes and in the Figueroa planes [5], [7]. The only known occurrences of the case (iv) are in the Desarguesian planes and in the Generalized Hughes planes when $\lambda = q+1$, i.e. $n = q^4$.

The strategy of the proof is the following. Firstly, we prove that G is irreducible on Π. Hence, Π consists of nontrivial G-orbits. If ψ is a Baer collineation of Π, we determine the general structure of the action of the group induced by $C_G(\psi)$ on $\mathrm{Fix}(\psi)$ by Theorem 2.1. This forces any admissible G-orbit on the points of Π to be divisible by either q^2 or $q\sqrt{q}$ for q square. So, n^2+n+1, i.e. the number of points of Π, is divisible by either q^2 or $q\sqrt{q}$ for q square, as Π consists of nontrivial G-orbits. This yields a Diophantine equation involving n^2+n+1 and either q^2 or $q\sqrt{q}$ for q square. However, such an equation has no admissible solutions by [13, Lemma 6.2]. Therefore, the involutions in G are homologies of Π. At this point, the proof of our result easily follows.

2. Background

The notation used in this paper is standard. For what concerns finite groups, the reader is referred to [11] and to [3]. The necessary background about finite projective planes may be found in [10].

Now, we collect some information about the structure of the groups $\mathsf{PSL}(2,q)$ and $\mathsf{PSL}(3,q)$ and some results on the projective planes admitting one of these as a collineation group. Based on the results of Lüneburg [14], Yaqub [22] and Moorhouse [19], the following theorem, due to Montinaro, determines the general structure of the projective planes of order up to q^2 admitting $\mathsf{PSL}(2,q)$, $q>3$, as a collineation group. Recall that a collineation group of a projective plane Π is said to be *irreducible on* Π if the group does not fix any point, line, triangle of Π. An irreducible collineation group of Π which does not fix any proper subplane of Π is said to be *strongly irreducible on* Π.

Theorem 2.1. *Let Π be a projective plane of order n admitting a collineation group $H\cong\mathsf{PSL}(2,q)$, $q>3$. If $n\le q^2$, then one of the following occurs:*

(1) *$n<q$ and one of the following occurs:*

   (a) *$n=4$, $\Pi\cong\mathsf{PG}(2,4)$ and $H\cong\mathsf{PSL}(2,5)$;*

   (b) *$n=2$ or 4, $\Pi\cong\mathsf{PG}(2,2)$ or $\mathsf{PG}(2,4)$, respectively, and $H\cong\mathsf{PSL}(2,7)$;*

   (c) *$n=4$, $\Pi\cong\mathsf{PG}(2,4)$ and $H\cong\mathsf{PSL}(2,9)$.*

(2) *$n=q$, $\Pi\cong\mathsf{PG}(2,q)$ and one of the following occurs:*

   (a) *H fixes a line or a point and q is even;*

   (b) *H is strongly irreducible and q is odd.*

(3) *$q<n<q^2$ and one of the following occurs:*

   (a) *H fixes a point or a line, and one of the following occurs:*

      (i) *$n=16$ and $H\cong\mathsf{PSL}(2,5)$;*

      (ii) *$n=16$, Π is the Lorimer-Rahilly plane or the Johnson-Walker plane, or their duals, and $H\cong\mathsf{PSL}(2,7)$.*

   (b) *H fixes a subplane Π_0 of Π, q is odd and one of the following occurs:*

      (i) *$n=16$, $\Pi_0\cong\mathsf{PG}(2,4)$ and $H\cong\mathsf{PSL}(2,5)$;*

      (ii) *$\Pi_0\cong\mathsf{PG}(2,2)$ or $\mathsf{PG}(2,4)$, and $H\cong\mathsf{PSL}(2,7)$;*

      (iii) *$\Pi_0\cong\mathsf{PG}(2,4)$ and $G\cong\mathsf{PSL}(2,9)$.*

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(c) *H is strongly irreducible and q is odd.*

(4) *$n = q^2$ and one of the following occurs:*

(a) *H fixes a point or a line, and one of the following occurs:*

(i) *$n = 25$ and $H \cong \mathsf{PSL}(2,5)$;*

(ii) *$n = 81$ and $H \cong \mathsf{PSL}(2,9)$;*

(iii) *$n = q^2$, q even, and $G \cong \mathsf{PSL}(2,q)$.*

(b) *H fixes a subplane Π_0 of Π, q is odd and one of the following occurs:*

(i) *$n = q^2$, $\Pi_0 \cong \mathsf{PG}(2,q)$ and $H \cong \mathsf{PSL}(2,q)$;*

(ii) *$n = 25$, $\Pi_0 \cong \mathsf{PG}(2,4)$ and $H \cong \mathsf{PSL}(2,5)$;*

(iii) *$n = 81$, $\Pi_0 \cong \mathsf{PG}(2,4)$ and $H \cong \mathsf{PSL}(2,9)$;*

(iv) *$n = 81$, Π_0 is a Hughes plane of order 9 and $H \cong \mathsf{PSL}(2,9)$.*

(c) *H is strongly irreducible.*

Proof. See [18, Theorem 1]. □

As we shall see, such a theorem will play a central role in our investigation due to the fact that the centralizer of an involution involves a group isomorphic to $\mathsf{PSL}(2,q)$.

Now, we recall some basic facts about the structure of the group $G \cong \mathsf{PSL}(3,q)$ (the reader is referred to [16]).

1. Let ψ and β be the involutions in G represented by $\mathrm{diag}(1,-1,-1)$ and $\mathrm{diag}(-1,1,-1)$, respectively. Then $\langle \psi, \beta \rangle \cong E_4$.

2. Let U be the Sylow p-subgroup of G represented by all the matrices

$$\begin{bmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (1)$$

where $x_1, x_2, x_3 \in \mathsf{GF}(q)$. Clearly, $|U| = q^3$. Let U_0 be the subgroup of U represented by the matrices in (1) having $x_1 = x_3 = 0$. Then U_0 has order q and $U_0 = Z(U) = U'$. Thus, U is a special p-group. Finally, let U^* be the subgroup of U represented by the matrices in (1) having $x_3 = 0$. Then U^* is elementary abelian of order q^2 which is normalized by ψ.

3. Let S be the Sylow p-subgroup of G represented by all the matrices

$$\begin{bmatrix} 1 & 0 & y_2 \\ y_3 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (2)$$

where $y_1, y_2, y_3 \in \mathsf{GF}(q)$, and let S_0 be the subgroup of S represented by those having $y_2 = y_3 = 0$. Then $S_0 = Z(S) = S'$. In particular, $U \cap S$ is an elementary abelian group of order q^2 containing S_0. Namely, $U \cap S$ consists of all the matrices in (2) having $y_3 = 0$.

4. The group $S_0.\langle \psi, \beta \rangle$ has order $4q$. In particular, ψ centralizes S_0, while β inverts S_0.

5. The group $C_G(\psi)$ consists of the matrices

$$\begin{bmatrix} e^{-1} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix},$$

where $a, b, c, d, e \in \mathsf{GF}(q)$, $e = ad - bc \neq 0$. Denote by Z_ψ the subgroup of $C_G(\psi)$ represented by all the matrices $\operatorname{diag}(d^{-2}, d, d)$, where $d \in \mathsf{GF}(q)^*$. Then $Z_\psi = Z(C_G(\psi))$. In particular, Z_ψ is a cyclic group of order $\frac{q-1}{\mu}$, where $\mu = (3, q-1)$ and $C_G(\psi) \cong Z_\psi.\mathsf{PGL}(2, q)$.

6. The group $U^* : C_G(\psi)$ is a maximal parabolic subgroup of G. Furthermore, $U^* \langle \psi \rangle \lhd U^* : C_G(\psi)$ and $C'_G(\psi) \cong \mathsf{SL}(2, q)$.

7. Let W^* be the subgroup of G represented by all the matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z_1 & z_2 & 1 \end{bmatrix},$$

where $z_1, z_2 \in \mathsf{GF}(q)$. Then W^* is an elementary abelian group of order q^2 which is normalized by $C_G(\psi)$. Moreover, the groups $U^* : C_G(\psi)$ and $W^* : C_G(\psi)$ are the representatives of the two distinct conjugate classes of maximal parabolic subgroups of G. The groups U^* and W^* are conjugate by the inverse-transpose automorphism.

We shall use the facts stated above without recalling them, unless it is explicitly required. In particular, since the Sylow p-subgroups of G are conjugate, we shall mainly refer either to U or to S. Furthermore, despite the fact that there are two distinct conjugate classes of maximal parabolic subgroups in G by (7), what we prove to be true for $U^* : C_G(\psi)$ can always be proven to be true for $W^* : C_G(\psi)$. Hence, for our purposes we may always refer to $U^* : C_G(\psi)$, without loss of generality.

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Lemma 2.2. *The group $G \cong \mathsf{PSL}(3,q)$ contains two distinct involutions ψ_1 and ψ_2 such that $C'_G(\psi_1) \cap C'_G(\psi_2) \neq \langle 1 \rangle$ and $\langle C'_G(\psi_1), C'_G(\psi_2) \rangle = G$.*

Proof. See [20, Lemma 4.1.vi]. □

Some geometrical results involving the group $G \cong \mathsf{PSL}(3,q)$ are in order. By using the results of Ostrom-Wagner [21], Lüneburg [15] and Dempwolff [4], Montinaro proved the following.

Theorem 2.3. *Let Π be a finite projective plane of order n and let G be a collineation group of Π inducing a group containing $\mathsf{PSL}(3,q)$ on a subplane Π_0 of order q. If $n \leq q^3$, then one of the following occurs:*

(1) *$\Pi_0 \cong \mathsf{PG}(2,q)$, $\mathsf{PSL}(3,q) \leq G$ and one of the following occurs:*

   (a) *$n = q$ and $\Pi = \Pi_0$;*

   (b) *$n = q^2$, Π is a Desarguesian plane or a Generalized Hughes plane and Π_0 is a Baer subplane of Π;*

   (c) *$n = q^3$.*

(2) *$\Pi_0 \cong \mathsf{PG}(2,7)$, Π is the generalized Hughes plane over the exceptional nearfield of order 7^2 and $\mathsf{SL}(3,7) \leq G$.*

Proof. See [17]. □

Finally, we quote this useful final result, due to Moorhouse [20], which inspired the present paper, and that allows to reduce our investigation to $n < q^4$ (when $q > 3$).

Theorem 2.4 (Moorhouse)**.** *Let Π be a projective plane of order q^4 admitting $G \cong \mathsf{PSL}(3,q)$, q odd. If $q > 3$, then the following must hold.*

(i) *G leaves invariant a Desarguesian subplane Π_0 of order q, on which G acts 2-transitively;*

(ii) *The involutions in G are homologies of Π, and those p-elements of G which induce elations of Π_0 are elations of Π.*

If $q = 3$ then the same two conclusions must hold, under the additional hypothesis that G acts irreducibly on Π.

Proof. See [20, Theorem 1.3]. □

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3. Preliminary reductions

The aim of this section is to show that G is irreducible on Π and that the involutions in G are perspectivities of Π, in order to apply Hering-Walker theory on the strong irreducibility (e.g. see [6], [8] and [9]).

In view of Theorem 2.1, we treat the cases $q = 3$ and $q > 3$ separately.

Lemma 3.1. *Let Π be a finite projective plane of order n that admits $G \cong \mathsf{PSL}(3,3)$ as a collineation group. If $n \leq 3^4$, then each involution in G is a perspectivity of Π.*

Proof. Assume that the involutions in G are Baer collineations of Π. Hence, $\sqrt{n} \leq 9$. Let J be a Sylow 2-subgroup of G. As n^2+n+1 is odd, then J fixes a secant s of $\mathrm{Fix}(\psi)$. Let $J_0 = J \cap C_G(\psi)$. Then $J_0 \cong Q_8$. Thus, J_0 is semiregular on $s - \mathrm{Fix}(\psi)$. So, $8 \mid \sqrt{n}(\sqrt{n}-1)$, since $|s - \mathrm{Fix}(\psi)| = \sqrt{n}(\sqrt{n}-1)$. Consequently, either $\sqrt{n} = 8$ or 9, as $\sqrt{n} \leq 9$. Note that $J = J_0.\langle\beta\rangle$ is known to be semidihedral of order 16. As J_0 is semiregular on $s - \mathrm{Fix}(\psi)$, then each J-orbit on $s - \mathrm{Fix}(\psi)$ has length either 8 or 16. Therefore, let x and y be the number of J-orbits on $s - \mathrm{Fix}(\psi)$ of length 8 and 16, respectively. It follows that

$$8x + 16y = \sqrt{n}(\sqrt{n}-1), \tag{3}$$

where $\sqrt{n} = 8$ or 9. As J is semidihedral of order 16, then J contains two distinct conjugate classes of involutions, one consisting of ψ and the other consisting of the four conjugates of β (including β). Furthermore, $C_J(\beta) \cong \langle\psi,\beta\rangle \cong E_4$. Thus, by [19, Relation (8)], the involution β fixes 2 and 0 points on the J-orbits on $s - \mathrm{Fix}(\psi)$ of length 8 and 16, respectively, since $\psi \in J_0$ and since J_0 is semiregular on $s - \mathrm{Fix}(\psi)$. Hence, β fixes exactly $2x$ points on $s - \mathrm{Fix}(\psi)$. If x is even, then $16 \mid \sqrt{n}(\sqrt{n}-1)$ by (3), which is impossible as $\sqrt{n} = 8$ or 9. Therefore, x is odd. Hence, β cannot induce either the identity or a perspectivity of axis s on $\mathrm{Fix}(\psi)$, otherwise $x = 0$, since β is a Baer collineation on Π (recall that $G \cong \mathsf{PSL}(3,3)$ has a unique conjugate class of involutions). Suppose that β induces a perspectivity on $\mathrm{Fix}(\psi)$ of axis distinct from s. Clearly, β induces on $\mathrm{Fix}(\psi)$ either an elation when $\sqrt{n} = 8$ or a homology when $\sqrt{n} = 9$. Then $x = \sqrt{n}$ or $\sqrt{n}-1$, respectively, again since β is a Baer collineation on Π . So, x is even in any case, which is a contradiction. Finally, assume that β induces a Baer collineation on $\mathrm{Fix}(\psi)$ when $\sqrt{n} = 9$. Arguing as above, we have that $x = \sqrt{n} - \sqrt[4]{n}$ which is even and we again obtain a contradiction. Thus, the involutions in G are perspectivities of Π. □

Proposition 3.2. *Let Π be a finite projective plane of order n that admits $G \cong \mathsf{PSL}(3,3)$ as a collineation group. If $n \leq 3^4$, then the following occurs:*

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(1) *There exists a subplane* $\Pi_0 \cong \mathsf{PG}(2,3)$ *of* Π *on which* G *acts in the natural way;*

(2) *The group* G *is irreducible on* Π*;*

(3) *The involutions in* G *are homologies of* Π*;*

(4) *The* 3*-elements that induce elations on* Π_0 *are elations of* Π.

Moreover, one of the following occurs:

(i) $n = 3$ *and* $\Pi = \Pi_0 \cong \mathsf{PG}(2,q)$*;*

(ii) $n = 3^2$*,* Π *is a Desarguesian plane or a Generalized Hughes plane and* Π_0 *is a Baer subplane of* Π*;*

(iii) $n = 3^3$*;*

(iv) $n = 3^4$.

Proof. Assume that $G \cong \mathsf{PSL}(3,3)$ fixes a line l of Π. As $n \leq 3^4$, then each nontrivial G-orbit on l has length divisible by 13 by [2]. Actually, G contains such orbits, since G acts faithfully, G being nonabelian simple. Let $X \in l$ such that $13 \mid \left|X^G\right|$. So, $G_X \leq E_9.\mathsf{GL}(2,3)$. Let $\mathcal{B}_X$ be the block of imprimitivity in X^G containing X. Clearly $\left|X^G\right| = 13\,|\mathcal{B}_X|$ ($|\mathcal{B}_X|$ might be 1). Furthermore, $E_9.\mathsf{GL}(2,3)$ acts transitively on $\mathcal{B}_X$. As the socle of $E_9.\mathsf{GL}(2,3)$ is E_9, then either $E_9 \trianglelefteq G_X$ or $13 \cdot 9 \mid \left|X^G\right|$ by [3, Theorem 4.1A]. Actually, the latter cannot occur, since $\left|X^G\right| \leq n+1$ and $n \leq 3^4$. Hence, each nontrivial G-orbit on l has length $\left|X^G\right| = 13\,|\mathcal{B}_X|$, where $|\mathcal{B}_X| \mid |\mathsf{GL}(2,3)|$. Actually, $|\mathcal{B}_X| = 1, 2, 3, 4$ or 6, since $|\mathcal{B}_{X_i}| \leq 6$, as $n \leq 3^4$. Since the blocks of imprimitivity are 13, then there exists a point P, lying in a nontrivial G-orbit on l, such that J_0 fixes $\mathcal{B}_P$, where J_0 is the 2-group isomorphic to Q_8 containing the involution ψ. Thus, ψ fixes $\mathcal{B}_P$ pointwise, since $|\mathcal{B}_P| \leq 6$. Then $|\mathcal{B}_P| \leq 2$, since ψ is a perspectivity of Π having axis distinct from l. If $|\mathcal{B}_P| = 1$, then P^G is a 2-transitive G-orbit. Hence, ψ fixes exactly 5 points on X^G. So, we arrive at a contradiction, since ψ is a perspectivity of Π having axis distinct from l. Thus, $|\mathcal{B}_P| = 2$ and hence $l = P^G$, since ψ fixes $\mathcal{B}_P$ pointwise. In particular, $n = 25$, since $\left|P^G\right| = 26$. Since G acts faithfully on l, there are no involutory homologies of axis l. Therefore, no involutions lie in a triangular configuration. In particular, since ψ is the unique central involution in J (recall that J is semidihedral of order 16), each involution in J has center and axis C_ψ and a_ψ, where C_ψ and a_ψ denote the center and the axis of ψ, respectively. So, each involution in J fixes exactly two points on l, namely C_ψ and $a_\psi \cap l$. Hence, J is semiregular on $l - \{C_\psi, W\}$, where $\{W\} = a_\psi \cap l$. Then $16 \mid n-1$, which is a contradiction, since $n = 25$

while $|J| = 16$. Therefore, G does not fix lines. By the dual of the previous proof, we obtain that G does not fix points. Finally, these two facts, combined with the fact that G is nonabelian simple, yield that G does not fix triangles of Π. Thus, G is irreducible on Π and hence the assertion (2).

Since $G \cong \mathrm{PSL}(3,3)$ is irreducible on Π, and since each involution in G is a perspectivity by Lemma 3.1, then G leaves invariant a subplane Π_0 on which it acts strongly irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong \mathrm{PG}(2,3)$ by [8, Theorem 1.1], and we obtain the assertion (1). Therefore, the involutions in G are homologies of Π and hence the assertion (3). For $n \leq 3^3$, the assertions (4) and (i)–(iii) follow by Theorem 2.3. Furthermore, for $n = 3^4$ the assertions (4) and (iv) follow by Theorem 2.4, since we proved the irreducibility of G on Π. Hence, assume that $3^3 < n < 3^4$. Note that G contains an elementary abelian group H of order 3^2 consisting of elations with the same axis r and distinct centres lying in $\Pi_0 \cap r$ by [10, Theorem 4.25]. As H is semiregular on $[Q] - \{l\}$, for any $Q \in r - \Pi_0$, then $3^2 \mid n$. So, $3^3 < n < 3^4$, n odd, and $3^2 \mid n$ yield that $n = 3^2 5$ or $3^2 7$. Let $\mathcal{E}$ be the set of external lines to Π_0. Easy computations yield $|\mathcal{E}| = 1512$ or 3240, respectively. Let R be any Sylow 2-subgroup of G. Then $|R| = 16$. Since each involution in G, and hence in R, is a homology of axis a secant to Π_0, then R is semiregular on $\mathcal{E}$. So, $16 \mid |\mathcal{E}|$, which is impossible as $|\mathcal{E}| = 1512$ or 3240. This completes the proof. □

It should be pointed out that the previous theorem extends the Theorem 2.4 also for $n = 3^4$. Indeed, Theorem 2.4 works for $q = 3$ under the additional assumption that G is irreducible on Π. In particular, Moorhouse shows that the irreducibility of G on Π implies that the involutions in G are homologies of Π. We, instead, prove that the involutions are perspectivities of Π and then we use this fact to prove that G is irreducible on Π.

From now on, we assume that $q > 3$.

Lemma 3.3. *The group G is irreducible on Π.*

Proof. Assume that G fixes a line l of Π. Then $\sqrt{n} < q^2$, since for $n = q^4$ the assertion follows by [20] (e.g. see the proof of Theorem 1.3). Let ψ be the involution in G defined in Section 2. Then, by Theorem 2.1 and by bearing in mind that q is odd and $\sqrt{n} < q^2$, one of the following occurs:

(1) $\sqrt{n} = 4$, $\mathrm{Fix}(\psi) \cong \mathrm{PG}(2,4)$ and $C_G(\psi)'/\langle\psi\rangle \cong \mathrm{PSL}(2,5)$;

(2) $\sqrt{n} = 16$ and $C_G(\psi)'/\langle\psi\rangle \cong \mathrm{PSL}(2,5)$;

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(3) $\sqrt{n} = 16$, $\mathrm{Fix}(\psi)$ is either the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C_G(\psi)'/\langle\psi\rangle \cong \mathsf{PSL}(2,7)$.

Assume that the case (1) occurs. Since $n+1 = 17$ and since these primitive permutation representations of G have a degree greater than 17 by [2], then G fixes l pointwise. That is, G is a group of perspectivities of axis l. So, G should be a Frobenius group by [10, Theorem 4.25], which is impossible as G is nonabelian simple.

We treat the cases (2)–(3) simultaneously. By a direct inspection of [2], it is plain that the unique nontrivial orbits on l under $G \cong \mathsf{PSL}(3,q)$, $q = 5$ or 7, are those of length a multiple of d_0, the minimal primitive permutation representation degree of G. By [2], such a d_0 is equal to 31 or 57, respectively. Let r be the minimal nonnegative integer such that $n+1 \equiv r_0 \mod d_0$. Easy computations yield that $r_0 = 9$, 29 or 6 in the cases (1)–(3), respectively. So, $6 \le r_0 < n+1$ and $\sqrt{n}+1 \not\equiv r_0 \mod d_0$ in any case. Therefore, G fixes at least 6 points on l in any case. Let P be any of these points. Now, by repeating the above argument with $[P]$ in the role of l, we obtain that G fixes at least 6 lines of $[P]$ (clearly, the line l is included). Again, by repeating the above argument for any for each of these 6 lines, we obtain that G fixes a subplane Σ of Π pointwise. Let r be the order of Σ. Then $r = r_0 + hd_0 - 1$, where $h \ge 0$. Note that $r_0 + hd_0 - 1 \le \sqrt{n}$ by [10, Theorem 3.7]. Hence, the case (3) is ruled out. Actually, $r_0 + hd_0 - 1 < \sqrt{n}$, since $\sqrt{n}+1 \not\equiv r_0 \mod d_0$. Thus, $\Sigma \subset \mathrm{Fix}(\psi)$, since $\Sigma \subseteq \mathrm{Fix}(\psi)$. Therefore, $(r_0 + hd_0 - 1)^2 \le \sqrt{n}$ by [10, Theorem 3.7]. This forces $h = 0$ in any admissible case. In particular, the case (2) is ruled out. Consequently, G is irreducible on Π. $\square$

Throughout this section, we assume that ψ is a Baer collineation of Π. Then $n < q^4$ by Theorem 2.4, as $q > 3$.

The following lemma determines the structure of the kernel K_ψ of the action of $C_G(\psi)$ on $\mathrm{Fix}(\psi)$.

Lemma 3.4. $\langle\psi\rangle \le K_\psi \le Z_\psi$.

Proof. Clearly, $\langle\psi\rangle \trianglelefteq K_\psi \trianglelefteq C_G(\psi)$. Recall that $C_G(\psi) \cong Z_\psi.\mathsf{PGL}(2,q)$. Since $K_\psi Z_\psi/Z_\psi \trianglelefteq \mathsf{PGL}(2,q)$, then either $K_\psi Z_\psi/Z_\psi = \langle 1\rangle$ or $\mathsf{PSL}(2,q) \le \bar{K}\bar{Z}_\psi/\bar{Z}_\psi$. Assume that the latter occurs. Then $C'_G(\psi) \le K_\psi$, since $C'_G(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2,q)$ and since $\langle\psi\rangle \trianglelefteq K_\psi \trianglelefteq C_G(\psi)$. Since for each involution $\beta \in G$ there exists $g \in G$ such that $\psi^g = \beta$, then $C'_G(\psi)^g = C'_G(\beta)$. Hence $C'_G(\beta)$ fixes $\mathrm{Fix}(\beta)$ pointwise for each involution β in G. By Lemma 2.2, there exist two involutions ψ_1 and ψ_2 such that $C'_G(\psi_1) \cap C'_G(\psi_2) \ne \langle 1\rangle$ and $\langle C'_G(\psi_1), C'_G(\psi_2)\rangle = G$. Since

$C'_G(\psi_i)$ fixes the Baer subplane $\mathrm{Fix}(\psi_i)$ pointwise for each $i = 1, 2$, and since $C'_G(\psi_1) \cap C'_G(\psi_2) \neq \langle 1 \rangle$, then $\mathrm{Fix}(\psi_1) = \mathrm{Fix}(\psi_2)$. Thus, $G = \langle C'_G(\psi_1), C'_G(\psi_2) \rangle$ fixes $\mathrm{Fix}(\psi_1)$ pointwise, which is impossible by Lemma 3.3. Consequently, $K_\psi Z_\psi / Z_\psi = \langle 1 \rangle$. That is, $K_\psi \leq Z_\psi$ and hence we obtain the assertion. □

For each subgroup X of $C_G(\psi)$, we denote by $\bar{X}$ the group XK_ψ/K_ψ.

Lemma 3.5. *For each point $X \in \Pi$ such that G_X lies in a maximal parabolic subgroup of G, one of the following occurs:*

(1) *X^G is a 2-transitive orbit;*

(2) *$\mathrm{Fix}(U^* \langle \psi \rangle)$ is either a flag, or an antiflag or a proper subplane of $\mathrm{Fix}(\psi)$. Furthermore, $\overline{C_G(\psi)}$ leaves $\mathrm{Fix}(U^* \langle \psi \rangle)$ invariant;*

(3) $q^2 \mid \left| X^G \right|$.

Proof. Let $X \in \Pi$ and assume that G_X lies in a maximal parabolic subgroup of G. As mentioned in Section 2, for our purposes we may reduce to study the case when $G_X \leq U^* : C_G(\psi)$, where $C_G(\psi) \cong Z_\psi.\mathrm{PGL}(2, q)$ and $Z_\psi \cong Z_{\frac{q-1}{\mu}}$, $\mu = (3, q-1)$. If $G_X = U^* : C_G(\psi)$, then X^G is a 2-transitive orbit and we obtain the assertion (1). If $G_X < U^* : C_G(\psi)$, denoted by $\mathcal{B}_X$ the block of imprimitivity in X^G containing X, we have $|\mathcal{B}_X| > 1$. Clearly, $U^* : C_G(\psi)$ acts on $\mathcal{B}_X$.

Assume that $U^* : C_G(\psi)$ does not act faithfully on $\mathcal{B}_X$, then U^* lies in the kernel of the action, since U^* is the socle of $U^* : C_G(\psi)$ by [3, Theorem 4.3B]. Thus, $\mathrm{Fix}(U^*) \neq \varnothing$. Since $U^* \lhd U^* : C_G(\psi)$, and since $\mathrm{Fix}(U^* : C_G(\psi)) = \varnothing$, being $G_X < U^* : C_G(\psi)$, either $\mathrm{Fix}(U^*) = \Delta$, where Δ is a triangle of Π, or $\mathrm{Fix}(U^*)$ is a subplane of Π by [6, Corollary 3.6]. This yields that $\mathrm{Fix}(U^* \langle \psi \rangle)$ consists of either a flag, or an antiflag or a plane. Clearly, $\mathrm{Fix}(U^* \langle \psi \rangle) \subseteq \mathrm{Fix}(\psi)$. Furthermore, $\overline{C_G(\psi)}$ acts on $\mathrm{Fix}(\psi)$ leaving $\mathrm{Fix}(U^* \langle \psi \rangle)$ invariant, since $U^* \langle \psi \rangle \lhd U^* : C'_G(\psi)$. If $\mathrm{Fix}(U^* \langle \psi \rangle) = \mathrm{Fix}(\psi)$, then $\mathrm{Fix}(U^*) = \mathrm{Fix}(\psi)$, since $\mathrm{Fix}(\psi)$ is a Baer subplane of Π. So, U^* is semiregular on $s - \mathrm{Fix}(U^*)$, where s is a secant of $\mathrm{Fix}(U^*)$. Therefore, $q^2 \mid n - \sqrt{n}$, since $|U^*| = q^2$. That is, either $q^2 \mid \sqrt{n} - 1$ or $q^2 \mid \sqrt{n}$, and we have a contradiction in any case since $n < q^4$ and $q > 3$. Thus, we obtain the assertion (2)

Assume that $U^* : C_G(\psi)$ acts faithfully on $\mathcal{B}_X$. Then $q^2 \mid |\mathcal{B}_X|$ by [3, Theorem 4.1A], since U^* is the socle of $U^* : C_G(\psi)$. Thus, $q^2 \mid \left| X^G \right|$ and we obtain the assertion (3). □

Lemma 3.6. *One of the following occurs:*

(I) *The groups $\overline{C'_G(\psi)}$ and $\overline{C_G(\psi)}$ are strongly irreducible on $\mathrm{Fix}(\psi)$;*

(II) $q = 5$ *and* $n = 4$*;*

(III) $q = 9$ *and* $9^2 < n < 9^4$.

Proof. Assume that the cases (II) and (III) do not occur. Note that $\overline{C'_G(\psi)} \cong \mathrm{PSL}(2,q)$, since $C'_G(\psi) \cap K_\psi = \langle \psi \rangle$ by Lemma 3.4. Suppose that the $\overline{C'_G(\psi)}$ is not strongly irreducible on $\mathrm{Fix}(\psi)$. The case $\sqrt{n} = q$ is ruled out by Theorem 2.1. As $\sqrt{n} < q^2$, then either $\sqrt{n} < q$ or $q < \sqrt{n} < q^2$. Then, again by Theorem 2.1 and bearing in mind that the cases (II) and (III) do not occur by our assumptions, one of the following occurs:

(1) $n = 4$ or 16, $\mathrm{Fix}(\psi) \cong \mathrm{PG}(2,2)$ or $\mathrm{PG}(2,4)$, respectively, and $\overline{C'_G(\psi)} \cong \mathrm{PSL}(2,7)$;

(2) $n = 16$, $\mathrm{Fix}(\psi) \cong \mathrm{PG}(2,4)$ and $\overline{C'_G(\psi)} \cong \mathrm{PSL}(2,9)$;

(3) $n = 16^2$, $\overline{C'_G(\psi)} \cong \mathrm{PSL}(2,5)$ fixes a subplane of $\mathrm{Fix}(\psi)$ isomorphic to $\mathrm{PG}(2,4)$;

(4) $7^2 < n < 49^2$, $\overline{C'_G(\psi)} \cong \mathrm{PSL}(2,7)$ fixes a subplane of $\mathrm{Fix}(\psi)$ isomorphic either to $\mathrm{PG}(2,2)$ or to $\mathrm{PG}(2,4)$.

Actually, in the cases (1)–(4), the group $\overline{C_G(\psi)} \cong \bar{Z}_\psi.\mathrm{PGL}(2,q)$ acts on $\mathrm{Fix}(\psi)$. The group $\overline{C_G(\psi)}$ fixes a subplane Π_0 of $\mathrm{Fix}(\psi)$ isomorphic either to $\mathrm{PG}(2,2)$ or to $\mathrm{PG}(2,4)$ for $q = 7$, or to $\mathrm{PG}(2,4)$ for $q \neq 7$ (note that it might be $\Pi_0 = \mathrm{Fix}(\psi)$).

Assume that $q = 7$. Then $\bar{Z}_\psi = \langle 1 \rangle$, since $Z_\psi = \langle \psi \rangle$. Therefore $\overline{C_G(\psi)} = \mathrm{PGL}(2,7)$ acts on Π_0. Then the case $\Pi_0 \cong \mathrm{PG}(2,2)$ is ruled out, since the full automorphism group of $\mathrm{PG}(2,2)$ is isomorphic to $\mathrm{PSL}(2,7)$. Hence, assume that $\Pi_0 \cong \mathrm{PG}(2,4)$ and $\overline{C'_G(\psi)} \cong \mathrm{PSL}(2,7)$. It is easy to see that $\mathrm{PSL}(2,7)$ fixes a subplane Π_1 of Π_0 which is isomorphic to $\mathrm{PG}(2,2)$. In particular, $\mathrm{PGL}(2,7)$ leaves Π_1 invariant. So, we arrive at a contradiction by the above argument with Π_1 in the role of Π_0. Therefore, $q \neq 7$ and hence the cases (1) and (4) are ruled out.

Assume that $q = 5$ or 9. Then $\Pi_0 \cong \mathrm{PG}(2,4)$ and hence $\overline{C_G(\psi)} \leq \mathrm{P\Gamma L}(3,4)$. Furthermore, $\bar{Z}_\psi = \langle 1 \rangle$ by [2]. Consequently, Z_ψ fixes $\mathrm{Fix}(\psi)$ pointwise and $\overline{C_G(\psi)} \cong \mathrm{PGL}(2,q)$ in any case. Since Z_ψ is semiregular $s - \mathrm{Fix}(\psi)$, then $\frac{q-1}{\mu} \mid n - \sqrt{n}$, where $\frac{q-1}{\mu} = |Z_\psi|$ and $\mu = (3, q-1)$. That is, $\frac{q-1}{\mu} \mid \sqrt{n}$ or $\frac{q-1}{\mu} \mid \sqrt{n} - 1$, since $q = 5$ or 9. Thus, the case (2) is ruled out, since $\sqrt{n} = 4$, while $\frac{q-1}{\mu} = 8$.

It remains to investigate the case (3). In this case, any subgroup Z_{31} of G fixes a subplane of Π of order $7 + 31k$, $k \geq 0$. Actually, $k = 0$ by [10, Theorem 3.7], since $n = 16^2$. Therefore, Z_{31} fixes exactly 57 points of Π. Note that $Z_{31} \leq$

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$G_X \leq Z_{31}.Z_3$ for any point X of Π fixed by Z_{31} by [2]. Moreover, $Z_{31}.Z_3$ is maximal in G. So, either $G_X = Z_{31}$, $\left|X^G\right| = 12000$ and Z_{31} fixes 3 points on X^G, or $G_X = Z_{31}.Z_3$, $\left|X^G\right| = 4000$ and Z_{31} fixes 1 point on X^G. Let x and y be the number of G-orbits on Π of length 12000 and 4000, respectively. Then $12000x + 4000y \leq 65793$, since $n^2 + n + 1 = 65793$. Furthermore, $3x + y = 57$, since Z_{31} fixes exactly 57 points of Π. By combining the previous relations involving x and y, we obtain a contradiction. Thus, $\overline{C'_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$. Then $\overline{C_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$, since $\overline{C'_G(\psi)} \leq \overline{C_G(\psi)}$. That is, the assertion (I) occurs. $\square$

Lemma 3.7. *The group G does not admit 2-transitive point-orbits on Π.*

Proof. Let $\mathcal{O}$ be a 2-transitive G-orbit on Π. Then $|\mathcal{O}| = q^2 + q + 1$. Clearly, $\mathcal{O}$ cannot be contained in a line by lemma 3.3. Then, it is a plain that, either $\mathcal{O}$ is an arc or $\mathcal{O} \cong \mathsf{PG}(2, q)$. Assume that the former occurs. Let U^* be the elementary abelian p-group defined in Section 2. Then $U^* \langle\psi\rangle$ fixes exactly $q+1$ points on $\mathcal{O}$. So $U^* \langle\psi\rangle$ is planar, since $\mathcal{O}$ is an arc. Since $U^* \langle\psi\rangle \lhd U^* C'_G(\psi)$ and since $C'_G(\psi)$ acts 2-transitively on $\mathrm{Fix}(U^* \langle\psi\rangle) \cap \mathcal{O}$, then $C'_G(\psi)$ acts as $\mathsf{PSL}(2, q)$ on $\mathrm{Fix}(U^* \langle\psi\rangle)$. Note that

$$|\mathrm{Fix}(U^* \langle\psi\rangle) \cap \mathcal{O}| = q + 1 \ \text{ and } \ |\mathrm{Fix}(\langle\psi\rangle) \cap \mathcal{O}| = q + 2\,,$$

as q is odd. Thus $\mathrm{Fix}(U^* \langle\psi\rangle) \subsetneq \mathrm{Fix}(\psi) \subsetneq \Pi$, with $o(\mathrm{Fix}(U^* \langle\psi\rangle)) \geq q - 1$. Assume that $o(\mathrm{Fix}(U^* \langle\psi\rangle)) = q$. Then $\sqrt{n} \geq q^2$ by [10, Theorem 3.7], since $\mathrm{Fix}(U^* \langle\psi\rangle) \subset \mathrm{Fix}(\psi)$, which is contrary to the assumption $\sqrt{n} < q^2$. So, $o(\mathrm{Fix}(U^* \langle\psi\rangle)) = q - 1$. Then $\mathrm{Fix}(U^* \langle\psi\rangle) \cap \mathcal{O}$ is a hyperoval of $\mathrm{Fix}(U^* \langle\psi\rangle)$, as $|\mathrm{Fix}(U^* \langle\psi\rangle) \cap \mathcal{O}| = q + 1$. Furthermore, $C_G(\psi)/\langle\psi\rangle \cong Z_\psi/\langle\psi\rangle.\mathsf{PGL}(2, q)$, where $|Z_\psi| = \frac{q-1}{\mu}$ and $\mu = (3, q-1)$, acts 2-transitively on $\mathrm{Fix}(U^* \langle\psi\rangle) \cap \mathcal{O}$. Then $q - 1 = 4$ and $C_G(\psi)/\langle\psi\rangle \leq S_6$ by [1], as $q > 3$. This implies that $\mathrm{Fix}(Z_\psi) = \mathrm{Fix}(\psi)$. So, $C_G(\psi)$ acts on $\mathrm{Fix}(\psi)$ as $\mathsf{PGL}(2, 5)$ leaving invariant a subplane $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$, which is impossible by Lemma 3.6, as $n > 4$.

Assume that $\mathcal{O} \cong \mathsf{PG}(2, q)$. As ψ is Baer collineation of Π and ψ induces a homology on $\mathcal{O}$, then $C'_G(\psi)$ acts on $\mathrm{Fix}(\psi)$ as $\mathsf{PSL}(2, q)$ and it also fixes an antiflag. Note that $q^3 < n < q^4$ by [17, Proposition 11], and since $n \neq q^4$ by our assumption. Then, by Theorem 2.1 (3a), either $\mathrm{Fix}(\psi)$ has order 16 and $C'_G(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2, 5)$, or $\mathrm{Fix}(\psi)$ is the Lorimer-Rahilly plane of order 16 or the Johnson-Walker plane of order 16, or their duals, and $C'_G(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2, 7)$. However, the same argument as in Lemma 3.6 rules out both these cases, since $C'_G(\psi)/\langle\psi\rangle$ fixes an antiflag. This completes the proof. $\square$

Lemma 3.8. *The groups $\overline{C'_G(\psi)}$ and $\overline{C_G(\psi)}$ are strongly irreducible on $\mathrm{Fix}(\psi)$.*

Proof. In order to prove the assertion, by Lemma 3.6, we need to analyze only the case $(q, n) = (5, 4)$ and $q = 9$ when $9^2 < n < 9^4$. Recall that $G \cong \mathsf{PSL}(3, q)$ is irreducible on Π by Lemma 3.3. Then Π consists of nontrivial G-orbits. Since each G-orbit has length $\lambda_j d_j(G)$, where $\lambda_j \geq 0$ and $d_j(G)$ is the degree of some primitive permutation representation of G, then

$$n^2 + n + 1 = \sum_{j \geq 0} \lambda_j d_j(G)\,. \tag{4}$$

That is, $n^2 + n + 1$ must admit a partition restricted to

$$D(G) = [d_0(G), d_1(G), ..d_k(G)]\ ,$$

the spectrum of the degrees of the primitive permutation representations of G. So, the case $(q, n) = (5, 4)$ is ruled out, since $n^2 + n + 1 = 21$, while $D(G) = [31, 3100, 3875, 4000]$ by [2].

Assume that $q = 9$ and $9^2 < n < 9^4$. As above, by [2], $n^2 + n + 1$ must admit a partition restricted to

$$D(G) = [91, 7020, 7560, 58968, 110565, 155520]\ .$$

Note that $9 \mid d_j(G)$ for each $j > 0$. If $\lambda_0 = 0$, then $9 \mid n^2 + n + 1$ by (4), while it is known that either $n^2 + n + 1 \equiv 1 \mod 3$ or $n^2 + n + 1 \equiv 3 \mod 9$. Hence, $\lambda_0 > 0$. So, there exists a point $X \in \Pi$ such that $G_X \leq U^* : C_G(\psi)$, where $C_G(\psi) \cong \mathsf{GL}(2, 9)$, by [2]. Since the group G does not admit 2-transitive point-orbits on Π for $n < 9^4$ by Lemma 3.7, then $G_X < U^* : C_G(\psi)$. Hence, by Lemma 3.5, either $9 \mid \left|X^G\right|$ or $\mathrm{Fix}(U^* \langle\psi\rangle)$ is either a flag, or an antiflag or a proper subplane of $\mathrm{Fix}(\psi)$. Furthermore, $\overline{C_G(\psi)}$ leaves $\mathrm{Fix}(U^* \langle\psi\rangle)$ invariant.

Assume that the latter occurs. If $\mathrm{Fix}(U^* \langle\psi\rangle)$ consists of a flag or an antiflag, again by Theorem 2.1, the case (3) inside the proof of Lemma 3.6 occurs, which leads to a contradiction, as we have seen. So, $\mathrm{Fix}(U^* \langle\psi\rangle)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Then $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ by Theorem 2.1. Note that either $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ is a Baer subplane of $\mathrm{Fix}(U^*)$ or $\mathrm{Fix}(U^* \langle\psi\rangle) = \mathrm{Fix}(U^*) \cong \mathsf{PG}(2, 4)$. Suppose that $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ is a Baer subplane of $\mathrm{Fix}(U^*)$. Note that $\bar{Z}_\psi = \langle 1\rangle$ by [2]. Consequently, Z_ψ fixes $\mathrm{Fix}(\psi)$ pointwise and $\overline{C_G(\psi)} \cong \mathsf{PGL}(2, 9)$. Hence, $\mathrm{Fix}(\langle\psi\rangle) = \mathrm{Fix}(Z_\psi)$. Thus, $\mathrm{Fix}(U^* \langle\psi\rangle) = \mathrm{Fix}(U^*.Z_\psi)$, as Z_ψ normalizes U^*. That is, $\mathrm{Fix}(U^*.Z_\psi) \cong \mathsf{PG}(2, 4)$ is a Baer subplane $\mathrm{Fix}(U^*)$. Then Z_ψ is semiregular on $s \cap (\mathrm{Fix}(U^*) - \mathrm{Fix}(U^*.Z_\psi))$, where s is a secant of $\mathrm{Fix}(U^*.Z_\psi)$. So $8 \mid 16 - 4$, since $o(\mathrm{Fix}(U^*)) = 16$, $\mathrm{Fix}(U^* \langle\psi\rangle) \cong \mathsf{PG}(2, 4)$ and since $|Z_\psi| = \frac{q-1}{\mu} = 8$; this is a contradiction. Thus, $\mathrm{Fix}(U^* \langle\psi\rangle) = \mathrm{Fix}(U^*) \cong \mathsf{PG}(2, 4)$. If there exists a nontrivial element ρ in U^* fixing a point in $\Pi - \mathrm{Fix}(U^*)$, then $\mathrm{Fix}(\rho)$ is a Baer subplane of Π, since $\mathrm{Fix}(U^*) \cong \mathsf{PG}(2, 4)$ and $n = 16^2$. Then each non trivial element in

U^* fixes a subplane of order 16 of Π, since the nontrivial elements in U^* are conjugate under $C_G(\psi) \cong \mathsf{GL}(2,9)$. Hence, if Q is a point fixed by U^*, then $9^2 \mid (9^2-1)(\sqrt{n}+1)+(n+1)$ by Cauchy-Frobenius Lemma, since $|U^*| = 9^2$. So, $9^2 \mid n - \sqrt{n}$, which is a contradiction, since $n = 16^2$. Therefore, U^* is semiregular on $r - \mathrm{Fix}(U^*)$, where r is a secant to U^*. Hence, $9^2 \mid n-4$ and we again obtain a contradiction, as $n = 16^2$. Thus, $9 \mid \left|X^G\right|$. Actually the previous argument can be repeated for each point $Y \in \Pi$ such that G_Y lies in a maximal parabolic subgroup of G. Consequently, any orbit divisible by $d_0(G)$ is actually divisible by $9d_0(G)$. Therefore, bearing in mind that $9 \mid d_j(G)$ for each $j > 0$, any admissible G-orbit has length divisible by 9. So, $9 \mid n^2+n+1$ by (4), and we obtain a contradiction as above. This completes the proof. □

Lemma 3.9. *The group $\overline{C_G(\psi)}$ contains Baer involutions of* $\mathrm{Fix}(\psi)$*. In particular, $\sqrt[4]{n}$ is an integer.*

Proof. Assume that all the involutions in $\overline{C_G(\psi)}$ are perspectivities of $\mathrm{Fix}(\psi)$. If $\sqrt{n}$ is even, then either $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,2)$ and $C_G^{'}(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2,7)$ or $\mathrm{Fix}(\psi) \cong \mathsf{PG}(2,4)$ and $C_G^{'}(\psi)/\langle\psi\rangle \cong \mathsf{PSL}(2,9)$ by [9]. However, both these cases cannot occur by the same argument as in Lemma 3.6. Hence, $\sqrt{n}$ is odd and the involutions in $\overline{C_G(\psi)}$ are homologies of $\mathrm{Fix}(\psi)$.

If $K = Z_\psi$, then $\overline{C_G(\psi)} \cong \mathsf{PGL}(2,q)$. Then $q \mid \sqrt{n}$ and $q-1 \mid \sqrt{n}-1$ by [12, Theorem C.ii]. As $q \mid \sqrt{n}$, then $\sqrt{n} = \lambda_1 q$ for some $\lambda_1 \geq 0$. Furthermore, $\lambda_1 = (q-1)\lambda_2+1$ for some $\lambda_2 \geq 0$, since $q-1 \mid \sqrt{n}-1$. Hence, $\sqrt{n} = q(q-1)\lambda_2+q$. However, this is impossible, since $n < q^4$ by our assumption.

If $K < Z_\psi$. Then $\bar{Z}_\psi \neq \langle 1\rangle$. Since $\overline{C_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$ by Lemma 3.8, and since each nontrivial subgroup of $\bar{Z}_\psi$ is normal in $\overline{C_G(\psi)}$, then $\bar{Z}_\psi$ is semiregular on $\mathrm{Fix}(\psi)$. Let $\bar{\sigma}$ be any involutory $(C_{\bar{\sigma}}, a_{\bar{\sigma}})$-homology of $\overline{C'_G(\psi)}$. Note that $\overline{C'_G(\psi)} \times \bar{Z}_\psi \lhd \overline{C_G(\psi)}$. That is, $\bar{Z}_\psi$ centralizes $\bar{\sigma}$ and hence $\bar{Z}_\psi$ fixes $(C_{\bar{\sigma}}, a_{\bar{\sigma}})$. This is impossible, since $\bar{Z}_\psi$ is semiregular on $\mathrm{Fix}(\psi)$. Therefore, $\overline{C_G(\psi)}$ contains Baer collineation of $\mathrm{Fix}(\psi)$ and hence $\sqrt[4]{n}$ is an integer. □

Proposition 3.10. *For each $X \in \Pi$ such that G_X lies in a maximal parabolic subgroup of G, then $q^2 \mid \left|X^G\right|$.*

Proof. Since $\overline{C_G(\psi)}$ is strongly irreducible on $\mathrm{Fix}(\psi)$ by Lemma 3.8 and since the group G does not admit 2-transitive point-orbits on Π by Lemma 3.7, the assertion follows by Lemma 3.5. □

Lemma 3.11. *One of the following occurs:*

(1) $q^2 \mid n^2+n+1$;

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page 16 / 22
go back
full screen
close
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(2) *q is a square, $q\sqrt{q} \mid n^2+n+1$, and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3,\sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3,\sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$.*

Proof. Since G is irreducible on Π, then Π consists of nontrivial G-orbits. By a direct inspection of the list of maximal subgroups of $\mathsf{PSL}(3,q)$ given in [16], we have that $q^2 \mid |X^G|$ for each point $X \in \Pi$, unless q is a square and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3,\sqrt{q})$, or $G_Y \leq \mathsf{PSU}(3,\sqrt{q})$, with $(|G_Y|, q\sqrt{q}) > q$, or $G_Y \leq E_{q^2} : C_G(\gamma)$ for some involution γ of G. Actually, if $G_Y \leq E_{q^2} : C_G(\gamma)$, then $q^2 \mid |Y^G|$ by Proposition 3.10.

If either there are no Z in Π such that $G_Z \leq \mathsf{PSL}(3,\sqrt{q})$ or $G_Z \leq \mathsf{PSU}(3,\sqrt{q})$ and $(|G_Z|, q\sqrt{q}) > q$, each admissible G-orbit on Π is divisible by q^2. Therefore, $q^2 \mid n^2+n+1$, since Π consists of nontrivial G-orbits. That is, the assertion (1).

If q is square and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3,\sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3,\sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$, each G-orbits is divisible by $q\sqrt{q}$ and hence $q\sqrt{q} \mid n^2+n+1$ by the above argument. That is, the assertion (2). □

Corollary 3.12. $p \neq 3$.

Proof. Assume that $p = 3$. As $q > 3$, then $9 \mid q$. Hence, $9 \mid n^2+n+1$ by Lemma 3.11. However, this is impossible, since it is known that either $n^2+n+1 \equiv 1 \mod 3$ or $n^2+n+1 \equiv 3 \mod 9$. □

Lemma 3.13. *Let S_0 be the p-group and let ψ and β be the involutions defined in Section 2. If q is a square and $q\sqrt{q} \mid n^2+n+1$, then one of the following occurs:*

(1) *The group S_0 is semiregular on Π and hence on $\mathrm{Fix}(\psi)$;*

(2) *$\mathrm{Fix}(S_0)$ is a subplane of Π. Furthermore, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$;*

(3) *There exists a nontrivial proper subgroup S^* of S_0 such that $\mathrm{Fix}(S^*)$ is a subplane of Π of order m and one of the following occurs:*

   (a) *$\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$ and the involution β induces a Baer collineation on it. In particular, $\sqrt[4]{m}$ is an integer.*

   (b) *$\mathrm{Fix}(S^*)$ is a proper subplane of $\mathrm{Fix}(\psi)$ and hence $m \leq \sqrt[4]{n}$.*

Furthermore, in the cases (3a)–(3b), the group S_0/S^* acts on $\mathrm{Fix}(S^*)$ and on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly.

Proof. Let S_0 be the p-group defined in Section 2. Recall that $p \neq 3$ by Corollary 3.12. Also, recall that ψ centralizes S_0, that β inverts S_0 and that $\langle \psi, \beta \rangle \cong$

E_4. Since $q\sqrt{q} \mid n^2+n+1$ and $n^2+n+1 = (n-\sqrt{n}+1)(n+\sqrt{n}+1)$, with $(n-\sqrt{n}+1, n+\sqrt{n}+1) = 1$, either $q\sqrt{q} \mid n-\sqrt{n}+1$ or $q\sqrt{q} \mid n+\sqrt{n}+1$.

Assume that $q\sqrt{q} \mid n-\sqrt{n}+1$. Thus $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi) \neq \varnothing$. In particular, $\mathrm{Fix}(S_0) \neq \varnothing$. As $p \neq 3$ and that $q\sqrt{q} \mid n^2+n+1$, we have that $(q,n) = (q, n\pm 1) = 1$. Therefore, $\mathrm{Fix}(S_0)$ is a subplane of Π. As ψ centralizes S_0, then ψ acts on $\mathrm{Fix}(S_0)$. Hence, $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi) \neq \varnothing$. Actually, $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a subplane of $\mathrm{Fix}(\psi)$, again since ψ centralizes S_0, $q\sqrt{q} \mid n+\sqrt{n}+1$ and $p \neq 3$. Moreover, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0) \subseteq \mathrm{Fix}(\psi)$. Assume that $\mathrm{Fix}(S_0) = \mathrm{Fix}(\psi)$. Then S_0 is semiregular on $s - \mathrm{Fix}(S_0)$, where s is a secant of $\mathrm{Fix}(S_0)$, since $\mathrm{Fix}(S_0)$ is a Baer subplane of Π. Therefore, $q \mid n - \sqrt{n}$, since $|S_0| = q$. That is, $q \mid \sqrt{n}(\sqrt{n}-1)$. So, we obtain a contradiction, since $(\sqrt{n}(\sqrt{n}-1), n-\sqrt{n}-1) = 1$. Thus, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$, and we obtain the assertion (2).

Assume that $q\sqrt{q} \mid n+\sqrt{n}+1$. If $\mathrm{Fix}(S_0) \neq \varnothing$, we still obtain the assertion (2) by the previous argument, by bearing in mind that $(\sqrt{n}(\sqrt{n}-1), n+\sqrt{n}+1) \mid 3$ and that $q > 3$. Hence, assume that $\mathrm{Fix}(S_0) = \varnothing$. At this point, either S_0 is semiregular on Π and we obtain the assertion (1), or there exists a nontrivial subgroup S_1 of S_0 such that $\mathrm{Fix}(S_1) \neq \varnothing$. By bearing in mind that ψ centralizes S_0 and hence S_1, that $(\sqrt{n}(\sqrt{n}-1), n+\sqrt{n}+1) \mid 3$ and that $p \neq 3$ by Corollary 3.12, the previous argument, with S_1 in the role of S_0, yields that either $\mathrm{Fix}(S_1) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_1)$ or $\mathrm{Fix}(S_1)$ is a proper subplane of $\mathrm{Fix}(\psi)$.

Let $\mathcal{S}$ be the set of the nontrivial subgroups of S_0 fixing a subplane of Π whose intersection with $\mathrm{Fix}(\psi)$ is in turn a subplane of this one. Clearly, $\mathcal{S} \neq \varnothing$, since $S_1 \in \mathcal{S}$. Let S^* be an element of $\mathcal{S}$ of maximal order. Hence, $\mathrm{Fix}(S^*)$ is a subplane of Π and $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a subplane of $\mathrm{Fix}(\psi)$. Moreover, either $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$ or $\mathrm{Fix}(S^*)$ is a proper subplane of $\mathrm{Fix}(\psi)$, again by the above argument with S^* in the role of S_1. Let m be the order of $\mathrm{Fix}(S^*)$. If $\mathrm{Fix}(S^*)$ is a proper subplane of $\mathrm{Fix}(\psi)$, then $m \leq \sqrt[4]{n}$ by [10, Theorem 3.7], and we obtain the assertion (3a). Hence, assume that $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$. Note that S_0/S^* is nontrivial and acts semiregularly on $\mathrm{Fix}(S^*)$ and on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$, since $\mathrm{Fix}(S_0) = \varnothing$, the group S^* is an element of $\mathcal{S}$ of maximal order, the group S_0 is abelian and since ψ centralizes S_0. Denote by m_ψ the order of $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Then $m = m_\psi^2$ by [10, Theorem 3.7], since $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$. As β inverts S_0 and as $\langle \psi, \beta \rangle \cong E_4$, then β normalizes $S^* \langle \psi \rangle$ and acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Denote by $S_0^+ = S_0/S^*$. Hence, S_0^+ is nontrivial and acts semiregularly on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$, as we have seen above. Furthermore, $S_0^+ \langle \beta \rangle$ acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Assume that β induces a perspectivity on $\mathrm{Fix}(S^*) \cap$

$\mathrm{Fix}(\psi)$. Let $\rho \in S_0^+$, $\rho \neq 1$. Then β^ρ is also a perspectivity of $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$, and $\mathrm{Fix}([\beta^\rho, \beta]) \cap \mathrm{Fix}(\psi) \neq \varnothing$ by [6, Lemma 5.1]. This is a contradiction, since $[\beta^\rho, \beta] \in S_0^+$, the group S_0^+ is nontrivial and acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly. Therefore, β induces a Baer collineation on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$. Then m_ψ is a square by [10, Theorem 3.7]. Consequently, $\sqrt[4]{m}$ is an integer, since we proved that $m = m_\psi^2$, and we obtain the assertion (3b). □

4. The proof of Theorem 1.1

Proposition 4.1. *The involutions in G are perspectivities of Π.*

Proof. We proceed with a series of steps to show that no one of the cases of Lemma 3.13 occurs, obtaining the assertion in this way.

Step I: The case (1) of Lemma 3.13 does not occur.

Assume that S_0 is semiregular on Π and on $\mathrm{Fix}(\psi)$. So, $q \mid n + \sqrt{n} + 1$. Recall that either $q^2 \mid n^2 + n + 1$ or, q is a square, $q\sqrt{q} \mid n^2 + n + 1$, and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3, \sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3, \sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$, by Lemma 3.11. In particular, either $q^2 \mid n + \sqrt{n} + 1$ or $q\sqrt{q} \mid n + \sqrt{n} + 1$, respectively, since $n^2 + n + 1 = (n + \sqrt{n} + 1)(n - \sqrt{n} + 1)$, $(n + \sqrt{n} + 1, n - \sqrt{n} + 1) = 1$, and since $q \mid n + \sqrt{n} + 1$. If $q^2 \mid n + \sqrt{n} + 1$, then we obtain a contradiction by [13, Lemma 6.2], since $\sqrt{n}$ is a square by Lemma 3.9. Thus, q is a square, $q\sqrt{q} \mid n + \sqrt{n} + 1$, and there exists a point $Y \in \Pi$ such that either $G_Y \leq \mathsf{PSL}(3, \sqrt{q})$ or $G_Y \leq \mathsf{PSU}(3, \sqrt{q})$, where $(|G_Y|, q\sqrt{q}) > q$.

Assume that there exists a point $Y \in \Pi$, such that either $G_Y \leq M$, where M is either $\mathsf{PSL}(3\sqrt{q})$ or $\mathsf{PSU}(3, \sqrt{q})$, and such that $(|G_Y|, q\sqrt{q}) > q$. Without loss of generality, we may assume that a Sylow p-subgroup of G_Y is contained in U, the group defined in Section 2. Set $U_Y = G_Y \cap U$ and $U(M) = M \cap U$. Clearly, $U_Y \leq U(M)$, with $(|U_Y|, q\sqrt{q}) > q$ and $|U(M)| = q\sqrt{q}$. In particular, $U(M)$ consists of matrices of type (1) given in Section 2 whose entries are all the elements of $\mathsf{GF}(\sqrt{q})$, while U_Y consists of some of these matrices. Let W be the subgroup of S_0, represented by the matrices type (2) given in Section 2, with $y_2 = y_3 = 0$ and with $y_1 \in \mathsf{GF}(\sqrt{q})$. Hence, $|W| = \sqrt{q}$ and $W \leq U(M)$. Therefore, $\langle U_Y, W\rangle \leq U(M)$, as $U_Y \leq U(M)$. Hence, $|\langle U_Y, W\rangle| \leq q\sqrt{q}$. On the other hand, $|\langle U_Y, W\rangle| \geq \frac{|U_Y||W|}{|U_Y \cap W|}$. Thus, $\frac{|U_Y||W|}{|U_Y \cap W|} \leq q\sqrt{q}$. Therefore, $p \mid |U_Y \cap W|$ since $(|U_Y|, q\sqrt{q}) > q$ and $|W| = \sqrt{q}$. So, $p \mid |U_Y \cap S_0|$, since $W \leq S_0$. Hence, we arrive at a contradiction, since S_0 is semiregular on Π.

Step II: The case (2) of Lemma 3.13 does not occur.

Recall that S, S_0 and ψ are defined as in Section 2. Hence, $Z(S) = S^{'} = S_0$. Furthermore, ψ normalizes S and S_0. Assume that the case (2) of Lemma 3.13 occurs. Hence, $\mathrm{Fix}(S_0)$ is a subplane of Π. Moreover, either $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$ or $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Clearly, S acts on $\mathrm{Fix}(S_0)$ (the action is unfaithful). Assume $S_0 < S_Q$ for some point $Q \in \mathrm{Fix}(S_0)$. Then S_Q lies in G_Q which, in turn, lies in a maximal parabolic subgroup of G by a direct inspection of the list of maximal subgroups of $G \cong \mathsf{PSL}(3, q)$, q odd, given in [16]. Then $q^2 \mid \left|Q^G\right|$ by Proposition 3.10. However, this is impossible, since $S_0 < S_Q \leq S$, while $|S_0| = q$ and $|S| = q^3$. Hence S induces the group S/S_0 on $\mathrm{Fix}(S_0)$ acting semiregularly. Thus $q^2 \mid h^2 + h + 1$, where h is the order of $\mathrm{Fix}(S_0)$.

Assume that $\mathrm{Fix}(S_0) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S_0)$. Then h is a square. Moreover $h \leq \sqrt{n}$ by [10, Theorem 3.7]. Hence $h^2+h+1 \leq q^4+q+1$, since $n \leq q^4$ by our assumption. However, this yields a contradiction, by [13, Lemma 6.2], since $q^2 \mid h^2 + h + 1$ and h is a square.

Assume that $\mathrm{Fix}(S_0)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Then $h \leq \sqrt[4]{n}$ and hence $h^2+h+1 \leq q^2+q+1$. Thus, $q^2 = h^2+h+1$, since $q^2 \mid h^2+h+1$, and we still obtain a contradiction by [13, Lemma 6.2].

Step III: The final contradiction.

By (I) and (II) it follows that only case (3) of Lemma 3.13 might occur. Hence, assume there exists a nontrivial proper subgroup S^* of S_0 such that S_0/S^* acts on $\mathrm{Fix}(S^*)$ and on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly. In particular, if m is the order of $\mathrm{Fix}(S^*)$, then $\sqrt[4]{m}$ is an integer. Since $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a Baer subplane of $\mathrm{Fix}(S^*)$ an since S_0/S^* is nontrivial and acts on $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ semiregularly, then $p \mid m + \sqrt{m} + 1$. Furthermore, as $S^* < S_0$, the group S centralizes S^* and acts on $\mathrm{Fix}(S^*)$. From the proof of Lemma 3.11, we actually obtain that any G-orbit has length divisible by either q^2 or, when q is a square, by $q\sqrt{q}$. This implies that each orbit under the group induced by S on $\mathrm{Fix}(S^*)$ has length divisible by either q^2 or, when q is a square, by $q\sqrt{q}$. So, we obtain that either $q^2 \mid m^2 + m + 1$, or $q\sqrt{q} \mid m^2 + m + 1$ when q is a square. Actually, either $q^2 \mid m + \sqrt{m} + 1$ or $q\sqrt{q} \mid m + \sqrt{m} + 1$, respectively, since $m^2 + m + 1 = (m + \sqrt{m} + 1)(m - \sqrt{m} + 1)$, $(m + \sqrt{m} + 1, m - \sqrt{m} + 1) = 1$, and since $p \mid m + \sqrt{m} + 1$.

Assume that $\mathrm{Fix}(S^*) \cap \mathrm{Fix}(\psi)$ is a proper subplane of $\mathrm{Fix}(\psi)$. Then $\sqrt{m} \leq \sqrt[4]{n}$ by [10, Theorem 3.7]. So, $m + \sqrt{m} + 1 \leq q^2 + q + 1$. This fact, in conjunction with either $q^2 \mid m + \sqrt{m} + 1$ or $q\sqrt{q} \mid m + \sqrt{m} + 1$, yields that $q\sqrt{q} = m + \sqrt{m} + 1$

with $q = 7$ and $\sqrt{m} = 18$ by [13, Lemma 6.2]. However, this is a contradiction, since $\sqrt{m}$ is a square.

Since none of the cases of Lemma 3.13 occurs, then ψ cannot be a Baer collineation of Π. Therefore, any involution of G is a perspectivity of Π, since $G \cong \mathsf{PSL}(3, q)$ contains a unique conjugate class of involutions. $\square$

Now, using Proposition 4.1, we prove our main result.

Proof of Theorem 1.1. The assertion follows by Proposition 3.2 for $q = 3$. Hence, assume that $q > 3$. Since $G \cong \mathsf{PSL}(3, q)$ is irreducible on Π by Lemma 3.3 and since each involution in G is a perspectivity of Π by Proposition 4.1, then G leaves invariant a subplane Π_0 on which it acts strong irreducibly by [6, Lemmas 5.2 and 5.3]. Then $\Pi_0 \cong \mathsf{PG}(2, q)$ by [8, Theorem 1.1]. If $n \le q^3$, the assertion follows from Theorem 2.3. Hence, assume that $q^3 < n \le q^4$. As the involutions in G are homologies of Π_0, they are also homologies of Π. Furthermore, each p-element inducing an elation on Π_0 is also an elation of Π by [10, Theorem 4.25]. Finally, by [12, Theorem C.ii], we have that $q^2 \mid n$, that $q - 1 \mid n - 1$ and that $q + 1 \mid n^2 - 1$. It is a straightforward computation to show that this numerical information yield that Π has order $n = \lambda q^3 + (1-\lambda)q^2$, where $1 < \lambda \le q + 1$ and $q + 1 \mid \lambda(\lambda - 1)$, since $q^3 < n \le q^4$. This completes the proof. $\square$

Remark 4.2. It seems to be tough proving that there are no planes of order $q^3 < n < q^4$ admitting $G \cong \mathsf{PSL}(3, q)$ as a collineation group. Indeed, although it is easy to show that a nontrivial stabilizer of a point has order odd and coprime to p, it is difficult to determine the exact orbital decomposition of the set of external lines to Π_0, especially when the stabilizer of a line of such a set is a subgroup of a Singer cycle of G.

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