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On the intersection of Hermitian surfaces

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Abstract

In [6] and [3] the authors determine the structure of the intersection of two Hermitian surfaces of $\mathsf{PG}(3, q^2)$ under the hypotheses that in the pencil they generate there is at least one degenerate surface. In [1] and [3] it is shown that under suitable hypotheses the intersection of two Hermitian surfaces generating a non-degenerate pencil is a pseudo-regulus. Here we completely determine all possible intersection configurations for two Hermitian surfaces of $\mathsf{PG}(3, q^2)$ generating a non-degenerate pencil.

Keywords: Hermitian curve, Hermitian surface

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1 Introduction

A *Hermitian variety* $\mathcal{H}$ in $\mathsf{PG}(n, q^2)$ is the set of absolute points for some Hermitian form defined on the underlying vector space. The variety $\mathcal{H}$ is called *degenerate* if the corresponding Hermitian form is degenerate; else, it is called *non-degenerate*. If $n = 2$, $\mathcal{H}$ is called a *Hermitian curve*, while if $n = 3$, $\mathcal{H}$ is called a *Hermitian surface*. A point P on $\mathcal{H}$ is called *singular* if any line through P either intersects $\mathcal{H}$ only in P or is contained in $\mathcal{H}$. The *vertex* of $\mathcal{H}$ is the set of all singular points of $\mathcal{H}$, and it is denoted by $V(\mathcal{H})$. It is clear that $V(\mathcal{H})$ is a projective subspace of $\mathsf{PG}(n, q^2)$, and the *rank* of $\mathcal{H}$ is the number $r(\mathcal{H}) = n - \dim(V(\mathcal{H}))$.

Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be two distinct Hermitian varieties of $\mathsf{PG}(n, q^2)$ with homogeneous equations $f_0 = 0$ and $f_1 = 0$, respectively. Then the *Hermitian pencil*

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$\mathcal{F}$ defined by $\mathcal{H}_0$ and $\mathcal{H}_1$ is the set of all Hermitian varieties with equations $a_0 f_0 + a_1 f_1 = 0$, as a_0 and a_1 vary over the subfield $\mathsf{GF}(q)$, not both zero. Note that there are $q+1$ distinct Hermitian varieties in the pencil $\mathcal{F}$, some of which may be degenerate. The set $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ is called the *base* of $\mathcal{F}$, and any two distinct varieties in $\mathcal{F}$ intersect precisely in $\mathcal{B}$.

We now restrict to the case $n = 3$, and let $\mathcal{H}_0$ and $\mathcal{H}_1$ denote two distinct Hermitian surfaces in $\mathsf{PG}(3, q^2)$ with associated polarities u_0 and u_1, respectively. If the pencil $\mathcal{F}$ generated by $\mathcal{H}_0$ and $\mathcal{H}_1$ contains at least one degenerate Hermitian surface, then the structure of the base $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ is completely known (see [6] and [3]).

Thus we are interested in the situation when all Hermitian surfaces in the pencil $\mathcal{F}$ are non-degenerate. In this case, since the Hermitian surfaces of a pencil cover all the points of $\mathsf{PG}(3, q^2)$, straightforward counting shows that $|\mathcal{B}| = (q^2+1)^2$. Conversely, if $q \geq 4$ and $|\mathcal{B}| = (q^2+1)^2$, then the pencil $\mathcal{F}$ generated by $\mathcal{H}_0$ and $\mathcal{H}_1$ necessarily contains only non-degenerate Hermitian surfaces. Indeed, going through the list in [6] or [3] of all possibilities for the cardinality of $\mathcal{B}$, when at least one of the surfaces in $\mathcal{F}$ is degenerate, we see that $|\mathcal{B}| = (q^2+1)^2$ only occurs for $q = 3$ and $|\mathcal{B}| = q^4 + q^3 - q^2 + 1 = 100 = (q^2+1)^2$ or $q = 2$ and $|\mathcal{B}| = q^3 + q^3 + 1 = 25 = (q^2+1)^2$. We thus have the following result.

Proposition 1.1. *For $q \geq 4$ the Hermitian pencil $\mathcal{F}$ contains only non-degenerate surfaces if and only if the base $\mathcal{B}$ has size $(q^2+1)^2$.*

In the sections that follow we prove the following result.

Theorem 1.2. *Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be two non-degenerate Hermitian surfaces in $\mathsf{PG}(3, q^2)$, and let $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ be the base of the Hermitian pencil $\mathcal{F}$ they generate. If $\mathcal{F}$ contains only non-degenerate surfaces, then one of the following holds:*

- *$\mathcal{B}$ contains exactly two skew lines and $q^4 - 1$ other points;*
- *$\mathcal{B}$ contains exactly two skew lines L and M, a third line N intersecting both L and M, and $q^4 - q^2$ other points;*
- *$\mathcal{B}$ contains exactly four lines forming a quadrangle and $q^4 - 2q^2 + 1$ other points;*
- *$\mathcal{B}$ is ruled by a pseudo-regulus.*

Moreover, all such cases occur.

2 Preliminary results

In [10] B. Segre defines two Hermitian surfaces in $\mathsf{PG}(3, q^2)$ to be *permutable* if and only if their associated polarities commute, and then he proves the following result.

Theorem 2.1 ([10]). *Let q be odd, and $\mathcal{H}_0$ and $\mathcal{H}_1$ be two permutable Hermitian surfaces in $\mathsf{PG}(3, q^2)$ with associated polarities u_0 and u_1, respectively. Then u_0u_1 is a projectivity with two skew lines, say L and M, of fixed points. That is, u_0u_1 is a biaxial harmonic involutorial collineation with fundamental lines L and M.*

The fundamental lines associated with two permutable Hermitian surfaces may or may not be lines lying on those surfaces. The lines completely contained in a Hermitian surface are called the *generators* of the surface, and a set of k mutually skew generators of a Hermitian surface $\mathcal{H}$ is called a *k-span* of $\mathcal{H}$. A k-span of $\mathcal{H}$ is called *$\mathcal{H}$-complete* if it is not contained in a $(k+1)$-span of $\mathcal{H}$. In [4] the following is proved.

Proposition 2.2 ([4]). *The q^2+1 generators meeting two skew generators of $\mathcal{H}$ form an $\mathcal{H}$-complete span. This $\mathcal{H}$-span has no further transversals.*

In general, any set of q^2+1 mutually skew lines in $\mathsf{PG}(3, q^2)$ with exactly two transversals is called a *pseudo-regulus*. This notion was introduced by J. Freeman in [5], where he proved that any pseudo-regulus can be extended to a spread of $\mathsf{PG}(3, q^2)$. The set of $(q^2+1)^2$ points covered by a pseudo-regulus is called a *hyperbolic $\mathcal{Q}_F$-set* in [2]. In this paper we see that this set of points naturally arises as one of the possible intersections $\mathcal{B}$ for $\mathcal{H}_0$ and $\mathcal{H}_1$. The following result is proved in [1].

Theorem 2.3 ([1]). *Let q be an odd prime power, and let $\mathcal{H}_0$ and $\mathcal{H}_1$ be permutable Hermitian surfaces in $\mathsf{PG}(3, q^2)$. If the fundamental lines L and M are contained in $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$, then $\mathcal{B}$ is a ruled determinantal variety consisting of the points on a pseudo-regulus. In particular, this pseudo-regulus is a complete (q^2+1)-span of both $\mathcal{H}_0$ and $\mathcal{H}_1$.*

The hypotheses in the previous theorem are weakened in [3], where it is shown that the point set of a pseudo-regulus can be obtained as the intersection of two Hermitian surfaces in the even characteristic case as well. In particular, the following result is proved.

Theorem 2.4 ([3]). *Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be two distinct Hermitian surfaces in $\mathsf{PG}(3, q^2)$ with associated polarities u_0 and u_1, respectively. Suppose that L and M are two skew lines contained in $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$. Then $\mathcal{B}$ is a hyperbolic $\mathcal{Q}_F$-set with transversals L and M if and only if u_0 and u_1 agree on the points of $L \cup M$.*

An investigation of other possible intersection configurations for $\mathcal{H}_0$ and $\mathcal{H}_1$ is started in [3], where the following three results appear.

Proposition 2.5. *If $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ contains a line L, then every plane π through L intersects $\mathcal{B}$ in one of the following configurations:*

- *the points of the line L;*
- *the points on a pair of distinct lines L and M;*
- *the points lying on a Baer subpencil of lines containing L;*
- *the points on a degenerate $\mathcal{C}_F$-set (that is, the union of L and an affine Baer subplane).*

Proposition 2.6. *If $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ contains a line L such that u_0 and u_1 agree on the points of L, then $\mathcal{B}$ is the union of q^2+1 pairwise skew lines, all intersecting L. If, in addition, $\mathcal{B}$ contains a line M skew with L, then necessarily $\mathcal{B}$ is a hyperbolic $\mathcal{Q}_F$-set.*

Proposition 2.7. *If $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ contains a line L such that u_0 and u_1 agree on the points of L and also on a point of $\mathcal{B} \setminus L$, then $\mathcal{B}$ is a hyperbolic $\mathcal{Q}_F$-set.*

In order to conduct our investigation of the possible configuration patterns for the base of a Hermitian pencil in $\mathsf{PG}(3, q^2)$ containing only non-degenerate Hermitian surfaces, it is useful to first recall the possible intersection configurations of two Hermitian curves in $\mathsf{PG}(2, q^2)$. This will be done in the next section.

3 The intersection of Hermitian curves

In this section $\mathcal{F}$ will be a Hermitian pencil of curves in $\mathsf{PG}(2, q^2)$. We refer to [8], [9] and [3] for the results catalogued here. A straightforward counting argument shows that if all the Hermitian curves in $\mathcal{F}$ are non-degenerate, then the base has size $q^2 - q + 1$. In fact, such a base is a complete arc, often called a *Kestenband* arc. At the other end of the spectrum, the following result lists the possibilities for the base when all the Hermitian curves in $\mathcal{F}$ are degenerate of rank 2.

Proposition 3.1. *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two rank 2 degenerate Hermitian curves in* $\mathsf{PG}(2, q^2)$. *If the pencil they generate contains only rank 2 degenerate Hermitian curves, then the base $\mathcal{B} = \mathcal{C}_0 \cap \mathcal{C}_1$ of $\mathcal{F}$ is one of the following: a line, a pair of distinct lines, or a degenerate $\mathcal{C}_F$-set (the union of a line and an affine Baer subplane).*

The general situation, when at least one of the Hermitian curves in $\mathcal{F}$ is non-degenerate, is described below in Proposition 3.2. We use the notation from [3]. A rank 1 Hermitian curve (that is, a line of $\mathsf{PG}(2, q^2)$ repeated $q + 1$ times) can meet a non-degenerate Hermitian curve in either one point or a Baer subline, thus accounting for the first two intersections mentioned in Proposition 3.2. As previously stated, if all Hermitian curves in $\mathcal{F}$ are non-degenerate, the base $\mathcal{C}_0 \cap \mathcal{C}_1$ is a Kestenband arc. Thus the remaining possibilities can be obtained by intersecting a non-degenerate Hermitian curve $\mathcal{C}$ with a Baer subpencil $\mathcal{P}$ of lines with center P (Hermitian curve of rank 2). There are four such configurations:

- Suppose $P \notin \mathcal{C}$ and exactly two of the lines in $\mathcal{P}$ are tangent to $\mathcal{C}$. Then $\mathcal{P} \cap \mathcal{C}$ contains the points on $q-1$ Baer sublines (through P) and two other points (one on each of the tangent lines to $\mathcal{C}$ in $\mathcal{P}$). Thus $\mathcal{B} = \mathcal{C} \cap \mathcal{P}$ has size $q^2 + 1$, and this base is called a $\mathcal{C}_F$*-set* in $\mathsf{PG}(2, q^2)$.
- Suppose $P \in \mathcal{C}$ and exaclty one of the lines in $\mathcal{P}$ is tangent to $\mathcal{C}$ (at P). Then $\mathcal{P} \cap \mathcal{C}$ contains the points on q Baer sublines (through P) and thus again has size $q^2 + 1$. This base is called a Γ*-set* in $\mathsf{PG}(2, q^2)$.
- Suppose $P \notin \mathcal{C}$ and exactly one of the lines in $\mathcal{P}$ is tangent to $\mathcal{C}$, say at the point Q. Then $\mathcal{P} \cap \mathcal{C}$ contains the points on q Baer sublines (through P) and the point Q. This base has size $q^2 + q + 1$, and is called a K*-set* in $\mathsf{PG}(2, q^2)$. It should be noted that this base can also be obtained as the intersection of $\mathcal{C}$ with a Baer subpencil through Q (and no tangent lines to $\mathcal{C}$), and thus there are $2q + 1$ Baer sublines contained in this base.
- Suppose $P \notin \mathcal{C}$ and none of the lines in $\mathcal{P}$ is tangent to $\mathcal{C}$. Then $\mathcal{P} \cap \mathcal{C}$ contains the points on $q + 1$ Baer sublines (through P) and thus has size $(q+1)^2$. This base is called an H*-set* in $\mathsf{PG}(2, q^2)$. It turns out that this base is partitioned by three different Baer subpencils through three distinct points, none of which are in the base, and thus there are $3(q + 1)$ Baer sublines contained in this base.

Diagrams for these last four intersections can be found on page 112 of [8], where Figures $1, 2, 3$, and 4 correspond to H-sets, K-sets, $\mathcal{C}_F$-sets, and Γ-sets, respectively.

The following result summarizes the above discussion.

Proposition 3.2. *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two Hermitian curves in $\mathsf{PG}(2, q^2)$. If the Hermitian pencil $\mathcal{F}$ they generate contains at least one non-degenerate Hermitian curve, then $\mathcal{C}_0 \cap \mathcal{C}_1$ is one of the following: a point, a Baer subline, a (complete) Kestenband $(q^2 - q + 1)$-arc, a $\mathcal{C}_F$-set, a Γ-set, a K-set, or an H-set.*

4 The intersection of Hermitian surfaces

We now return to our fundamental problem of determining the possible intersection patterns for the base $\mathcal{B}$ of a Hermitian pencil $\mathcal{F}$ in $\mathsf{PG}(3, q^2)$ containing all non-degenerate Hermitian surfaces. As stated previously, this is the only remaining open case. We will denote by $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_q$ the $q+1$ Hermitian surfaces of the pencil $\mathcal{F}$ generated by $\mathcal{H}_0$ and $\mathcal{H}_1$, and by $u_0, u_1, \ldots, u_q$ their associated polarities. Recall that in this case the base $\mathcal{B}$ necessarily has size $(q^2+1)^2$.

We first consider the situation where $\mathcal{B}$ contains a line L, and refine the possibilities listed in Proposition 2.5.

Proposition 4.1. *Assume the Hermitian pencil $\mathcal{F}$ contains all non-degenerate Hermitian surfaces, and that the base $\mathcal{B}$ contains a line L. Then any plane π through L intersects $\mathcal{B}$ nontrivially, and $\mathcal{B} \cap \pi$ is one of the following:*

- *the set of points on a pair of distinct (intersecting) lines L and M;*
- *the set of points on a degenerate $\mathcal{C}_F$-set (the union of L and an affine Baer subplane).*

Proof. Suppose π is a plane through L that meets $\mathcal{B}$ in the points on a Baer subpencil $\mathcal{P}$ of lines, one of which is necessarily L, and let P be the center of this pencil. Then $|\pi \cap (\mathcal{B} \setminus L)| = q^3$ and $|\mathcal{B} \setminus L| = (q^2+1)q^2$. Since every plane through L intersects $\mathcal{B} \setminus L$ in $0, q^2$ or q^3 points by Proposition 2.5, the existence of π implies the existence of another plane π' through L intersecting $\mathcal{B}$ only in L.

We now work with the plane π'. Since $\pi' \cap \mathcal{B} = L$, there necessarily exists a point Q on $L \setminus \{P\}$ such that $\pi' = Q^{u_i}$ for $i = 0, 1, 2, \ldots, q$ (otherwise, π' cannot intersect $\mathcal{B}$ only in L). Hence π' contains $q+1$ Baer subpencils of lines through Q that share only the line L. This yields $q(q+1)+1$ lines through Q in π', a contradiction. The result now follows from Proposition 2.5. $\square$

Remark 4.2. Note that if a plane π intersects $\mathcal{B}$ in two lines, say L and M, then $\pi = P^{u_i}$ for every i, where $P = L \cap M$. Conversely, if there exists a point P on L such that two of the polarities agree on P, then all of the polarities agree on P and $\pi = P^{u_0}$ meets $\mathcal{B}$ in L and another line, say M, passing through P.

Proposition 4.3. *If $\mathcal{B}$ contains a line L, then one of the following cases occurs:*

- *all planes through L intersect $\mathcal{B}$ in a degenerate $\mathcal{C}_F$-set (the union of L and an affine Baer subplane);*
- *there exists a unique plane through L intersecting $\mathcal{B}$ in L and another line;*

- *there exist exactly two planes through L intersecting $\mathcal{B}$ in L and another line;*
- *all planes through L intersect $\mathcal{B}$ in L and another line.*

Proof. By Proposition 4.1 every plane through L intersects $\mathcal{B}$ either in L and another line, or in L and an affine Baer subplane. If there is a point Q on L on which the $q+1$ polarities u_i agree, then the plane Q^{u_0} ($= Q^{u_i}$ for all i) intersects $\mathcal{B}$ in L and another line, and conversely. Hence the multiset of intersection configurations of $\mathcal{B}$ and the various planes through L depends on the number of points of L on which the $q+1$ polarities u_i agree. But this is the same as the number of fixed points of the projectivity $u_0u_1^{-1}$ on the line L by Remark 4.2. From the fundamental theorem of projective geometry, this number must be one of the following: $0, 1, 2$ or q^2+1 (see [12], for instance). Thus the four configurations in the statement of the proposition are the only possibilities. □

If $\mathcal{B}$ contains two intersecting lines, then more can be said.

Proposition 4.4. *If $\mathcal{B}$ contains two intersecting lines, say L and M, then one of the following occurs:*

(1) *$\mathcal{B}$ contains exactly three lines, and the third line meets either L or M (but not both);*

(2) *$\mathcal{B}$ contains exactly four lines which form a quadrilateral;*

(3) *$\mathcal{B}$ contains exactly q^2+3 lines, forming a pseudo-regulus and its two transversals.*

Proof. The stabilizer of a Hermitian surface in $\mathsf{PG}(3, q^2)$ acts transitively on pairs of intersecting generators (see [11], for instance). Thus, after choosing our favorite Hermitian surface, say $\mathcal{H}_0 : X_0X_3^q + X_1X_2^q + X_2X_1^q + X_3X_0^q = 0$, we may also choose our favorite pair of intersecting lines in $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$, say $L : X_0 = X_1 = 0$ and $M : X_1 = X_3 = 0$. If H_1 is the Hermitian matrix giving the equation of $\mathcal{H}_1$, then H_1 is of the following form:

$$H_1 = \begin{pmatrix} a & \alpha & \beta & \gamma \\ \alpha^q & b & \delta & \epsilon \\ \beta^q & \delta^q & c & \nu \\ \gamma^q & \epsilon^q & \nu^q & d \end{pmatrix},$$

where the diagonal elements a, b, c, d are in $\mathsf{GF}(q)$ and the other elements in $\mathsf{GF}(q^2)$.

Since $\mathcal{B}$ contains the points $P : (0, 0, 1, 0)$, $Q : (0, 0, 0, 1)$ and $R : (1, 0, 0, 0)$, we necessarily have $a = c = d = 0$. Moreover, the tangent plane to $\mathcal{H}_0$ at the point $P = L \cap M$ is $P^{u_0} : X_1 = 0$, and this plane also must be the tangent plane to $\mathcal{H}_1$ at P. This further implies that $\beta = \nu = 0$, and the matrix H_1 must have the following restricted form:

$$H_1 = \begin{pmatrix} 0 & \alpha & 0 & \gamma \\ \alpha^q & b & \delta & \epsilon \\ 0 & \delta^q & 0 & 0 \\ \gamma^q & \epsilon^q & 0 & 0 \end{pmatrix}.$$

Note that $\gamma = 0$ implies $\det(H_1) = 0$, contradicting our fundamental assumption that the pencil $\mathcal{F}$ contains only non-degenerate Hermitian surfaces. So, $\gamma \neq 0$. Next observe that the polarities u_1 and u_0 agree on the point $Q : (0,0,0,1)$ of the line L if and only if $\epsilon = 0$. Similarly, these polarities agree on the point $X : (0,0,1,\zeta)$ of the line L, where $\zeta \neq 0$, if and only if $\epsilon\zeta^q = \gamma - \delta$. Thus we have the following possibilities:

(i) If $\epsilon \neq 0$ and $\gamma \neq \delta$, then for any choice of γ, ϵ and δ there is a unique point X of $L \setminus \{P\}$ on which the polarities u_0 and u_1 agree;

(ii) If $\epsilon \neq 0$ and $\gamma = \delta$, then there is no point of $L \setminus \{P\}$ on which the polarities agree;

(iii) If $\epsilon = 0$ and $\gamma \neq \delta$, then Q is the unique point of $L \setminus \{P\}$ on which the polarities agree;

(iv) If $\epsilon = 0$ and $\gamma = \delta$, then the polarities agree on all the points of L.

Similarly, the polarities u_1 and u_0 agree on the point $R : (1,0,0,0)$ of M if and only if $\alpha = 0$, and they agree on the point $Y : (1,0,\zeta,0)$, where $\zeta \neq 0$, if and only if $\zeta^q(\gamma^q - \delta) = \alpha^q$. Thus we again have four possibilities:

(i)' If $\alpha \neq 0$ and $\delta \neq \gamma^q$, then there is a unique point Y of $M \setminus \{P\}$ on which the polarities u_0 and u_1 agree;

(ii)' If $\alpha \neq 0$ and $\delta = \gamma^q$, then there is no point of $M \setminus \{P\}$ on which the polarities agree;

(iii)' If $\alpha = 0$ and $\delta \neq \gamma^q$, then R is the unique point of $M \setminus \{P\}$ on which the polarities agree;

(iv)' If $\alpha = 0$ and $\delta = \gamma^q$, then the polarities agree on all the points of M.

Hence on $L \setminus \{P\}$ the polarities u_0 and u_1 agree on no point, one point, or all points. A similar statement holds for $M \setminus \{P\}$. If the polarities agree on exactly one point of $(L \cup M) \setminus \{P\}$, then Proposition 4.1 (and the ensuing remark) implies that $\mathcal{B}$ contains exactly three lines as described in statement (1).

If the polarities agree on exactly one point of $L \setminus \{P\}$ and agree on exactly one point of $M \setminus \{P\}$, then Proposition 4.1 implies that $\mathcal{B}$ contains exactly four lines. In particular, $\mathcal{B}$ contains a line M' meeting L (but not M) and a line L' meeting M (but not L). Consider the plane π generated by L and M', which by Proposition 4.1 meets $\mathcal{B}$ precisely in $L \cup M'$. Since the line L' necessarily meets π in a point, this point must lie on M' and we get the configuration described in statement (2).

If the polarities agree on no point of $(L \cup M) \setminus \{P\}$, then from the above analysis of the entries in the matrix H_1 we have $\gamma = \delta = \gamma^q$, and thus $\gamma \in \mathsf{GF}(q)$. In this case the determinant of the matrix representing a generic Hermitian surface in the pencil $\mathcal{F}$ is $(\gamma+t)^4$, where t is a generic element of $\mathsf{GF}(q)$. Thus, by choosing $t = -\gamma$, we get a degenerate Hermitian surface in $\mathcal{F}$, a contradiction. Hence this possibility does not occur. Moreover, we get the same contradiction if the polarities agree on all points of $L \cup M$, or if the polarities agree on all points of L (respectively, M) but on no point of $M \setminus \{P\}$ (respectively, $L \setminus \{P\}$).

Thus the only remaining case to consider is if the polarities agree on all points of L (respectively, M) and on exactly one point of $M \setminus \{P\}$ (respectively, $L \setminus \{P\}$). In this case Proposition 2.7 implies that $\mathcal{B}$ is the point set of a pseudo-regulus, and we get the configuration described in statement (3). This completes the proof. □

Straightforward computations show that all possibilities listed in the above proposition indeed do occur. Moreover, it now follows from Proposition 4.4 that if $\mathcal{B}$ contains any lines, then the only remaining case to be studied is when those lines are pairwise skew.

Proposition 4.5. *If the lines contained in $\mathcal{B}$ are pairwise skew, then $\mathcal{B}$ contains at most two lines.*

Proof. Suppose $\mathcal{B}$ contains three pairwise skew lines, say L_0, L_1 and L_2. Then $\mathcal{H}_0$ contains the union of two "quadratically extended" subreguli (see [7]), say $L_0, L_1, L_2, \ldots, L_q$ and $M_0, M_1, M_2, \ldots, M_q$. Similarly, for any $i \neq 0$, $\mathcal{H}_i$ contains the union of two "quadratically extended" subreguli, say $L_0, L_1, L_2, \ldots, L_q$ and $M'_0, M'_1, M'_2, \ldots, M'_q$. Note that the first subregulus is the same since it is uniquely determined by L_0, L_1 and L_2. Let $\mathcal{R}$ be the regulus of $\mathsf{PG}(3, q^2)$ containing $\{L_0, L_1, L_2, \ldots, L_q\}$, so that $\mathcal{R}^{\mathrm{opp}}$ is the regulus containing both $\{M_0, M_1, M_2, \ldots, M_q\}$ and $\{M'_0, M'_1.M'_2, \ldots, M'_q\}$.

Now suppose that $\{M_0, M_1, M_2, \ldots, M_q\} \cap \{M'_0, M'_1.M'_2, \ldots, M'_q\} = \emptyset$ for every choice of $i \neq 0$. Then the $q+1$ Hermitian surfaces in the pencil $\mathcal{F}$ will contain $q+1$ mutually disjoint subreguli of $\mathcal{R}^{\mathrm{opp}}$, a contradiction. Thus $\mathcal{B}$ must contain at least one (actually, two) lines of $\mathcal{R}^{\mathrm{opp}}$ as well as the lines in $\{L_0, L_1, L_2, \ldots, L_q\}$, implying that $\mathcal{B}$ contains intersecting lines. This final contradiction proves the result. □

We next show that the possible configuration for $\mathcal{B}$ consisting of exactly two skew lines and $q^4 - 1$ other points is always possible. Without loss of generality we may assume that the Hermitian surface $\mathcal{H}_0$ has equation $X_0X_3^q + X_1X_2^q + X_2X_1^q + X_3X_0^q = 0$. Then $L : X_0 = X_1 = 0$ and $M : X_2 = X_3 = 0$ are two skew generators contained in $\mathcal{H}_0$. If there exists another Hermitian surface $\mathcal{H}_1$ containing L and M such that u_0 and u_1 agree on no point of $L \cup M$, then from Remark 4.2 and Proposition 4.5 we know that $\mathcal{B}$ contains exactly two lines.

We now show that such a Hermitian surface $\mathcal{H}_1$ exists. Namely, consider the Hermitian surface

$$\mathcal{H}_1 : (X_2 + X_3)X_0^q + \alpha^q X_3 X_1^q + X_0 X_2^q + (X_0 + \alpha X_1)X_3^q = 0$$

with α a nonzero element of $\mathsf{GF}(q^2)$ (otherwise it is degenerate). Straightforward computations show that $\mathcal{H}_1$ contains $L \cup M$ and, moreover, u_0 and u_1 do not agree on either $P = (0,0,0,1)$ or $Q = (0,1,0,0)$.

Next consider an arbitrary point $R = (0,0,1,\zeta)$ of $L\setminus\{P\}$, where $\zeta \neq 0$. Then R^{u_0} has equation $\zeta^q X_0 + X_1 = 0$, and R^{u_1} has equation $(1+\zeta^q)X_0 + \alpha\zeta^q X_1 = 0$. Hence $R^{u_0} = R^{u_1}$ if and only if $1 + \zeta^q = \alpha\zeta^{2q}$; that is, if and only if $\alpha x^2 - x - 1$ has a root in $\mathsf{GF}(q^2)$. This occurs precisely when $\mathrm{Tr}_0(\alpha) = 0$ for q is even or when $1 + 4\alpha$ is a square in $\mathsf{GF}(q^2)$ for q odd. Here Tr_0 denotes the absolute trace of $\mathsf{GF}(q^2)$ over $\mathsf{GF}(2)$ when q is even.

Finally, consider an arbitrary point $S = (1,\zeta,0,0)$ of $M \setminus \{Q\}$, where again $\zeta \neq 0$. Then S^{u_0} has equation $\zeta^q X_2 + X_3 = 0$, and S^{u_1} has equation $X_2 + (1 + \alpha^q\zeta^q)X_3 = 0$. Thus $S^{u_0} = S^{u_1}$ if and only if $\alpha^q\zeta^{2q} + \zeta^q = 1$; that is, if and only if $\alpha^q x^2 + x - 1$ has a root in $\mathsf{GF}(q^2)$. Again this occurs precisely when $\mathrm{Tr}_0(\alpha) = 0$ for q even or when $1 + 4\alpha^q$ is a square in $\mathsf{GF}(q^2)$ for q odd.

Hence u_1 and u_0 agree on no points of $L \cup M$ if and only if $\mathrm{Tr}_0(\alpha) = 1$ when q is even, and if and only if $1 + 4\alpha$ is a non-square of $\mathsf{GF}(q^2)$ when q is odd. Choosing such an element $\alpha \in \mathsf{GF}(q^2)$, we obtain a non-degenerate Hermitian surface $\mathcal{H}_1$ such that $B = \mathcal{H}_1 \cap \mathcal{H}_0$ contains exactly two skew lines and $q^4 - 1$ other points.

5 Proof of Theorem 1.2

From the results in the previous section, it remains to study two cases: when $\mathcal{B}$ contains exactly one line, and when $\mathcal{B}$ contains no line. In this section we will prove that both cases are not possible. We make extensive use of the results recalled in Section 3 on the intersection of two Hermitian curves. We start with three observations.

Observation 5.1. *Let π be a plane. Then $\pi \cap \mathcal{H}_i$ is either a Baer subpencil of lines or a non-degenerate Hermitian curve. Moreover $\pi \cap \mathcal{B}$ is the common intersection of all the Hermitian curves $\pi \cap \mathcal{H}_i$. Since none of those curves is a rank 1 Hermitian curve, we have that $\pi \cap \mathcal{B}$ cannot be a Baer subline.*

Proof. If the intersection of two Hermitian curves $\mathcal{C}$ and $\mathcal{C}'$ is a Baer subline ℓ, then the line L containing ℓ, counted $q+1$ times, is an Hermitian curve (of rank 1) contained in the pencil generated by $\mathcal{C}$ and $\mathcal{C}'$. □

Observation 5.2. *No plane π intersects $\mathcal{B}$ in a non-degenerate Hermitian curve.*

Proof. Suppose that $\mathcal{H}_0$ and $\mathcal{H}_1$ contain the non-degenerate Hermitian curve $\mathcal{C}$, whose equation we may assume is $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$. Thus both $\mathcal{H}_1$ and $\mathcal{H}_2$ meet the plane with equation $X_0 = 0$ in the curve $\mathcal{C}$. Hence these Hermitian surfaces have equations of the form $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} + a_i X_0^{q+1} + b_i X_0 X_1^q + c_i X_0 X_2^q + d_i X_0 X_3^q + b_i^q X_0^q X_1 + c_i^q X_0^q X_2 + d_i^q X_0^q X_3 = 0$, for $i = 1, 2$. If $\mathcal{F}$ is the pencil generated by $\mathcal{H}_0$ and $\mathcal{H}_1$, then $\mathcal{H}_0 - \mathcal{H}_1$ must be in $\mathcal{F}$. But a straightforward determinant argument shows that $\mathcal{H}_0 - \mathcal{H}_1$ is a degenerate Hermitian surface, contradicting our assumption on $\mathcal{F}$. □

Observation 5.3. *Through every point P of $\mathcal{B}$ there is at least one Baer subline contained in $\mathcal{B}$, and every Baer subline contained in $\mathcal{B}$ is contained in a generator of some Hermitian surface $\mathcal{H}_i$.*

Proof. Through every point P of $\mathcal{B}$ there is a Baer subline of $\mathcal{B}$ since $|\mathcal{B}| = (q^2+1)^2$. Moreover, let ℓ be a Baer subline contained in $\mathcal{B}$, and let L be the line containing ℓ. If L is contained in $\mathcal{B}$, then it is a generator of $\mathcal{H}_i$ for every i. Otherwise, we assume L is not contained in $\mathcal{B}$, and consider some plane π through L. If π meets each $\mathcal{H}_i$ in a degenerate Hermitian curve, then by Proposition 3.1 we only need consider the case where $\pi \cap \mathcal{B}$ is a degenerate $\mathcal{C}_F$-set. But in this case L is necessarily a generator of some $\pi \cap \mathcal{H}_i$, and hence a generator of $\mathcal{H}_i$. Finally, in the case when every plane π meets at least one $\mathcal{H}_i$ is a non-degenerate Hermitian curve, then by Observation 5.1, Observation 5.2, and Proposition 3.2 we may assume that the plane π meets $\mathcal{B}$ in one of the following: an H-set, a K-set, a Γ-set, or a $\mathcal{C}_F$-set. In all these cases the line L is a generator of some Hermitian surface $\mathcal{H}_i$. □

We are now ready to show that $\mathcal{B}$ must contain at least two lines. We begin with a weaker proposition.

Proposition 5.4. *If $q \geq 4$, then $\mathcal{B}$ contains at least a line.*

Proof. Suppose $\mathcal{B}$ contains no line, and let ℓ be a Baer subline contained in $\mathcal{B}$. Then from Observation 5.3 we know that the line L containing ℓ is a generator of exactly one of the Hermitian surfaces, say $\mathcal{H}_0$, in the pencil $\mathcal{F}$. We also know from Proposition 3.1, Proposition 3.2, and Observation 5.1 that every plane through L intersects $\mathcal{B}$ in one of the following: an H-set, a K-set, a Γ-set, or a $\mathcal{C}_F$-set. Let a be the number of planes through L intersecting $\mathcal{B}$ in an H-set, and let b be the number of planes through L intersecting $\mathcal{B}$ either in a Γ-set or in a $\mathcal{C}_F$-set. Then

$$q^4 + 2q^2 + 1 = |\mathcal{B}| = a(q^2 + q) + b(q^2 - q) + (q^2 + 1 - a - b)q^2 + q + 1,$$

which implies that $a - b = q - 1$.

Now fix some point P of ℓ, and count the number of Baer sublines contained in $\mathcal{B}$ through P. Note that planes through L yield different numbers of Baer sublines through P contained in $\mathcal{B}$, depending upon the planar intersection with $\mathcal{B}$. Indeed, if such a plane π intersects $\mathcal{B}$ in an H-set, then π yields two Baer sublines through P contained in $\mathcal{B}$, other than ℓ. If π intersects $\mathcal{B}$ in a $\mathcal{C}_F$-set, then we obtain no Baer subline, other than ℓ, passing through P and contained in $\mathcal{B}$. If π intersects $\mathcal{B}$ in a Γ-set, then we obtain either 0 or $q - 1$ Baer sublines, other than ℓ, contained in $\mathcal{B}$ and incident with P. Finally, if π intersects $\mathcal{B}$ in a K-set, then we obtain either 1 or q Baer sublines, other than ℓ, contained in $\mathcal{B}$ and passing through P.

The classification in [8] and the geometry of tangent planes to Hermitian surfaces shows that the tangent plane π_0 to $\mathcal{H}_0$ at P is either a Γ-set with center P or a K-set with center P. Moreover, π_0 is the only plane through L that can meet $\mathcal{B}$ in either a Γ-set or a K-set with center P. Thus, since $q \geq 4$, π_0 is the only plane through L whose intersection with $\mathcal{B}$ contains at least three Baer sublines through P.

Suppose first that $\pi_0 \cap \mathcal{B}$ is a K-set with center P, and thus contains q Baer sublines through P other than ℓ. Planes, other than π_0, through L which meet $\mathcal{B}$ in a K-set do not have center P and thus provide exactly one additional Baer subline through P in $\mathcal{B}$. There are $q^2 - a - b$ such planes. All planes through L meeting $\mathcal{B}$ in a Γ-set provide no additional Baer sublines through P in $\mathcal{B}$, as is true for all planes through L meeting $\mathcal{B}$ in a $\mathcal{C}_F$-set. Hence, counting the total number of Baer sublines through P, other than ℓ, that are contained in $\mathcal{B}$, we obtain

$$1 \cdot q + a \cdot 2 + b \cdot 0 + (q^2 - a - b) \cdot 1 = q^2 + q + a - b = q^2 + 2q - 1$$

since $a - b = q - 1$.

Suppose now that $\pi_0 \cap \mathcal{B}$ is a Γ-set with center P, and thus contains $q-1$ Baer sublines through P other than ℓ. Planes, other than π_0, through L which meet $\mathcal{B}$ in a Γ-set do not have center P and thus provide no additional Baer sublines through P in $\mathcal{B}$, as is true for all planes through L meeting $\mathcal{B}$ in a $\mathcal{C}_F$-set. The planes through L meeting $\mathcal{B}$ in a K-set, of which there are $q^2 - a - (b-1) = q^2 + 1 - a - b$, each provide one additional Baer subline through P in $\mathcal{B}$. Again, counting the total number of Baer sublines through P, other than ℓ, that are contained in $\mathcal{B}$, we obtain

$$1 \cdot (q-1) + a \cdot 2 + (q^2 + 1 - a - b) \cdot 1 = q^2 + q + a - b = q^2 + 2q - 1,$$

the same count as in the previous case. Hence, after including the Baer subline ℓ, in all cases we obtain precisely $q^2 + 2q$ Baer sublines through P contained in $\mathcal{B}$.

Finally, counting in two ways the total number of flags (P, ℓ), where ℓ is a Baer subline contained in $\mathcal{B}$, we see that the number of Baer sublines in $\mathcal{B}$ is $(q^2 + 2q)(q^2 + 1)^2/(q + 1)$. Since this must be an integer, we have $(q + 1) \mid 4$ and hence $q = 3$, a contradiction. □

Remark 5.5. Note that for $q = 3$ there is an example of a base $\mathcal{B}$ of size $q^4 + q^3 - q^2 + 1 = 100 = (q^2 + 1)^2$ containing no lines. However, the associated pencil $\mathcal{F}$ contains a degenerate Hermitian surface.

Proposition 5.6. *If $q \geq 4$, then $\mathcal{B}$ contains at least two lines.*

Proof. The argument is a refinement of the above proof. Suppose that $\mathcal{B}$ contains exactly one line, say M. Then every plane through M intersects $\mathcal{B}$ in a degenerate $\mathcal{C}_F$-set (that is, the union of M and an affine Baer subplane) by Proposition 4.1. Let ℓ be a Baer subline contained in $\mathcal{B}$ which intersects M in one point (through every point $P \in \mathcal{B}$ there is at least one such a Baer subline), and let L be the line containing ℓ. Every plane through L intersects $\mathcal{B}$ in one of the following: an H-set, a K-set, a Γ-set, a $\mathcal{C}_F$-set, or a degenerate $\mathcal{C}_F$-set. Let a be the number of planes through L intersecting $\mathcal{B}$ in an H-set, and let b be the number of planes through L intersecting $\mathcal{B}$ either in a Γ-set or in a $\mathcal{C}_F$-set. Then

$$q^4 + 2q^2 + 1 = |\mathcal{B}| = a(q^2 + q) + b(q^2 - q) + (q^2 - a - b)q^2 + 2q^2 + 1,$$

implying that $a - b = 0$ and hence $a = b$.

Now let P be the point $\ell \cap M$, and count the total number of Baer sublines contained in $\mathcal{B}$ which are incident with P but are not contained in M. A computation analogous to that made in the proof of Proposition 5.4 shows that this number is

$$2a + (q^2 - a - b) + q = q^2 + q,$$

since $a = b$. Note that the tangent plane to $\mathcal{H}_0$ at P, where $\mathcal{H}_0$ is the unique Hermitian surface in the pencil $\mathcal{F}$ which contains the line L, is the plane intersecting $\mathcal{B}$ in a degenerate $\mathcal{C}_F$-set.

Next let Q be a point of ℓ different from $P = \ell \cap M$. Counting the number of Baer sublines incident with Q and contained in $\mathcal{B}$, we obtain

$$2a + (q^2 - a - b) + (q - 1) + (q + 1) = q^2 + 2q,$$

again using $a = b$. Note that the tangent plane to $\mathcal{H}_0$ at Q is not the plane intersecting $\mathcal{B}$ in a degenerate $\mathcal{C}_F$-set, and hence meets $\mathcal{B}$ either in a Γ-set or a K-set with center Q. The count is the same in both cases, as in the proof of Proposition 5.4.

Counting in two ways the number of flags (P, ℓ), where ℓ is a Baer subline contained in $\mathcal{B}$ and not contained in M, we see that the total number of such sublines is $((q^2 + q)(q^2 + 1) + (q^2 + 2q)(q^4 + q^2))/(q + 1)$. As this must be an integer, we have that $(q + 1) \mid 2$, a contradiction. □

Theorem 1.2 now follows from Propositions 4.4, 4.5, and 5.6 when $q \geq 4$. For $q = 2$ and $q = 3$, the result follows from an exhaustive computer search.

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