Affine sets arising from spreads

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Abstract

Certain affine sets arising from spreads of the projective space $\mathsf{PG}(3,q)$ are investigated. The affine set arising from a Lüneburg spread is studied in detail.

Keywords: spread, Lüneburg spread, translation hyperoval

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1. Introduction

A *spread* $\mathcal{S}$ of a 3-dimensional projective space over $\mathsf{GF}(q)$ is a set of q^2+1 mutually skew lines partitioning the point-set of the space.

Any spread $\mathcal{S}$ of $\mathsf{PG}(3,q)$ defines a translation plane $\pi(\mathcal{S})$ of order q^2 via the construction of André/Bruck and Bose [1].

A spread $\mathcal{S}$ is said to be *Desarguesian* if $\pi(\mathcal{S})$ is a Desarguesian plane. A *regulus* of $\mathsf{PG}(3,q)$ is one ruling of a non-singular hyperbolic quadric $Q^+(3,q)$ of $\mathsf{PG}(3,q)$. If ℓ, m and n are three pairwise disjoint lines of $\mathsf{PG}(3,q)$, there is a unique regulus $\mathcal{R}(\ell,m,n)$ of $\mathsf{PG}(3,q)$ containing ℓ, m and n. A spread $\mathcal{S}$ is said to be *regular* if $\mathcal{R}(\ell,m,n)$ is contained in $\mathcal{S}$, for any triple ℓ, m and n of distinct lines of $\mathcal{S}$. If $q>2$, a spread $\mathcal{S}$ of $\mathsf{PG}(3,q)$ is *regular* if and only if $\mathcal{S}$ is Desarguesian.

The spread $\mathcal{S}$ is said to be *symplectic* if its lines turn out to be totally isotropic with respect to a symplectic polarity of $\mathsf{PG}(3,q)$. A spread $\mathcal{S}$ is said to be a *semifield spread* with respect to a line r of $\mathcal{S}$ if there exists a collineation group

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of $\mathrm{PG}(3,q)$ fixing the line r pointwise and acting sharply transitively on $\mathcal{S}\setminus\{r\}$. If $\mathcal{S}$ is a Desarguesian spread, then $\mathcal{S}$ is both a symplectic spread and a semifield spread with respect to all of its lines.

Under the Plücker correspondence, to any spread $\mathcal{S}$ of $\mathrm{PG}(3,q)$ there corresponds an ovoid $\mathcal{O}(\mathcal{S})$ of $Q^+(5,q)$. If $\mathcal{S}$ is a symplectic spread then the ovoid $\mathcal{O}(\mathcal{S})$ lies in a parabolic quadric $Q(4,q)$. If $\mathcal{S}$ is Desarguesian, then $\mathcal{O}(\mathcal{S})$ is an elliptic quadric $Q^-(3,q)$ embedded in $Q^+(5,q)$.

In [8, Sec. 3] it has been proven[1] that if $\mathbb{O}$ is an ovoid of $Q^+(5,q)$, then to any point x of $\mathbb{O}$ there corresponds a set $\mathcal{A}_x(\mathbb{O})$ of q^2 points of a 4-dimensional projective space $\Omega' = \mathrm{PG}(4,q)$ such that

(i) $\mathcal{A}_x(\mathbb{O}) \subset \Omega'\setminus\Omega$, where Ω is an hyperplane of Ω';

(ii) the line joining two points of $\mathcal{A}_x(\mathbb{O})$ is disjoint from the hyperbolic quadric $Q^+(3,q) = \Omega \cap Q^+(5,q)$.

More precisely, the set $\mathcal{A}_x(\mathbb{O})$ is obtained by projecting $\mathbb{O}$ from the point x to any hyperplane Ω' of $\mathrm{PG}(5,q)$ not containing x and $\Omega = \Omega' \cap x^\perp$, where $\perp$ denotes the orthogonal polarity arising from $Q^+(5,q)$. Conversely, if $\mathcal{A}$ is a set of q^2 points of $\Omega'\setminus\Omega$ satisfying (ii), then the set

$$\mathbb{O} = \{xy \cap Q^+(5,q) \mid y \in \mathcal{A}\}$$

is an ovoid of $Q^+(5,q)$ and the set $\mathcal{A}_x(\mathbb{O})$ obtained by projecting $\mathbb{O}$ from the point x to the hyperplane Ω' coincides with $\mathcal{A}$.

Let κ denote the Plücker map from the set of lines of $\mathrm{PG}(3,q)$ to the set of points of $Q^+(5,q)$. If $\mathcal{S}$ is a spread of $\mathrm{PG}(3,q)$ and ℓ is any line of $\mathcal{S}$, then $\kappa(\mathcal{S})$ is an ovoid of $Q^+(5,q)$ containing the point $\kappa(\ell)$. We denote by $\mathcal{A}_\ell(\mathcal{S})$ the set $\mathcal{A}_{\kappa(\ell)}(\kappa(\mathcal{S}))$ described above and we will refer to it as *the affine set arising from $\mathcal{S}$ with respect to ℓ*, or *affine set* for short.

In this note we study the affine sets $\mathcal{A}_\ell(\mathcal{S})$ arising from the Lüneburg spreads, proving that the sets $\mathcal{A}_\ell(\mathcal{S})$ are unions of q q-arcs and each such an arc can be completed to a translation hyperoval.

2. Preliminaries

Let $\Sigma = \mathrm{PG}(3,q)$, $q = p^h$, p prime, $h \geq 1$, be the three-dimensional projective space over $\mathrm{GF}(q)$ and let X_0, X_1, X_2, X_3 be homogeneous projective coordinates of Σ. Let $\mathcal{S}$ be a spread of Σ and ℓ be a fixed line of $\mathcal{S}$. We can always assume

[1] Note that in [8] the construction involves any orthogonal polar space.

that the line $\ell = \ell_\infty$ has equations $X_0 = X_1 = 0$. In this case, for any line m of $\mathcal{S}$ different from ℓ_∞, there is a unique 2×2 matrix J_m over $\mathsf{GF}(q)$ such that

$$m = \{(a, b, c, d) \mid (c, d) = (a, b)J_m,\ a, b \in \mathsf{GF}(q)\}\,.$$

The set $\mathcal{C}_\mathcal{S} = \mathcal{C}_\mathcal{S}(\ell_\infty) = \{J_m \mid m \in \mathcal{S}\}$ has the following properties:

(1) $\mathcal{C}_\mathcal{S}$ has q^2 elements;

(2) $X - Y$ is a non-singular matrix for all $X, Y \in \mathcal{C}_\mathcal{S}$, $X \neq Y$.

Such a set $\mathcal{C}_\mathcal{S}$ is a *spread set* associated with $\mathcal{S}$ with respect to ℓ_∞ (see [4]). On the other hand, starting from a set $\mathcal{C}$ of 2×2 matrices over $\mathsf{GF}(q)$ satisfying (1) and (2), the set of lines $\mathcal{S} = \{\ell_M \mid M \in \mathcal{C}\} \cup \{\ell_\infty\}$ where

$$\ell_M = \{(a, b, c, d) \mid (c, d) = (a, b)M,\ a, b \in \mathsf{GF}(q)\}$$

is a spread of $\mathsf{PG}(3, q)$ and $\mathcal{C}_\mathcal{S} = \mathcal{C}$. Note that if $\ell_0 : X_2 = X_3 = 0$ is a line of $\mathcal{S}$, then any line of $\mathcal{S} \setminus \{\ell_0\}$ is of type

$$\ell^N = \{(a, b, c, d) \mid (a, b) = (c, d)N,\ c, d \in \mathsf{GF}(q)\}\,,$$

where N is a 2×2 matrix over $\mathsf{GF}(q)$. Since the map $(a, b, c, d) \mapsto (c, d, a, b)$ sends ℓ_∞ to ℓ_0, the set $\mathcal{C}_\mathcal{S}(\ell_0) = \{N \mid \ell^N \in \mathcal{S} \setminus \{\ell_0\}\}$ is a spread set associated with $\mathcal{S}$ with respect to ℓ_0.

Let $\mathcal{S}$ be a spread of Σ containing the line ℓ_∞ and let $\mathcal{C}_\mathcal{S}$ be a spread set associated with $\mathcal{S}$. Then we have

Proposition 2.1. *The affine set arising from $\mathcal{S}$ with respect to ℓ_∞ can be written*

$$\mathcal{A}_{\ell_\infty}(\mathcal{S}) = \left\{(1, a, b, c, d) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_\mathcal{S}\right\}.$$

Proof. Let κ be the Plücker map from the set of lines of $\mathsf{PG}(3, q)$ to the point-set of the Klein quadric $\mathcal{Q} = Q^+(5, q) : Y_0Y_5 + Y_1Y_4 + Y_2Y_3 = 0$ of $\Lambda = \mathsf{PG}(5, q)$. Then $\kappa(\ell_\infty) = x = (0, 0, 0, 0, 0, 1)$ and $\kappa(\ell) = (1, c, d, -a, b, ad - bc)$, where

$$\ell = \left\{(X_0, X_1, X_2, X_3) \mid (X_2, X_3) = (X_0, X_1)\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ X_0, X_1 \in \mathsf{GF}(q)\right\}$$

is a line of $\mathcal{S}$. Hence the corresponding ovoid $\mathcal{O}$ of $\mathcal{Q}$ is

$$\mathcal{O} = \left\{(1, c, d, -a, b, ad - bc) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_\mathcal{S}\right\} \cup \{x\}.$$

Let $\varphi\colon \Lambda \to \Lambda$ be the collineation with equations

$$Y_0' = Y_0 \qquad Y_2' = Y_4 \qquad Y_4' = Y_2$$
$$Y_1' = -Y_3 \qquad Y_3' = Y_1 \qquad Y_5' = Y_5\,.$$

Then $\varphi(\mathcal{O}) = \{(1, a, b, c, d, ad - bc) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_{\mathcal{S}}\} \cup \{x\}$ is the ovoid of the quadric $\varphi(\mathcal{Q})$: $Y_0Y_5 - Y_1Y_4 + Y_2Y_3 = 0$. By projecting $\varphi(\mathcal{O})$ from x onto the hyperplane Ω' : $Y_5 = 0$ we get

$$\mathcal{A}_{\ell_\infty}(\mathcal{S}) = \{(1, a, b, c, d) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_{\mathcal{S}}\} \cup \{x\}\,. \qquad \square$$

Now, let $\mathcal{A}$ be a set of q^2 points of $\Omega' = \mathsf{PG}(4, q)$ with properties (i) and (ii). Choose projective coordinates X_0, X_1, X_2, X_3, X_4 in Ω' in such a way that Ω is the hyperplane with equation $X_0 = 0$ and $Q^+(3, q)$ is the hyperbolic quadric of Ω with equation $X_1X_4 - X_2X_3 = 0$. In this case we have that each point of $\mathcal{A}$ has coordinates $(1, a, b, c, d)$. Moreover,if

$$\mathcal{C} = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (1, a, b, c, d) \in \mathcal{A} \right\},$$

then by property (ii) for any $X, Y \in \mathcal{C}$, $X \neq Y$, the matrix $X - Y$ is non-singular, i.e. $\mathcal{C}$ is a spread set. If $\mathcal{S}$ is the spread of $\mathsf{PG}(3, q)$ defined by $\mathcal{C}$, it is clear that $\mathcal{A} = \mathcal{A}_{\ell_\infty}(\mathcal{S})$.

Remark 2.2. The affine sets arising from a Desarguesian spread with respect to any line are affine planes whose line at infinity is disjoint from the quadric $Q^+(3, q)$.

Remark 2.3. It should be noticed that the notion of affine set given above coincides with the notion of geometric spread set introduced by M. Law and T. Penttila in [7, p. 29].

3. Affine sets in PG$(4, q)$

Let $\mathcal{Q} = Q^+(3, q)$ be the hyperbolic quadric of a 3-dimensional projective space Ω embedded in $\Omega' = \mathsf{PG}(4, q)$, which is disjoint from the secant lines of two affine sets $\mathcal{A}$ and $\mathcal{A}'$ of Ω'. Let G denote the subgroup of $\mathsf{P\Gamma O}^+(4, q) = \mathrm{Aut}(\mathcal{Q})$ fixing the reguli of $\mathcal{Q}$. We give the following

Definition 3.1. The affine sets $\mathcal{A}$ and $\mathcal{A}'$ are said to be *equivalent* if there exists a collineation φ of $\mathsf{P\Gamma L}(5, q)$ fixing the hyperplane Ω, such that $\varphi_{|\Omega} \in G$ and $\varphi(\mathcal{A}) = \mathcal{A}'$.

Proposition 3.2. *Let $\mathcal{A} = \mathcal{A}_\ell(\mathcal{S})$ and $\mathcal{A}' = \mathcal{A}_\ell(\mathcal{S}')$ be two affine sets arising from two spreads $\mathcal{S}$ and $\mathcal{S}'$ sharing a common line ℓ. Then $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if and only if there exists a collineation Φ of $\mathsf{PG}(3,q)$, fixing the line ℓ and such that $\Phi(\mathcal{S}) = \mathcal{S}'$.*

Proof. Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent. Then, there exists a collineation φ of $\Omega' = \mathsf{PG}(4,q)$ such that $\varphi(\Omega) = \Omega$, $\varphi_{|\Omega} \in G$ and $\varphi(\mathcal{A}) = \mathcal{A}'$.

Recall that

$$\mathcal{A} = \left\{(1,a,b,c,d) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_\mathcal{S}\right\}$$

and

$$\mathcal{A}' = \left\{(1,a',b',c',d') \mid \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathcal{C}_{\mathcal{S}'}\right\}.$$

Since $\varphi(\Omega) = \Omega$, if σ is its companion automorphism, then φ sends the point $(0,z_1,z_2,z_3,z_4)$ of Ω to the point $(0,z_1',z_2',z_3',z_4')$ where

$$\begin{pmatrix} 0 \\ z_1' \\ z_2' \\ z_3' \\ z_4' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ d_1 & m_{11} & m_{12} & m_{13} & m_{14} \\ d_2 & m_{21} & m_{22} & m_{23} & m_{24} \\ d_3 & m_{31} & m_{32} & m_{33} & m_{34} \\ d_4 & m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} 0 \\ z_1^\sigma \\ z_2^\sigma \\ z_3^\sigma \\ z_4^\sigma \end{pmatrix}$$

with $d_i, m_{ij} \in \mathsf{GF}(q)$. Also, since $\varphi_{|\Omega} \in G$, we have

$$\begin{pmatrix} z_1' & z_2' \\ z_3' & z_4' \end{pmatrix} = A \begin{pmatrix} z_1^\sigma & z_2^\sigma \\ z_3^\sigma & z_4^\sigma \end{pmatrix} B$$

where $A = (a_{ij})$, $B = (b_{ij})$ are non-singular 2×2 matrices over $\mathsf{GF}(q)$ (see [6, p.28]), i.e.

$$M = (m_{ij}) = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21} \\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21} \\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{pmatrix}.$$

Hence φ sends the point $(1,z_1,z_2,z_3,z_4)$ of $\Omega' \setminus \Omega$ to the point $(1,z_1',z_2',z_3',z_4')$ such that

$$\begin{pmatrix} z_1' & z_2' \\ z_3' & z_4' \end{pmatrix} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} + A \begin{pmatrix} z_1^\sigma & z_2^\sigma \\ z_3^\sigma & z_4^\sigma \end{pmatrix} B. \tag{1}$$

Since $\varphi(\mathcal{A}) = \mathcal{A}'$, if $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$ we have that

$$\mathcal{C}_{\mathcal{S}'} = \{D + AM^\sigma B \mid M \in \mathcal{C}_\mathcal{S}\}.$$

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Now, if $(X_0, X_1, X_2, X_3) = (\underline{X}, \underline{Y})$ are the projective coordinates of $\mathsf{PG}(3, q)$ and $\underline{X}_t = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}$ denotes the transpose of $\underline{X}$, the collineation Φ defined by

$$\Phi\colon \begin{pmatrix} \underline{X}'_t \\ \underline{Y}'_t \end{pmatrix} = \begin{pmatrix} A_t^{-1} & 0 \\ D_t A_t^{-1} & B_t \end{pmatrix} \begin{pmatrix} \underline{X}^\sigma_t \\ \underline{Y}^\sigma_t \end{pmatrix} \tag{2}$$

fixes the line $\ell_\infty = \ell$ and sends the line

$$\ell_M = \{(\underline{X}, \underline{X}M) \mid \underline{X} \in \mathsf{GF}(q) \times \mathsf{GF}(q)\}$$

of $\mathcal{S}$ to the line $\ell_{D+AM^\sigma B}$ of $\mathcal{S}'$, i.e. $\Phi(\mathcal{S}) = \mathcal{S}'$. Indeed,

$$\begin{aligned} \Phi(\ell_M) &= \{(\underline{X}^\sigma A^{-1}, \underline{X}^\sigma(A^{-1}D + M^\sigma B)) \mid \underline{X} \in \mathsf{GF}(q) \times \mathsf{GF}(q)\} \\ &= \{(\underline{X}', \underline{X}'(D + AM^\sigma B)) \mid \underline{X}' \in \mathsf{GF}(q) \times \mathsf{GF}(q)\} \\ &= \ell_{D+AM^\sigma B}\,. \end{aligned}$$

Conversely, suppose that there exists a collineation Φ of $\mathsf{PG}(3, q)$ such that $\Phi(\ell) = \ell$ and $\Phi(\mathcal{S}) = \mathcal{S}'$ and let $\ell = \ell_\infty$. It is easy to see that a collineation fixing the line ℓ_∞ can be written as in (2); since $\Phi(\mathcal{S}) = \mathcal{S}'$, we have $\mathcal{C}_{\mathcal{S}'} = \{D + AM^\sigma B \mid M \in \mathcal{C}_\mathcal{S}\}$. Now, the matrices A, B, D and the field automorphism σ define in $\mathsf{PG}(4, q)$ a collineation φ given by (1) such that $\varphi(\mathcal{A}) = \mathcal{A}'$. Since $\varphi(\Omega) = \Omega$ and $\varphi_{|\Omega} \in G$, we are done. □

By the previous proposition equivalent affine sets produce isomorphic spreads. The converse is not true. Indeed, isomorphic spreads could produce non-equivalent affine sets. So it makes sense to ask how many non-equivalent affine sets arise from a given spread. We have the following proposition.

Proposition 3.3. *Let $\mathcal{S}$ be a spread of $\Sigma = \mathsf{PG}(3, q)$ and let H be the subgroup of $\mathsf{P\Gamma L}(4, q)$ leaving invariant the spread $\mathcal{S}$. The number of non-equivalent affine sets arising from $\mathcal{S}$ equals the number of H-orbits on the lines of $\mathcal{S}$.*

Proof. Let ℓ, m be two lines of the spread $\mathcal{S}$ of $\mathsf{PG}(3, q)$ and choose projective coordinates in $\mathsf{PG}(3, q)$ in such a way that $\ell = \ell_\infty : X_0 = X_1 = 0$ and $m = \ell_0 : X_2 = X_3 = 0$ and let $(X_0, X_1, X_2, X_3) = (\underline{X}, \underline{Y})$. Suppose that there exists a collineation Ψ such that $\Psi(\mathcal{S}) = \mathcal{S}$ and $\Psi(\ell_\infty) = \ell_0$. Then Ψ can be written as

$$\Psi\colon \begin{pmatrix} \underline{X}'_t \\ \underline{Y}'_t \end{pmatrix} = \begin{pmatrix} C_t A_t^{-1} & B_t \\ A_t^{-1} & 0 \end{pmatrix} \begin{pmatrix} \underline{X}^\sigma_t \\ \underline{Y}^\sigma_t \end{pmatrix},$$

where A, B, C are 2×2 matrices over $\mathsf{GF}(q)$, A and B non-singular and $\sigma \in \mathrm{Aut}(\mathsf{GF}(q))$. Here, Ψ sends the line

$$\ell_M = \{(\underline{X}, \underline{X}M) \mid \underline{X} = (X_0, X_1) \in \mathsf{GF}(q) \times \mathsf{GF}(q)\}$$

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to the line

$$\begin{aligned}\{(\underline{X}^\sigma(A^{-1}C+M^\sigma B),\underline{X}^\sigma A^{-1}) \mid \underline{X}\in \mathsf{GF}(q)\times\mathsf{GF}(q)\} \\ =\{(\underline{X}'(C+AM^\sigma B),\underline{X}') \mid \underline{X}'\in \mathsf{GF}(q)\times\mathsf{GF}(q)\} = \ell_{C+AM^\sigma B}\end{aligned}$$

(see Section 2 for notation). Since $\Psi(\mathcal{S})=\mathcal{S}$, we have that a spread set of $\mathcal{S}$ associated with ℓ_0 is given by $\mathcal{C}_{\mathcal{S}}(\ell_0)=\{C+AM^\sigma B \mid M\in\mathcal{C}_{\mathcal{S}}\}$.

As in the previous proof, the matrices $D=C$, A and B and the automorphism σ define in $\mathsf{PG}(4,q)$ a collineation φ of type (1) such that $\varphi(\mathcal{A}_{\ell_\infty}(\mathcal{S}))=\mathcal{A}_{\ell_0}(\mathcal{S})$.

Now, suppose that $\mathcal{A}_{\ell_\infty}(\mathcal{S})$ and $\mathcal{A}_{\ell_0}(\mathcal{S})$ are equivalent. Then, as in the previous proof, there exist three 2×2 matrices D, A, B, with A and B non-singular and $\sigma\in\mathrm{Aut}(\mathsf{GF}(q))$ such that

$$\mathcal{C}_{\mathcal{S}}(\ell_0)=\{D+AM^\sigma B \mid M\in\mathcal{C}_{\mathcal{S}}(\ell_\infty)\},$$

i.e. $\mathcal{S}$ consists of the lines

$$\mathcal{S}=\Big\{\{(\underline{X}(D+AM^\sigma B),\underline{X}) \mid \underline{X}\in\mathsf{GF}(q)\times\mathsf{GF}(q)\} \mid M\in\mathcal{C}_{\mathcal{S}}(\ell_\infty)\Big\}\cup\{\ell_0\}\,.$$

The collineation Ψ with equations

$$\begin{pmatrix}\underline{X}'_t\\ \underline{Y}'_t\end{pmatrix}=\begin{pmatrix}D_tA_t^{-1} & B_t\\ A_t^{-1} & 0\end{pmatrix}\begin{pmatrix}\underline{X}^\sigma_t\\ \underline{Y}^\sigma_t\end{pmatrix}$$

leaves $\mathcal{S}$ invariant and sends ℓ_∞ to ℓ_0. This concludes the proof. □

From the previous proposition we have that if $\mathcal{S}$ is a transitive spread (i.e. $\mathrm{Aut}(\mathcal{S})$ is transitive on $\mathcal{S}$), then there is a unique affine set arising from it, up to equivalence.

4. Affine sets of symplectic spreads

Let $\mathcal{S}$ be a symplectic spread of $\Sigma=\mathsf{PG}(3,q)$ and let ℓ be a line of $\mathcal{S}$. The corresponding affine set $\mathcal{A}_\ell(\mathcal{S})$ lies in a 3-dimensional projective subspace of $\mathsf{PG}(4,q)$. Indeed, since $\mathcal{S}$ is symplectic the corresponding ovoid $\mathcal{O}$ of the Klein quadric $Q^+(5,q)$ is contained in a parabolic quadric $Q(4,q)$ which lies in a hyperplane $\Lambda'=\mathsf{PG}(4,q)$. Hence if x is the point of $\mathcal{O}$ corresponding under the Plücker map to the line ℓ of $\mathcal{S}$, then by projecting $\mathcal{O}$ from the point x to a 4-dimensional projective subspace $\Lambda=\mathsf{PG}(4,q)$ as in Section 1, we get that $\mathcal{A}_\ell(\mathcal{S})$ is contained in the 3-dimensional projective subspace $\Gamma=\Lambda\cap\Lambda'$. Conversely, if $\mathcal{A}_\ell(\mathcal{S})$ is contained in a 3-dimensional subspace Γ of $\Lambda=\mathsf{PG}(4,q)$, the corresponding ovoid of $Q^+(5,q)$ constructed as in Section 1, is contained in a parabolic quadric $Q(4,q)$ and $\mathcal{S}$ is symplectic. So we have proven the following

Proposition 4.1. *A spread $\mathcal{S}$ is symplectic if and only if there exists a line ℓ of $\mathcal{S}$ such that $\mathcal{A}_\ell(\mathcal{S})$ is contained in a 3-dimensional projective space.* □

From the previous proposition it follows that an affine set $\mathcal{A} = \mathcal{A}_\ell(\mathcal{S})$ of a symplectic spread is a set of q^2 points of a 3-dimensional projective space Γ such that

(i') $\mathcal{A} \subset \Gamma \setminus \pi$, where π is a plane of Γ and

(ii') the secant lines of $\mathcal{A}$ are disjoint from a given non-degenerate conic $\mathcal{C}$ of π.

Hence the set $D(\mathcal{A}_\ell(\mathcal{S}))$ consisting of the intersection points of the secant lines of $\mathcal{A}_\ell(\mathcal{S})$ with the plane $\pi = \Gamma \cap \Omega$ is disjoint from the conic $\mathcal{C}$, so $|D(\mathcal{A}_\ell(\mathcal{S}))| \leq q^2$. If q is even this upper bound becomes $q^2 - 1$, as we prove in the following proposition.

Proposition 4.2. *Let q be even and let N be the nucleus of the conic $\mathcal{C}$. Then $N \notin D(\mathcal{A}_\ell(\mathcal{S}))$ and hence $|D(\mathcal{A}_\ell(\mathcal{S}))| \leq q^2 - 1$.*

Proof. Let $\mathcal{A} = \mathcal{A}_\ell(\mathcal{S})$ be a symplectic affine set. This means that $\mathcal{A}$ is a set of q^2 points of a 3-dimensional projective space $\Gamma = \mathrm{PG}(3,q)$, q even, such that

(1) $\mathcal{A} \subset \Gamma \setminus \pi$, where π is a plane of Γ;

(2) $D(\mathcal{A}) \cap \mathcal{C} = \emptyset$, where $\mathcal{C}$ is a non-degenerate conic of π.

Let N be the nucleus of $\mathcal{C}$ and embed Γ in a 4-dimensional projective space Λ in such a way that $\mathcal{C}$ is contained in a parabolic quadric $\mathcal{Q}(4,q)$ of Γ having nucleus N. Let x be a point of $Q(4,q)$ such that $\pi \subset x^\perp$ (where $\perp$ is the orthogonal polarity induced by $Q(4,q)$) and let $\mathbb{O}$ be the ovoid of $Q(4,q)$ containing x such that $\mathcal{A}_x(\mathbb{O}) = \mathcal{A}$ (see Introduction). By way of contradiction, suppose that $N \in D(\mathcal{A})$ and let y and z be two points of $\mathcal{A}$ such that $N \in \langle y, z\rangle$. Also, let y_1 and z_1 be the two points of $\mathbb{O}$ projected by x onto y and z, respectively. Let $\mathbb{P}' \cong \mathrm{PG}(3,q)$ be a 3-dimensional projective space of Λ not on N. By projecting the ovoid $\mathbb{O}$ from N onto $\mathbb{P}'$ we get an ovoid of $\mathbb{P}'$ [5, Ch. 7] containing the three collinear points which are the projection of x, y_1 and z_1 from the nucleus N; a contradiction. □

So far, the known symplectic spreads of $\mathrm{PG}(3,q)$ which are not Desarguesian are the Kantor semifield spreads for $q = p^h$ (p odd prime and $h > 1$), the Payne-Thas semifield spreads for $q = 3^h$ ($h > 2$), the sporadic semifield spread of Penttila-Williams when $q = 3^5$, the Ree-Tits spreads for $q = 3^{2h+1}$ ($h > 0$) and the Lüneburg spread for $q = 2^{2h+1}$ ($h > 0$).

If $\mathcal{S}$ is a symplectic semifield spread with respect to the line $\ell_\infty \in \mathcal{S}$, then the corresponding affine set $\mathcal{A}_{\ell_\infty}(\mathcal{S})$ can be written as

$$\mathcal{A}_{\ell_\infty}(\mathcal{S}) = \left\{(1,u,v,v,f(u,v)) \mid \begin{pmatrix} u & v \\ v & f(u,v)\end{pmatrix} \in \mathcal{C}_\mathcal{S}\right\},$$

where $f\colon \mathsf{GF}(q) \times \mathsf{GF}(q) \to \mathsf{GF}(q)$ is an additive map such that $f(0,0) = 0$. Hence $D(\mathcal{A}_{\ell_\infty}(\mathcal{S}))$ is induced by the set of non-zero vectors

$$\{(0,t,s,s,f(t,s)) \mid (t,s) \in \mathsf{GF}(q) \times \mathsf{GF}(q),\ (t,s) \neq (0,0)\}.$$

So, in this case if $\mathsf{GF}(q')$ is the maximal subfield of $\mathsf{GF}(q)$ with respect to which $f(t,s)$ is linear (i.e. $f(\lambda t, \lambda s) = \lambda f(t,s)$ for each $\lambda \in \mathsf{GF}(q')$ and $t,s \in \mathsf{GF}(q)$), and $q = q'^n$, then $D(\mathcal{A}_{\ell_\infty}(\mathcal{S}))$ is a so-called $\mathsf{GF}(q')$-linear set of rank $2n$ and hence $|D(\mathcal{A}_{\ell_\infty}(\mathcal{S}))| \leq \frac{q'^{2n}-1}{q'-1} = \frac{q^2-1}{q'-1}$ [2, Eq. (5)]. Hence if $q' > 2$, then the number of directions determined by $\mathcal{A}_{\ell_\infty}(\mathcal{S})$ is less than q^2-1. On the other hand, if q is even, the only symplectic semifield spread of $\mathsf{PG}(3,q)$ is the Desarguesian one [3, Thm.1], say $\mathcal{F}$, and in such a case the number of directions of $\mathcal{A}_\ell(\mathcal{F})$ is $q+1$, for any $\ell \in \mathcal{S}$.

The only symplectic spreads of $\mathsf{PG}(3,q)$ which are not semifield spreads are the Ree-Tits spread and the Lüneburg spread.

The stabilizer of the Ree-Tits spread $\mathcal{S}_{RT}$ of $\mathsf{PG}(3,q)$, $q = 3^{2h+1}$ $(h > 0)$ has three orbits on $\mathcal{S}_{RT}$ of length 1, q and $q(q-1)$, respectively (see, for example, [9, §2.2]). If $\ell \in \mathcal{S}_{RT}$ is the line fixed by the stabilizer of $\mathcal{S}_{RT}$, then when $h = 1$ or 2, computational results show that $D(\mathcal{A}_{\ell_\infty}(\mathcal{S}_{RT})) = q(q-1) + \frac{q-1}{2} < q^2-1$.

In the next section we will prove that the affine sets arising from the Lüneburg spread attain the upper bound of Proposition 4.2.

5. The affine set of the Lüneburg spread

The Lüneburg spread $\mathcal{S}$ is a transitive sympelctic spread, i.e. there exists a collineation group which fixes $\mathcal{S}$ and acts transitively on the elements of $\mathcal{S}$. So there exists, up to equivalence, a unique affine set arising from $\mathcal{S}$. In this section we study this affine set. Let X_0, X_1, X_2, X_3 be homogeneous projective coordinates in $\Sigma = \mathsf{PG}(3,q)$, $q = 2^{2e+1}$, $e \geq 1$ and let $\sigma\colon x \in \mathsf{GF}(q) \mapsto x^{2^{e+1}} \in \mathsf{GF}(q)$. Then $\mathcal{S} = \{\ell_{uv} \mid u,v \in \mathsf{GF}(q)\} \cup \{\ell_\infty\}$, where

$$\ell_{uv} = \left\{(a,b,c,d) \mid (c,d) = (a,b)\begin{pmatrix} u^\sigma & u + v^{\sigma+1} \\ u + v^{\sigma+1} & v^\sigma \end{pmatrix}, a,b \in \mathsf{GF}(q)\right\}$$

and $\ell_\infty = \{(0,0,c,d) \mid c,d \in \mathsf{GF}(q)\}$, is the Lüneburg spread of $\mathsf{PG}(3,q)$. Hence

$$\mathcal{A}_{\ell_\infty}(\mathcal{S}) = \{(1, u^\sigma, u + v^{\sigma+1}, u + v^{\sigma+1}, v^\sigma) \mid u,v \in \mathsf{GF}(q)\}$$

and it lies in the 3-dimensional projective space Γ with equation $X_2 = X_3$. Denoting by Z_0, Z_1, Z_2, Z_3 the homogeneous projective coordinates in Γ, the set $D(\mathcal{A}_{\ell_\infty}(\mathcal{S}))$ is contained in the plane $\pi : Z_0 = 0$ and it is disjoint from the conic

$$\mathcal{C} : \begin{cases} Z_0 = 0 \\ Z_1 Z_3 - Z_2^2 = 0 \,. \end{cases}$$

Let $N = (0,0,1,0)$ be the nucleus of the conic $\mathcal{C}$ and let Q be the point of $\mathcal{C}$ with coordinates $(0,1,0,0)$. Let $\alpha_a : Z_3 = aZ_0$, where $a \in \mathsf{GF}(q)$, be the generic plane of Γ passing through the line $\langle N, Q\rangle$ and different from π. Then the set $\{\alpha_a\}_{a\in\mathsf{GF}(q)}$ partitions the affine set $\mathcal{A}_{\ell_\infty}(\mathcal{S})$ and the intersection between α_a and $\mathcal{A}_{\ell_\infty}(\mathcal{S})$ is the set

$$\mathcal{O}_a = \{(1, u^\sigma, u + a^{2^e+1}, a) \mid u \in \mathsf{GF}(q)\} \,.$$

Note that $\mathcal{O}_0$ is a q-arc that can be completed to a translation hyperoval $\mathcal{H}_0 = \mathcal{O}_0 \cup \{Q, N\}$ (see [5, Ch.4, §3]). Also, the collineation $\varphi_a\colon (Z_0, Z_1, Z_2, Z_3) \mapsto (Z_0, Z_1, Z_2 + a^{2^e+1} Z_0, Z_3 + aZ_0)$ maps $\mathcal{H}_0$ into $\mathcal{H}_a = \mathcal{O}_a \cup \{Q, N\}$. Hence we have proved the following proposition.

Proposition 5.1. *The affine set arising from a Lüneburg spread is the union of q q-arcs, and each of them can be completed to a translation hyperoval.* □

Proposition 5.2. *Let $\mathcal{S}$ be the Lüneburg spread of $\Sigma = \mathsf{PG}(3,q)$ and let $\mathcal{A}(\mathcal{S})$ be the affine set arising from $\mathcal{S}$. Then $|D(\mathcal{A}(\mathcal{S}))| = q^2 - 1$.*

Proof. The plane $\pi_m : Z_1 = mZ_3$, with $m \in \mathsf{GF}(q)$, intersects the plane π in the tangent line t_m to the conic $\mathcal{C}$ at the point $P_m = (0, m, \sqrt{m}, 1)$ and the plane $\pi_\infty : Z_3 = 0$ intersects the plane π in the tangent line t_∞ to the conic $\mathcal{C}$ at the point $P_\infty = (0,1,0,0)$. Let $I_m = \pi_m \cap \mathcal{A}(\mathcal{S})$ for any $m \in J = \mathsf{GF}(q) \cup \{\infty\}$. Since $\{\pi_j \mid j \in J\}$ is the pencil of planes through the line $Z_3 = Z_1 = 0$, we have $\mathcal{A}(\mathcal{S}) = \bigcup_{m\in J} I_m$. Hence in order to prove the result, it is sufficient to show that $D(I_m) = t_m \setminus \{P_m, N\}$ for each $m \in J$. Recall that $D(\mathcal{A}(\mathcal{S})) \cap (\mathcal{C} \cup \{N\}) = \emptyset$. If $m \in \mathsf{GF}(q)$, we have

$$I_m = \{(1, ma, a^{2^e+1} + m^{2^e} a^{2^e}, a) \mid a \in \mathsf{GF}(q)\} \,.$$

Since $D(I_m)$ is induced by the set of non-zero vectors

$$\left\{(0, m(a+a'), a^{2^e+1} + a'^{2^e+1} + m^{2^e}(a+a')^{2^e}, a+a') \mid a, a' \in \mathsf{GF}(q), a \neq a'\right\},$$

we can write

$$D(I_m)=\left\{\left(0,m,\frac{a^{2^e+1}+a'^{2^e+1}}{a+a'}+m^{2^e}(a+a')^{2^e-1},1\right)\mid a,a'\in \mathsf{GF}(q),a\neq a'\right\},$$

and putting $t=a+a'$ (hence $t\in\mathsf{GF}(q)^*$) we get

$$\begin{aligned}\frac{a^{2^e+1}+a'^{2^e+1}}{a+a'}+m^{2^e}(a+a')^{2^e-1}&=\frac{a^{2^e+1}+(t+a)^{2^e+1}}{t}+m^{2^e}t^{2^e-1}\\&=t^{2^e}+at^{2^e-1}+a^{2^e}+m^{2^e}t^{2^e-1}, \qquad (3)\end{aligned}$$

i.e. $D(I_m)=\{(0,m,t^{2^e}+at^{2^e-1}+a^{2^e}+m^{2^e}t^{2^e-1},1)\mid a,t\in\mathsf{GF}(q),t\neq 0\}$. If $a=m^{2^e}$, then (3) becomes $t^{2^e}+m^{2^{2e}}=t^{2^e}+\sqrt{m}$, hence

$$D(I_m)=\{(0,m,b,1)\mid b\in\mathsf{GF}(q),b\neq\sqrt{m}\}=t_m\setminus\{P_m,N\},$$

i.e. $|D(I_m)|=|t_m|-2=q-1$. Finally, $I_\infty=\{(1,u^\sigma,u,0)\mid u\in\mathsf{GF}(q)\}$ and hence $D(I_\infty)=\{(0,a^{\sigma-1},1,0)\mid a\in\mathsf{GF}(q),a\neq 0\}=t_\infty\setminus\{P_\infty,N\}$, i.e. $|D(I_\infty)|=q-1$. □

Remark 5.3. We end the paper with a question that could be of some interest. As we have seen, the affine sets arising from the Lüneburg spreads attain the upper bound of Proposition 4.2. Does this property characterize Lüneburg spreads?

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