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On the intersection of Hermitian surfaces

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Abstract

In [6] and [3] the authors determine the structure of the intersection of two Hermitian surfaces of $\text{PG}(3, q^2)$ under the hypotheses that in the pencil they generate there is at least one degenerate surface. In [1] and [3] it is shown that under suitable hypotheses the intersection of two Hermitian surfaces generating a non-degenerate pencil is a pseudo-regulus. Here we completely determine all possible intersection configurations for two Hermitian surfaces of $\text{PG}(3, q^2)$ generating a non-degenerate pencil.

Keywords: Hermitian curve, Hermitian surface

MSC 2000: 51E20, 05B25

1. Introduction

A Hermitian variety \mathcal{H} in $\text{PG}(n, q^2)$ is the set of absolute points for some Hermitian form defined on the underlying vector space. The variety \mathcal{H} is called *degenerate* if the corresponding Hermitian form is degenerate; else, it is called *non-degenerate*. If $n = 2$, \mathcal{H} is called a *Hermitian curve*, while if $n = 3$, \mathcal{H} is called a *Hermitian surface*. A point P on \mathcal{H} is called *singular* if any line through P either intersects \mathcal{H} only in P or is contained in \mathcal{H} . The *vertex* of \mathcal{H} is the set of all singular points of \mathcal{H} , and it is denoted by $V(\mathcal{H})$. It is clear that $V(\mathcal{H})$ is a projective subspace of $\text{PG}(n, q^2)$, and the *rank* of \mathcal{H} is the number $r(\mathcal{H}) = n - \dim(V(\mathcal{H}))$.

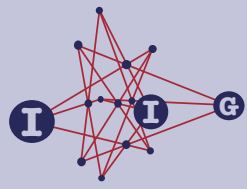
Let \mathcal{H}_0 and \mathcal{H}_1 be two distinct Hermitian varieties of $\text{PG}(n, q^2)$ with homogeneous equations $f_0 = 0$ and $f_1 = 0$, respectively. Then the *Hermitian pencil*

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\mathcal{F} defined by \mathcal{H}_0 and \mathcal{H}_1 is the set of all Hermitian varieties with equations $a_0f_0 + a_1f_1 = 0$, as a_0 and a_1 vary over the subfield $\text{GF}(q)$, not both zero. Note that there are $q + 1$ distinct Hermitian varieties in the pencil \mathcal{F} , some of which may be degenerate. The set $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ is called the *base* of \mathcal{F} , and any two distinct varieties in \mathcal{F} intersect precisely in \mathcal{B} .

We now restrict to the case $n = 3$, and let \mathcal{H}_0 and \mathcal{H}_1 denote two distinct Hermitian surfaces in $\text{PG}(3, q^2)$ with associated polarities u_0 and u_1 , respectively. If the pencil \mathcal{F} generated by \mathcal{H}_0 and \mathcal{H}_1 contains at least one degenerate Hermitian surface, then the structure of the base $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ is completely known (see [6] and [3]).

Thus we are interested in the situation when all Hermitian surfaces in the pencil \mathcal{F} are non-degenerate. In this case, since the Hermitian surfaces of a pencil cover all the points of $\text{PG}(3, q^2)$, straightforward counting shows that $|\mathcal{B}| = (q^2 + 1)^2$. Conversely, if $q \geq 4$ and $|\mathcal{B}| = (q^2 + 1)^2$, then the pencil \mathcal{F} generated by \mathcal{H}_0 and \mathcal{H}_1 necessarily contains only non-degenerate Hermitian surfaces. Indeed, going through the list in [6] or [3] of all possibilities for the cardinality of \mathcal{B} , when at least one of the surfaces in \mathcal{F} is degenerate, we see that $|\mathcal{B}| = (q^2 + 1)^2$ only occurs for $q = 3$ and $|\mathcal{B}| = q^4 + q^3 - q^2 + 1 = 100 = (q^2 + 1)^2$ or $q = 2$ and $|\mathcal{B}| = q^3 + q^3 + 1 = 25 = (q^2 + 1)^2$. We thus have the following result.

Proposition 1.1. *For $q \geq 4$ the Hermitian pencil \mathcal{F} contains only non-degenerate surfaces if and only if the base \mathcal{B} has size $(q^2 + 1)^2$.*

In the sections that follow we prove the following result.

Theorem 1.2. *Let \mathcal{H}_0 and \mathcal{H}_1 be two non-degenerate Hermitian surfaces in $\text{PG}(3, q^2)$, and let $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ be the base of the Hermitian pencil \mathcal{F} they generate. If \mathcal{F} contains only non-degenerate surfaces, then one of the following holds:*

- \mathcal{B} contains exactly two skew lines and $q^4 - 1$ other points;
- \mathcal{B} contains exactly two skew lines L and M , a third line N intersecting both L and M , and $q^4 - q^2$ other points;
- \mathcal{B} contains exactly four lines forming a quadrangle and $q^4 - 2q^2 + 1$ other points;
- \mathcal{B} is ruled by a pseudo-regulus.

Moreover, all such cases occur.

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2. Preliminary results

In [10] B. Segre defines two Hermitian surfaces in $\text{PG}(3, q^2)$ to be *permutable* if and only if their associated polarities commute, and then he proves the following result.

Theorem 2.1 ([10]). *Let q be odd, and \mathcal{H}_0 and \mathcal{H}_1 be two permutable Hermitian surfaces in $\text{PG}(3, q^2)$ with associated polarities u_0 and u_1 , respectively. Then u_0u_1 is a projectivity with two skew lines, say L and M , of fixed points. That is, u_0u_1 is a biaxial harmonic involutorial collineation with fundamental lines L and M .*

The fundamental lines associated with two permutable Hermitian surfaces may or may not be lines lying on those surfaces. The lines completely contained in a Hermitian surface are called the *generators* of the surface, and a set of k mutually skew generators of a Hermitian surface \mathcal{H} is called a k -span of \mathcal{H} . A k -span of \mathcal{H} is called \mathcal{H} -complete if it is not contained in a $(k + 1)$ -span of \mathcal{H} . In [4] the following is proved.

Proposition 2.2 ([4]). *The $q^2 + 1$ generators meeting two skew generators of \mathcal{H} form an \mathcal{H} -complete span. This \mathcal{H} -span has no further transversals.*

In general, any set of $q^2 + 1$ mutually skew lines in $\text{PG}(3, q^2)$ with exactly two transversals is called a *pseudo-regulus*. This notion was introduced by J. Freeman in [5], where he proved that any pseudo-regulus can be extended to a spread of $\text{PG}(3, q^2)$. The set of $(q^2 + 1)^2$ points covered by a pseudo-regulus is called a *hyperbolic \mathcal{Q}_F -set* in [2]. In this paper we see that this set of points naturally arises as one of the possible intersections \mathcal{B} for \mathcal{H}_0 and \mathcal{H}_1 . The following result is proved in [1].

Theorem 2.3 ([1]). *Let q be an odd prime power, and let \mathcal{H}_0 and \mathcal{H}_1 be permutable Hermitian surfaces in $\text{PG}(3, q^2)$. If the fundamental lines L and M are contained in $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$, then \mathcal{B} is a ruled determinantal variety consisting of the points on a pseudo-regulus. In particular, this pseudo-regulus is a complete $(q^2 + 1)$ -span of both \mathcal{H}_0 and \mathcal{H}_1 .*

The hypotheses in the previous theorem are weakened in [3], where it is shown that the point set of a pseudo-regulus can be obtained as the intersection of two Hermitian surfaces in the even characteristic case as well. In particular, the following result is proved.

Theorem 2.4 ([3]). *Let \mathcal{H}_0 and \mathcal{H}_1 be two distinct Hermitian surfaces in $\text{PG}(3, q^2)$ with associated polarities u_0 and u_1 , respectively. Suppose that L and M are two skew lines contained in $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$. Then \mathcal{B} is a hyperbolic \mathcal{Q}_F -set with transversals L and M if and only if u_0 and u_1 agree on the points of $L \cup M$.*



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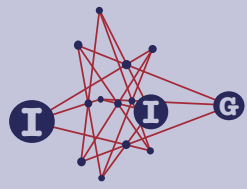
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An investigation of other possible intersection configurations for \mathcal{H}_0 and \mathcal{H}_1 is started in [3], where the following three results appear.

Proposition 2.5. *If $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ contains a line L , then every plane π through L intersects \mathcal{B} in one of the following configurations:*

- the points of the line L ;
- the points on a pair of distinct lines L and M ;
- the points lying on a Baer subpencil of lines containing L ;
- the points on a degenerate \mathcal{C}_F -set (that is, the union of L and an affine Baer subplane).

Proposition 2.6. *If $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ contains a line L such that u_0 and u_1 agree on the points of L , then \mathcal{B} is the union of $q^2 + 1$ pairwise skew lines, all intersecting L . If, in addition, \mathcal{B} contains a line M skew with L , then necessarily \mathcal{B} is a hyperbolic \mathcal{Q}_F -set.*

Proposition 2.7. *If $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$ contains a line L such that u_0 and u_1 agree on the points of L and also on a point of $\mathcal{B} \setminus L$, then \mathcal{B} is a hyperbolic \mathcal{Q}_F -set.*

In order to conduct our investigation of the possible configuration patterns for the base of a Hermitian pencil in $\text{PG}(3, q^2)$ containing only non-degenerate Hermitian surfaces, it is useful to first recall the possible intersection configurations of two Hermitian curves in $\text{PG}(2, q^2)$. This will be done in the next section.

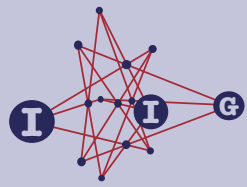
3. The intersection of Hermitian curves

In this section \mathcal{F} will be a Hermitian pencil of curves in $\text{PG}(2, q^2)$. We refer to [8], [9] and [3] for the results catalogued here. A straightforward counting argument shows that if all the Hermitian curves in \mathcal{F} are non-degenerate, then the base has size $q^2 - q + 1$. In fact, such a base is a complete arc, often called a *Kestenband* arc. At the other end of the spectrum, the following result lists the possibilities for the base when all the Hermitian curves in \mathcal{F} are degenerate of rank 2.

Proposition 3.1. *Let \mathcal{C}_0 and \mathcal{C}_1 be two rank 2 degenerate Hermitian curves in $\text{PG}(2, q^2)$. If the pencil they generate contains only rank 2 degenerate Hermitian curves, then the base $\mathcal{B} = \mathcal{C}_0 \cap \mathcal{C}_1$ of \mathcal{F} is one of the following: a line, a pair of distinct lines, or a degenerate \mathcal{C}_F -set (the union of a line and an affine Baer subplane).*

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The general situation, when at least one of the Hermitian curves in \mathcal{F} is non-degenerate, is described below in Proposition 3.2. We use the notation from [3]. A rank 1 Hermitian curve (that is, a line of $\text{PG}(2, q^2)$ repeated $q + 1$ times) can meet a non-degenerate Hermitian curve in either one point or a Baer subline, thus accounting for the first two intersections mentioned in Proposition 3.2. As previously stated, if all Hermitian curves in \mathcal{F} are non-degenerate, the base $\mathcal{C}_0 \cap \mathcal{C}_1$ is a Kestenband arc. Thus the remaining possibilities can be obtained by intersecting a non-degenerate Hermitian curve \mathcal{C} with a Baer subpencil \mathcal{P} of lines with center P (Hermitian curve of rank 2). There are four such configurations:

- Suppose $P \notin \mathcal{C}$ and exactly two of the lines in \mathcal{P} are tangent to \mathcal{C} . Then $\mathcal{P} \cap \mathcal{C}$ contains the points on $q - 1$ Baer sublines (through P) and two other points (one on each of the tangent lines to \mathcal{C} in \mathcal{P}). Thus $\mathcal{B} = \mathcal{C} \cap \mathcal{P}$ has size $q^2 + 1$, and this base is called a \mathcal{C}_F -set in $\text{PG}(2, q^2)$.
- Suppose $P \in \mathcal{C}$ and exactly one of the lines in \mathcal{P} is tangent to \mathcal{C} (at P). Then $\mathcal{P} \cap \mathcal{C}$ contains the points on q Baer sublines (through P) and thus again has size $q^2 + 1$. This base is called a Γ -set in $\text{PG}(2, q^2)$.
- Suppose $P \notin \mathcal{C}$ and exactly one of the lines in \mathcal{P} is tangent to \mathcal{C} , say at the point Q . Then $\mathcal{P} \cap \mathcal{C}$ contains the points on q Baer sublines (through P) and the point Q . This base has size $q^2 + q + 1$, and is called a K -set in $\text{PG}(2, q^2)$. It should be noted that this base can also be obtained as the intersection of \mathcal{C} with a Baer subpencil through Q (and no tangent lines to \mathcal{C}), and thus there are $2q + 1$ Baer sublines contained in this base.
- Suppose $P \notin \mathcal{C}$ and none of the lines in \mathcal{P} is tangent to \mathcal{C} . Then $\mathcal{P} \cap \mathcal{C}$ contains the points on $q + 1$ Baer sublines (through P) and thus has size $(q + 1)^2$. This base is called an H -set in $\text{PG}(2, q^2)$. It turns out that this base is partitioned by three different Baer subpencils through three distinct points, none of which are in the base, and thus there are $3(q + 1)$ Baer sublines contained in this base.

Diagrams for these last four intersections can be found on page 112 of [8], where Figures 1, 2, 3, and 4 correspond to H -sets, K -sets, \mathcal{C}_F -sets, and Γ -sets, respectively.

The following result summarizes the above discussion.

Proposition 3.2. *Let \mathcal{C}_0 and \mathcal{C}_1 be two Hermitian curves in $\text{PG}(2, q^2)$. If the Hermitian pencil \mathcal{F} they generate contains at least one non-degenerate Hermitian curve, then $\mathcal{C}_0 \cap \mathcal{C}_1$ is one of the following: a point, a Baer subline, a (complete) Kestenband $(q^2 - q + 1)$ -arc, a \mathcal{C}_F -set, a Γ -set, a K -set, or an H -set.*

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4. The intersection of Hermitian surfaces

We now return to our fundamental problem of determining the possible intersection patterns for the base \mathcal{B} of a Hermitian pencil \mathcal{F} in $\text{PG}(3, q^2)$ containing all non-degenerate Hermitian surfaces. As stated previously, this is the only remaining open case. We will denote by $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_q$ the $q + 1$ Hermitian surfaces of the pencil \mathcal{F} generated by \mathcal{H}_0 and \mathcal{H}_1 , and by u_0, u_1, \dots, u_q their associated polarities. Recall that in this case the base \mathcal{B} necessarily has size $(q^2 + 1)^2$.

We first consider the situation where \mathcal{B} contains a line L , and refine the possibilities listed in Proposition 2.5.

Proposition 4.1. *Assume the Hermitian pencil \mathcal{F} contains all non-degenerate Hermitian surfaces, and that the base \mathcal{B} contains a line L . Then any plane π through L intersects \mathcal{B} nontrivially, and $\mathcal{B} \cap \pi$ is one of the following:*

- the set of points on a pair of distinct (intersecting) lines L and M ;
- the set of points on a degenerate \mathcal{C}_F -set (the union of L and an affine Baer subplane).

Proof. Suppose π is a plane through L that meets \mathcal{B} in the points on a Baer subpencil \mathcal{P} of lines, one of which is necessarily L , and let P be the center of this pencil. Then $|\pi \cap (\mathcal{B} \setminus L)| = q^3$ and $|\mathcal{B} \setminus L| = (q^2 + 1)q^2$. Since every plane through L intersects $\mathcal{B} \setminus L$ in $0, q^2$ or q^3 points by Proposition 2.5, the existence of π implies the existence of another plane π' through L intersecting \mathcal{B} only in L .

We now work with the plane π' . Since $\pi' \cap \mathcal{B} = L$, there necessarily exists a point Q on $L \setminus \{P\}$ such that $\pi' = Q^{u_i}$ for $i = 0, 1, 2, \dots, q$ (otherwise, π' cannot intersect \mathcal{B} only in L). Hence π' contains $q + 1$ Baer subpencils of lines through Q that share only the line L . This yields $q(q + 1) + 1$ lines through Q in π' , a contradiction. The result now follows from Proposition 2.5. \square

Remark 4.2. Note that if a plane π intersects \mathcal{B} in two lines, say L and M , then $\pi = P^{u_i}$ for every i , where $P = L \cap M$. Conversely, if there exists a point P on L such that two of the polarities agree on P , then all of the polarities agree on P and $\pi = P^{u_0}$ meets \mathcal{B} in L and another line, say M , passing through P .

Proposition 4.3. *If \mathcal{B} contains a line L , then one of the following cases occurs:*

- all planes through L intersect \mathcal{B} in a degenerate \mathcal{C}_F -set (the union of L and an affine Baer subplane);
- there exists a unique plane through L intersecting \mathcal{B} in L and another line;



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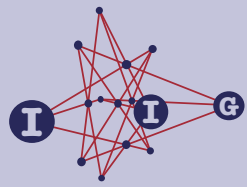
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- there exist exactly two planes through L intersecting \mathcal{B} in L and another line;
- all planes through L intersect \mathcal{B} in L and another line.

Proof. By Proposition 4.1 every plane through L intersects \mathcal{B} either in L and another line, or in L and an affine Baer subplane. If there is a point Q on L on which the $q+1$ polarities u_i agree, then the plane Q^{u_0} ($= Q^{u_i}$ for all i) intersects \mathcal{B} in L and another line, and conversely. Hence the multiset of intersection configurations of \mathcal{B} and the various planes through L depends on the number of points of L on which the $q+1$ polarities u_i agree. But this is the same as the number of fixed points of the projectivity $u_0 u_1^{-1}$ on the line L by Remark 4.2. From the fundamental theorem of projective geometry, this number must be one of the following: 0, 1, 2 or $q^2 + 1$ (see [12], for instance). Thus the four configurations in the statement of the proposition are the only possibilities. \square

If \mathcal{B} contains two intersecting lines, then more can be said.

Proposition 4.4. *If \mathcal{B} contains two intersecting lines, say L and M , then one of the following occurs:*

- (1) \mathcal{B} contains exactly three lines, and the third line meets either L or M (but not both);
- (2) \mathcal{B} contains exactly four lines which form a quadrilateral;
- (3) \mathcal{B} contains exactly $q^2 + 3$ lines, forming a pseudo-regulus and its two transversals.

Proof. The stabilizer of a Hermitian surface in $\text{PG}(3, q^2)$ acts transitively on pairs of intersecting generators (see [11], for instance). Thus, after choosing our favorite Hermitian surface, say $\mathcal{H}_0 : X_0 X_3^q + X_1 X_2^q + X_2 X_1^q + X_3 X_0^q = 0$, we may also choose our favorite pair of intersecting lines in $\mathcal{B} = \mathcal{H}_0 \cap \mathcal{H}_1$, say $L : X_0 = X_1 = 0$ and $M : X_1 = X_3 = 0$. If H_1 is the Hermitian matrix giving the equation of \mathcal{H}_1 , then H_1 is of the following form:

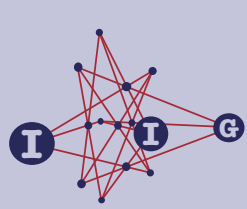
$$H_1 = \begin{pmatrix} a & \alpha & \beta & \gamma \\ \alpha^q & b & \delta & \epsilon \\ \beta^q & \delta^q & c & \nu \\ \gamma^q & \epsilon^q & \nu^q & d \end{pmatrix},$$

where the diagonal elements a, b, c, d are in $\text{GF}(q)$ and the other elements in $\text{GF}(q^2)$.

Since \mathcal{B} contains the points $P : (0, 0, 1, 0)$, $Q : (0, 0, 0, 1)$ and $R : (1, 0, 0, 0)$, we necessarily have $a = c = d = 0$. Moreover, the tangent plane to \mathcal{H}_0 at the

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point $P = L \cap M$ is $P^{u_0} : X_1 = 0$, and this plane also must be the tangent plane to \mathcal{H}_1 at P . This further implies that $\beta = \nu = 0$, and the matrix H_1 must have the following restricted form:

$$H_1 = \begin{pmatrix} 0 & \alpha & 0 & \gamma \\ \alpha^q & b & \delta & \epsilon \\ 0 & \delta^q & 0 & 0 \\ \gamma^q & \epsilon^q & 0 & 0 \end{pmatrix}.$$

Note that $\gamma = 0$ implies $\det(H_1) = 0$, contradicting our fundamental assumption that the pencil \mathcal{F} contains only non-degenerate Hermitian surfaces. So, $\gamma \neq 0$. Next observe that the polarities u_1 and u_0 agree on the point $Q : (0, 0, 0, 1)$ of the line L if and only if $\epsilon = 0$. Similarly, these polarities agree on the point $X : (0, 0, 1, \zeta)$ of the line L , where $\zeta \neq 0$, if and only if $\epsilon\zeta^q = \gamma - \delta$. Thus we have the following possibilities:

- (i) If $\epsilon \neq 0$ and $\gamma \neq \delta$, then for any choice of γ, ϵ and δ there is a unique point X of $L \setminus \{P\}$ on which the polarities u_0 and u_1 agree;
- (ii) If $\epsilon \neq 0$ and $\gamma = \delta$, then there is no point of $L \setminus \{P\}$ on which the polarities agree;
- (iii) If $\epsilon = 0$ and $\gamma \neq \delta$, then Q is the unique point of $L \setminus \{P\}$ on which the polarities agree;
- (iv) If $\epsilon = 0$ and $\gamma = \delta$, then the polarities agree on all the points of L .

Similarly, the polarities u_1 and u_0 agree on the point $R : (1, 0, 0, 0)$ of M if and only if $\alpha = 0$, and they agree on the point $Y : (1, 0, \zeta, 0)$, where $\zeta \neq 0$, if and only if $\zeta^q(\gamma^q - \delta) = \alpha^q$. Thus we again have four possibilities:

- (i)' If $\alpha \neq 0$ and $\delta \neq \gamma^q$, then there is a unique point Y of $M \setminus \{P\}$ on which the polarities u_0 and u_1 agree;
- (ii)' If $\alpha \neq 0$ and $\delta = \gamma^q$, then there is no point of $M \setminus \{P\}$ on which the polarities agree;
- (iii)' If $\alpha = 0$ and $\delta \neq \gamma^q$, then R is the unique point of $M \setminus \{P\}$ on which the polarities agree;
- (iv)' If $\alpha = 0$ and $\delta = \gamma^q$, then the polarities agree on all the points of M .

Hence on $L \setminus \{P\}$ the polarities u_0 and u_1 agree on no point, one point, or all points. A similar statement holds for $M \setminus \{P\}$. If the polarities agree on exactly one point of $(L \cup M) \setminus \{P\}$, then Proposition 4.1 (and the ensuing remark) implies that \mathcal{B} contains exactly three lines as described in statement (1).



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If the polarities agree on exactly one point of $L \setminus \{P\}$ and agree on exactly one point of $M \setminus \{P\}$, then Proposition 4.1 implies that \mathcal{B} contains exactly four lines. In particular, \mathcal{B} contains a line M' meeting L (but not M) and a line L' meeting M (but not L). Consider the plane π generated by L and M' , which by Proposition 4.1 meets \mathcal{B} precisely in $L \cup M'$. Since the line L' necessarily meets π in a point, this point must lie on M' and we get the configuration described in statement (2).

If the polarities agree on no point of $(L \cup M) \setminus \{P\}$, then from the above analysis of the entries in the matrix H_1 we have $\gamma = \delta = \gamma^q$, and thus $\gamma \in \text{GF}(q)$. In this case the determinant of the matrix representing a generic Hermitian surface in the pencil \mathcal{F} is $(\gamma+t)^4$, where t is a generic element of $\text{GF}(q)$. Thus, by choosing $t = -\gamma$, we get a degenerate Hermitian surface in \mathcal{F} , a contradiction. Hence this possibility does not occur. Moreover, we get the same contradiction if the polarities agree on all points of $L \cup M$, or if the polarities agree on all points of L (respectively, M) but on no point of $M \setminus \{P\}$ (respectively, $L \setminus \{P\}$).

Thus the only remaining case to consider is if the polarities agree on all points of L (respectively, M) and on exactly one point of $M \setminus \{P\}$ (respectively, $L \setminus \{P\}$). In this case Proposition 2.7 implies that \mathcal{B} is the point set of a pseudo-regulus, and we get the configuration described in statement (3). This completes the proof. \square

Straightforward computations show that all possibilities listed in the above proposition indeed do occur. Moreover, it now follows from Proposition 4.4 that if \mathcal{B} contains any lines, then the only remaining case to be studied is when those lines are pairwise skew.

Proposition 4.5. *If the lines contained in \mathcal{B} are pairwise skew, then \mathcal{B} contains at most two lines.*

Proof. Suppose \mathcal{B} contains three pairwise skew lines, say L_0, L_1 and L_2 . Then \mathcal{H}_0 contains the union of two “quadratically extended” subreguli (see [7]), say $L_0, L_1, L_2, \dots, L_q$ and $M_0, M_1, M_2, \dots, M_q$. Similarly, for any $i \neq 0$, \mathcal{H}_i contains the union of two “quadratically extended” subreguli, say $L_0, L_1, L_2, \dots, L_q$ and $M'_0, M'_1, M'_2, \dots, M'_q$. Note that the first subregulus is the same since it is uniquely determined by L_0, L_1 and L_2 . Let \mathcal{R} be the regulus of $\text{PG}(3, q^2)$ containing $\{L_0, L_1, L_2, \dots, L_q\}$, so that \mathcal{R}^{opp} is the regulus containing both $\{M_0, M_1, M_2, \dots, M_q\}$ and $\{M'_0, M'_1, M'_2, \dots, M'_q\}$.

Now suppose that $\{M_0, M_1, M_2, \dots, M_q\} \cap \{M'_0, M'_1, M'_2, \dots, M'_q\} = \emptyset$ for every choice of $i \neq 0$. Then the $q + 1$ Hermitian surfaces in the pencil \mathcal{F} will contain $q + 1$ mutually disjoint subreguli of \mathcal{R}^{opp} , a contradiction. Thus \mathcal{B} must contain at least one (actually, two) lines of \mathcal{R}^{opp} as well as the lines

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in $\{L_0, L_1, L_2, \dots, L_q\}$, implying that \mathcal{B} contains intersecting lines. This final contradiction proves the result. \square

We next show that the possible configuration for \mathcal{B} consisting of exactly two skew lines and $q^4 - 1$ other points is always possible. Without loss of generality we may assume that the Hermitian surface \mathcal{H}_0 has equation $X_0X_3^q + X_1X_2^q + X_2X_1^q + X_3X_0^q = 0$. Then $L : X_0 = X_1 = 0$ and $M : X_2 = X_3 = 0$ are two skew generators contained in \mathcal{H}_0 . If there exists another Hermitian surface \mathcal{H}_1 containing L and M such that u_0 and u_1 agree on no point of $L \cup M$, then from Remark 4.2 and Proposition 4.5 we know that \mathcal{B} contains exactly two lines.

We now show that such a Hermitian surface \mathcal{H}_1 exists. Namely, consider the Hermitian surface

$$\mathcal{H}_1 : (X_2 + X_3)X_0^q + \alpha^q X_3X_1^q + X_0X_2^q + (X_0 + \alpha X_1)X_3^q = 0$$

with α a nonzero element of $\text{GF}(q^2)$ (otherwise it is degenerate). Straightforward computations show that \mathcal{H}_1 contains $L \cup M$ and, moreover, u_0 and u_1 do not agree on either $P = (0, 0, 0, 1)$ or $Q = (0, 1, 0, 0)$.

Next consider an arbitrary point $R = (0, 0, 1, \zeta)$ of $L \setminus \{P\}$, where $\zeta \neq 0$. Then R^{u_0} has equation $\zeta^q X_0 + X_1 = 0$, and R^{u_1} has equation $(1 + \zeta^q)X_0 + \alpha\zeta^q X_1 = 0$. Hence $R^{u_0} = R^{u_1}$ if and only if $1 + \zeta^q = \alpha\zeta^{2q}$; that is, if and only if $\alpha x^2 - x - 1$ has a root in $\text{GF}(q^2)$. This occurs precisely when $\text{Tr}_0(\alpha) = 0$ for q is even or when $1 + 4\alpha$ is a square in $\text{GF}(q^2)$ for q odd. Here Tr_0 denotes the absolute trace of $\text{GF}(q^2)$ over $\text{GF}(2)$ when q is even.

Finally, consider an arbitrary point $S = (1, \zeta, 0, 0)$ of $M \setminus \{Q\}$, where again $\zeta \neq 0$. Then S^{u_0} has equation $\zeta^q X_2 + X_3 = 0$, and S^{u_1} has equation $X_2 + (1 + \alpha^q \zeta^q)X_3 = 0$. Thus $S^{u_0} = S^{u_1}$ if and only if $\alpha^q \zeta^{2q} + \zeta^q = 1$; that is, if and only if $\alpha^q x^2 + x - 1$ has a root in $\text{GF}(q^2)$. Again this occurs precisely when $\text{Tr}_0(\alpha) = 0$ for q even or when $1 + 4\alpha^q$ is a square in $\text{GF}(q^2)$ for q odd.

Hence u_1 and u_0 agree on no points of $L \cup M$ if and only if $\text{Tr}_0(\alpha) = 1$ when q is even, and if and only if $1 + 4\alpha$ is a non-square of $\text{GF}(q^2)$ when q is odd. Choosing such an element $\alpha \in \text{GF}(q^2)$, we obtain a non-degenerate Hermitian surface \mathcal{H}_1 such that $B = \mathcal{H}_1 \cap \mathcal{H}_0$ contains exactly two skew lines and $q^4 - 1$ other points.

5. Proof of Theorem 1.2

From the results in the previous section, it remains to study two cases: when \mathcal{B} contains exactly one line, and when \mathcal{B} contains no line. In this section we will prove that both cases are not possible. We make extensive use of the results

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recalled in Section 3 on the intersection of two Hermitian curves. We start with three observations.

Observation 5.1. *Let π be a plane. Then $\pi \cap \mathcal{H}_i$ is either a Baer subpencil of lines or a non-degenerate Hermitian curve. Moreover $\pi \cap \mathcal{B}$ is the common intersection of all the Hermitian curves $\pi \cap \mathcal{H}_i$. Since none of those curves is a rank 1 Hermitian curve, we have that $\pi \cap \mathcal{B}$ cannot be a Baer subline.*

Proof. If the intersection of two Hermitian curves \mathcal{C} and \mathcal{C}' is a Baer subline ℓ , then the line L containing ℓ , counted $q + 1$ times, is an Hermitian curve (of rank 1) contained in the pencil generated by \mathcal{C} and \mathcal{C}' . \square

Observation 5.2. *No plane π intersects \mathcal{B} in a non-degenerate Hermitian curve.*

Proof. Suppose that \mathcal{H}_0 and \mathcal{H}_1 contain the non-degenerate Hermitian curve \mathcal{C} , whose equation we may assume is $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0$. Thus both \mathcal{H}_1 and \mathcal{H}_2 meet the plane with equation $X_0 = 0$ in the curve \mathcal{C} . Hence these Hermitian surfaces have equations of the form $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} + a_i X_0^{q+1} + b_i X_0 X_1^q + c_i X_0 X_2^q + d_i X_0 X_3^q + b_i^q X_0^q X_1 + c_i^q X_0^q X_2 + d_i^q X_0^q X_3 = 0$, for $i = 1, 2$. If \mathcal{F} is the pencil generated by \mathcal{H}_0 and \mathcal{H}_1 , then $\mathcal{H}_0 - \mathcal{H}_1$ must be in \mathcal{F} . But a straightforward determinant argument shows that $\mathcal{H}_0 - \mathcal{H}_1$ is a degenerate Hermitian surface, contradicting our assumption on \mathcal{F} . \square

Observation 5.3. *Through every point P of \mathcal{B} there is at least one Baer subline contained in \mathcal{B} , and every Baer subline contained in \mathcal{B} is contained in a generator of some Hermitian surface \mathcal{H}_i .*

Proof. Through every point P of \mathcal{B} there is a Baer subline of \mathcal{B} since $|\mathcal{B}| = (q^2 + 1)^2$. Moreover, let ℓ be a Baer subline contained in \mathcal{B} , and let L be the line containing ℓ . If L is contained in \mathcal{B} , then it is a generator of \mathcal{H}_i for every i . Otherwise, we assume L is not contained in \mathcal{B} , and consider some plane π through L . If π meets each \mathcal{H}_i in a degenerate Hermitian curve, then by Proposition 3.1 we only need consider the case where $\pi \cap \mathcal{B}$ is a degenerate \mathcal{C}_F -set. But in this case L is necessarily a generator of some $\pi \cap \mathcal{H}_i$, and hence a generator of \mathcal{H}_i . Finally, in the case when every plane π meets at least one \mathcal{H}_i in a non-degenerate Hermitian curve, then by Observation 5.1, Observation 5.2, and Proposition 3.2 we may assume that the plane π meets \mathcal{B} in one of the following: an H -set, a K -set, a Γ -set, or a \mathcal{C}_F -set. In all these cases the line L is a generator of some Hermitian surface \mathcal{H}_i . \square

We are now ready to show that \mathcal{B} must contain at least two lines. We begin with a weaker proposition.

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Proposition 5.4. *If $q \geq 4$, then \mathcal{B} contains at least a line.*

Proof. Suppose \mathcal{B} contains no line, and let ℓ be a Baer subline contained in \mathcal{B} . Then from Observation 5.3 we know that the line L containing ℓ is a generator of exactly one of the Hermitian surfaces, say \mathcal{H}_0 , in the pencil \mathcal{F} . We also know from Proposition 3.1, Proposition 3.2, and Observation 5.1 that every plane through L intersects \mathcal{B} in one of the following: an H -set, a K -set, a Γ -set, or a \mathcal{C}_F -set. Let a be the number of planes through L intersecting \mathcal{B} in an H -set, and let b be the number of planes through L intersecting \mathcal{B} either in a Γ -set or in a \mathcal{C}_F -set. Then

$$q^4 + 2q^2 + 1 = |\mathcal{B}| = a(q^2 + q) + b(q^2 - q) + (q^2 + 1 - a - b)q^2 + q + 1,$$

which implies that $a - b = q - 1$.

Now fix some point P of ℓ , and count the number of Baer sublines contained in \mathcal{B} through P . Note that planes through L yield different numbers of Baer sublines through P contained in \mathcal{B} , depending upon the planar intersection with \mathcal{B} . Indeed, if such a plane π intersects \mathcal{B} in an H -set, then π yields two Baer sublines through P contained in \mathcal{B} , other than ℓ . If π intersects \mathcal{B} in a \mathcal{C}_F -set, then we obtain no Baer subline, other than ℓ , passing through P and contained in \mathcal{B} . If π intersects \mathcal{B} in a Γ -set, then we obtain either 0 or $q - 1$ Baer sublines, other than ℓ , contained in \mathcal{B} and incident with P . Finally, if π intersects \mathcal{B} in a K -set, then we obtain either 1 or q Baer sublines, other than ℓ , contained in \mathcal{B} and passing through P .

The classification in [8] and the geometry of tangent planes to Hermitian surfaces shows that the tangent plane π_0 to \mathcal{H}_0 at P is either a Γ -set with center P or a K -set with center P . Moreover, π_0 is the only plane through L that can meet \mathcal{B} in either a Γ -set or a K -set with center P . Thus, since $q \geq 4$, π_0 is the only plane through L whose intersection with \mathcal{B} contains at least three Baer sublines through P .

Suppose first that $\pi_0 \cap \mathcal{B}$ is a K -set with center P , and thus contains q Baer sublines through P other than ℓ . Planes, other than π_0 , through L which meet \mathcal{B} in a K -set do not have center P and thus provide exactly one additional Baer subline through P in \mathcal{B} . There are $q^2 - a - b$ such planes. All planes through L meeting \mathcal{B} in a Γ -set provide no additional Baer sublines through P in \mathcal{B} , as is true for all planes through L meeting \mathcal{B} in a \mathcal{C}_F -set. Hence, counting the total number of Baer sublines through P , other than ℓ , that are contained in \mathcal{B} , we obtain

$$1 \cdot q + a \cdot 2 + b \cdot 0 + (q^2 - a - b) \cdot 1 = q^2 + q + a - b = q^2 + 2q - 1$$

since $a - b = q - 1$.

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Suppose now that $\pi_0 \cap \mathcal{B}$ is a Γ -set with center P , and thus contains $q-1$ Baer sublines through P other than ℓ . Planes, other than π_0 , through L which meet \mathcal{B} in a Γ -set do not have center P and thus provide no additional Baer sublines through P in \mathcal{B} , as is true for all planes through L meeting \mathcal{B} in a \mathcal{C}_F -set. The planes through L meeting \mathcal{B} in a K -set, of which there are $q^2 - a - (b-1) = q^2 + 1 - a - b$, each provide one additional Baer subline through P in \mathcal{B} . Again, counting the total number of Baer sublines through P , other than ℓ , that are contained in \mathcal{B} , we obtain

$$1 \cdot (q-1) + a \cdot 2 + (q^2 + 1 - a - b) \cdot 1 = q^2 + q + a - b = q^2 + 2q - 1,$$

the same count as in the previous case. Hence, after including the Baer subline ℓ , in all cases we obtain precisely $q^2 + 2q$ Baer sublines through P contained in \mathcal{B} .

Finally, counting in two ways the total number of flags (P, ℓ) , where ℓ is a Baer subline contained in \mathcal{B} , we see that the number of Baer sublines in \mathcal{B} is $(q^2 + 2q)(q^2 + 1)^2 / (q + 1)$. Since this must be an integer, we have $(q + 1) \mid 4$ and hence $q = 3$, a contradiction. \square

Remark 5.5. Note that for $q = 3$ there is an example of a base \mathcal{B} of size $q^4 + q^3 - q^2 + 1 = 100 = (q^2 + 1)^2$ containing no lines. However, the associated pencil \mathcal{F} contains a degenerate Hermitian surface.

Proposition 5.6. *If $q \geq 4$, then \mathcal{B} contains at least two lines.*

Proof. The argument is a refinement of the above proof. Suppose that \mathcal{B} contains exactly one line, say M . Then every plane through M intersects \mathcal{B} in a degenerate \mathcal{C}_F -set (that is, the union of M and an affine Baer subplane) by Proposition 4.1. Let ℓ be a Baer subline contained in \mathcal{B} which intersects M in one point (through every point $P \in \mathcal{B}$ there is at least one such a Baer subline), and let L be the line containing ℓ . Every plane through L intersects \mathcal{B} in one of the following: an H -set, a K -set, a Γ -set, a \mathcal{C}_F -set, or a degenerate \mathcal{C}_F -set. Let a be the number of planes through L intersecting \mathcal{B} in an H -set, and let b be the number of planes through L intersecting \mathcal{B} either in a Γ -set or in a \mathcal{C}_F -set. Then

$$q^4 + 2q^2 + 1 = |\mathcal{B}| = a(q^2 + q) + b(q^2 - q) + (q^2 - a - b)q^2 + 2q^2 + 1,$$

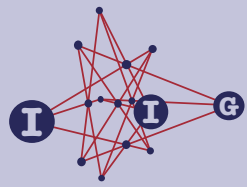
implying that $a - b = 0$ and hence $a = b$.

Now let P be the point $\ell \cap M$, and count the total number of Baer sublines contained in \mathcal{B} which are incident with P but are not contained in M . A computation analogous to that made in the proof of Proposition 5.4 shows that this number is

$$2a + (q^2 - a - b) + q = q^2 + q,$$

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since $a = b$. Note that the tangent plane to \mathcal{H}_0 at P , where \mathcal{H}_0 is the unique Hermitian surface in the pencil \mathcal{F} which contains the line L , is the plane intersecting \mathcal{B} in a degenerate \mathcal{C}_F -set.

Next let Q be a point of ℓ different from $P = \ell \cap M$. Counting the number of Baer sublines incident with Q and contained in \mathcal{B} , we obtain

$$2a + (q^2 - a - b) + (q - 1) + (q + 1) = q^2 + 2q,$$

again using $a = b$. Note that the tangent plane to \mathcal{H}_0 at Q is not the plane intersecting \mathcal{B} in a degenerate \mathcal{C}_F -set, and hence meets \mathcal{B} either in a Γ -set or a K -set with center Q . The count is the same in both cases, as in the proof of Proposition 5.4.

Counting in two ways the number of flags (P, ℓ) , where ℓ is a Baer subline contained in \mathcal{B} and not contained in M , we see that the total number of such sublines is $((q^2 + q)(q^2 + 1) + (q^2 + 2q)(q^4 + q^2))/(q + 1)$. As this must be an integer, we have that $(q + 1) \mid 2$, a contradiction. \square

Theorem 1.2 now follows from Propositions 4.4, 4.5, and 5.6 when $q \geq 4$. For $q = 2$ and $q = 3$, the result follows from an exhaustive computer search.

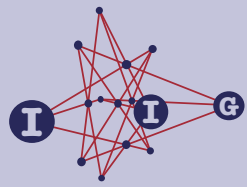
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