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The classification of spreads of $T_2(\mathcal{O})$ and α-flocks over small fields

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Abstract

We classify spreads of the Tits quadrangles $T_2(\mathcal{O})$, for $\mathcal{O}$ an oval in $\mathsf{PG}(2,q)$, for $q = 2,4,8,16$ and 32, using a computer for the last three cases. Along the way, we classify α-flocks of $\mathsf{PG}(3,32)$, and so flocks of the quadratic cone in $\mathsf{PG}(3,32)$. Perhaps our most striking results are that, for many ovals $\mathcal{O}$ in $\mathsf{PG}(2,32)$, including all 12 O'Keefe-Penttila ovals, $T_2(\mathcal{O})$ has no spreads, and that $T_2(\mathcal{O})$ is a proper subGQ of a GQ of order $(s,32)$ for precisely 6 of the 35 ovals $\mathcal{O}$ of $\mathsf{PG}(2,32)$, all of which were previously known to be subquadrangles of a (flock or dual Tits) GQ of order $(1024,32)$. Also $T_2(\mathcal{O})$ is not a proper subGQ of a GQ of order (s,q) or of a GQ of order (q,t) for $\mathcal{O}$ a pointed conic in $\mathsf{PG}(2,q)$, for $q = 16,32$.

Keywords: generalized quadrangle, spread, flock, subquadrangle, oval

MSC 2000: 51E12, 51E23, 51E21, 51B15, 51E20

1 Introduction

This paper is a sequel to [6] (and to [28]), and will use the definitions and notation therein without comment. (See also the newly published book [8] for information about flocks in characteristic 2.) It is also a sequel to [15, 17, 19] in that it generalizes the results of those papers (which are equivalent to classifying the spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ a conic of $\mathsf{PG}(2,q)$, $q = 16$ or 32) to classifying the spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ an oval of $\mathsf{PG}(2,q)$, $q = 16$ (Theorem 4.11) or 32 (Theorem 5.6). Other worthwhile results we obtain in the process include the classification of all flocks of the quadratic cone over the field of order 32 (Theorem 5.3) and the proof that only those Tits quadrangles $T_2(\mathcal{O})$ already known to be proper subquadrangles of a GQ of order (s, q) *are* proper subquadrangles of a GQ of order (s, q) for $q \leq 32$ (Corollary 5.9). We also disprove a conjecture of Cherowitzo [9] about the O'Keefe-Penttila hyperoval (see the remark after Corollary 5.7), by classifying α-flocks over the field of order 32 (Theorems 5.3, 5.5). For completeness, we also include the corresponding results for fields of even order at most 8 in Section 3. As to odd order, we note that, by the celebrated theorem of Segre, all ovals $\mathcal{O}$ of $\mathsf{PG}(2, q)$, q odd, are conics, so the Tits quadrangle $T_2(\mathcal{O})$ is isomorphic to the classical quadrangle $Q(4, q)$ [20, Theorem 3.2.2]. This GQ has no spreads [20, Theorem 3.4.1(i)]. For q prime, the only ovoids of $Q(4, q)$ are elliptic quadrics [1]. For $q = 9$, the only ovoids of $Q(4, q)$ are the elliptic quadrics and the Kantor ovoids [13] (see also [30, p. 51]). A survey on ovoids of $Q(4, q)$ is given in [23]. All our computer calculations took place in the computer algebra package Magma [3].

2 Equivalence of α-flocks and isomorphism of spreads

In [6], it is shown that every α-flock gives rise to a spread of $T_2(\mathcal{O})$, where $\mathcal{O}$ is the oval constructed from the α-flock by Cherowitzo [9]. In order to classify spreads of $T_2(\mathcal{O})$, it is therefore necessary to classify α-flocks of $\mathsf{PG}(3, q)$, and to deal with isomorphism. A subtle point occurs here. As originally shown in [12], each α-flock gives a flock of the cone subtended by the hyperoval, and so a $\frac{1}{\alpha}$-flock. If $\alpha \neq 2, \frac{1}{2}$, then two α-flocks are equivalent if and only if the corresponding $\frac{1}{\alpha}$-flocks are equivalent. But the excluded cases are exceptional:

Theorem 2.1. *Each 2-flock gives rise to a $\frac{1}{2}$-flock for every orbit of its stabiliser on generators of the quadratic cone. $\frac{1}{2}$-flocks arising from different orbits are inequivalent.*

Proof. This follows from a simple calculation similar to the one below. □

Let $q = 2^e$, $F_q = \mathsf{GF}(q)$, and $\alpha = 2^i$, $(i, e) = 1$. In $\mathsf{PG}(3, q)$ let K_α be the cone

$$K_\alpha : x_1^\alpha = x_0 x_2^{\alpha-1} \text{ with vertex } V(0,0,0,1),$$
$$\text{and nuclear generator } \langle V(0,0,0,1), (0,1,0,0) \rangle .$$

It also follows that $\langle V(0,0,0,1), (1,0,0,0) \rangle$ is an axial generator, the unique one if $\alpha \neq 2$.

Theorem 2.2. *The subgroup of $\mathsf{P\Gamma L}(4, q)$ leaving invariant the cone K_α consists of the following collineations:*

$$\theta : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3)^\sigma M, \quad M = \begin{pmatrix} a^{\alpha\sigma} & 0 & 0 & x \\ 0 & a^\sigma & 0 & y \\ (as)^{\alpha\sigma} & (as)^\sigma & 1 & z \\ 0 & 0 & 0 & w \end{pmatrix}. \tag{1}$$

Here a, s, x, y, z, w are elements of F_q with a and w not zero, and σ is any automorphism of F_q.

For convenience in computing the images of planes we give the inverse of M.

$$M^{-1} = \begin{pmatrix} a^{-\alpha\sigma} & 0 & 0 & xa^{-\alpha\sigma}w^{-1} \\ 0 & a^{-\sigma} & 0 & ya^{-\sigma}w^{-1} \\ s^{\alpha\sigma} & s^\sigma & 1 & xs^{\alpha\sigma}w^{-1} + ys^\sigma w^{-1} + zw^{-1} \\ 0 & 0 & 0 & w^{-1} \end{pmatrix}. \tag{2}$$

Suppose that a and s have been fixed with $a \neq 0$, and that $[r, v, t, 1]$ is any plane not on the vertex $V(0,0,0,1)$. Let w be any non-zero element of F_q and put $x = wr^\sigma$, $y = wv^\sigma$, $z = wt^\sigma$. Then θ maps the plane $[r, v, t, 1]$ to $[0, 0, 0, 1]$. Hence we may move any plane not on the vertex $V(0,0,0,1)$ to the plane $[0,0,0,1]$ without moving the cone or its axial generator or its nuclear generator. And the collineations fixing the cone and the plane $[0,0,0,1]$ are given by $\sigma \in \mathrm{Aut}(F_q)$, $a, w, s \in F_q$, $a \neq 0 \neq w$, with

$$M = \begin{pmatrix} a^{\alpha\sigma} & 0 & 0 & 0 \\ 0 & a^\sigma & 0 & 0 \\ (as)^{\alpha\sigma} & (as)^\sigma & 1 & 0 \\ 0 & 0 & 0 & w \end{pmatrix}, \text{ and } M^{-1} = \begin{pmatrix} a^{-\alpha\sigma} & 0 & 0 & 0 \\ 0 & a^{-\sigma} & 0 & 0 \\ s^{\alpha\sigma} & s^\sigma & 1 & 0 \\ 0 & 0 & 0 & w^{-1} \end{pmatrix}. \tag{3}$$

If α is different from 2, then the unique axial generator of K_α is the line $\langle V(0,0,0,1), Y(1,0,0,0)\rangle$. The general plane through the axial generator is $[0, x, y, 0]$. We name the plane $[0,0,1,0]$ as $\mathsf{PG}(2,q) = [0,0,1,0]$. It is fixed by the collineations indicated in equation (3). The plane $[0,1,y,0]$, $y \in F_q$, is mapped to $[0, a^{-\sigma}, (s+y)^\sigma, 0]$. If we pick $s = y$, then the plane $[0,1,y,0]$ is mapped to $[0,1,0,0]$ without moving the cone, its axial generator, or the plane $\mathsf{PG}(2,q)$.

At this point we are still free to pick nonzero a and w and an automorphism σ and leave invariant the cone and its vertex and both its axial and nuclear generators, the planes $\zeta = [0,1,0,0], \mathsf{PG}(2,q) = [0,0,1,0]$ and $\pi = [0,0,0,1]$. Suppose a is fixed and $0 \neq \lambda \in F_q$. If $(1,0,0,\lambda^\alpha), \lambda \neq 0$, is an arbitrary point of the axial generator different from $V(0,0,0,1)$ and $Y(1,0,0,0)$, put

$w = (a\lambda^{-1})^{\sigma\alpha}$. Then $(1,0,0,\lambda^\alpha)$ is mapped to $(1,0,0,1)$ without moving any of the structures so carefully arranged above. At this point we can still choose nonzero a. The cone meets the plane $\pi : x_3 = 0$ in the oval $x_1^\alpha + x_0x_2^{\alpha-1} = x_3 = 0$ with nucleus $(0,1,0,0)$, containing the point $Y(1,0,0,0)$ and with the line $\langle Y(1,0,0,0),(0,1,0,0)\rangle$ as an axis (unique if $\alpha \neq 2$). For arbitrary nonzero $a \in F_q$, the collineation

$$(x_0,x_1,x_2,x_3) \mapsto (x_0,x_1,x_2,x_3)^\sigma \begin{pmatrix} a^{\sigma\alpha} & 0 & 0 & 0 \\ 0 & a^\sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{\sigma\alpha} \end{pmatrix}$$

fixes the structure set up above, and it maps $(1,1,0,0)$ to $(a^{\sigma\alpha}, a^\sigma, 0, 0) \equiv (1,(a^\sigma)^{1-\alpha},0,0)$. Since $a \mapsto a^{1-\alpha}$ is a permutation of the nonzero elements of F_q, we may map $(1,1,0,0)$ to any point of the axis $\langle Y(1,0,0,0),(0,1,0,0)\rangle$ other than $Y(1,0,0,0)$ or $(0,1,0,0)$. So far we have not used the field automorphism σ. Hence we have proved the following theorem.

Theorem 2.3. *Let K_α be an α-cone in $\mathsf{PG}(3,q)$ with vertex V, with axial generator L_1 and nuclear line L_2. (This means that for any plane π not containing the vertex, the oval $\mathcal{O} = \pi \cap K_\alpha$ has nucleus $N = L_2 \cap \pi$, contains the point $Q = L_1 \cap \pi$, and the line $\langle Q, N\rangle$ is an axis of $\mathcal{O}$.) Let π_3 be an arbitrary but fixed plane not containing the vertex V. Let π_2 be the plane containing the axial generator and the nuclear generator of K_α. Let π_1 be any other plane containing the axial generator. Let Y be the point of π_3 on the axial generator, and let U be any point of the axial generator different from V and from Y (i.e., not in π_3). Finally, let P be any point of $\pi_2 \cap \pi_3$ not on the axial generator or nuclear generator of K_α. Then the coordinates for $\mathsf{PG}(3,q)$ may be chosen so that the following hold:*

$$\begin{array}{llll} V=(0,0,0,1), & \pi_3=[0,0,0,1], & \pi_2=[0,0,1,0], & \pi_1=[0,1,0,0], \\ Y=(1,0,0,0), & U=(1,0,0,1), & P=(1,1,0,0), & L_2\cap\pi_3=(0,1,0,0). \end{array}$$

Now we suppose that the automorphism α generates the Galois group of F_q but $\alpha \neq 2$. Hence each oval equivalent to $\mathcal{O}_\alpha : x_1^\alpha = x_0x_2^{\alpha-1}$ has a unique axis, so the cone K_α has a unique axial generator. Let $\mathcal{O}$ be an oval of $\mathsf{PG}(2,q)$ identified as the hyperplane $\pi_2 : x_2 = 0$ of $\mathsf{PG}(3,q)$. It is uniquely extended to a hyperoval $\mathcal{O}^+$ and we may assume that the hyperoval has any four points of $\mathsf{PG}(2,q)$ in general position that we please. So suppose it has among its points those of the fundamental quadrangle. Then there is an o-polynomial f such that

$$\mathcal{O}^+ = \{(t,1,0,f(t)) : t \in F_q\} \cup \{(1,0,0,0),(0,0,0,1)\}.$$

If we apply the collineation (elation with axis $\pi_1 : x_1 = 0$) $(x,y,0,z) \mapsto (x,y,0,z+y)$, then the image $\mathcal{O}_f^+$ has points $\{(t,1,0,1+f(t)) : t \in F_q\} \cup$

$\{((1,0,0,0)(0,0,0,1)\}$. And if we use the automorphism α just mentioned, we find

$$(\mathcal{O}_f^+)^{\frac{1}{\alpha}} = \{(t^{\frac{1}{\alpha}}, 1, (1+f(t))^{\frac{1}{\alpha}})\} \cup \{(0,0,1),(1,0,0)\}\,.$$

Given the o-polynomial f we are free to choose either $V(0,0,0,1)$ or $Y(1,0,0,0)$ to be the nucleus of the remaining $q+1$-arc. Let $\mathcal{O}_V$ be the oval containing V and $\mathcal{O}_Y$ be the oval containing Y. Then in the construction of the GQ $T_2(\mathcal{O}_V)$, we know that given any spread consisting of q^2 lines of $\mathsf{PG}(3,q)$ plus the "line" V we can use the same q^2 lines of $\mathsf{PG}(3,q)$ plus the "line" Y as a spread for $T_2(\mathcal{O}_Y)$. Now suppose we have a spread (containing Y as a "line") of $T_2(\mathcal{O}_Y)$ associated with a generalized f-fan and α-flock. This means there is a permutation polynomial g with $g(0)=0$ and $g(1)=1$, and a constant a with $\mathrm{tr}(a)=1$, such that the q^2 lines of the associated spread (different from the "line" $Y(1,0,0,0)$) are of the form

$$\Big\langle (t^{\frac{1}{\alpha}}, 1, 0, (1+f(t))^{\frac{1}{\alpha}}), ((1+f(t))s^\alpha + t^{\frac{1}{\alpha}} + ag(t), s, 1, 0) \Big\rangle : t, s \in F_q\,.$$

For a fixed $t \in F_q$ the cone with vertex $X_t = (t^{\frac{1}{\alpha}}, 1, 0, (1+f(t))^{\frac{1}{\alpha}})$ and base q-arc the oval $\mathcal{O}'_{g(t)} = \{(r^\alpha + ag(t), 0, 1, r) : r \in F_q\} \cup \{Y(1,0,0,0)\}$ minus the point $Y(1,0,0,0)$ has q lines of the associated spread. For $t \neq 1$, (and put $r = s(1+f(t))^{\frac{1}{\alpha}}$) these lines meet the plane $\pi_3 : x_3 = 0$ in the q-arc

$$\begin{aligned}\{s(t^{\frac{1}{\alpha}}, 1, 0, (1+f(t))^{\frac{1}{\alpha}}) + ((1+f(t))s^\alpha + ag(t), 0, 1, s(1+f(t))^{\frac{1}{\alpha}}) : s \in F_q\} \\ = \{((1+f(t))s^\alpha + t^{\frac{1}{\alpha}}s + ag(t), s, 1, 0) : s \in F_q\}\,.\end{aligned}$$

The points $(r^\alpha + ag(t), 0, 1, r)$, $r \in F_q$, together with the point $Y(1,0,0,0)$ give a linear axial pencil of ovals with nucleus $V(0,0,0,1)$ that constitute a generalized f-fan). The points where the spread lines intersect the plane π_3 (along with the line $w : x_3 = x_0 + x_1 + ax_2 = 0$) give a planar representation of an α-flock, i.e., a flock of the given alpha cone which consists of the planes $\mathcal{F}_\alpha = \{\pi_t = [f(t), t^{\frac{1}{\alpha}}, ag(t), 1] : t \in F_q\}$. Projecting the planes of this flock from the point $U(1,0,0,1)$ onto the plane π_3 gives the ovals

$$\mathcal{O}_t = \{((1+f(t))s^\alpha + t^{\frac{1}{\alpha}} + ag(t), s, 1, 0) : s \in F_q\} \cup \{Y(1,0,0,0)\}$$

(with nucleus $(t^{\frac{1}{\alpha}}, 1, 0, 0)$), as long as $t \neq 1$, plus the line $w : x_3 = x_0 + x_1 + ax_2 = 0$ corresponding to the case $t = 1$. Suppose we have this setup for two o-polynomials f_1 and f_2, along with a_1, a_2, g_1, g_2, such that $\{\pi_t^i = [f_i(t), t^{\frac{1}{\alpha}}, a_i g_i(t), 1] : t \in F_q\}$ is an α-flock. Note that with this π_t notation $\pi_0^i = \pi_3$ for both $i=1$ and $i=2$. We want to suppose that there is a collineation of $\mathsf{PG}(3,q)$ mapping $T_2(\mathcal{O}_{f_1})$ to $T_2(\mathcal{O}_{f_2})$ and mapping the spread in the first case to the spread in the second case.

In both cases ($i = 1,\ i = 2$) we can set up the coordinates so that π_2 is the plane of the oval (embed $\mathsf{PG}(2,q)$ in $\mathsf{PG}(3,q)$ by $(x,y,z) \mapsto (x,y,0,z)$). The plane of the generalized f-fan , i.e., the linear axial pencil of ovals is π_1, and π_3 is the plane in which is given the planar representation of the flock of the cone. The α-cone is just as in the previous section: $U(1,0,0,1)$ is the point from which we project the ovals of the α-flocks, and the point of both ovals $\mathcal{O}_{f_1}$ and $\mathcal{O}_{f_2}$ on the nuclear generator is $(0,1,0,1)$, so they both project to $P = (1,1,0,0)$.

> *Now we start by assuming that there is a collineation θ of $\mathsf{PG}(3,q)$ mapping $T_2(\mathcal{O}_{f_1})$ to $T_2(\mathcal{O}_{f_2})$ in such a way that the spread lines in the first GQ map to the spread lines of the second GQ.*

We also assume that $\mathcal{O}_{f_i}$ is not a conic, so the point $(\infty)_1$ is mapped to $(\infty)_2$, so the unique spread "line" $Y(1,0,0,0)$ incident with $(\infty)_i$ is mapped to itself.

So we have a field automorphism σ and a matrix M such that

$$\theta : (x_0,x_1,x_2,x_3) \mapsto (x_0,x_1,x_2,x_3)^\sigma M$$

has the following effect. First, the oval $\mathcal{O}_{f_1}$ is mapped to the oval $\mathcal{O}_{f_2}$, so the plane π_2 is mapped to itself, the vertex $V(0,0,0,1)$ (i.e., the nucleus of $\mathcal{O}_{f_i}$, $i = 1,2$) is mapped to itself. The unique oval point $Y(1,0,0,0)$ serving as a "line" of the spread in each case is mapped to itself.

If the plane $\pi_1 = [0,1,0,0]$ is mapped to some other plane $[0,1,y,0]$ through the axial generator, follow the original θ with the elation having matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y^\alpha & y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This elation with axis π_2 maps one axial linear pencil of ovals in the plane $[0,1,y,0]$ to another in the plane π_1, but it leaves the oval $\mathcal{O}_{f_2}$ fixed pointwise and it leaves the cone K_α invariant. It does move the spread lines to a projectively equivalent spread of $T_2(\mathcal{O}_{f_2})$, but now we may assume that π_1 is mapped to itself.

These assumptions quickly force the matrix M to have the following form.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & y & 0 & z \\ u & 0 & v & w \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \text{ and } M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x/y & 1/y & 0 & z/\lambda y \\ u/v & 0 & 1/v & w/\lambda v \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

There must be a permutation $t \mapsto \bar{t}$ of the elements of F_q for which the point $(t^{\frac{1}{\alpha}},1,0,1+f_1(t)^{\frac{1}{\alpha}})$ is mapped to $(\bar{t}^{\frac{1}{\alpha}},1,0,1+f_2(\bar{t})^{\frac{1}{\alpha}})$. Hence

$$(t^{\sigma/\alpha}+x,y,0,z+\lambda(1+f_1(t)^{\sigma/\alpha})=y(\bar{t}^{\frac{1}{\alpha}},1,0,1+f_2(\bar{t})^{\frac{1}{\alpha}}). \quad (4)$$

From this it follows that

$$\bar{t}=y^{-\alpha}(t^{\sigma}+x^{\alpha}),\ \text{which is equivalent to } t=y^{\alpha/\sigma}\bar{t}^{\frac{1}{\sigma}}+x^{\alpha/\sigma}. \quad (5)$$

Put this value of $\bar{t}$ into equation (4) to get

$$f_2(y^{-\alpha}(t^{\sigma}+x^{\alpha}))=(\lambda/y)^{\alpha}f_1(t)^{\sigma}+1+y^{-\alpha}(z^{\alpha}+\lambda^{\alpha}). \quad (6)$$

For a fixed t, the cone with vertex $X_t=(t^{\frac{1}{\alpha}},1,0,(1+f_1(t))^{\frac{1}{\alpha}})$ and base oval $\{(r^{\alpha}+a_1g_1(t),0,1,r):r\in F_q\}\cup\{Y(1,0,0,0)\}$ gets mapped to the cone with vertex $\bar{X}_{(y^{-\alpha}(t^{\sigma}+x^{\alpha}))^{\frac{1}{\alpha}}}=((y^{-\alpha}(t^{\sigma}+x^{\alpha}))^{\frac{1}{\alpha}},1,0,(1+f_2((y^{-\alpha}(t^{\sigma}+x^{\alpha}))^{\frac{1}{\alpha}})$ and base oval $\{(\bar{r}^{\alpha}+a_2g_2(t^{\sigma}),0,1,\bar{r}):\bar{r}\in F_q\}\cup\{Y(1,0,0,0)\}$, where $r\mapsto\bar{r}$ is a permutation of the elements of F_q that might depend on t. Since the plane $\pi_1:x_1=0$ isIt now follows that mapped to itself, for a **fixed** t there must be a permutation $r\mapsto\bar{r}$ and a nonzero scalar μ such that

$$(r^{\sigma\alpha}+a_1^{\sigma}g_1^{\sigma}(t)+u,0,v,w+\lambda r^{\sigma})=\mu(\bar{r}^{\alpha}+a_2g_2(\bar{t})),0,1,\bar{r}).$$

Hence $\mu=v$ and $\bar{r}=v^{-1}(w+\lambda r^{\sigma})$. Note that this does not depend on t after all! Put in these values of μ and $\bar{r}$ to get $v=\lambda^{\frac{\alpha}{\alpha-1}}$ and

$$\bar{r}=\frac{w}{\lambda^{\frac{\alpha}{\alpha-1}}}+\frac{1}{\lambda^{\frac{1}{\alpha-1}}}r^{\sigma}. \quad (7)$$

It now follows that θ can be written as

$$\begin{aligned}(r^{\alpha}+a_1g_1(t),0,1,r)\mapsto&\left(\frac{r^{\sigma\alpha}+a_1^{\sigma}g_1^{\sigma}(t)+u}{\lambda^{\frac{\alpha}{\alpha-1}}},0,1,\frac{w+\lambda r^{\sigma}}{\lambda^{\frac{\alpha}{\alpha-1}}}\right)\\ &=\left(\left(\frac{w+\lambda r^{\sigma}}{\lambda^{\frac{\alpha}{\alpha-1}}}\right)^{\alpha}+a_2g_2(\bar{t}),0,1,\frac{w+\lambda r^{\sigma}}{\lambda^{\frac{\alpha}{\alpha-1}}}\right).\end{aligned}$$

This is just a reindexing of the original f_2-fan. Hence in the special situation we are considering, spread-equivalent fans are projectively equivalent.

Thus we have proved:

Theorem 2.4. *Suppose $\alpha\neq 2,\frac{1}{2}$ and the α-flocks F_i give rise to the spreads S_i of $T_2(\mathcal{O}_i)$, for $i=1,2$. Then F_1 and F_2 are equivalent if and only if there is an isomorphism from $T_2(\mathcal{O}_1)$ onto $T_2(\mathcal{O}_2)$ mapping S_1 to S_2.*

3 Spreads of $T_2(\mathcal{O})$ for fields of orders 2, 4 and 8

Since the only ovals in $\mathsf{PG}(2,2)$ and $\mathsf{PG}(2,4)$ are conics [25], the classification of spreads of $T_2(\mathcal{O})$ for fields of orders 2,4 is equivalent to the classification of ovoids of $\mathsf{PG}(3,2)$ and of $\mathsf{PG}(3,4)$ — they are elliptic quadrics [2, 26]. This also implies the classification of flocks of the quadratic cone in $\mathsf{PG}(3,2)$ and $\mathsf{PG}(3,4)$ [27] — they are linear. Indeed, in these cases the GQs are uniquely determined by their orders [20]. In summary,

Theorem 3.1 ([20, 6.1.2])**.** *There is a unique spread of the unique GQ of order 2.*

Theorem 3.2. *There is a unique spread of the unique GQ of order 4.*

The only ovals in $\mathsf{PG}(2,8)$ are conics and pointed conics [25]. The classification of spreads of $T_2(\mathcal{O})$ for $\mathcal{O}$ a conic of $\mathsf{PG}(2,8)$ is equivalent to the classification of ovoids of $\mathsf{PG}(3,8)$ — they are Tits ovoids [29] and elliptic quadrics [11, 21]. Since for $\mathcal{O}$ a pointed conic of $\mathsf{PG}(2,8)$, $T_2(\mathcal{O})$ is self-dual, it follows that the classification of spreads of $T_2(\mathcal{O})$ is equivalent to the classification of ovoids of $T_2(\mathcal{O})$, a result already obtained [7], see [24, III.17.8]. However, using the theoretical machinery of [6, Sections 4 and 5], it is possible to reduce the amount of computation required. Indeed, the only generalized fans satisying the hypothesis of [6, Theorem 5.8] arise from Tits ovoids, by a small computation, so every other generalized fan arises from an α-flock. Since the flocks of the quadratic cone in $\mathsf{PG}(3,8)$ were classified in [27], the classification of spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of $\mathsf{PG}(2,8)$ now follows. The flocks of the quadratic cone in $\mathsf{PG}(3,8)$ are the linear and Fisher-Thas-Walker flocks. However, there are three 4-flocks in $\mathsf{PG}(3,8)$ as the Fisher-Thas-Walker flocks give rise to two inequivalent 4-flocks. These three 4-flocks in $\mathsf{PG}(3,8)$ give rise to three spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of $\mathsf{PG}(2,8)$, with the nucleus of the conic as an element of the spread, and so three ovoids of $T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of $\mathsf{PG}(2,8)$, on (∞). These are the ovoids labelled III, IV and V in result III.17.8 of [24], with IV arising from the linear flock. (The orders of the groups agree with those in [24], each being 8 times the group order of the corresponding 4-flock stabiliser, since the ovoids are translation ovoids.) The Tits ovoid generalized fan arising from a line not in the Luneburg spread also gives rise by nucleus switching (applied to the fan) to two spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of $\mathsf{PG}(2,8)$, this time with the spread *not* containing the nucleus of the conic, and so to two ovoids, *not* on (∞). These are the ovoids labelled I and II in result III.1.7.8 of [24]. In summary,

Theorem 3.3. *There are two spreads of $T_2(\mathcal{O})$ for $\mathcal{O}$ a conic of $\mathsf{PG}(2,8)$, the regular spread and the Lüneburg spread.*

Theorem 3.4. *There are five spreads of $T_2(\mathcal{O})$ for $\mathcal{O}$ a pointed conic of* $\mathsf{PG}(2,8)$*, with groups of orders 168, 168, 168, 1344 and 2688.*

Corollary 3.5. *There are five ovoids of $T_2(\mathcal{O})$ for $\mathcal{O}$ a pointed conic of* $\mathsf{PG}(2,8)$*, with groups of orders 168, 168, 168, 1344 and 2688.*

4 Spreads of $T_2(\mathcal{O})$ for the field of order 16

This section follows [15, 17] closely and is best read in that context. These two papers determine all fans in $\mathsf{PG}(2,16)$ (in order to classify ovoids in $\mathsf{PG}(3,16)$ — equivalently, spreads of $T_2(\mathcal{O})$, where $\mathcal{O}$ is a conic of $\mathsf{PG}(2,16)$). In this section, we determine all *generalized* fans in $\mathsf{PG}(2,16)$ in order to classify spreads of $T_2(\mathcal{O})$, where $\mathcal{O}$ is an *oval* of $\mathsf{PG}(2,16)$.

By [14] (see also [16] for a computer-free proof), the only hyperovals of $\mathsf{PG}(2,16)$ are the regular and Lunelli-Sce hyperovals. Hence the only ovals of $\mathsf{PG}(2,16)$ are the conic, the pointed conic and the Lunelli-Sce oval.

Let L be a Lunelli-Sce oval in $\mathsf{PG}(2,16)$. There are three orbits of the stabiliser of L on tangent lines to L. Let l be the line $x = 0$ and (L_1, l), (L_2, l), (L_3, l) be particular representatives of the three orbits on (Lunelli-Sce oval,tangent line) pairs, with each of L_1, L_2 and L_3 having nucleus $(0,0,1)$ and meeting l in $(0,1,0)$. (Here we choose L_1 so that the stabiliser of L_1 fixes l.) Let P_s be the point $(0,1,s)$. Let PC be the pointed conic $\{(1,t,t^{14}) : t \in \mathsf{GF}(16)\} \cup \{(0,1,0)\}$.

Lemma 4.1. *[ ]*

- (P_s, PC) *matches only with one point* (P_t, L_3)*;*
- *10 of the points* (P_s, L_2) *match only with* (X, L_2) *for some* X*;*
- *5 of the points* (P_s, L_2) *match only with* (X, L_3) *for some* X*;*
- *10 of the points* (P_s, L_3) *match only with* (X, L_3) *for some* X*;*
- *1 of the points* (P_s, L_3) *match only with* (X, L_2) *for some* X*;*
- *4 of the points* (P_s, L_3) *match both with* (X, L_2) *for some* X *and with* (X, L_2) *for some* X.

Proof. By a computer calculation. □

Lemma 4.2. *There is no generalized fan in* $\mathsf{PG}(2,16)$ *containing a pointed conic with the common tangent line of the fan not an axis of the pointed conic.*

Proof. Since the only match for (a point not on the axis, pointed conic) is of type L_3, the remaining 15 ovals of such a generalized fan would have to be of type L_3. But one of the points when paired with an L_3 type oval matches only with L_2, a contradiction. □

Lemma 4.3. *There is no generalized fan in* $\mathsf{PG}(2,16)$ *containing a Lunelli-Sce oval for which the common tangent line is not fixed by the stabiliser of the oval.*

Proof. Putting in L_2 forces there to be 11 ovals of type L_2 and 5 of type L_3 in the generalized fan. Putting in L_3 forces there to be between 11 and 15 ovals of type L_3 and between 1 and 5 ovals of type L_2. These two conditions contradict one another. □

Lemma 4.4. *If there is a standard generalized f-fan in* $\mathsf{PG}(2,16)$ *with O_0 being the canonical Lunelli-Sce oval $L = L_1$ then for all s not equal to 0 or 1, $O_s = g_s L$ for some homography g_s with axis* $[1,0,0]$.

Proof. (Compare with [17, Lemma 3.4].) By the matching data, O_s is an image of L under a collineation g_s fixing l, for all s not equal to 0 or 1. Moreover, since the index of $\mathsf{PGL}(3,16)_{L,l}$ in $\mathsf{P\Gamma L}(3,16)_{L,l}$ is $4 = h$, we may assume that g_s is a homography. Hence since $O_0 = L$ and $O_s = g_s L$ are compatible at $P_{f(s)/s}$, it follows that $(P_{f(s)/s}, L)$ and $(g_s^{-1} P_{f(s)/s}, L)$ match and so, by (a slight correction to) [17, Lemma 3.3], $g_s^{-1} P f(s)/s = P_{f(s)/s}$ or $P_{s/f(s)}$. If g_s does not have axis l, then the latter alternative occurs, and so $g_s^{-1} P_x = P_{(s/f(s))^2 x}$ for all x. Now the proof of [17, Lemma 3.4] can be followed mutatis mutandis and a contradiction occurs. □

Lemma 4.5. *Let b_s denote what it denotes in [17]. Then, under the hypotheses of Lemma 4.4,*

$$\begin{aligned} &(f(t)/t + f(u)/u)b_s + (f(u)/u + f(s)/s)b_t + (f(s)/s + f(t)/t)b_u \\ &= f(s)/sd_{(f(t)+f(u))/(t+u)} + f(t)/td_{(f(s)+f(u))/(s+u)} + f(u)/ud_{(f(s)+f(t))/(s+t)}\,, \end{aligned}$$

where $d_r = b_r$ or $b_r + r + 1$, for all distinct s, t, u in $\mathsf{GF}(16) \setminus \{0,1\}$.

Lemma 4.6. *There is no generalised fan in* $\mathsf{PG}(2,16)$ *containing a Lunelli-Sce oval.*

Proof. By computer, there are no solutions to the equation of Lemma 4.5 for any o-polynomial f. □

Lemma 4.7. *Every generalised fan in* $\mathsf{PG}(2,16)$ *arises from an α-flock.*

Proof. We have ruled out all ovals other than conics and pointed conics for which the common tangent is an axis. So now [6, Theorem 4.6] applies. □

Theorem 4.8 ([10]). *The only quadratic flocks in* PG$(3, 16)$ *are the linear flocks and the De Clerck-Herssens (= Subiaco) flocks.*

Proof. We classify the herds using the magic action of [18]. It is enough to find the herds containing x^8, x^2 or the Lunelli-Sce o-polynomial

$$ls = x^8 + (d^2(x^4 + x) + (d^2(d^2 + d + 1)(x^3 + x^2))/(x^2 + dx + 1)^2 .$$

where d is fixed with $\mathrm{trace}(1/d) = 1$. (Here trace is the absolute trace with image GF(2).) By inspection of the o-polynomials, if f is one of these 3 and g is an o-polynomial with $f + x^8 + g$ an o-polynomial, then $f = g = x^8$ and we have the classical herd or $f = ls$ and g is in the Subiaco herd. Hence the only herds are the classical and Subiaco herds for $q = 16$. □

Corollary 4.9. *The only* $\frac{1}{2}$*-flocks in* PG$(3, 16)$ *are the linear flocks and the three De Clerck-Herssens flocks.*

Proof. The group of the De Clerck-Herssens flock is cyclic of order 8 and has three orbits on generators of the quadratic cone, of lengths 1, 8 and 8. □

Theorem 4.10. *The only generalized fans in* PG$(2, 16)$ *are*

(i) *the fan of conics;*

(i)' *a generalized* $x^{\frac{1}{2}}$*-fan of pointed conics;*

(ii) *the generalized* L*-fan of conics;*

(ii)' *3 generalized* L*-fans of pointed conics.*

Proof. By Theorems 4.7 and 4.8 and Corollary 4.9, noting that (i) corresponds to the linear quadratic flock (and so to the elliptic quadric), (i)' arises by nucleus switching applied to (i) (and so is an axial fan corresponding to the linear $(1/2)$-flock), (ii) corresponds to the De Clerck-Herssens quadratic flock, and (ii)' arise by nucleus switching applied to (ii) (and so are axial fans corresponding to the 3 De Clerck-Herssens $(1/2)$-flocks). □

Thus we have shown:

Theorem 4.11. (a) *There is a unique spread of* $T_2(\mathcal{O})$*, for* $\mathcal{O}$ *a conic in* PG$(2, 16)$.

(a)' *There is a unique ovoid of* $T_2(\mathcal{O})$*, for* $\mathcal{O}$ *a conic in* PG$(2, 16)$.

(b) *There is a unique spread of $T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic in* $\mathsf{PG}(2,16)$. *It contains the nucleus of the conic.*

(b)' *There is a unique ovoid of $T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic in* $\mathsf{PG}(2,16)$. *It is on* (∞) *(indeed it is planar).*

(c) *There are many spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ a Lunelli-Sce oval in* $\mathsf{PG}(2,16)$. *For each point P of $\mathcal{O}$, there is a spread containing P. All of these spreads are either subtended (by the De Clerck-Herssens flock GQ) or are obtained via nucleus switching from subtended spreads.*

Corollary 4.12. *$T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of* $\mathsf{PG}(2,16)$, *is not a subquadrangle of a generalized quadrangle of order* $(16,256)$, *nor of one of order* $(256,16)$.

Proof. It has no spreads that do not contain the nucleus of the conic, and no ovoids not on (∞). □

5 Spreads of $T_2(\mathcal{O})$ for the field of order 32

By [22], there are six hyperovals of $\mathsf{PG}(2,32)$, namely, the regular, translation, Segre-Bartocci, Payne, Cherowitzo and O'Keefe-Penttila hyperovals. These lead to 35 ovals of $\mathsf{PG}(2,32)$.

Lemma 5.1. *The ovals contained in a generalized fan are contained in translation hyperovals and the common tangent is an axis to all or principal spoke to all of the ovals in the fan.*

Proof. Computer data, following the method of [19]. A first pass shows that ovals which have a tangent line with matches at all points are contained in translation hyperovals. A second pass, eliminating matches with ovals not contained in translation hyperovals, shows that the tangent line is an axis or principal spoke. □

Lemma 5.2. *There are no solutions to equation (6) of [6] that do not arise from a Tits ovoid [29] for $q = 32$.*

Proof. Computer search over the 742 ovals with a distinguished point. □

Theorem 5.3. *There are exactly 5 flocks of the quadratic cone in* $\mathsf{PG}(3,32)$, *namely the linear, Fisher-Thas-Walker, Subiaco and the two Payne flocks.*

Proof. Computer run over 35 ovals, each giving an o-polynomial f, determining herds. Except for the linear case, the only choices for g were elements of the herd. □

Corollary 5.4. *There are exactly 17 $\frac{1}{2}$-flocks in* $\mathsf{PG}(3,32)$.

Proof. The group of the Subiaco flock has 5 orbits on generators of the quadratic cone, of lengths 1, 2, 10, 10 and 10. The group of the Fisher-Thas-Walker flock has 2 orbits on generators of the quadratic cone, of lengths 1 and 32. The group of the first Payne flock $P1$ has 3 orbits on generators of the quadratic cone, of lengths 1,1 and 31. The group of the second Payne flock $P2$ has 6 orbits on generators of the quadratic cone, of lengths 1,2,10,10 and 10. □

Theorem 5.5. *There are exactly 6 4-flocks in* $\mathsf{PG}(3,32)$*, namely the linear, Fisher-Thas, and the four Cherowitzo flocks of Propositions 6, 7, and 8 and proof of Corollary 10 of [9].*

Proof. Computer search over the 742 ovals with a distinguished point, each giving an o-polynomial f. Except for the linear case, only choices for g were elements of the herd (in the sense of [9]). □

Theorem 5.6. *The spreads of $T_2(\mathcal{O})$, for $\mathcal{O}$ an oval of* $\mathsf{PG}(2,32)$*, are known.*

Proof. Apply Theorems 5.1, 5.2, 5.3, 5.5, Corollary 5.4 and [6, Theorem 4.6]. □

Corollary 5.7. *$T_2(\mathcal{O})$ has no spreads for all 12 O'Keefe-Penttila ovals and 8 of the 10 Cherowitzo ovals.*

Remark 5.8. This shows that the O'Keefe-Penttila hyperoval does not arise from an α-flock, disproving a conjecture of Cherowitzo [9].

Corollary 5.9. *$T_2(O)$ is not a proper subGQ of a GQ of order $(s,32)$ for all 12 O'Keefe-Penttila ovals, all 10 Cherowitzo ovals, 4 of the 6 Payne ovals, 1 of the 2 Segre-Bartocci ovals, the non-translation oval contained in the irregular translation hyperoval and the pointed conic in* $\mathsf{PG}(2,32)$.

Proof. In every case, the GQ has a line on no spread. □

Remark 5.10. The 6 ovals $\mathcal{O}$ for which $T_2(\mathcal{O})$ is not ruled out as a proper subGQ do arise as proper subGQs of GQ of order (1024,32), 5 of them from flock GQs and the remaining one from the dual of the Tits quadrangle $T_3(\Omega)$ arising from the Tits ovoid Ω in $\mathsf{PG}(3,32)$.

Corollary 5.11. *The ovoids of $T_2(\mathcal{O})$, for $\mathcal{O}$ a translation oval of* $\mathsf{PG}(2,32)$ *are known.*

Proof. These GQs are self-dual. □

Corollary 5.12. *$T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of* $\mathsf{PG}(2, 32)$*, is not a proper subGQ of a GQ of order $(32, t)$, nor of a GQ of order $(s, 32)$.*

Proof. It has no ovoids that are not on (∞), and no spreads that do not contain the nucleus of the conic. □

The evidence above for the fields of orders 16 and 32, coupled with the results of [4] that a pointed conic cannot be the section of an ovoid for fields of order bigger than 8 and of [5] that a pseudo-pointed conic cannot occur as part of a pseudo-ovoid leads us to give the following conjecture.

Conjecture 5.13. *$T_2(\mathcal{O})$, for $\mathcal{O}$ a pointed conic of* $\mathsf{PG}(2, q)$*, is not a proper subGQ of a GQ of order (q, t), nor of a GQ of order (s, q), for even $q > 8$.*

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