



Sharply transitive decompositions of complete graphs into generalized Petersen graphs

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Abstract

A decomposition of the complete graph K_v into copies of a subgraph Γ is called a sharply transitive Γ -decomposition if it is left invariant by an automorphism group acting sharply transitively on the vertex-set of K_v . For suitable values of v we construct examples of sharply transitive Γ -decompositions when Γ is either a Petersen graph, a generalized Petersen graph or a prism.

Keywords: decomposition, (generalized) Petersen graph, prism, sharply transitive permutation group

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1 Introduction

In this paper we shall only deal with undirected simple graphs. For each graph T we shall denote by $V(T)$ and $E(T)$ its vertex-set and edge-set, respectively. A decomposition of a graph T into copies of a given graph Γ is a set \mathcal{C} of subgraphs isomorphic to Γ , whose edges partition, altogether, the edge-set of T . Such a decomposition is generally called a Γ -decomposition of T . A Γ -decomposition of K_v , the complete graph on v vertices, is often called a Γ -design; see [2, 4]. This terminology is suggested by the circumstance that a standard 2 - $(v, k, 1)$ design is a Γ -design for $\Gamma = K_k$, once the blocks are regarded as complete graphs on k vertices.

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A group G is an *automorphism group* of the Γ -decomposition \mathcal{C} if it is a permutation group on the vertices of T leaving \mathcal{C} invariant. The action of G is said to be *sharply transitive* on the vertex-set $V(T)$ if for any given pair of (not necessarily distinct) vertices $x, y \in V(T)$ there exists a unique automorphism g in G mapping x to y . In the theory of finite permutation groups a sharply transitive action is often called a *regular* action.

We shall speak of a *sharply transitive Γ -decomposition* of T if \mathcal{C} admits an automorphism group G acting sharply transitively on the vertex-set of T . In particular, we shall speak of a *cyclic* or of an *elementary abelian Γ -decomposition* if the action of G is sharply transitive and G is cyclic or elementary abelian, respectively. In this paper, when speaking of a group G we will use the additive notation, unless differently specified.

If the group G acts sharply transitively on the vertex-set of K_v , then G has order v and we can identify the vertex set of K_v with G . In this case we shall usually denote the complete graph by K_G rather than by K_v . In this manner each group-element $g \in G$ is identified with the permutation $V(K_G) \rightarrow V(K_G): x \mapsto x+g$. This action of G induces actions on the subsets of $V(K_G)$ and on sets of such subsets. Hence if $[x, y]$ is an edge, then it is mapped to the edge $[x+g, y+g]$ by g and we write $[x, y]+g = [x+g, y+g]$. In particular, if U is a collection of edges of K_G then $U+g = \{[x+g, y+g] \mid [x, y] \in U\}$. In the present paper we shall usually denote by 0 the zero element of an additive group G and by 1 the identity element in multiplicative notation.

Let n, k be positive integers, with $n \geq 3$ and $1 \leq k \leq n-2$. Denote by $P(n, k)$ the graph with vertex-set

$$V(P(n, k)) = \{x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}\}$$

and edge-set

$$E(P(n, k)) = \bigcup_{i=0}^{n-1} \{[x_i, x_{i+1}], [y_i, y_{i+k}], [x_i, y_i]\},$$

where subscripts are meant modulo n . In particular, $P(5, 2)$ is the *standard Petersen graph*, $P(n, 1)$ is a *prism* and if $n > 5$, $1 < k < n-1$ the graph $P(n, k)$ is a *generalized Petersen graph*.

For each fixed choice of the graph Γ , the values of v for which a Γ -decomposition exists form the so-called *spectrum*. Clearly the values of v for which a sharply transitive Γ -decomposition exists must be in the spectrum. The same is true if we fix the isomorphism type of the automorphism group G which is assumed to act sharply transitively on the vertices of K_v (say, for instance, cyclic or elementary abelian).

It may well happen that there exists no such sharply transitive Γ -decomposition of K_v with the group G of the required isomorphism type even though v is in the spectrum. That is trivially the case if no finite group of order v of the required isomorphism type exists. For example if Γ is the Petersen graph $P(5, 2)$ then Adams and Bryant have shown in [2] that v is in the spectrum of Γ -decompositions of K_v if and only if $v \equiv 1$ or $10 \pmod{15}$, $v > 10$. Consequently, although $v = 40$ is in this spectrum, it does not even make sense to look for an elementary abelian Γ -decomposition of K_{40} because 40 is not a prime power. It may also happen that v *does* lie in the spectrum and a finite group of order v of the desired isomorphism type *does* exist while a sharply transitive Γ -decomposition with group of the required type *does not* exist. Probably the most famous example for this situation is when v is a power of 2, $v \geq 8$, the group G is cyclic and Γ is a 1-factor of K_v . Hartman and Rosa have namely proved in [13] that a cyclic 1-factorization of K_v does not exist for such values of v .

The problem we approach in the present paper is thus a special instance of the determination of what we might call a “restricted” spectrum: given the graph Γ and the isomorphism type of the group G , determine the values of v for which there exists a Γ -decomposition of K_v admitting an automorphism group that is isomorphic to G and acts sharply transitively on the vertices of K_v .

In this paper we construct sharply transitive decompositions of complete graphs into Petersen graphs, generalized Petersen graphs and prisms. Our main results are the following:

- If p is an odd prime and $q \equiv 1 \pmod{6}$ is a power of p with $q > p$, then for each k with $2 \leq k \leq p - 2$ we construct an elementary abelian $P(p, k)$ -decomposition of K_q .
- Let $n > 3$ be an odd integer and let $v > 13$ be an odd integer all of whose prime factors are congruent to 1 $\pmod{6n}$. For each k with $1 \leq k \leq \frac{n-1}{2}$ we construct a cyclic $P(n, k)$ -decomposition of K_v .

In the final section we construct examples of sharply transitive decompositions of a complete graph into different types of generalized Petersen graphs and prisms. That can somehow be viewed in analogy with the situation in which a block design contains blocks of different sizes, a situation which is often considered in design theory. In our situation the “blocks” are subgraphs with different “shapes”.

In [18] Rosa introduced the concept of *graceful labelings* and α -*labelings* of graphs. These are assignments of integers to the vertices of the graph subject to certain conditions. A necessary condition for a graph to have an α -labeling is the property of being bipartite. We refer to [11] for an extensive survey on this topic.

Rosa showed how graphs with graceful labelings or α -labelings are useful in the development of the theory of graph decompositions. For instance he proved in [18] that if Γ is a graph possessing a graceful labeling, then there exists a cyclic Γ -decomposition of K_v , where $v = 2|E(\Gamma)| + 1$. In the same paper he proved that if Γ is a graph possessing an α -labeling, then there exists a cyclic Γ -decomposition of K_v , for each $v \equiv 1 \pmod{2|E(\Gamma)|}$. Redl [17] has shown that $P(n, k)$ has a graceful labeling for $n = 5, 6, 7, 8, 9$ and 10 . Vietri [20, 21] proved that $P(4t, 3)$ has a graceful labeling for all t 's. The problem of determining whether generalized Petersen graphs admit graceful labelings is still open, while it is proved in [10] that $P(n, 1)$ has an α -labeling if and only if n is even. Our constructions work on graphs which are not bipartite, hence they cannot be obtained from α -labelings. Our existence results can thus be regarded as an extension of the following proposition which can be obtained from [18] and [10]: if $n > 3$ is an even integer and v is an odd integer with $v \equiv 1 \pmod{6n}$, then a cyclic $P(n, 1)$ -decomposition of K_v does exist.

As to our notation, for a given prime p we denote by Z_p the finite field of order p , that is the field of residue classes modulo p . More generally, for any positive integer v we denote by Z_v the ring of residue classes modulo v as well as its additive cyclic group; for any positive integer n we denote by $\text{GF}(p^n)$ the finite field of order p^n , that is the n -dimensional field-extension of Z_p (so in particular, $Z_p = \text{GF}(p)$). The notation $\text{GF}(p^n)^*$ stands for $\text{GF}(p^n) - \{0\}$, in the same manner $Z_p^* = Z_p - \{0\}$. When dealing with vector spaces, vectors will generally be denoted by bold letters.

2 Preliminary results

Our constructions are essentially based on the method of partial differences, a useful tool in many circumstances. See for example [5], [6] and [8], where the method is used for constructing cycle systems, and [1] for constructions in classical design theory. We begin with a brief review of this method in our special situation. In particular, we work in K_G where G is a group of *odd* order. In this situation each edge orbit under the action of G has length equal to the order of G .

Given a simple graph Γ with vertices in K_G , the *list $\Delta\Gamma$ of differences* of Γ is defined as follows:

$$\Delta\Gamma = \{x - y, y - x \mid [x, y] \in E(\Gamma)\}.$$

In general, $\Delta\Gamma$ contains repeated elements, so it is a multiset. Moreover, if the stabilizer of Γ in G has order h , then each element of $\Delta\Gamma$ is repeated at least

h times. Let us denote by H the stabilizer of Γ in G and let Γ^* be a chosen subgraph of Γ whose edges form a complete system of representatives for the H -orbits of the edges of Γ . In what follows we will set $\partial\Gamma = \Delta\Gamma^*$. In general $\partial\Gamma$ can itself be a multiset.

Theorem 2.1. *Let $\mathcal{C} = \{\Gamma_1, \dots, \Gamma_t\}$ be a set of subgraphs of K_G which are isomorphic to a given graph Γ . The set \mathcal{C} is a complete system of representatives for the G -orbits of a sharply transitive Γ -decomposition of K_G if and only if $\partial\Gamma_1 \cup \dots \cup \partial\Gamma_t$ is a repetition-free cover of $G - \{0\}$.*

Proof. Let v be the order of G and let n be the number of edges in Γ . Let h_i denote the order of the G -stabilizer of the subgraph Γ_i , $i = 1, \dots, t$. Suppose \mathcal{C} to be a complete system of representatives for the G -orbits of a sharply transitive Γ -decomposition of K_G . The total number of edges of $\text{orb}_G(\Gamma_1) \cup \dots \cup \text{orb}_G(\Gamma_t)$ is $\frac{nv}{h_1} + \dots + \frac{nv}{h_t} = \frac{v(v-1)}{2}$. This equality implies $\frac{2n}{h_1} + \dots + \frac{2n}{h_t} = v - 1$. Observe that $v - 1 = |G - \{0\}|$ and $\frac{2n}{h_i} = |\partial\Gamma_i|$. Therefore, for the proof of the first part of the theorem, it is sufficient to show that each element of G appears at least once in the list $\partial\Gamma_1 \cup \dots \cup \partial\Gamma_t$. Given $x \in G - \{0\}$, the edge $[0, x]$ is in $\Gamma_i + g$ for a suitable $i \in \{1, \dots, t\}$ and $g \in G$. That implies $[0, x] = [y, z] + g$ with $[y, z] \in E(\Gamma_i)$ and $x = z - y$ or $x = y - z$, which proves the assertion.

Let conversely the multiset equality $\partial\Gamma_1 \cup \dots \cup \partial\Gamma_t = G - \{0\}$ hold without repetitions. Since $|\partial\Gamma_i| = \frac{2n}{h_i}$, the graph $\text{orb}_G(\Gamma_i)$ contains $\frac{nv}{h_i}$ edges and $\text{orb}_G(\Gamma_1) \cup \dots \cup \text{orb}_G(\Gamma_t)$ has the same number of edges as K_G . We will thus obtain a decomposition of K_G if we can prove that each edge of K_G is an edge of a suitable $\Gamma_i + g$, $g \in G$. If $[x, y]$ is an edge of K_G , then $y - x \in \partial\Gamma_i$, for some i . Therefore, there exists $[a, b] \in E(\Gamma_i)$ with $b - a = y - x$, which implies $[a, b] - a + x = [x, y]$ and $[x, y] \in \Gamma_i - a + x$. \square

In the particular case in which all the graphs of \mathcal{C} have trivial stabilizer in G , one says that \mathcal{C} is a $(G, \Gamma, 1)$ -difference family [7]. In the proof of Theorem 4.2 below, we shall need the result of the next Proposition, which was obtained in [7].

Proposition 2.2. *If there exists a $(Z_{v_i}, \Gamma, 1)$ -difference family for $i = 1, \dots, n$ and if the chromatic number $\chi(\Gamma)$ does not exceed the smallest prime factor of the product $v_1 v_2 \dots v_n$, then there exists a cyclic Γ -decomposition of $K_{v_1 v_2 \dots v_n}$.*

3 Elementary abelian decompositions of the complete graph into (generalized) Petersen graphs

Lemma 3.1. *Let $p > 3$ be a prime and let V be the n -dimensional vector space over the field Z_p with $|V| \equiv 1 \pmod{6}$. Then, for every fixed $k \in Z_p - \{0, 1, -1\}$, it is possible to partition $V - \{0\}$ into $(p^n - 1)/6$ sextuples of the form $\{\pm \mathbf{u}, \pm k\mathbf{u}, \pm \mathbf{v}\}$ where \mathbf{u} and \mathbf{v} are linearly independent vectors of V .*

Proof. Set $(p^n - 1)/(p - 1) = t$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t\}$ be a complete system of representatives for the 1-dimensional vector subspaces of V so that we have $V - \{0\} = \{a\mathbf{v}_j \mid 1 \leq a \leq p-1; 1 \leq j \leq t\}$. Let H be the multiplicative subgroup of Z_p^* consisting of all elements of the form $\pm k^i$. Let $2d$ be the order of H , and let S be a complete system of representatives for the cosets of H in Z_p^* . We distinguish two cases according to whether $p \equiv 1$ or $-1 \pmod{6}$.

1st case: $p \equiv 1 \pmod{6}$.

It is straightforward to check that the set

$$\mathcal{Q} = \left\{ \{\pm k^{2i}s, \pm k^{2i+1}s\} \mid 0 \leq i < \lfloor d/2 \rfloor; s \in S \right\}$$

consists of pairwise disjoint quadruples of elements of $Z_p - \{0\}$. Since we have $|\mathcal{Q}| = \lfloor d/2 \rfloor \cdot |S| \geq |H| \cdot |S|/6 = (p-1)/6$, we can choose a $(p-1)/6$ -subset \mathcal{A} of \mathcal{Q} . In view of the form of the quadruples of \mathcal{Q} we can write

$$\mathcal{A} = \left\{ \{\pm a_i, \pm k a_i\} \mid 1 \leq i \leq (p-1)/6 \right\}.$$

The set $B = Z_p^* - \bigcup_{i=1}^{(p-1)/6} \{\pm a_i, \pm k a_i\}$ has size $2(p-1)/6$ and, obviously, if $b \in B$, then $-b \in B$. Thus we can write $B = \{\pm b_i \mid 1 \leq i \leq (p-1)/6\}$, and the sextuples

$$\{\pm a_i, \pm k a_i, \pm b_i\}, \quad 1 \leq i \leq (p-1)/6$$

form a partition of Z_p^* .

It is now clear that $V - \{0\}$ can be partitioned into the following $(p^n - 1)/6$ sextuples of the required form:

$$\{\pm a_i \mathbf{v}_j, \pm k a_i \mathbf{v}_j, \pm b_i \mathbf{v}_{j+1}\}, \quad 1 \leq i \leq (p-1)/6; 1 \leq j \leq t$$

where we agree that $\mathbf{v}_{t+1} = \mathbf{v}_1$.

2nd case: $p \equiv 5 \pmod{6}$.

In this case t is divisible by 3. Distinguish two subcases according to whether d is even or odd.

Subcase (a): d is even, say $d = 2e$.

We claim that $V - \{0\}$ can be partitioned into the following $(p^n - 1)/6$ sixtuples of the required form:

$$\begin{aligned} & \{ \pm k^{2i} s \mathbf{v}_{3j+1}, \pm k^{2i+1} s \mathbf{v}_{3j+1}, \pm k^{2i} s \mathbf{v}_{3j+3} \}, \\ & \{ \pm k^{2i} s \mathbf{v}_{3j+2}, \pm k^{2i+1} s \mathbf{v}_{3j+2}, \pm k^{2i+1} s \mathbf{v}_{3j+3} \}, \\ & 0 \leq i \leq e - 1; s \in S; 0 \leq j \leq t/3 - 1. \end{aligned}$$

In fact, it is easy to check that for each fixed value of j the above sixtuples form a partition of $\{a \mathbf{v}_{3j+\ell} \mid 1 \leq a \leq p - 1; \ell = 1, 2, 3\}$.

Subcase (b): d is odd, say $d = 2e + 1$.

We claim that $V - \{0\}$ can be partitioned into the following $(p^n - 1)/6$ sixtuples of the required form:

$$\begin{aligned} & \{ \pm k^{2i} s \mathbf{v}_{3j+1}, \pm k^{2i+1} s \mathbf{v}_{3j+1}, \pm k^{2i} s \mathbf{v}_{3j+3} \}, \\ & \{ \pm k^{2i} s \mathbf{v}_{3j+2}, \pm k^{2i+1} s \mathbf{v}_{3j+2}, \pm k^{2i+1} s \mathbf{v}_{3j+3} \}, \\ & 0 \leq i \leq e - 2; s \in S; 0 \leq j \leq t/3 - 1; \\ & \{ \pm k^{2e-2} s \mathbf{v}_{3j+1}, \pm k^{2e-1} s \mathbf{v}_{3j+1}, \pm k^{2e} s \mathbf{v}_{3j+2} \}, \\ & \{ \pm k^{2e-1} s \mathbf{v}_{3j+3}, \pm k^{2e} s \mathbf{v}_{3j+3}, \pm k^{2e} s \mathbf{v}_{3j+1} \}, \\ & \{ \pm k^{2e-2} s \mathbf{v}_{3j+2}, \pm k^{2e-1} s \mathbf{v}_{3j+2}, \pm k^{2e-2} s \mathbf{v}_{3j+3} \}, \\ & s \in S; 0 \leq j \leq t/3 - 1. \end{aligned}$$

Again it is enough to check that for each fixed value of j the above sixtuples form a partition of $\{a \mathbf{v}_{3j+\ell} \mid 1 \leq a \leq p - 1; \ell = 1, 2, 3\}$. The assertion follows. \square

Theorem 3.2. *Let $q \equiv 1 \pmod{6}$ be a power of an odd prime p and let $2 \leq k \leq p - 2$. Then there exists an elementary abelian $P(p, k)$ -decomposition of K_q .*

Proof. Set $q = p^n$ and let V be the n -dimensional vector space over Z_p . By Lemma 3.1 there exists a partition of $V - \{0\}$ of the form

$$V - \{0\} = \bigcup_{i=1}^{(p^n-1)/6} \{ \pm \mathbf{u}_i, \pm k \mathbf{u}_i, \pm \mathbf{v}_i \} \quad (1)$$

where \mathbf{u}_i and \mathbf{v}_i are linearly independent for each i .

For $i = 1, \dots, (p^n - 1)/6$, let Γ_i be the graph defined as follows.

$$V(\Gamma_i) = \{\mathbf{0}, \mathbf{u}_i, 2\mathbf{u}_i, \dots, (p-1)\mathbf{u}_i, \mathbf{v}_i, \mathbf{v}_i + \mathbf{u}_i, \mathbf{v}_i + 2\mathbf{u}_i, \dots, \mathbf{v}_i + (p-1)\mathbf{u}_i\};$$

$$E(\Gamma_i) = \bigcup_{j=0}^{p-1} \{[j\mathbf{u}_i, (j+1)\mathbf{u}_i], [\mathbf{v}_i + j\mathbf{u}_i, \mathbf{v}_i + (j+k)\mathbf{u}_i], [j\mathbf{u}_i, \mathbf{v}_i + j\mathbf{u}_i]\}.$$

Observe that $V(\Gamma_i)$ has size $2p$ since \mathbf{u}_i and \mathbf{v}_i are linearly independent and hence Γ_i is isomorphic to $P(p, k)$.

The differences arising from the edges $[j\mathbf{u}_i, (j+1)\mathbf{u}_i]$, $[\mathbf{v}_i + j\mathbf{u}_i, \mathbf{v}_i + (j+k)\mathbf{u}_i]$ and $[j\mathbf{u}_i, \mathbf{v}_i + j\mathbf{u}_i]$ are $\pm\mathbf{u}_i$, $\pm k\mathbf{u}_i$ and $\pm\mathbf{v}_i$, respectively, and that is independent of j . It follows that $\Delta\Gamma_i$ is p times the set $\{\pm\mathbf{u}_i, \pm k\mathbf{u}_i, \pm\mathbf{v}_i\}$. Consequently, noting that the stabilizer of Γ_i under the natural action of V has order p (it is namely the 1-dimensional subspace generated by \mathbf{u}_i) we have:

$$\partial\Gamma_i = \{\pm\mathbf{u}_i, \pm k\mathbf{u}_i, \pm\mathbf{v}_i\}.$$

By (1), we have thus $\bigcup_{i=1}^{(p^n-1)/6} \partial\Gamma_i = V - \{\mathbf{0}\}$. Using Theorem 2.1 we finally conclude that $\Gamma_1, \Gamma_2, \dots, \Gamma_{(p^n-1)/6}$ yield an elementary abelian $P(p, k)$ -decomposition of K_q . \square

We present an alternative construction which is somewhat simpler than the one of the previous theorem, even though it only works under additional constraints on the parameters.

Theorem 3.3. *Let $p \equiv 1 \pmod{6}$ be a prime and let k be an integer, with $1 < k < p - 1$, such that k is not a cube when interpreted as a field element in Z_p . Let n be an integer greater than 1 which is not divisible by 3 and set $q = p^n$. There exists an elementary abelian $P(p, k)$ -decomposition of K_q .*

Proof. We construct a copy Γ of $P(p, k)$ as follows. Let x be a chosen cube in $\text{GF}(q) - \text{GF}(p)$. We define

$$V(\Gamma) = \{0, 1, \dots, p-1\} \cup \{k^2x, k^2x+1, \dots, k^2x+(p-1)\} \text{ and}$$

$$E(\Gamma) = \{[i, i+1], [i, k^2x+i], [k^2x+i, k^2x+i+k] \mid i = 0, 1, \dots, p-1\}.$$

The list $\partial\Gamma$ consists precisely of the six elements $\pm 1, \pm k, \pm k^2x$. Since n is not divisible by 3, the subgroup Z_p^* is not contained in the subgroup C of cubes in $\text{GF}(q)^*$ and then k is not a cube in $\text{GF}(q)$. This fact, together with our choice of x , assures that the triple $\{1, k, k^2x\}$ forms a complete system of representatives for the subgroup C in $\text{GF}(q)^*$.

Let now S be a complete system of representatives for the subgroup $\{1, -1\}$ in the multiplicative group of cubes in $\text{GF}(q)^*$. For each $s \in S$ consider the subgraph $s\Gamma$ which is obtained from Γ by replacing each vertex u of Γ by su . The list of partial differences of $s\Gamma$ is then clearly $s\partial\Gamma$ and, consequently, the list of partial differences which are covered by the family $\{s\Gamma \mid s \in S\}$ is given by $S \{\pm 1\} \{1, k, k^2 x\} = C \{1, k, k^2 x\} = \text{GF}(q) - \{0\}$. We can conclude that this family is a complete system of representatives for the orbits of an elementary abelian Γ -decomposition of K_q . \square

Remark 3.4. If the prime p of the previous Theorem 3.3 is also congruent to 7 (mod 12), then we can choose a complete system S of representatives for $\{1, -1\}$ in C as the subgroup of the 6-th powers in $\text{GF}(q)^*$. In this manner the group of affine linear transformations over $\text{GF}(q)$ of the form $x \mapsto sx + t$, $s \in S$, $t \in \text{GF}(q)$ is a group of automorphisms of the $P(p, k)$ -decomposition of K_q . The action of this group is transitive on the subgraphs of the decomposition.

4 Cyclic decompositions of the complete graph into (generalized) Petersen graphs and prisms

Let $q = p^\alpha$ be a prime power with $\alpha \geq 1$ and $q = ef + 1$. Let γ denote a generator of the multiplicative group of $\text{GF}(q)$, i.e. $\text{GF}(q)^* = \langle \gamma \rangle$. Let C_0 be the subgroup of the e -th powers in $\text{GF}(q)^*$ and let C_i be the coset of C_0 which contains γ^i , $0 \leq i \leq e - 1$. The *cyclotomic number* A_{ij} of order e counts the field elements x in C_i such that $x - 1$ is in C_j . The determination of cyclotomic numbers is quite difficult. For the particular case $e = 3$, the problem was first considered by Gauss [12], and later on by Dickson [9], under the restriction $\alpha = 1$. For the case $\alpha > 1$ we recall results by Hall [14], Storer [19] and Katre and Raiwade [16].

It is proved in [15] that all cyclotomic numbers A_{ij} satisfy the following relation:

$$e^2 A_{ij} \geq q - (e - 1)(e - 2)\sqrt{q} - (3e - 1).$$

For $e = 3$ we get

$$9A_{ij} \geq (\sqrt{q} - 1)^2 - 9$$

and hence all cyclotomic numbers of order 3 are positive if $q > 16$ (i.e. $q > 13$ if q is odd).

Lemma 4.1. *Let $q = 6nt + 1$, with $q = p^\alpha$, $\alpha \geq 1$, $q > 13$ and n odd. Let ζ be a primitive n -th root of unity in $\text{GF}(q)$. For each k , with $1 \leq k \leq \frac{n-1}{2}$, there exists an element $x \in \text{GF}(q)$, with $x \notin \langle \zeta \rangle$, such that $\zeta - 1$, $x(\zeta^k - 1)$, $x - 1$ form a complete system of representatives for the subgroup of cubes in $\text{GF}(q)^*$.*

Proof. Let C_0 be the subgroup of cubes in $\text{GF}(q)^*$ and let C_1 and C_2 be its cosets in $\text{GF}(q)^*$. Since ζ has order n and $q - 1 = 6nt$, we have $\langle \zeta \rangle \subseteq C_0$. If $\zeta - 1 \in C_i$ and $\zeta^k - 1 \in C_j$, let $x \in C_{h-j}$ and $x - 1 \in C_l$, where h is different from i and $j \pmod{3}$, and l is the unique element of $\{0, 1, 2\}$ which is distinct from i and h . The existence of x is assured by the fact that all cyclotomic numbers of order 3 are positive. Moreover, $C_{h-j} \neq C_0$ and then $x \notin \langle \zeta \rangle$. In this manner we get $\zeta - 1 \in C_i$, $x - 1 \in C_l$, $x(\zeta^k - 1) \in C_h$ and the assertion follows. \square

We are now able to prove the following:

Theorem 4.2. *Let $n \geq 3$ be an odd integer and let v be an odd integer whose prime factors are congruent to 1 $\pmod{6n}$. For each k , with $1 \leq k \leq \frac{n-1}{2}$, there exists a cyclic $P(n, k)$ -decomposition of K_v .*

Proof. Let $p = 6nt + 1$ be a prime, with $p > 13$. Follow the previous Lemma 4.1 in the particular case $\alpha = 1$ and take $x \in Z_p$, with $x \notin \langle \zeta \rangle$, and such that $\zeta - 1, x(\zeta^k - 1), x - 1$ are a complete system of representatives of the subgroup of cubes in Z_p^* . The subgroup $\langle -\zeta \rangle$ is contained in C_0 and it has order $2n$. Let $\{s_1, \dots, s_t\}$ be a complete system of representatives for the cosets of $\langle -\zeta \rangle$ in C_0 . For each s_i , let $\bar{\Gamma}_i$ be the graph defined as follows:

$$\begin{aligned} V(\bar{\Gamma}_i) &= \{s_i, s_i\zeta, s_i\zeta^2, \dots, s_i\zeta^{n-1}, xs_i, xs_i\zeta, xs_i\zeta^2, \dots, xs_i\zeta^{n-1}\}, \\ E(\bar{\Gamma}_i) &= \{[s_i\zeta^j, s_i\zeta^{j+1}], [xs_i\zeta^j, xs_i\zeta^{j+k}], [s_i\zeta^j, xs_i\zeta^j] \mid j = 0, \dots, n-1\}. \end{aligned}$$

The graph $\bar{\Gamma}_i$ is isomorphic to $P(n, k)$, it has trivial stabilizer in the additive group of Z_p and $\partial\bar{\Gamma}_i = s_i\{\zeta - 1, x(\zeta^k - 1), x - 1\}\langle -\zeta \rangle$. We conclude that the set $\{\bar{\Gamma}_1, \dots, \bar{\Gamma}_t\}$ is a complete system of representatives for a cyclic $P(n, k)$ -decomposition of K_p .

Since each $\bar{\Gamma}_i$ has trivial stabilizer, the set $\{\bar{\Gamma}_1, \dots, \bar{\Gamma}_t\}$ is a $(Z_p, \bar{\Gamma}, 1)$ -difference family with $\bar{\Gamma} = P(n, k)$. For each odd integer v whose prime factors are all congruent to 1 $\pmod{6n}$, $n \geq 3$, since the graph $P(n, k)$ has chromatic number 3, an application of Proposition 2.2 yields the existence of a cyclic $P(n, k)$ -decomposition of K_v . \square

Remark 4.3. Consider the $P(n, k)$ -decomposition of K_p constructed in the previous Theorem 4.2. The group of affine linear transformations over the field Z_p of the form $x \mapsto rx + b$, $r \in \langle \zeta \rangle$, $b \in Z_p$ is an automorphism group of the decomposition. Moreover, if t and $2n$ are coprime, the subgroup C_0 is the direct product of $\langle -\zeta \rangle$ together with a subgroup H of order t . It is thus clear that the elements of H can be seen as the complete system $\{s_1, \dots, s_t\}$ chosen in the construction of the theorem. In this case, the group of affine linear transformations $x \mapsto ax + b$, $a \in \langle \zeta \rangle H$, $b \in Z_p$ is again an automorphism group of

the decomposition. In particular, the subgroup of affine linear transformations $x \mapsto hx + b$, $h \in H$, $b \in Z_p$ has a sharply transitive action on the graphs of the decomposition.

Remark 4.4. In the first part of the proof of Theorem 4.2 a cyclic $P(n, k)$ -decomposition of K_p is constructed via Lemma 4.1. This construction can be repeated in $\text{GF}(q)$, with $q = p^\alpha = 6nt + 1$, n odd and $\alpha > 1$. In this manner we obtain an elementary abelian $P(n, k)$ -decomposition of K_q , $q > 13$.

5 Sharply transitive mixed decompositions

We now turn our attention to the problem of finding decompositions of complete graphs into subgraphs of different types, a problem which has also received considerable attention in the literature. An instance of this situation comes from the decompositions of complete graphs into cycles of varying specified lengths, see [3] for a recent survey.

In this section we construct examples of sharply transitive decompositions of complete graphs into copies of different generalized Petersen graphs and prisms. As before, G denotes a finite additive group of odd order. By $\text{EA}(q)$ we denote the elementary abelian group of order q . Theorem 2.1 can be generalized as follows.

Theorem 5.1. *Let $\mathcal{C} = \{\Gamma_1, \dots, \Gamma_t\}$ be a set of subgraphs of K_G , each of which is isomorphic to one of the given graphs R_1, \dots, R_s . The set \mathcal{C} is a complete system of representatives for the G -orbits of a sharply transitive decomposition of K_G into copies of R_1, \dots, R_s if and only if $\partial\Gamma_1 \cup \dots \cup \partial\Gamma_t = G - \{0\}$.*

Proof. The steps of the proof follow closely those of Theorem 2.1. We omit the details. \square

We call such a decomposition a *sharply transitive (R_1, \dots, R_s) -decomposition* of K_v .

Theorem 5.2. *Let $q \equiv 1 \pmod{6}$ be a power of an odd prime p and let $2 \leq k \leq p - 2$. Let $n \geq 3$ be an odd integer, let v be an odd integer whose prime factors are congruent to $1 \pmod{6n}$ and let h be an integer with $1 \leq h \leq \frac{n-1}{2}$. Let $P_1 = P(p, k)$ and let either $P_2 = P(2p, k)$ or $P_2 = P(2p, p + k)$ according to whether k is odd or even, respectively. Similarly, let $P_3 = P(n, h)$ and let either $P_4 = P(2n, h)$ or $P_4 = P(2n, n + h)$ according to whether h is odd or even, respectively.*

- There exists an elementary abelian (P_1, P_2) -decomposition of K_{q^2} .

- There exists a sharply transitive (P_3, P_4) -decomposition of $K_{Z_v \oplus Z_v}$.
- There exists a sharply transitive (P_1, P_2, P_3) -decomposition of $K_{\text{EA}(q) \oplus Z_v}$.
- There exists a sharply transitive (P_1, P_3, P_4) -decomposition of $K_{Z_v \oplus \text{EA}(q)}$.

Proof. Let G be a group of odd order and let $\Gamma = P(s, r)$, $s \geq 3$ odd and $1 \leq r \leq s-1$, be a subgraph of K_G . Denote by $\{a_0, a_1, \dots, a_{s-1}, b_0, b_1, \dots, b_{s-1}\}$ its vertex-set and let $\{[a_i, a_{i+1}], [b_i, b_{i+r}], [a_i, b_i], i = 0, \dots, s-1\}$ be its edge-set, where indices are modulo s . Denote by S the stabilizer of Γ in G . Suppose S to be either the trivial group or the group generated by a rotation of the form $a_i \mapsto a_{i+t}, b_i \mapsto b_{i+t}$, for some fixed t .

Let H be an additive group of odd order and partition its elements into three sets: $\{0\}, H_Y, -H_Y$, in such a way that $y \in H_Y$ if and only if $-y \in -H_Y$.

For each $y \in H_Y$ let Γ^y be the graph defined as follows. The vertices are obtained by pairing the elements of the sequence $(y, -y, y, -y, \dots, y, -y)$ with the elements of the sequence $(a_0, a_1, \dots, a_{s-1}, a_0, a_1, \dots, a_{s-1})$ and by pairing the elements of the sequence $(-y, y, -y, y, \dots, -y, y)$ with the elements of the sequence $(b_0, b_1, \dots, b_{s-1}, b_0, b_1, \dots, b_{s-1})$, respectively. We have namely:

$$V(\Gamma^y) = \{(a_0, y), (a_1, -y), \dots, (a_{s-1}, y), (a_0, -y), (a_1, y), \dots, (a_{s-1}, -y)\} \\ \cup \{(b_0, -y), (b_1, y), \dots, (b_{s-1}, -y), (b_0, y), (b_1, -y), \dots, (b_{s-1}, y)\}.$$

The edge-set is defined as follows:

$$E(\Gamma^y) = \bigcup_{i=0}^{s-1} \left\{ [(a_i, y), (a_{i+1}, -y)], [(a_i, -y), (a_{i+1}, y)], [(a_i, y), (b_i, -y)], \right. \\ \left. [(a_i, -y), (b_i, y)], [(b_i, -y), (b_{i+r}, y)], [(b_i, y), (b_{i+r}, -y)] \right\}.$$

Clearly, the graph Γ^y is isomorphic to either $P(2s, r)$ or $P(2s, s+r)$ according to whether r is odd or even, respectively.

The graph Γ^y has vertices in $K_{G \oplus H}$, where $G \oplus H$ denotes the direct sum of G and H . Moreover its stabilizer in $G \oplus H$ is $S \oplus \{0\}$ and $\partial\Gamma^y = \partial\Gamma \times \{2y, -2y\}$.

Observe also that it is always possible to construct two copies of Γ , namely Γ^{10} and Γ^{01} in such a way that the stabilizer of Γ^{10} is $S \oplus \{0\}$ and $\partial\Gamma^{10} = \partial\Gamma \times \{0\}$, the stabilizer of Γ^{01} is $\{0\} \oplus S$ and $\partial\Gamma^{01} = \{0\} \times \partial\Gamma$. In particular the graph Γ^{10} is obtained by relabeling each vertex x of Γ with $(x, 0)$, while Γ^{01} is obtained by relabeling each vertex x of Γ with $(0, x)$.

In what follows we repeat these constructions in four particular situations. In these situations, we choose the groups G and H to be either cyclic or elementary abelian groups.

Set $q = p^m$, $q \equiv 1 \pmod{6}$. Follow the construction of Theorem 3.2 and let $\{\Gamma_1, \dots, \Gamma_{(p^m-1)/6}\}$ be a complete system of representatives for the elementary abelian P_1 -decomposition of K_q . Follow the construction of Theorem 4.2 and let $\{\bar{\Gamma}_1, \dots, \bar{\Gamma}_t\}$ be a complete system of representatives for the cyclic P_3 -decomposition of K_v .

The set $\{\Gamma_i^{10}, \Gamma_i^{01}, \Gamma_i^y \mid i = 1, \dots, (p^m - 1)/6, y \in \text{EA}(q)_Y\}$ is a complete system of representatives for an elementary abelian (P_1, P_2) -decomposition of K_{q^2} .

The set $\{\bar{\Gamma}_j^{10}, \bar{\Gamma}_j^{01}, \bar{\Gamma}_j^y \mid j = 1, \dots, t, y \in Z_{vY}\}$ is a complete system of representatives for a sharply transitive (P_3, P_4) -decomposition of $K_{Z_v \oplus Z_v}$.

The set $\{\Gamma_i^{10}, \Gamma_i^y, \bar{\Gamma}_j^{01} \mid i = 1, \dots, (p^m - 1)/6, y \in Z_{vY}, j = 1, \dots, t\}$ is a complete system of representatives for a sharply transitive (P_1, P_2, P_3) -decomposition of $K_{\text{EA}(q) \oplus Z_v}$.

The set $\{\bar{\Gamma}_j^{10}, \bar{\Gamma}_j^y, \Gamma_i^{01} \mid j = 1, \dots, t, y \in \text{EA}(q)_Y, i = 1, \dots, (p^m - 1)/6\}$ is a complete system of representatives for a sharply transitive (P_1, P_3, P_4) -decomposition of $K_{Z_v \oplus \text{EA}(q)}$. \square

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