Sharply transitive decompositions of complete graphs into generalized Petersen graphs

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Abstract

A decomposition of the complete graph K_v into copies of a subgraph Γ is called a sharply transitive Γ-decomposition if it is left invariant by an automorphism group acting sharply transitively on the vertex-set of K_v. For suitable values of v we construct examples of sharply transitive Γ-decompositions when Γ is either a Petersen graph, a generalized Petersen graph or a prism.

`Keywords:` decomposition, (generalized) Petersen graph, prism, sharply transitive permutation group
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1. Introduction

In this paper we shall only deal with undirected simple graphs. For each graph T we shall denote by $V(T)$ and $E(T)$ its vertex-set and edge-set, respectively. A decomposition of a graph T into copies of a given graph Γ is a set $\mathcal{C}$ of subgraphs isomorphic to Γ, whose edges partition, altogether, the edge-set of T. Such a decomposition is generally called a Γ-*decomposition* of T. A Γ-decomposition of K_v, the complete graph on v vertices, is often called a Γ-*design*; see [2, 4]. This terminology is suggested by the circumstance that a standard 2-$(v, k, 1)$ design is a Γ-design for $\Gamma = K_k$, once the blocks are regarded as complete graphs on k vertices.

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A group G is an *automorphism group* of the Γ-decomposition $\mathcal{C}$ if it is a permutation group on the vertices of T leaving $\mathcal{C}$ invariant. The action of G is said to be *sharply transitive* on the vertex-set $V(T)$ if for any given pair of (not necessarily distinct) vertices x, $y \in V(T)$ there exists a unique automorphism g in G mapping x to y. In the theory of finite permutation groups a sharply transitive action is often called a *regular* action.

We shall speak of a *sharply transitive* Γ*-decomposition* of T if $\mathcal{C}$ admits an automorphism group G acting sharply transitively on the vertex-set of T. In particular, we shall speak of a *cyclic* or of an *elementary abelian* Γ-decomposition if the action of G is sharply transitive and G is cyclic or elementary abelian, respectively. In this paper, when speaking of a group G we will use the additive notation, unless differently specified.

If the group G acts sharply transitively on the vertex-set of K_v, then G has order v and we can identify the vertex set of K_v with G. In this case we shall usually denote the complete graph by K_G rather than by K_v. In this manner each group-element $g \in G$ is identified with the permutation $V(K_G) \to V(K_G)\colon x \mapsto x+g$. This action of G induces actions on the subsets of $V(K_G)$ and on sets of such subsets. Hence if $[x,y]$ is an edge, then it is mapped to the edge $[x+g, y+g]$ by g and we write $[x,y]+g = [x+g, y+g]$. In particular, if U is a collection of edges of K_G then $U+g = \{[x+g, y+g] \mid [x,y] \in U\}$. In the present paper we shall usually denote by 0 the zero element of an additive group G and by 1 the identity element in multiplicative notation.

Let n, k be positive integers, with $n \geq 3$ and $1 \leq k \leq n-2$. Denote by $P(n,k)$ the graph with vertex-set

$$V\big(P(n,k)\big) = \{x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1}\}$$

and edge-set

$$E\big(P(n,k)\big) = \bigcup_{i=0}^{n-1} \big\{[x_i, x_{i+1}], [y_i, y_{i+k}], [x_i, y_i]\big\},$$

where subscripts are meant modulo n. In particular, $P(5,2)$ is the *standard Petersen graph*, $P(n,1)$ is a *prism* and if $n > 5$, $1 < k < n-1$ the graph $P(n,k)$ is a *generalized Petersen graph*.

For each fixed choice of the graph Γ, the values of v for which a Γ-decomposition exists form the so-called *spectrum*. Clearly the values of v for which a sharply transitive Γ-decomposition exists must be in the spectrum. The same is true if we fix the isomorphism type of the automorphism group G which is assumed to act sharply transitively on the vertices of K_v (say, for instance, cyclic or elementary abelian).

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It may well happen that there exists no such sharply transitive Γ-decomposition of K_v with the group G of the required isomorphism type even though v is in the spectrum. That is trivially the case if no finite group of order v of the required isomorphism type exists. For example if Γ is the Petersen graph $P(5,2)$ then Adams and Bryant have shown in [2] that v is in the spectrum of Γ-decompositions of K_v if and only if $v \equiv 1$ or $10 \pmod{15}$, $v > 10$. Consequently, although $v = 40$ is in this spectrum, it does not even make sense to look for an elementary abelian Γ-decomposition of K_{40} because 40 is not a prime power. It may also happen that v *does* lie in the spectrum and a finite group of order v of the desired isomorphism type *does* exist while a sharply transitive Γ-decomposition with group of the required type *does not* exist. Probably the most famous example for this situation is when v is a power of 2, $v \geq 8$, the group G is cyclic and Γ is a 1-factor of K_v. Hartman and Rosa have namely proved in [13] that a cyclic 1-factorization of K_v does not exist for such values of v.

The problem we approach in the present paper is thus a special instance of the determination of what we might call a "restricted" spectrum: given the graph Γ and the isomorphism type of the group G, determine the values of v for which there exists a Γ-decomposition of K_v admitting an automorphism group that is isomorphic to G and acts sharply transitively on the vertices of K_v.

In this paper we construct sharply transitive decompositions of complete graphs into Petersen graphs, generalized Petersen graphs and prisms. Our main results are the following:

- If p is an odd prime and $q \equiv 1 \pmod 6$ is a power of p with $q > p$, then for each k with $2 \leq k \leq p-2$ we construct an elementary abelian $P(p,k)$-decomposition of K_q.
- Let $n > 3$ be an odd integer and let $v > 13$ be an odd integer all of whose prime factors are congruent to $1 \pmod{6n}$. For each k with $1 \leq k \leq \frac{n-1}{2}$ we construct a cyclic $P(n,k)$-decomposition of K_v.

In the final section we construct examples of sharply transitive decompositions of a complete graph into different types of generalized Petersen graphs and prisms. That can somehow be viewed in analogy with the situation in which a block design contains blocks of different sizes, a situation which is often considered in design theory. In our situation the "blocks" are subgraphs with different "shapes".

In [18] Rosa introduced the concept of *graceful labelings* and *α-labelings* of graphs. These are assignments of integers to the vertices of the graph subject to certain conditions. A necessary condition for a graph to have an α-labeling is the property of being bipartite. We refer to [11] for an extensive survey on this topic.

Rosa showed how graphs with graceful labelings or α-labelings are useful in the development of the theory of graph decompositions. For instance he proved in [18] that if Γ is a graph possessing a graceful labeling, then there exists a cyclic Γ-decomposition of K_v, where $v = 2|E(\Gamma)| + 1$. In the same paper he proved that if Γ is a graph possessing an α-labeling, then there exists a cyclic Γ-decomposition of K_v, for each $v \equiv 1 \pmod{2|E(\Gamma)|}$. Redl [17] has shown that $P(n, k)$ has a graceful labeling for $n = 5$, 6, 7, 8, 9 and 10. Vietri [20, 21] proved that $P(4t, 3)$ has a graceful labeling for all t's. The problem of determining whether generalized Petersen graphs admit graceful labelings is still open, while it is proved in [10] that $P(n, 1)$ has an α-labeling if and only if n is even. Our constructions work on graphs which are not bipartite, hence they cannot be obtained from α-labelings. Our existence results can thus be regarded as an extension of the following proposition which can be obtained from [18] and [10]: if $n > 3$ is an even integer and v is an odd integer with $v \equiv 1 \pmod{6n}$, then a cyclic $P(n, 1)$-decomposition of K_v does exist.

As to our notation, for a given prime p we denote by Z_p the finite field of order p, that is the field of residue classes modulo p. More generally, for any positive integer v we denote by Z_v the ring of residue classes modulo v as well as its additive cyclic group; for any positive integer n we denote by $\mathrm{GF}(p^n)$ the finite field of order p^n, that is the n-dimensional field-extension of Z_p (so in particular, $Z_p = \mathrm{GF}(p)$). The notation $\mathrm{GF}(p^n)^*$ stands for $\mathrm{GF}(p^n) - \{0\}$, in the same manner $Z_p^* = Z_p - \{0\}$. When dealing with vector spaces, vectors will generally be denoted by bold letters.

2. Preliminary results

Our constructions are essentially based on the method of partial differences, a useful tool in many circumstances. See for example [5], [6] and [8], where the method is used for constructing cycle systems, and [1] for constructions in classical design theory. We begin with a brief review of this method in our special situation. In particular, we work in K_G where G is a group of *odd* order. In this situation each edge orbit under the action of G has length equal to the order of G.

Given a simple graph Γ with vertices in K_G, the *list* $\Delta\Gamma$ *of differences* of Γ is defined as follows:

$$\Delta\Gamma = \{x - y,\ y - x \mid [x, y] \in E(\Gamma)\}\,.$$

In general, $\Delta\Gamma$ contains repeated elements, so it is a multiset. Moreover, if the stabilizer of Γ in G has order h, then each element of $\Delta\Gamma$ is repeated at least

h times. Let us denote by H the stabilizer of Γ in G and let Γ^* be a chosen subgraph of Γ whose edges form a complete system of representatives for the H-orbits of the edges of Γ. In what follows we will set $\partial\Gamma = \Delta\Gamma^*$. In general $\partial\Gamma$ can itself be a multiset.

Theorem 2.1. *Let $\mathcal{C} = \{\Gamma_1, \ldots, \Gamma_t\}$ be a set of subgraphs of K_G which are isomorphic to a given graph Γ. The set $\mathcal{C}$ is a complete system of representatives for the G-orbits of a sharply transitive Γ-decomposition of K_G if and only if $\partial\Gamma_1 \cup \cdots \cup \partial\Gamma_t$ is a repetition-free cover of $G - \{0\}$.*

Proof. Let v be the order of G and let n be the number of edges in Γ. Let h_i denote the order of the G-stabilizer of the subgraph Γ_i, $i = 1, \ldots, t$. Suppose $\mathcal{C}$ to be a complete system of representatives for the G-orbits of a sharply transitive Γ-decomposition of K_G. The total number of edges of $\mathrm{orb}_G(\Gamma_1) \cup \cdots \cup \mathrm{orb}_G(\Gamma_t)$ is $\frac{nv}{h_1} + \cdots + \frac{nv}{h_t} = \frac{v(v-1)}{2}$. This equality implies $\frac{2n}{h_1} + \cdots + \frac{2n}{h_t} = v - 1$. Observe that $v - 1 = |G - \{0\}|$ and $\frac{2n}{h_i} = |\partial\Gamma_i|$. Therefore, for the proof of the first part of the theorem, it is sufficient to show that each element of G appears at least once in the list $\partial\Gamma_1 \cup \cdots \cup \partial\Gamma_t$. Given $x \in G - \{0\}$, the edge $[0, x]$ is in $\Gamma_i + g$ for a suitable $i \in \{1, \ldots, t\}$ and $g \in G$. That implies $[0, x] = [y, z] + g$ with $[y, z] \in E(\Gamma_i)$ and $x = z - y$ or $x = y - z$, which proves the assertion.

Let conversely the multiset equality $\partial\Gamma_1 \cup \cdots \cup \partial\Gamma_t = G - \{0\}$ hold without repetitions. Since $|\partial\Gamma_i| = \frac{2n}{h_i}$, the graph $\mathrm{orb}_G(\Gamma_i)$ contains $\frac{nv}{h_i}$ edges and $\mathrm{orb}_G(\Gamma_1) \cup \cdots \cup \mathrm{orb}_G(\Gamma_t)$ has the same number of edges as K_G. We will thus obtain a decomposition of K_G if we can prove that each edge of K_G is an edge of a suitable $\Gamma_i + g$, $g \in G$. If $[x, y]$ is an edge of K_G, then $y - x \in \partial\Gamma_i$, for some i. Therefore, there exists $[a, b] \in E(\Gamma_i)$ with $b - a = y - x$, which implies $[a, b] - a + x = [x, y]$ and $[x, y] \in \Gamma_i - a + x$. □

In the particular case in which all the graphs of $\mathcal{C}$ have trivial stabilizer in G, one says that $\mathcal{C}$ is a $(G, \Gamma, 1)$-*difference family* [7]. In the proof of Theorem 4.2 below, we shall need the result of the next Proposition, which was obtained in [7].

Proposition 2.2. *If there exists a $(Z_{v_i}, \Gamma, 1)$-difference family for $i = 1, \ldots, n$ and if the chromatic number $\chi(\Gamma)$ does not exceed the smallest prime factor of the product $v_1 v_2 \ldots v_n$, then there exists a cyclic Γ-decomposition of $K_{v_1 v_2 \ldots v_n}$.*

3. Elementary abelian decompositions of the complete graph into (generalized) Petersen graphs

Lemma 3.1. *Let $p > 3$ be a prime and let V be the n-dimensional vector space over the field Z_p with $|V| \equiv 1 \pmod 6$. Then, for every fixed $k \in Z_p - \{0, 1, -1\}$, it is possible to partition $V - \{\mathbf{0}\}$ into $(p^n - 1)/6$ sixtuples of the form $\{\pm\mathbf{u}, \pm k\mathbf{u}, \pm\mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors of V.*

Proof. Set $(p^n - 1)/(p - 1) = t$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ be a complete system of representatives for the 1-dimensional vector subspaces of V so that we have $V - \{\mathbf{0}\} = \{a\mathbf{v}_j \mid 1 \le a \le p-1; 1 \le j \le t\}$. Let H be the multiplicative subgroup of Z_p^* consisting of all elements of the form $\pm k^i$. Let $2d$ be the order of H, and let S be a complete system of representatives for the cosets of H in Z_p^*. We distinguish two cases according to whether $p \equiv 1$ or $-1 \pmod 6$.

1st case: $p \equiv 1 \pmod 6$.

It is straightforward to check that the set

$$\mathcal{Q} = \Big\{\{\pm k^{2i}s, \pm k^{2i+1}s\} \mid 0 \le i < \lfloor d/2 \rfloor; s \in S\Big\}$$

consists of pairwise disjoint quadruples of elements of $Z_p - \{0\}$. Since we have $|\mathcal{Q}| = \lfloor d/2 \rfloor \cdot |S| \ge |H| \cdot |S|/6 = (p-1)/6$, we can choose a $(p-1)/6$-subset $\mathcal{A}$ of $\mathcal{Q}$. In view of the form of the quadruples of $\mathcal{Q}$ we can write

$$\mathcal{A} = \Big\{\{\pm a_i, \pm k a_i\} \mid 1 \le i \le (p-1)/6\Big\}.$$

The set $B = Z_p^* - \bigcup_{i=1}^{(p-1)/6} \{\pm a_i, \pm k a_i\}$ has size $2(p-1)/6$ and, obviously, if $b \in B$, then $-b \in B$. Thus we can write $B = \{\pm b_i \mid 1 \le i \le (p-1)/6\}$, and the sixtuples

$$\{\pm a_i, \pm k a_i, \pm b_i\}\,, \quad 1 \le i \le (p-1)/6$$

form a partition of Z_p^*.

It is now clear that $V - \{\mathbf{0}\}$ can be partitioned into the following $(p^n - 1)/6$ sixtuples of the required form:

$$\{\pm a_i\mathbf{v}_j, \pm k a_i\mathbf{v}_j, \pm b_i\mathbf{v}_{j+1}\}\,, \quad 1 \le i \le (p-1)/6; 1 \le j \le t$$

where we agree that $\mathbf{v}_{t+1} = \mathbf{v}_1$.

2nd case: $p \equiv 5 \pmod 6$.

In this case t is divisible by 3. Distinguish two subcases according to whether d is even or odd.

Subcase (a): *d is even,* say $d = 2e$.

We claim that $V - \{\mathbf{0}\}$ can be partitioned into the following $(p^n - 1)/6$ sixtuples of the required form:

$$\begin{aligned}&\left\{\pm k^{2i} s\mathbf{v}_{3j+1}, \pm k^{2i+1} s\mathbf{v}_{3j+1}, \pm k^{2i} s\mathbf{v}_{3j+3}\right\},\\ &\left\{\pm k^{2i} s\mathbf{v}_{3j+2}, \pm k^{2i+1} s\mathbf{v}_{3j+2}, \pm k^{2i+1} s\mathbf{v}_{3j+3}\right\},\\ &\qquad\qquad\qquad 0 \le i \le e-1; s \in S; 0 \le j \le t/3 - 1\,.\end{aligned}$$

In fact, it is easy to check that for each fixed value of j the above sixtuples form a partition of $\{a\mathbf{v}_{3j+\ell} \mid 1 \le a \le p-1; \ell = 1, 2, 3\}$.

Subcase (b): *d is odd,* say $d = 2e + 1$.

We claim that $V - \{\mathbf{0}\}$ can be partitioned into the following $(p^n - 1)/6$ sixtuples of the required form:

$$\begin{aligned}&\left\{\pm k^{2i} s\mathbf{v}_{3j+1}, \pm k^{2i+1} s\mathbf{v}_{3j+1}, \pm k^{2i} s\mathbf{v}_{3j+3}\right\},\\ &\left\{\pm k^{2i} s\mathbf{v}_{3j+2}, \pm k^{2i+1} s\mathbf{v}_{3j+2}, \pm k^{2i+1} s\mathbf{v}_{3j+3}\right\},\\ &\qquad\qquad\qquad 0 \le i \le e-2; s \in S; 0 \le j \le t/3 - 1\,;\end{aligned}$$

$$\begin{aligned}&\left\{\pm k^{2e-2} s\mathbf{v}_{3j+1}, \pm k^{2e-1} s\mathbf{v}_{3j+1}, \pm k^{2e} s\mathbf{v}_{3j+2}\right\},\\ &\left\{\pm k^{2e-1} s\mathbf{v}_{3j+3}, \pm k^{2e} s\mathbf{v}_{3j+3}, \pm k^{2e} s\mathbf{v}_{3j+1}\right\},\\ &\left\{\pm k^{2e-2} s\mathbf{v}_{3j+2}, \pm k^{2e-1} s\mathbf{v}_{3j+2}, \pm k^{2e-2} s\mathbf{v}_{3j+3}\right\},\\ &\qquad\qquad\qquad s \in S; 0 \le j \le t/3 - 1\,.\end{aligned}$$

Again it is enough to check that for each fixed value of j the above sixtuples form a partition of $\{a\mathbf{v}_{3j+\ell} \mid 1 \le a \le p-1; \ell = 1, 2, 3\}$. The assertion follows. □

Theorem 3.2. *Let $q \equiv 1 \pmod 6$ be a power of an odd prime p and let $2 \le k \le p-2$. Then there exists an elementary abelian $P(p,k)$-decomposition of K_q.*

Proof. Set $q = p^n$ and let V be the n-dimensional vector space over Z_p. By Lemma 3.1 there exists a partition of $V - \{\mathbf{0}\}$ of the form

$$V - \{\mathbf{0}\} = \bigcup_{i=1}^{(p^n-1)/6} \{\pm\mathbf{u}_i, \pm k\mathbf{u}_i, \pm\mathbf{v}_i\} \tag{1}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are linearly independent for each i.

For $i = 1, \ldots, (p^n - 1)/6$, let Γ_i be the graph defined as follows.

$$V(\Gamma_i) = \{\mathbf{0}, \mathbf{u}_i, 2\mathbf{u}_i, \ldots, (p-1)\mathbf{u}_i, \mathbf{v}_i, \mathbf{v}_i + \mathbf{u}_i, \mathbf{v}_i + 2\mathbf{u}_i, \ldots, \mathbf{v}_i + (p-1)\mathbf{u}_i\}\,;$$
$$E(\Gamma_i) = \bigcup_{j=0}^{p-1} \big\{[j\mathbf{u_i}, (j+1)\mathbf{u_i}], [\mathbf{v}_i + j\mathbf{u_i}, \mathbf{v}_i + (j+k)\mathbf{u_i}], [j\mathbf{u_i}, \mathbf{v}_i + j\mathbf{u_i}]\big\}\,.$$

Observe that $V(\Gamma_i)$ has size $2p$ since $\mathbf{u}_i$ and $\mathbf{v}_i$ are linearly independent and hence Γ_i is isomorphic to $P(p,k)$.

The differences arising from the edges $[j\mathbf{u_i}, (j+1)\mathbf{u_i}]$, $[\mathbf{v}_i + j\mathbf{u_i}, \mathbf{v}_i + (j+k)\mathbf{u_i}]$ and $[j\mathbf{u_i}, \mathbf{v}_i + j\mathbf{u_i}]$ are $\pm\mathbf{u}_i$, $\pm k\mathbf{u}_i$ and $\pm\mathbf{v}_i$, respectively, and that is independent of j. It follows that $\Delta\Gamma_i$ is p times the set $\{\pm\mathbf{u}_i, \pm k\mathbf{u}_i, \pm\mathbf{v}_i\}$. Consequently, noting that the stabilizer of Γ_i under the natural action of V has order p (it is namely the 1-dimensional subspace generated by $\mathbf{u}_i$) we have:

$$\partial\Gamma_i = \{\pm\mathbf{u}_i, \pm k\mathbf{u}_i, \pm\mathbf{v}_i\}.$$

By (1), we have thus $\bigcup_{i=1}^{(p^n-1)/6} \partial\Gamma_i = V - \{\mathbf{0}\}$. Using Theorem 2.1 we finally conclude that $\Gamma_1, \Gamma_2, \ldots, \Gamma_{(p^n-1)/6}$ yield an elementary abelian $P(p,k)$-decomposition of K_q. □

We present an alternative construction which is somewhat simpler than the one of the previous theorem, even though it only works under additional constraints on the parameters.

Theorem 3.3. *Let $p \equiv 1 \pmod 6$ be a prime and let k be an integer, with $1 < k < p - 1$, such that k is not a cube when interpreted as a field element in Z_p. Let n be an integer greater than 1 which is not divisible by 3 and set $q = p^n$. There exists an elementary abelian $P(p,k)$-decomposition of K_q.*

Proof. We construct a copy Γ of $P(p,k)$ as follows. Let x be a chosen cube in $\mathrm{GF}(q) - \mathrm{GF}(p)$. We define

$$V(\Gamma) = \big\{0, 1, \ldots, p-1\big\} \cup \big\{k^2 x, k^2 x + 1, \ldots, k^2 x + (p-1)\big\} \text{ and}$$
$$E(\Gamma) = \big\{[i, i+1], [i, k^2 x + i], [k^2 x + i, k^2 x + i + k] \mid i = 0, 1, \ldots, p-1\big\}\,.$$

The list $\partial\Gamma$ consists precisely of the six elements ± 1, $\pm k$, $\pm k^2 x$. Since n is not divisible by 3, the subgroup Z_p^* is not contained in the subgroup C of cubes in $\mathrm{GF}(q)^*$ and then k is not a cube in $\mathrm{GF}(q)$. This fact, together with our choice of x, assures that the triple $\{1, k, k^2 x\}$ forms a complete system of representatives for the subgroup C in $\mathrm{GF}(q)^*$.

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Let now S be a a complete system of representatives for the subgroup $\{1, -1\}$ in the multiplicative group of cubes in $\mathsf{GF}(q)^*$. For each $s \in S$ consider the subgraph $s\Gamma$ which is obtained from Γ by replacing each vertex u of Γ by su. The list of partial differences of $s\Gamma$ is then clearly $s\,\partial\Gamma$ and, consequently, the list of partial differences which are covered by the family $\{s\Gamma \mid s \in S\}$ is given by $S\,\{\pm 1\}\,\{1, k, k^2\,x\} = C\,\{1, k, k^2\,x\} = \mathsf{GF}(q) - \{0\}$. We can conclude that this family is a complete system of representatives for the orbits of an elementary abelian Γ-decomposition of K_q. □

Remark 3.4. If the prime p of the previous Theorem 3.3 is also congruent to $7 \pmod{12}$, then we can choose a complete system S of representatives for $\{1, -1\}$ in C as the subgroup of the 6-th powers in $\mathsf{GF}(q)^*$. In this manner the group of affine linear transformations over $\mathsf{GF}(q)$ of the form $x \mapsto sx + t$, $s \in S$, $t \in \mathsf{GF}(q)$ is a group of automorphisms of the $P(p, k)$-decomposition of K_q. The action of this group is transitive on the subgraphs of the decomposition.

4. Cyclic decompositions of the complete graph into (generalized) Petersen graphs and prisms

Let $q = p^\alpha$ be a prime power with $\alpha \geq 1$ and $q = ef + 1$. Let γ denote a generator of the multiplicative group of $\mathsf{GF}(q)$, i.e. $\mathsf{GF}(q)^* = \langle\gamma\rangle$. Let C_0 be the subgroup of the e-th powers in $\mathsf{GF}(q)^*$ and let C_i be the coset of C_0 which contains γ^i, $0 \leq i \leq e - 1$. The *cyclotomic number* A_{ij} of order e counts the field elements x in C_i such that $x - 1$ is in C_j. The determination of cyclotomic numbers is quite difficult. For the particular case $e = 3$, the problem was first considered by Gauss [12], and later on by Dickson [9], under the restriction $\alpha = 1$. For the case $\alpha > 1$ we recall results by Hall [14], Storer [19] and Katre and Raiwade [16].

It is proved in [15] that all cyclotomic numbers A_{ij} satisfy the following relation:

$$e^2 A_{ij} \geq q - (e-1)(e-2)\sqrt{q} - (3e-1)\,.$$

For $e = 3$ we get

$$9A_{ij} \geq (\sqrt{q} - 1)^2 - 9$$

and hence all cyclotomic numbers of order 3 are positive if $q > 16$ (i.e. $q > 13$ if q is odd).

Lemma 4.1. *Let $q = 6nt + 1$, with $q = p^\alpha$, $\alpha \geq 1$, $q > 13$ and n odd. Let ζ be a primitive n-th root of unity in $\mathsf{GF}(q)$. For each k, with $1 \leq k \leq \frac{n-1}{2}$, there exists an element $x \in \mathsf{GF}(q)$, with $x \notin \langle\zeta\rangle$, such that $\zeta - 1$, $x(\zeta^k - 1)$, $x - 1$ form a complete system of representatives for the subgroup of cubes in $\mathsf{GF}(q)^*$.*

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Proof. Let C_0 be the subgroup of cubes in $\mathsf{GF}(q)^*$ and let C_1 and C_2 be its cosets in $\mathsf{GF}(q)^*$. Since ζ has order n and $q-1=6nt$, we have $\langle\zeta\rangle \subseteq C_0$. If $\zeta-1\in C_i$ and $\zeta^k-1\in C_j$, let $x\in C_{h-j}$ and $x-1\in C_l$, where h is different from i and j $\pmod 3$, and l is the unique element of $\{0,1,2\}$ which is distinct from i and h. The existence of x is assured by the fact that all cyclotomic numbers of order 3 are positive. Moreover, $C_{h-j}\neq C_0$ and then $x\notin\langle\zeta\rangle$. In this manner we get $\zeta-1\in C_i$, $x-1\in C_l$, $x(\zeta^k-1)\in C_h$ and the assertion follows. □

We are now able to prove the following:

Theorem 4.2. *Let $n\geq 3$ be an odd integer and let v be an odd integer whose prime factors are congruent to $1 \pmod{6n}$. For each k, with $1\leq k\leq \frac{n-1}{2}$, there exists a cyclic $P(n,k)$-decomposition of K_v.*

Proof. Let $p=6nt+1$ be a prime, with $p>13$. Follow the previous Lemma 4.1 in the particular case $\alpha=1$ and take $x\in Z_p$, with $x\notin\langle\zeta\rangle$, and such that $\zeta-1$, $x(\zeta^k-1)$, $x-1$ are a complete system of representatives of the subgroup of cubes in Z_p^*. The subgroup $\langle-\zeta\rangle$ is contained in C_0 and it has order $2n$. Let $\{s_1,\dots,s_t\}$ be a complete system of representatives for the cosets of $\langle-\zeta\rangle$ in C_0. For each s_i, let $\overline{\Gamma}_i$ be the graph defined as follows:

$$V(\overline{\Gamma}_i)=\{s_i,s_i\zeta,s_i\zeta^2,\dots,s_i\zeta^{n-1},xs_i,xs_i\zeta,xs_i\zeta^2,\dots,xs_i\zeta^{n-1}\},$$
$$E(\overline{\Gamma}_i)=\{[s_i\zeta^j,s_i\zeta^{j+1}],[xs_i\zeta^j,xs_i\zeta^{j+k}],[s_i\zeta^j,xs_i\zeta^j]\mid j=0,\dots,n-1\}.$$

The graph $\overline{\Gamma}_i$ is isomorphic to $P(n,k)$, it has trivial stabilizer in the additive group of Z_p and $\partial\overline{\Gamma}_i=s_i\{\zeta-1,x(\zeta^k-1),x-1\}\langle-\zeta\rangle$. We conclude that the set $\{\overline{\Gamma}_1,\dots,\overline{\Gamma}_t\}$ is a complete system of representatives for a cyclic $P(n,k)$-decomposition of K_p.

Since each $\overline{\Gamma}_i$ has trivial stabilizer, the set $\{\overline{\Gamma}_1,\dots,\overline{\Gamma}_t\}$ is a $(Z_p,\overline{\Gamma},1)$-difference family with $\overline{\Gamma}=P(n,k)$. For each odd integer v whose prime factors are all congruent to $1 \pmod{6n}$, $n\geq 3$, since the graph $P(n,k)$ has chromatic number 3, an application of Proposition 2.2 yields the existence of a cyclic $P(n,k)$-decomposition of K_v. □

Remark 4.3. Consider the $P(n,k)$-decomposition of K_p constructed in the previous Theorem 4.2. The group of affine linear transformations over the field Z_p of the form $x\mapsto rx+b$, $r\in\langle\zeta\rangle$, $b\in Z_p$ is an automorphism group of the decomposition. Moreover, if t and $2n$ are coprime, the subgroup C_0 is the direct product of $\langle-\zeta\rangle$ together with a subgroup H of order t. It is thus clear that the elements of H can be seen as the complete system $\{s_1,\dots,s_t\}$ chosen in the construction of the theorem. In this case, the group of affine linear transformations $x\mapsto ax+b$, $a\in\langle\zeta\rangle H$, $b\in Z_p$ is again an automorphism group of

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the decomposition. In particular, the subgroup of affine linear transformations $x \mapsto hx + b$, $h \in H$, $b \in Z_p$ has a sharply transitive action on the graphs of the decomposition.

Remark 4.4. In the first part of the proof of Theorem 4.2 a cyclic $P(n,k)$-decomposition of K_p is constructed via Lemma 4.1. This construction can be repeated in $\mathsf{GF}(q)$, with $q = p^\alpha = 6nt + 1$, n odd and $\alpha > 1$. In this manner we obtain an elementary abelian $P(n,k)$-decomposition of K_q, $q > 13$.

5. Sharply transitive mixed decompositions

We now turn our attention to the problem of finding decompositions of complete graphs into subgraphs of different types, a problem which has also received considerable attention in the literature. An instance of this situation comes from the decompositions of complete graphs into cycles of varying specified lengths, see [3] for a recent survey.

In this section we construct examples of sharply transitive decompositions of complete graphs into copies of different generalized Petersen graphs and prisms. As before, G denotes a finite additive group of odd order. By $\mathrm{EA}(q)$ we denote the elementary abelian group of order q. Theorem 2.1 can be generalized as follows.

Theorem 5.1. *Let $\mathcal{C} = \{\Gamma_1, \ldots, \Gamma_t\}$ be a set of subgraphs of K_G, each of which is isomorphic to one of the given graphs $R_1, \ldots, R_s$. The set $\mathcal{C}$ is a complete system of representatives for the G-orbits of a sharply transitive decomposition of K_G into copies of $R_1, \ldots, R_s$ if and only if $\partial\Gamma_1 \cup \cdots \cup \partial\Gamma_t = G - \{0\}$.*

Proof. The steps of the proof follow closely those of Theorem 2.1. We omit the details. □

We call such a decomposition a *sharply transitive* $(R_1, \ldots, R_s)$*-decomposition* of K_v.

Theorem 5.2. *Let $q \equiv 1 \pmod 6$ be a power of an odd prime p and let $2 \leq k \leq p - 2$. Let $n \geq 3$ be an odd integer, let v be an odd integer whose prime factors are congruent to $1 \pmod{6n}$ and let h be an integer with $1 \leq h \leq \frac{n-1}{2}$. Let $P_1 = P(p,k)$ and let either $P_2 = P(2p,k)$ or $P_2 = P(2p,p+k)$ according to whether k is odd or even, respectively. Similarly, let $P_3 = P(n,h)$ and let either $P_4 = P(2n,h)$ or $P_4 = P(2n,n+h)$ according to whether h is odd or even, respectively.*

- *There exists an elementary abelian (P_1, P_2)-decomposition of K_{q^2}.*

- *There exists a sharply transitive (P_3, P_4)-decomposition of $K_{Z_v \oplus Z_v}$.*
- *There exists a sharply transitive (P_1, P_2, P_3)-decomposition of $K_{\mathrm{EA}(q) \oplus Z_v}$.*
- *There exists a sharply transitive (P_1, P_3, P_4)-decomposition of $K_{Z_v \oplus \mathrm{EA}(q)}$.*

Proof. Let G be a group of odd order and let $\Gamma = P(s,r)$, $s \geq 3$ odd and $1 \leq r \leq s-1$, be a subgraph of K_G. Denote by $\{a_0, a_1, \ldots, a_{s-1}, b_0, b_1, \ldots, b_{s-1}\}$ its vertex-set and let $\{[a_i, a_{i+1}],\ [b_i, b_{i+r}],\ [a_i, b_i],\quad i = 0, \ldots, s-1\}$ be its edge-set, where indices are modulo s. Denote by S the stabilizer of Γ in G. Suppose S to be either the trivial group or the group generated by a rotation of the form $a_i \mapsto a_{i+t},\ \ b_i \mapsto b_{i+t}$, for some fixed t.

Let H be an additive group of odd order and partition its elements into three sets: $\{0\}$, H_Y, $-H_Y$, in such a way that $y \in H_Y$ if and only if $-y \in -H_Y$.

For each $y \in H_Y$ let Γ^y be the graph defined as follows. The vertices are obtained by pairing the elements of the sequence $(y, -y, y, -y, \ldots, y, -y)$ with the elements of the sequence $(a_0, a_1, \ldots, a_{s-1}, a_0, a_1, \ldots, a_{s-1})$ and by pairing the elements of the sequence $(-y, y, -y, y, \ldots, -y, y)$ with the elements of the sequence $(b_0, b_1, \ldots, b_{s-1}, b_0, b_1, \ldots, b_{s-1})$, respectively. We have namely:

$$\begin{aligned} V(\Gamma^y) = \{&(a_0, y), (a_1, -y), \ldots, (a_{s-1}, y), (a_0, -y), (a_1, y), \ldots, (a_{s-1}, -y)\} \\ &\cup \{(b_0, -y), (b_1, y), \ldots, (b_{s-1}, -y), (b_0, y), (b_1, -y), \ldots (b_{s-1}, y)\}\,. \end{aligned}$$

The edge-set is defined as follows:

$$\begin{aligned} E(\Gamma^y) = \bigcup_{i=0}^{s-1} \Big\{&[(a_i, y), (a_{i+1}, -y)], [(a_i, -y), (a_{i+1}, y)], [(a_i, y), (b_i, -y)], \\ &[(a_i, -y), (b_i, y)], [(b_i, -y), (b_{i+r}, y)], [(b_i, y), (b_{i+r}, -y)]\Big\}. \end{aligned}$$

Clearly, the graph Γ^y is isomorphic to either $P(2s, r)$ or $P(2s, s+r)$ according to whether r is odd or even, respectively.

The graph Γ^y has vertices in $K_{G \oplus H}$, where $G \oplus H$ denotes the direct sum of G and H. Moreover its stabilizer in $G \oplus H$ is $S \oplus \{0\}$ and $\partial\Gamma^y = \partial\Gamma \times \{2y, -2y\}$.

Observe also that it is always possible to construct two copies of Γ, namely Γ^{10} and Γ^{01} in such a way that the stabilizer of Γ^{10} is $S \oplus \{0\}$ and $\partial\Gamma^{10} = \partial\Gamma \times \{0\}$, the stabilizer of Γ^{01} is $\{0\} \oplus S$ and $\partial\Gamma^{01} = \{0\} \times \partial\Gamma$. In particular the graph Γ^{10} is obtained by relabeling each vertex x of Γ with $(x, 0)$, while Γ^{01} is obtained by relabeling each vertex x of Γ with $(0, x)$.

In what follows we repeat these constructions in four particular situations. In these situations, we choose the groups G and H to be either cyclic or elementary abelian groups.

Set $q = p^m$, $q \equiv 1 \pmod 6$. Follow the construction of Theorem 3.2 and let $\{\Gamma_1, \dots, \Gamma_{(p^m-1)/6}\}$ be a complete system of representatives for the elementary abelian P_1-decomposition of K_q. Follow the construction of Theorem 4.2 and let $\{\overline{\Gamma}_1, \dots \overline{\Gamma}_t\}$ be a complete system of representatives for the cyclic P_3-decomposition of K_v.

The set $\{\Gamma_i^{10}, \Gamma_i^{01}, \Gamma_i^{y} \mid i = 1, \dots, (p^m-1)/6,\ y \in \mathrm{EA}(q)_Y\}$ is a complete system of representatives for an elementary abelian (P_1, P_2)-decomposition of K_{q^2}.

The set $\{\overline{\Gamma}_j^{10}, \overline{\Gamma}_j^{01}, \overline{\Gamma}_j^{y} \mid j = 1, \dots, t,\ y \in Z_{vY}\}$ is a complete system of representatives for a sharply transitive (P_3, P_4)-decomposition of $K_{Z_v \oplus Z_v}$.

The set $\{\Gamma_i^{10}, \Gamma_i^{y}, \overline{\Gamma}_j^{01} \mid i = 1, \dots, (p^m-1)/6,\ y \in Z_{vY},\ j = 1, \dots, t\}$ is a complete system of representatives for a sharply transitive (P_1, P_2, P_3)-decomposition of $K_{\mathrm{EA}(q) \oplus Z_v}$.

The set $\{\overline{\Gamma}_j^{10}, \overline{\Gamma}_j^{y}, \Gamma_i^{01} \mid j = 1, \dots, t,\ y \in \mathrm{EA}(q)_Y,\ i = 1, \dots, (p^m-1)/6\}$ is a complete system of representatives for a sharply transitive (P_1, P_3, P_4)-decomposition of $K_{Z_v \oplus \mathrm{EA}(q)}$. □

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