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# A $t \pmod{p}$ result on weighted multiple $(n - k)$ -blocking sets in $\text{PG}(n, q)$

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## Abstract

In this article, we prove a  $t \pmod{p}$  result for minimal weighted  $t$ -fold  $(n - k)$ -blocking sets in  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ ,  $n \geq 2$ . Such a theorem plays a crucial role in characterizing minimal weighted  $t$ -fold  $(n - k)$ -blocking sets. Our result is based on generalizations of earlier theorems on blocking sets in  $\text{PG}(2, q)$  to weighted blocking sets of higher dimensions.

**Keywords:** weighted multiple blocking sets,  $t \pmod{p}$  result

**MSC 2000:** 05B25, 51E20, 51E21

## 1. Introduction

Throughout this paper,  $\text{PG}(n, q)$  and  $\text{AG}(n, q)$  will respectively denote the  $n$ -dimensional projective and affine space over the Galois field  $\text{GF}(q)$ , where  $q = p^h$ ,  $p$  prime,  $h \geq 1$ .

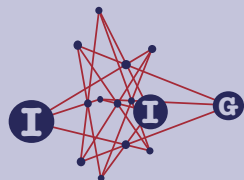
A  $t$ -fold  $(n - k)$ -blocking set  $B$  of  $\text{PG}(n, q)$ , with  $0 < k < n$ , is a set of points of  $\text{PG}(n, q)$  intersecting every  $k$ -dimensional subspace of  $\text{PG}(n, q)$  in at least  $t$  points.

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A point  $P$  of  $B$  is called *essential* if there is a  $k$ -dimensional subspace through  $P$  intersecting  $B$  in exactly  $t$  points. A  $t$ -fold blocking set  $B$  is called *minimal* if all of its points are essential. A 1-fold  $(n - k)$ -blocking set is also called an  $(n - k)$ -*blocking set*. A  $t$ -fold 1-blocking set in  $\text{PG}(2, q)$  is also called a  $t$ -fold *blocking set*, or a  $t$ -fold *planar blocking set*. A 1-fold blocking set in  $\text{PG}(2, q)$  is simply called a *blocking set* in  $\text{PG}(2, q)$ .

These latter  $t$ -fold planar blocking sets have been studied in great detail. General bounds can be found in Ball [1], and are mentioned in the following table. In this table, and in the following tables,  $p$  is a prime, and  $c_2 = c_3 = 2^{-1/3}$ , where  $c_p = 1$  if  $p > 3$ . In the first table, the first two columns give the conditions on  $q$  and  $t$ , while the third column gives the lower bound on  $k = |B| - t(q + 1)$ .

$q$	conditions	$k =  B  - t(q + 1)$
$q$	no line in $B$	$\geq \sqrt{tq} + 1 - t$
$p > 3$	$1 < t < p/2$	$\geq (p + 1)/2$
$p > 3$	$t > p/2$	$\geq p - t$

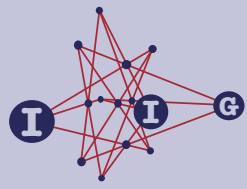
The following table contains what was proved for minimal  $t$ -fold blocking sets of  $\text{PG}(2, q)$  in [3]. The last two columns give the structure of  $B$ , plus an implied lower bound on the value  $k$ .

$q$	$t, k =  B  - t(q + 1)$	implies $k$	$B$
$p^{2d+1}$	$t = 1, k < c_p q^{2/3}$		line
$p^{2d+1}$	$1 < t < q/2 - c_p q^{2/3}/2$	$\geq c_p q^{2/3}$	
$p^{2d} > 4$	$t = 1, k < c_p q^{2/3}$		line or Baer subplane
$p^{2d} > 4$	$1 < t < c_p q^{1/6}, k < c_p q^{2/3}$	$\geq t\sqrt{q}$	union of $t$ disjoint Baer subplanes
$p^2$	$t = 1, k < p \lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \rceil$		line or Baer subplane
$p^2$	$1 < t < q^{1/4}/2, k < p \lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \rceil$	$\geq t\sqrt{q}$	union of $t$ disjoint Baer subplanes

The next two tables summarize the results of [2] for minimal  $t$ -fold blocking sets of  $\text{PG}(2, q)$ . The third and fourth column give the implied lower bounds on  $k$ , the information on the structure of  $B$ , plus some extra remarks.

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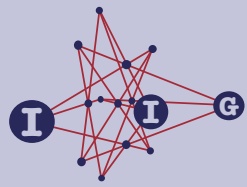
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$q$	$t, k =  B  - t(q + 1),$ other conditions	implies $k$	remark
$p^{6m}$	$2 \leq t < p^{3m/2}/4, k < p^{4m}\sqrt{p}/2$ no Baer subplane in $B$	$\geq tp^{4m} - 4t^2p^{2m}$	
$p^{6m+1}$	$2 \leq t < p^{3m/2+1/4}/4,$ $k < p^{4m+1} - 2p^{2m+1}$	$\geq \max(tp^{4m} - 4t^2p^{2m-1},$ $p^{4m+1} - p^{4m} - p^{2m+1}/2)$	$m \geq 1$
$p^{6m+2}$	$2 \leq t < p^3/(4(p+1))$ if $m = 1$ $2 \leq t < p^{(3m+1)/2}/4$ if $m > 1$ $k < p^{4m+2}/2$ no Baer subplane in $B$	$\geq tp^{4m+1} - 4t^2p^{2m}$	$m \geq 1$ $p \geq 5$
$p^{6m+3}$	$2 \leq t < p^{(6m+3)/4}/4,$ $k < p^{4m+2}\sqrt{p}/2$	$\geq tp^{4m+2} - 4t^2p^{2m+1}$	$p \geq 23$ ( $m = 0$ ) $p \geq 3$ ( $m = 1$ )
$p^{6m+4}$	$2 \leq t < p^{(3m+2)/2}/4,$ $k < p^{4m+3} - 2p^{2m+2}$ no Baer subplane in $B$	$\geq \max(tp^{4m+2} - 4t^2p^{2m},$ $p^{4m+3} - p^{4m+2} - p^{2m+2}/2)$	$m \geq 1$
$p^{6m+5}$	$k < p^{4m+4}/2,$ $2 \leq t < p^{3m/2+5/4}/4$ for $m > 0$ $2 \leq t \leq (p-3)/4$ for $m = 0$	$\geq \max(tp^{4m+3} - 4t^2p^{2m+1},$ $p^{4m+7/2} - p^{4m+3} - p^{2m+2}/2)$	$p \geq 5$

$q$	$t, k =  B  - t(q + 1),$ other conditions	$B$
$p^{6m}$	$2 \leq t < p^{3m/2}/4, k < \min(p^{4m}\sqrt{p}/2,$ $2p^{4m} + (t-2)p^{3m} - 16p^{2m})$	$t - 1$ disjoint Baer subplanes union a $t$ -th minimal blocking set
$p^{6m+2}$	$m \geq 1, 2 \leq t < p^{3m/2+1/2}/4, k < \min(p^{4m+2}/2,$ $2p^{4m+1} + (t-2)p^{3m+1} - 16p^{2m})$	union of $t$ disjoint Baer subplanes
$p^{6m+4}$	$2 \leq t < p^{(3m+2)/2}/4,$ $k < \min(p^{4m+3} - 2p^{2m+2}, (t-2)p^{3m+2} +$ $\max(2p^{4m+2} - 16p^{2m}, p^{4m+3} - p^{4m+2} - p^{2m+2}/2))$	union of $t$ disjoint Baer subplanes

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For a  $t$ -fold blocking set  $B$ , a “ $t \pmod{p}$  result” states that every line intersects  $B$  in  $t \pmod{p}$  points. In the theory of  $t$ -fold planar blocking sets,  $t \pmod{p}$  results for *small* minimal  $t$ -fold planar blocking sets play an important role.

**Definition 1.1.** A blocking set of  $\text{PG}(2, q)$  is called *small* when it has less than  $3(q + 1)/2$  points.

If  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , the *exponent*  $e$  of the minimal blocking set  $B$  of  $\text{PG}(2, q)$  is the maximal integer  $e$  such that every line intersects  $B$  in  $1 \pmod{p^e}$  points.

**Theorem 1.2** (Szőnyi [10], Sziklai [9]). *Let  $B$  be a small minimal 1-fold blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ . Then  $B$  intersects every line in  $1 \pmod{p}$  points, so for the exponent  $e$  of  $B$  we have  $1 \leq e \leq h$ . In fact this exponent is a divisor of  $h$ .*

The Linearity Conjecture (see Sziklai [9]) states that a small minimal blocking set is always a  $\text{GF}(p^e)$ -linear blocking set, i.e.  $\text{GF}(p^e)$  is a subfield of  $\text{GF}(q)$  and the blocking set is a projected image of a suitable subgeometry  $\text{PG}(h/e, p^e)$ .

Let’s see how these notions were generalized for higher dimensions and for  $t$ -fold blocking sets.

**Definition 1.3.** A 1-fold  $(n - k)$ -blocking set of  $\text{PG}(n, q)$  is called *small* when it has less than  $3(q^{n-k} + 1)/2$  points.

If  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , the *exponent*  $e$  of a minimal 1-fold  $(n - k)$ -blocking set  $B$  in  $\text{PG}(n, q)$  is the maximal integer  $e$  such that every  $k$ -dimensional space intersects  $B$  in  $1 \pmod{p^e}$  points.

Szőnyi and Weiner [11] proved a  $1 \pmod{p}$  result for small minimal 1-fold  $(n - k)$ -blocking sets in  $\text{PG}(n, q)$ .

**Theorem 1.4** (Szőnyi and Weiner [11]). *A minimal 1-fold  $(n - k)$ -blocking set in  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p > 2$  prime,  $h \geq 1$ , of size less than  $\frac{3}{2}(q^{n-k} + 1)$  intersects every subspace in zero points or in  $1 \pmod{p}$  points.*

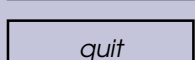
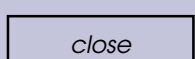
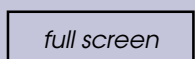
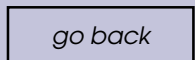
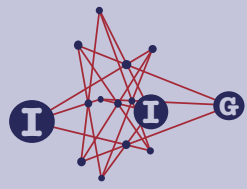
The  $1 \pmod{p}$  result in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , was extended by Blokhuis *et al.* to a  $t \pmod{p}$  result on *small* minimal  $t$ -fold blocking sets in  $\text{PG}(2, q)$ .

**Definition 1.5.** A  $t$ -fold blocking set of  $\text{PG}(2, q)$  is called *small* when it has less than  $tq + (q + 3)/2$  points.

If  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , the *exponent*  $e$  of the minimal  $t$ -fold blocking set  $B$  in  $\text{PG}(2, q)$  is the maximal integer  $e$  such that every line intersects  $B$  in  $t \pmod{p^e}$  points.

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**Theorem 1.6** (Blokhuis *et al.* [2]). *Let  $B$  be a small minimal  $t$ -fold blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , then  $B$  intersects every line in  $t \pmod{p}$  points.*

For a multiset  $B$  in  $\text{PG}(n, q)$ , we call the *multiplicity* of a point of  $B$  also the *weight* of that point. A point of  $B$  is called *simple* if it has weight one. A *multiple* point of  $B$  has weight larger than one. A *weighted  $t$ -fold  $(n - k)$ -blocking set*  $B$  of  $\text{PG}(n, q)$ , with  $0 < k < n$ , is a multiset of points of  $\text{PG}(n, q)$  intersecting every  $k$ -dimensional subspace of  $\text{PG}(n, q)$  in at least  $t$  points, counted with weights.

A point  $P$  of a weighted  $t$ -fold  $(n - k)$ -blocking set  $B$  is called *essential* if there is a  $k$ -dimensional subspace through  $P$  intersecting  $B$  in  $t$  points, counted with weights. A weighted  $t$ -fold  $(n - k)$ -blocking set  $B$  is called *minimal* if all of its points are essential.

The General Linearity Conjecture for  $t$ -fold blocking sets (see Sziklai [9]) states that (if  $t$  is small enough then) a small minimal  $t$ -fold  $(n - k)$ -blocking set in  $\text{PG}(n, q)$  is always the (not necessarily disjoint) union of  $\text{GF}(p^{e_i})$ -linear (possibly multiple)  $(n - k)$ -blocking sets, i.e. for each of the  $(n - k)$ -blocking sets  $\text{GF}(p^{e_i})$  is a subfield of  $\text{GF}(q)$  and it is a projected image of a suitable subgeometry  $\text{PG}(m_i, p^{e_i})$ .

The goal of this article is to prove a  $t \pmod{p}$  result on weighted minimal  $t$ -fold  $(n - k)$ -blocking sets in  $\text{PG}(n, q)$ ,  $n \geq 2$ .

Once such a  $t \pmod{p}$  result has been proved, characterization results can be obtained. We illustrate this in [5] by characterizing minimal  $t$ -fold  $(n - k)$ -blocking sets in  $\text{PG}(n, q)$ ,  $q$  square.

We prove in the following section a  $t \pmod{p}$  result on weighted minimal  $t$ -fold blocking sets in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ . This result is then used to obtain a  $t \pmod{p}$  result on weighted minimal  $t$ -fold  $(n - k)$ -blocking sets in  $\text{PG}(n, q)$ ,  $n > 2$ . Here the idea is based on the generalization of [11].

As a supplementary result, we also prove that small minimal weighted  $t$ -fold blocking sets in  $\text{PG}(2, q)$ , containing a line  $\ell$ , are the sum of this line  $\ell$  and a minimal  $(t - 1)$ -fold blocking set. This implies that, when characterizing small  $t$ -fold blocking sets in  $\text{PG}(2, q)$ , it is possible to assume that they do not contain any lines.

## 2. A $t \pmod{p}$ result

Let  $B$  be a *minimal* weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$ , with  $|B| = tq + t + k$ , where  $t + k < q$ .

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Assume that the line  $l_\infty$  is an  $m$ -secant to  $B$ . Consider  $\text{PG}(2, q)$  as the affine plane  $\text{AG}(2, q)$  with  $l_\infty$  as the line at infinity. Assume that  $B \cap l_\infty = D = \{(\infty), \dots, (\infty), (y_1), \dots, (y_{m-s})\}$ , where  $(\infty)$  is a point of weight  $s$  of  $B$  ( $1 \leq s \leq t$ ), where some of the other points of  $D$  might be multiple points of  $B$ , and that  $U = B \setminus D = \{(a_i, b_i) : i = 1, \dots, tq + t + k - m\}$ , where  $U$  is a multiset when  $B$  has affine multiple points.

We first define the Rédei polynomial associated to the  $t$ -fold blocking set  $B$ . The last equation in the following definition follows from the fact that this Rédei polynomial is  $t$  times zero everywhere in  $\text{GF}(q) \times \text{GF}(q)$  [4].

**Definition 2.1** (The Rédei polynomial of the set  $B$ ).

$$\begin{aligned}
 H(X, Y) &= \prod_{j=1}^{m-s} (Y - y_j) \prod_{i=1}^{tq+t+k-m} (X + a_i Y - b_i) \\
 &= \prod_{j=1}^{m-s} (Y - y_j) \sum_{i=0}^{tq+t+k-m} X^{tq+t+k-m-i} h_i(Y) \tag{1}
 \end{aligned}$$

$$\begin{aligned}
 &= (X^q - X)^t f_0(X, Y) + (X^q - X)^{t-1} (Y^q - Y) f_1(X, Y) \\
 &\quad + \dots + (Y^q - Y)^t f_t(X, Y), \tag{2}
 \end{aligned}$$

where  $\deg(h_i) \leq i$ ,  $i = 0, \dots, tq + t + k - m$ , and  $\deg(f_i) \leq k + t - s$ ,  $i = 0, \dots, t$ .

It is well-known that this polynomial encodes the intersection properties of  $B$  with lines: e.g. a line with equation  $Y = mX + b$  intersects  $B$  in  $r$  points if and only if the point  $(b, m)$  in the *dual* plane has multiplicity  $r$  on the curve  $H(X, Y)$  (i.e.  $r$  linear factors of  $H$  go through  $(b, m)$ ).

Choose a point  $(b, m)$ ,  $b \notin \{b_j \mid (0, b_j) \in U\}$ ,  $m \neq y_j$ . Consider  $H(X, m) = (X^q - X)^t f_0(X, m)$ . By the properties of the Rédei polynomial, the line  $Y = mX + b$  intersects  $U$  in more than  $t$  points if and only if  $X = b$  is a root of  $H(X, m)$  with multiplicity  $\geq t + 1$  if and only if  $(b, m)$  is a point of the algebraic curve  $f_0(X, Y)$ . Considering  $H(b, Y) = (Y^q - Y)^t f_t(b, Y)$  instead, we get that the line  $Y = mX + b$  intersects  $U$  in more than  $t$  points if and only if  $(b, m)$  is a point of the algebraic curve  $f_t(X, Y)$ .

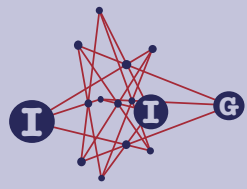
Therefore, these two algebraic curves  $f_0$  and  $f_t$  have almost the same set of  $\text{GF}(q)$ -rational points.

If  $m = y_j$  or  $b \in \{b_j \mid (0, b_j) \in U\}$ , and the line  $Y = mX + b$  intersects  $U$  in more than  $t$  points, then  $f_0(b, m) = f_t(b, m) = 0$  holds again. As  $H(X, m)$  (or  $H(b, Y)$ ) is identically zero in this case,  $f_0(b, m) = 0$  or  $f_t(b, m) = 0$  does not imply that  $Y = mX + b$  intersects  $U$  in more than  $t$  points.

Later on in this section, we will assume that there is no line contained in  $B$ . As the following theorems will show, this is no restriction when  $2t + k < q + 2$ .

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**Theorem 2.2.** Let  $B$  be a minimal weighted  $t$ -fold blocking set of  $\text{PG}(2, q)$ , with  $|B| = tq + t + k$ , where  $2t + k < q + 2$ , containing a line  $\ell$ . Then  $B$  is the sum of the line  $\ell$  and the minimal weighted  $(t - 1)$ -fold blocking set  $B^*$ , obtained from  $B$  by reducing the weight of every point  $P$  of  $\ell$  by one.

*Proof.* Since  $\ell \subseteq B$ ,  $|\ell \cap B| \geq q + 1$ .

If  $|\ell \cap B| \geq q + t$ , then after reducing the weight of every point of  $\ell$  by one, a new weighted set  $B^*$  is obtained which still intersects every line in at least  $t - 1$  points. Since  $B$  is a minimal weighted  $t$ -fold blocking set, also  $B^*$  is a minimal weighted  $(t - 1)$ -fold blocking set.

Assume now that  $q + 1 \leq |B \cap \ell| < q + t$ . Reduce again the weight of every point on  $\ell$  by one, and add a minimal number of simple points  $P_1, \dots, P_r$  of  $\ell$  back, until a weighted  $(t - 1)$ -fold blocking set  $B^*$  is obtained, hence  $|B^* \cap \ell| = t - 1$ . We need to add at most  $r \leq t - 1$  points to achieve this, hence  $|B^*| \leq tq + t + k - (q + 1) + t - 1 = (t - 1)q + 2(t - 1) + k$ . A particular feature of a point  $P_i$ ,  $i = 1, \dots, r$ , is that the line  $\ell$  is the only  $(t - 1)$ -secant to  $B^*$  passing through  $P_i$ .

Finally, we show that through  $P_i$ , there pass at least two  $(t - 1)$ -secants, hence the above case cannot occur. Now we choose our coordinate system in such a way that  $(\infty) \in B$ ,  $P_i$  is an affine point  $(a, b)$ , and  $\ell_\infty \cap \ell \notin B^*$  and  $(\infty)$  has multiplicity  $s$ . Suppose that  $|\ell_\infty \cap B^*| = m$  and write up the Rédei polynomial. Since  $B^*$  is a  $(t - 1)$ -fold blocking set, using a suitable indexing we get that

$$\begin{aligned} H^*(X, Y) &= \prod_{j=1}^{m-s} (Y - y_j) \prod_{i=q+2}^{tq+t+k+r-m} (X + a_i Y - b_i) \\ &= (X^q - X)^{t-1} f_0^*(X, Y) + (X^q - X)^{t-2} (Y^q - Y) f_1^*(X, Y) \\ &\quad + \dots + (Y^q - Y)^{t-1} f_{t-1}^*(X, Y), \end{aligned} \quad (3)$$

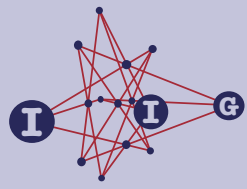
where  $\deg(f_i^*) \leq |B^*| - q(t - 1) - s \leq 2(t - 1) + k - s$ ,  $i = 0, \dots, t - 1$ .

The argument before this theorem shows that if a line  $Y = mX + b$  intersects  $B^*$  in more than  $(t - 1)$  points, then  $(b, m)$  is a point of the curve  $f_0^*$ . Each line except  $\ell$  through the point  $P_i = (a, b)$  intersects  $B^*$  in at least  $t$  points. These lines are points of the line  $X + aY - b$  in the dual plane. Hence  $X + aY - b$  intersects  $f_0^*$  in at least  $q - 1$  points (we do not see the vertical line here). Since  $\deg f_0^* < q - 1$ , Bézout's theorem implies that the line  $X + aY - b$  is a component of  $f_0^*$ . Suppose that  $\ell$  is the line  $\ell = Y + m'X + b'$ . Then  $f_0^*(b', m') = 0$  and since  $\ell \cap \ell_\infty \notin B^*$ ,  $\ell$  intersects  $B^*$  in at least  $t$  points. This is a contradiction, hence  $q + 1 \leq |B \cap \ell| < q + t$  does not occur.  $\square$

As the next example shows, the above theorem is sharp.

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**Example 2.3.** Let  $S$  be the set of points lying on the lines of a dual hyperoval in  $\text{PG}(2, q)$ ,  $q$  even. Then  $S$  is a  $(\frac{q}{2} + 1)$ -fold blocking set of size  $(\frac{q}{2} + 1)q + (\frac{q}{2} + 1)$  (each point in  $S$  has multiplicity one). Note that now  $t = \frac{q}{2} + 1$ ,  $k = 0$  and  $2t + k = q + 2$ . If we delete a line of  $S$ , then the resulting point set is not a  $\frac{q}{2}$ -fold blocking set.

**Remark 2.4.** Theorem 2.2 has some straightforward applications.

- (1) It first of all shows that when characterizing minimal weighted  $t$ -fold blocking sets of size  $tq + t + k$ , where  $2t + k < q + 2$ , in  $\text{PG}(2, q)$ , it is possible to assume that they do not contain any lines.
- (2) Moreover, also when proving the  $t \pmod{p}$  result for a minimal weighted  $t$ -fold blocking set  $B$ ,  $|B| = tq + t + k$ , where  $2t + k < q + 2$ , it is possible to assume that there are no lines contained in  $B$ . If there is a line  $\ell$  contained in  $B$ , then Theorem 2.2 implies that you can reduce the weight of every point of  $\ell$  by one in order to obtain a new minimal weighted  $(t - 1)$ -fold blocking set  $B^*$ . Proving the  $t \pmod{p}$  result for  $B$  is now reduced to proving the  $(t - 1) \pmod{p}$  result for  $B^*$ .
- (3) Now we are also able to characterize weighted minimal  $t$ -fold blocking sets of size  $tq + t$ , with  $2t < q + 2$ , and to exclude the existence of weighted minimal  $t$ -fold blocking sets of size  $tq + t + 1$ , with  $2t + 1 < q + 2$ .

**Theorem 2.5.** A weighted  $t$ -fold blocking set  $B$  in  $\text{PG}(2, q)$ , of size  $|B| = tq + t$ , where  $2t < q + 2$ , is a sum of  $t$  lines.

There does not exist a weighted minimal  $t$ -fold blocking set  $B$  in  $\text{PG}(2, q)$  of size  $|B| = tq + t + 1$ ,  $2t + 1 < q + 2$ .

*Proof.* Suppose that  $tq + t \leq |B| \leq tq + t + 1$ . Then counting the incidences of the points of  $B$  with the lines through a point  $R$  not in  $B$ , we have that through  $R$  all the lines are  $t$ -secants if  $|B| = tq + t$  and there is exactly one  $(t + 1)$ -secant and  $q - t$   $t$ -secants through  $R$  if  $|B| = tq + t + 1$ .

Now count the incidences of the points of  $B$  with the lines through a point  $R' \in B$ . Assume first of all that  $|B| = t(q + 1)$ . Then we get in total  $wt(R') + (q + 1)(t - wt(R'))$  incidences if we assume that  $R'$  only lies on  $t$ -secants to  $B$ . Since  $|B| = t(q + 1)$ , we obtain that  $t(q + 1) = t(q + 1) - qwt(R')$ , hence  $wt(R') = 0$ , but then  $R' \notin B$ . So we get that  $R'$  lies on at least one line  $\ell$  completely contained in  $B$  when  $|B| = tq + t$ .

Secondly, assume that  $|B| = t(q + 1) + 1$ , let  $R' \in B$ , and assume that  $R'$  does not lie on a line  $\ell$  completely contained in  $B$ , then  $R'$  only lies on  $t$ - and  $(t + 1)$ -secants to  $B$ . If we would assume that  $R'$  only lies on  $t$ -secants to  $B$ , then counting the incidences of the lines through  $R'$  with the points of  $B$ , we obtain  $wt(R') + (q + 1)(t - wt(R'))$  incidences. So, there still remain  $t(q + 1) + 1 -$

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$wt(R') - t(q+1) + (q+1)wt(R') = 1 + qwt(R')$  incidences of the lines through  $R'$  with the points of  $B$ . Since we assume that  $R'$  does not lie on a line completely contained in  $B$ , these lines can share at most one extra point with  $B$ . There are  $q+1$  lines through  $R'$  and there remain  $1 + qwt(R')$  incidences. This implies that  $wt(R') = 1$  and that  $R'$  lies on  $q+1$   $(t+1)$ -secants, when  $|B| = tq + t + 1$ . This latter case means that  $B$  is not minimal. Hence we can assume that each point of  $B$  lies on at least one line completely contained in  $B$ .

Now the  $t$  points of any  $t$ -secant (which must exist) and Theorem 2.2 show that  $B$  contains the sum of  $t$  lines, which is a  $t$ -fold blocking set already, of size  $tq + t$ .  $\square$

**Remark 2.6.** One can observe now that a weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$ , of size  $tq + t$ , where  $2t < q + 2$ , intersects every line in  $t \pmod{p}$  points; also that through any point of it there pass at least  $q + 1 - t$   $t$ -secants.

**Lemma 2.7.** The polynomial  $\prod_{j=1}^{t-s} (Y - y_j)$  divides  $f_0(X, Y)$  if  $k + t < q$ .

*Proof.* By (1),

$$H(X, Y) = \sum_{i=0}^{tq+k} \left( h_i(Y) \cdot \prod_{j=1}^{t-s} (Y - y_j) \right) X^{tq+k-i}.$$

So every coefficient polynomial of a term  $X^{tq+k-i}$  is divisible by  $\prod_{j=1}^{t-s} (Y - y_j)$ . By (2), the high degree part

$$\prod_{j=1}^{t-s} (Y - y_j) \cdot X^{tq+k} + \dots + h_k(Y) \cdot \prod_{j=1}^{t-s} (Y - y_j) \cdot X^{tq}$$

must be equal to  $X^{tq} f_0(X, Y)$ , when one compares the  $X$ -degrees of the two expressions (1) and (2) for  $H(X, Y)$ . So  $\prod_{j=1}^{t-s} (Y - y_j)$  divides  $f_0(X, Y)$ .  $\square$

If  $X = 0$  intersects  $U$  in the, possible weighted, points  $(0, b_j)$ ,  $j = 1, \dots, z$ , then a similar argument shows that  $\prod_{j=1}^z (X - b_j)$  divides  $f_t(X, Y)$ , where the product is taken over the values  $b_j$ , according to their weights.

**Theorem 2.8.** Let  $B$  be a minimal weighted  $t$ -fold blocking set of  $\text{PG}(2, q)$ , with  $|B| = tq + t + k < (t+1)q$ . Then every point of  $B$  lies on at least  $q + 1 - k - t$  different  $t$ -secants.

*Proof.* Let  $P = (a, b) \in U$  and suppose that  $(\infty) \in B$ ,  $|l_\infty \cap B| = t$ . Assume that  $P$  lies on more than  $k + t$  different lines sharing at least  $t + 1$  points with  $B$ . Then more than  $k$  of those lines intersect  $l_\infty$  in a point not belonging to  $B$ .

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Each of these latter lines defines a point of  $f_0(X, Y) / \prod_{j=1}^{t-s} (Y - y_j)$ . More precisely, they define intersection points, in the dual plane, of the algebraic curve  $f_0(X, Y) / \prod_{j=1}^{t-s} (Y - y_j) = 0$  with the line  $X + aY - b = 0$ . The polynomial  $f_0(X, Y) / \prod_{j=1}^{t-s} (Y - y_j)$  has at most degree  $k$ , so by Bézout's theorem, the linear term  $X + aY - b$  is a factor of  $f_0(X, Y) / \prod_{j=1}^{t-s} (Y - y_j)$ .

Consider a line through  $P$  with slope  $m \neq y_j$ ,  $m \neq \infty$ , so that we can use the arguments above.

Evaluating  $H(X, Y)$  at  $Y = m$ , we get

$$H(X, m) = \prod_{j=1}^{t-s} (m - y_j) \prod_{i=1}^{tq+k} (X + a_i m - b_i) = (X^q - X)^t f_0(X, m).$$

The fact that  $X + aY - b$  is a linear factor of  $f_0$  means geometrically that the lines through  $P$  with slope  $m \neq y_j$ ,  $m \neq \infty$ , intersect  $U$  in at least  $t + 1$  points.

We have shown that every line joining  $P$  to a point of  $l_\infty \setminus B$  is a  $\geq (t + 1)$ -secant. But  $l_\infty$  is an arbitrary  $t$ -secant, so for any  $t$ -secant  $l$  incident with  $P$  we just need to find a  $t$ -secant incident with a point of  $l \setminus B$ . A point  $Q \notin B$  is incident with at least  $q + 1 - k > t$  different  $t$ -secants, and so at least one of them meets  $l$  in a point not in  $B$ .  $\square$

**Corollary 2.9.** *Let  $B$  be a weighted  $t$ -fold blocking set of  $\text{PG}(2, q)$ , with  $|B| = tq + t + k < (t + 1)q$ . Assume that  $P$  is an essential point of  $B$ . Then there are at least  $q + 1 - k - t$  different  $t$ -secants through  $P$ .*

*Proof.* Delete the non-essential points of  $B$  one-by-one until a minimal  $t$ -fold blocking set  $B'$  is obtained. By Theorem 2.8, there will be at least  $q + 1 - (|B'| - tq)$  different  $t$ -secants of  $B'$  through  $P$ . Now if we add back the points of  $B \setminus B'$ , then through  $P$ , we will see at least  $q + 1 - (|B'| - tq) - |B \setminus B'|$   $t$ -secants to  $B$ .  $\square$

We will now adapt the results of [2, 8] to the case when there are multiple points. In this section, from now on, we suppose that  $|B| < tq + (q + 3)/2$ . The cases of  $|B| = t(q + 1)$  and  $|B| = t(q + 1) + 1$ , with  $|B| < tq + (q + 3)/2$ , are all discussed in Theorem 2.5, except for the case  $|B| = t(q + 1)$  with  $t = q/2 + 1$  for  $q$  even.

The weighted  $t$ -fold blocking sets  $B$  in  $\text{PG}(2, q)$ ,  $q$  even, of size  $t(q + 1)$  with  $t = q/2 + 1$ , have been classified in [6] and [7]. They are either a sum of  $t = q/2 + 1$  lines or equal to the  $(q/2 + 1)$ -fold blocking set of Example 2.3. So we can consider all the  $t$ -fold blocking sets of sizes  $t(q + 1)$  and  $t(q + 1) + 1$ , with  $|B| < tq + (q + 3)/2$ , to be classified. So from now on, we assume that  $tq + (q + 3)/2 > |B| = tq + t + k \geq tq + t + 2$ .

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Note that since  $k \geq 2$ , we still have  $2t + k < q + 2$  so that Theorem 2.2 is valid. Hence, we also can assume that  $B$  does not contain any line.

Furthermore, we choose our coordinate system so that  $\ell_\infty$  is a  $t$ -secant and the point  $(\infty)$  in  $B$  has multiplicity  $s$ , where  $1 \leq s \leq t$ .

The following lemma can be proved in the same way as [2, Lemma 3.2]. Let  $B$  be a minimal weighted  $t$ -fold blocking set of size  $tq + t + k$ , where  $t + k < (q + 3)/2$  and  $k \geq 2$ . Recall the definition of the Rédei polynomial from the beginning of this section.

**Lemma 2.10.** *If a line  $Y = mX + b$  intersects  $B \cap U$  in more than  $t$  points, then  $f_0(b, m) = \dots = f_t(b, m) = 0$ .*

The following lemma is similar to [2, Lemma 3.3].

**Lemma 2.11.** *The algebraic curve  $f_0(X, Y) = 0$  does not have linear components depending on the variable  $X$ .*

*Proof.* Such a linear component depending on  $X$  should have the form  $X + aY - b = 0$ . The proof of Theorem 2.8 then shows that the point  $P = (a, b)$  is a non-essential point of  $B$ ; which contradicts the minimality of  $B$ .  $\square$

**Lemma 2.12.** *If  $B$  is minimal, then the polynomials  $f_0, \dots, f_t$  cannot have a common divisor different from  $Y - y_j$ .*

*Proof.* Such a polynomial would divide  $H(X, Y)$ ; so would be linear. This can only be of the form  $Y - y_j$ .  $\square$

We now come to the main theorem of this section: the proof of the  $t \pmod p$  result.

**Theorem 2.13.** *Let  $B$  be a minimal weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , with  $|B| = tq + t + k$ ,  $t + k < (q + 3)/2$ ,  $k \geq 2$ . Then every line intersects  $B$  in  $t \pmod p$  points.*

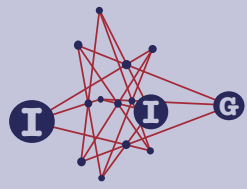
*Proof.* By Remark 2.4, it is possible to assume that  $B$  does not contain any lines. We will assume that the line at infinity intersects  $B$  in  $t$  points. In this way, we can use the beginning of the proof of [2, Theorem 3.1].

So let  $h(X, Y)$  be an absolutely irreducible component of the polynomial  $f_0(X, Y) / \prod_{j=1}^{t-s} (Y - y_j)$  of degree larger than one. The arguments of the proof of [2, Theorem 3.1] imply that  $h'_X \equiv 0$ .

If  $Y = m \neq y_i$ , we obtain  $H(X, m) = (X^q - X)^t f_0(X, m)$ , having  $t \pmod p$  solutions since  $f_0(X, m)$  is a  $p$ -th power. So every line  $Y = mX + b$ , not containing a point of  $B$  at infinity, intersects  $B$  in  $t \pmod p$  points.

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For every line  $\ell$  of which we are not yet sure that it intersects  $B$  in  $t \pmod{p}$  points, it is possible to find a new line at infinity intersecting  $B$  in  $t$  points and intersecting  $\ell$  in a point not belonging to  $B$ . Repeating the previous arguments now shows that also  $\ell$  intersects  $B$  in  $t \pmod{p}$  points.  $\square$

The next corollary follows from Theorem 2.8 and Remark 2.6.

**Corollary 2.14.** *Let  $B$  be a weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , with  $|B| = tq + t + k$ ,  $t + k < (q + 3)/2$ ,  $2t < q + 2$ . Assume that all the points of  $B$  on the line  $\ell$  are essential. Then  $\ell$  intersects  $B$  in  $t \pmod{p}$  points.  $\square$*

When each line intersects  $B$  in  $t \pmod{q}$  points, then the characterization of  $B$  is immediate.

**Proposition 2.15.** *Let  $B$  be a minimal weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$  of size  $tq + t + k$ , where  $t + k < (q + 3)/2$ ,  $k \geq 2$ . Assume that each line intersects  $B$  in  $t \pmod{q}$  points. Then  $B$  is a sum of  $t$  (not necessarily different) lines.*

*Proof.* Let  $\ell$  be a line of  $\text{PG}(2, q)$  not contained in  $B$ . Let  $P \in \ell \setminus B$ . Since all the lines, different from  $\ell$ , through  $P$  contain at least  $t$  points of  $B$ ,  $\ell$  contains at most  $t + k$  points of  $B$ .

Every point  $R$  of  $B$  lies on at least one line containing more than  $t$  points of  $B$ , so on a line  $\ell$  containing at least  $t + q$  points of  $B$ . Since  $t + k < t + q$ , the preceding paragraph implies that  $\ell$  is contained in  $B$ . By Theorem 2.2,  $B$  is the sum of this line  $\ell$  and a  $(t - 1)$ -fold blocking set  $B^*$  intersecting every line in  $(t - 1) \pmod{q}$  points. Repeating the above argument shows that  $B$  is a sum of  $t$  lines.  $\square$

### 3. A lower bound on the size of $B$

We now determine a lower bound on the size of a minimal weighted  $t$ -fold blocking set  $B$  in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ .

We again assume that  $B$  does not contain any lines, for it is trivially possible to construct a minimal weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$  by taking a sum  $B$  of  $t$  lines. Then  $|B| = t(q + 1)$ .

**Theorem 3.1.** *Let  $B$  be a minimal weighted  $t$ -fold blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , with  $|B| = tq + t + k$ ,  $t + k < (q + 3)/2$ , containing no lines.*

*Assume that  $h(X, Y)$  is a component of  $f_0$ , which can be written as  $h(X, Y) = g(X^{p^e}, Y)$  with  $g'_X \neq 0$ . Then  $k \geq \frac{q+p^e}{p^e+1} - t + 1$ .*

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*Proof.* This can be proved in the same way as [2, Prop. 3.6]. □

## 4. A $t \pmod{p}$ result in higher dimensions

**Theorem 4.1.** *A minimal weighted  $t$ -fold 1-blocking set  $B$  in  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , of size  $|B| = tq + t + k$ ,  $t + k \leq (q - 1)/2$ , intersects every hyperplane in  $t \pmod{p}$  points.*

*Proof.* The proof goes by induction on  $n$ . For  $n = 2$ , see Theorem 2.13 and Remark 2.6. Assume now that the theorem is true for  $n - 1$  dimensions, we are going to prove it for  $n$  dimensions. We will adapt the ideas of [11].

**Part 1.** We embed  $\Pi_n = \text{PG}(n, q)$  into  $\Pi_{2n-2} = \text{PG}(2n - 2, q)$ . Let  $H$  be a hyperplane of  $\Pi_n$ .

By the induction hypothesis, we can assume that  $B$  is not contained in  $H$ . Assume therefore that  $B \cap H$  is a weighted  $\alpha$ -fold blocking set in  $H$  with respect to hyperplanes of  $H$  and of cardinality  $\alpha(q + 1) + \beta$ , where  $0 \leq \alpha < t$ .

Consider an  $(n - 2)$ -dimensional subspace  $L$  in  $H$  sharing  $\alpha$  points with  $B$ . A counting argument shows that we can find an  $(n - 1)$ -dimensional subspace  $H^* \neq H$  of  $\Pi_n$ , through  $L$ , containing exactly  $t$  points  $P_i, i = 1, \dots, t$ , of  $B$ .

We construct in  $\Pi_{2n-2}$  the cone  $\mathcal{C}$  with vertex  $P$ , where  $P$  is an  $(n - 3)$ -dimensional space skew to  $\Pi_n$ , and base  $B \cup \{Q\}$ , with  $Q$  a point of  $H^* \setminus H$ ,  $Q \notin B$ .

By [11, Remark 2.1], there exists a regular  $(n - 2)$ -spread  $W$  of the hyperplane  $\langle H^*, P \rangle$  of  $\Pi_{2n-2}$  so that it contains  $\langle P, Q \rangle$  and  $L$ . Let  $\pi^W$  denote the projective plane defined by the  $(n - 2)$ -spread  $W$  and let  $\mathcal{C}'$  denote the image of  $\mathcal{C}$  in  $\pi^W$ .

**Part 2.** We first discuss the structure of  $\mathcal{C}'$  on the line at infinity of  $\pi^W$ .

The points of the cone with vertex  $P$  and base  $B$  in  $\langle P, H^* \rangle$  are the points of  $t$ , not necessarily different,  $(n - 2)$ -dimensional spaces  $\langle P, P_i \rangle, i = 1, \dots, t$ . The space  $\langle P, Q \rangle$  belongs to the  $(n - 2)$ -spread  $W$  and is given weight  $t$  in the weighted set  $\mathcal{C}'$ . The other elements of  $W$  are skew to  $\langle P, Q \rangle$  and share at most one point with each of the spaces  $\langle P, P_i \rangle, i = 1, \dots, t$ . If an element of  $W \setminus \{\langle P, Q \rangle\}$  contains  $\gamma$  points of the spaces  $\langle P, P_i \rangle, i = 1, \dots, t$ , then we give this element weight  $\gamma$  in  $\mathcal{C}'$ .

Hence the size of  $\mathcal{C}'$  is  $|B|q^{n-2} + t$ .

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**Part 3.** We prove that the set  $C'$  is a  $t$ -fold blocking set in  $\pi^W$ . The ideal point corresponding to the spread element  $\langle P, Q \rangle$  has multiplicity  $t$  and so the lines in  $\pi^W$  through this point are blocked at least  $t$  times by  $C'$ . Now take an arbitrary line  $\ell'$  of  $\pi^W$  not through this ideal point. The  $(n-1)$ -dimensional subspace  $\ell$  of  $\Pi_{2n-2}$  corresponding to this line is skew to  $P$ . The projection  $\ell^*$  of  $\ell$  from  $P$  to  $\Pi_n$  is an  $(n-1)$ -dimensional subspace in  $\Pi_n$  and so it contains at least  $t$  points of  $B$ . If  $S$  is in  $\ell^* \cap B$  then  $\langle P, S \rangle \subset C'$ , hence the intersection point of  $\ell$  and  $\langle P, S \rangle$  is a point of  $C'$ . Therefore  $\ell$  contains at least  $t$  points of  $C'$ .

So, for  $|B| = tq + t + k$ ,  $t + k \leq (q-1)/2$ , we have  $|C'| = tq^{n-1} + (t+k)q^{n-2} + t = tq^{n-1} + k' + t$  in  $\pi^W = \text{PG}(2, q^{n-1})$ , with  $t + k' < (q^{n-1} + 3)/2$ .

*Note that in  $\pi^W$ , the subspace  $H$  corresponds to a line  $h$ . In the rest of the proof we will show that the points of  $h \cap C'$  are all essential to  $C'$ . By Corollary 2.14, this will imply that  $h$  shares  $t \pmod{p}$  points with  $C'$ , and equivalently, that  $H$  shares  $t \pmod{p}$  points with  $B$ .*

**Part 4.** The ideal point  $L'$  of  $\pi^W$  corresponding to  $L$  is essential to  $C'$ . To see this, note that we can find a second  $(n-1)$ -dimensional subspace through  $L$ , not lying in  $\langle H^*, P \rangle$ , containing  $t$  points of  $B$ . Hence the corresponding line in  $\pi^W$  will be a  $t$ -secant through  $L'$ , which proves that the point  $L'$  is essential for  $C'$ .

Finally we show that the points of  $h \setminus L'$  are all essential to  $C'$ .

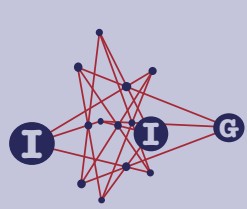
**Part 5.** First we show that through each point  $R_i$  of  $(H \setminus L) \cap B$  there is an  $(n-1)$ -space  $H_{R_i}$  of  $\Pi_n$  containing  $t$  points of  $B$  but not containing  $Q$ . Let  $H_{R_i}^*$  be an  $(n-1)$ -space of  $\Pi_n$  through  $R_i$  containing  $t$  points of  $B$  and containing  $Q$  as well.

We show that there is an  $(n-3)$ -dimensional subspace  $\Pi_{n-3}$  in  $H_{R_i}^*$  skew to  $B$ , such that  $\langle \Pi_{n-3}, R_i \rangle \neq \langle \Pi_{n-3}, Q \rangle$  and such that  $\langle \Pi_{n-3}, R_i \rangle$  only contains the point  $R_i$  of  $B$ . To obtain this, project the points of  $(B \cap H_{R_i}^*) \setminus \{R_i\}$  from  $R_i$  to an  $(n-2)$ -space  $T$  of  $H_{R_i}^*$  through  $Q$ . Since  $|(B \cap H_{R_i}^*) \setminus \{R_i\}| \leq t-1$ , the projection will contain at most  $t-1 < q$  different points, hence we can choose an  $(n-3)$ -space  $M$  in  $T$  not containing  $Q$  nor any of the projections of the points of  $(B \cap H_{R_i}^*) \setminus \{R_i\}$ . So  $\langle M, R_i \rangle$  intersects  $B \cup \{Q\}$  in  $R_i$  only, hence for  $\Pi_{n-3}$  we can choose any of the  $(n-3)$ -spaces of  $\langle M, R_i \rangle$  that are skew to  $R_i$ .

We now project  $B$  from  $\Pi_{n-3}$  onto a plane  $\pi$  of  $\Pi_n$ . We obtain a weighted  $t$ -fold blocking set  $B^*$  in  $\pi$ , of size  $|B^*| = tq + t + k$ ,  $t + k \leq (q-1)/2$ , where  $R_i$  is projected onto a point  $R_i^*$  having the same weight as  $R_i$  and where  $H_{R_i}^*$  is projected onto a  $t$ -secant through  $R_i^*$ . Hence  $R_i$  is an essential point of  $B^*$  and

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so, by Corollary 2.9, we can choose a  $t$ -secant  $\ell$  through  $R_i^*$ , but not through  $Q$ . Then the  $(n - 1)$ -space  $H_{R_i} = \langle \Pi_{n-3}, \ell \rangle$  contains  $t$  points of  $B$  but does not contain  $Q$ .

**Part 6.** Finally we show that the point  $R'_i$  in  $\pi^W$  (corresponding to  $R_i$ ) is essential to  $C'$ . A particular property of an  $(n - 2)$ -spread in  $\langle H^*, P \rangle = \text{PG}(2n - 3, q)$  is that every hyperplane of  $\langle H^*, P \rangle$  contains exactly one element of the  $(n - 2)$ -spread. The hyperplane  $\langle H_{R_i}, P \rangle$  of  $\Pi_{2n-2}$  intersects  $\langle H^*, P \rangle$  in a  $(2n - 4)$ -dimensional subspace, so it contains one element  $w$  of  $W$ . The point  $Q \notin H_{R_i}$ , hence  $w \neq \langle P, Q \rangle$ . We show that  $\langle w, R_i \rangle$  corresponds to a  $t$ -secant in  $\pi^W$ . As in Part 4, projecting the points of  $\langle w, R_i \rangle$  from  $P$  to  $H_{R_i}$ , we get a one-to-one correspondence between the points of  $H_{R_i} \cap B$  and  $\langle w, R_i \rangle$ , which proves that  $R'_i$  is essential to  $C'$ .  $\square$

**Theorem 4.2.** Let  $B$  be a minimal weighted  $t$ -fold  $(n - k)$ -blocking set of  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , of size  $|B| = tq^{n-k} + t + k'$ , with  $t + k' \leq (q^{n-k} - 1)/2$ . Then  $B$  intersects every  $k$ -dimensional subspace in  $t \pmod{p}$  points.

*Proof.* This proof is similar to the proof of [11, Theorem 2.7]. We include it since it makes clear where the upper bound on the size of  $B$  comes from.

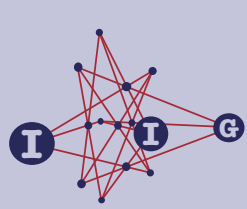
Case  $k = n - 1$  is proved in Theorem 4.1. Now let  $k < n - 1$ . Embed  $\text{PG}(n, q)$  in  $\text{PG}(n, q^{n-k})$  as a subgeometry. Consider  $\text{PG}(n, q^{n-k})$  as an  $(n + 1)(n - k)$ -dimensional vector space  $V$  over  $\text{GF}(q)$ . A hyperplane of  $\text{PG}(n, q^{n-k})$  is an  $n(n - k)$ -dimensional vector space and  $\text{PG}(n, q)$  is an  $(n + 1)$ -dimensional vector space in  $V$ . Hence, a hyperplane of  $\text{PG}(n, q^{n-k})$  contains at least a  $k$ -dimensional subspace of  $\text{PG}(n, q)$ . Therefore,  $B$  is a  $t$ -fold blocking set with respect to the hyperplanes of  $\text{PG}(n, q^{n-k})$ .

Then  $B$  is a minimal  $t$ -fold blocking set with respect to the hyperplanes of  $\text{PG}(n, q^{n-k})$ . Namely, consider a point  $P$  of  $B$ . Since  $B$  was minimal as a  $t$ -fold  $(n - k)$ -blocking set in  $\text{PG}(n, q)$ , there exists a  $k$ -dimensional subspace  $K$  of  $\text{PG}(n, q)$  through  $P$  that intersects  $B$  in exactly  $t$  points. Any hyperplane of  $\text{PG}(n, q^{n-k})$  through  $K$  that intersects  $\text{PG}(n, q)$  exactly in  $K$  proves that  $P$  is essential for  $B$  as  $t$ -fold blocking set with respect to the hyperplanes of  $\text{PG}(n, q^{n-k})$ .

To prove the  $t \pmod{p}$  result, every  $k$ -dimensional space  $K$  of  $\text{PG}(n, q)$  can be extended to a hyperplane of  $\text{PG}(n, q^{n-k})$  intersecting  $\text{PG}(n, q)$  in precisely this  $k$ -dimensional space  $K$ . Since  $|B| = tq^{n-k} + t + k'$ , with  $t + k' \leq (q^{n-k} - 1)/2$ , it is possible to apply Theorem 4.1. This hyperplane shares  $t \pmod{p}$  points with  $B$ , so  $B$  shares  $t \pmod{p}$  points with  $K$ .  $\square$

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**Lemma 4.3.** *Let  $B$  be a minimal weighted  $t$ -fold 1-blocking set of  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , of size  $|B| = tq + t + k'$ , with  $t + k' \leq (q - 1)/2$ .*

*By Theorem 4.1, each hyperplane intersects  $B$  in  $t \pmod{p^e}$  points for some  $e \geq 1$ , with  $e$  the maximal integer for which this is true. Then for  $0 \leq s \leq n - 1$  and every  $s$ -dimensional subspace  $\Pi_s$ ,  $|B \cap \Pi_s| \in \{0, 1, \dots, t\} \pmod{p^e}$ .*

*Proof.* Note that we can assume  $t < p^e - 1$ , otherwise the statement is obvious. Consider  $\Pi_s$  with  $0 \leq s \leq n - 2$ , and suppose to the contrary that  $|B \cap \Pi_s| \in \{t + 1, \dots, p^e - 1\} \pmod{p^e}$ . Then each hyperplane through  $\Pi_s$  contains at least  $t + 1$  further points from  $B \setminus \Pi_s$ .

There are  $|\text{PG}(n - 1 - s, q)|$  hyperplanes through  $\Pi_s$ , so the number of incidences of the points of  $B \setminus \Pi_s$  with the hyperplanes through  $\Pi_s$  is at least  $(t + 1)(q^{n-s} - 1)/(q - 1)$ . As every point of  $B \setminus \Pi_s$  takes part in  $(q^{n-s-1} - 1)/(q - 1)$  incidences, we have  $|B| \geq |B \setminus \Pi_s| \geq (t + 1)(q^{n-s} - 1)/(q^{n-s-1} - 1) \geq (t + 1)q$ , which is false.  $\square$

**Theorem 4.4.** *Let  $B$  be a minimal weighted  $t$ -fold  $(n - k)$ -blocking set of  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , of size  $|B| = tq^{n-k} + t + k'$ , with  $t + k' \leq (q^{n-k} - 1)/2$ .*

*Let  $e \geq 1$  be the largest integer such that each  $k$ -dimensional subspace intersects  $B$  in  $t \pmod{p^e}$  points. Then, for  $0 \leq s \leq k$  and every  $s$ -dimensional subspace  $\Pi_s$ , we have  $|B \cap \Pi_s| \in \{0, 1, \dots, t\} \pmod{p^e}$ .*

*Proof.* As in the proof of Theorem 4.2, embed  $\text{PG}(n, q)$  in  $\text{PG}(n, q^{n-k})$  as a subgeometry and note again that  $B$  is a minimal  $t$ -fold blocking set with respect to hyperplanes of  $\text{PG}(n, q^{n-k})$ . Now apply Lemma 4.3.  $\square$

We note that all the known small minimal weighted  $t$ -fold  $(n - k)$ -blocking sets are unions of (not necessarily disjoint) linear  $(n - k)$ -blocking sets (if  $t \leq p^e$ , then linear 1-fold  $(n - k)$ -blocking sets), satisfying the General Linearity Conjecture for small minimal  $t$ -fold blocking sets. As these examples suggest, we think that for  $0 \leq s \leq k - 1$ ,  $|B \cap \Pi_s| \equiv 0 \pmod{p^e}$  can only occur if  $B \cap \Pi_s$  is in fact empty (some assumption for  $t$  might be needed). For  $t = 1$ , this was proved in [11].

## 5. Intervals on the sizes of minimal $t$ -fold $(n - k)$ -blocking sets in $\text{PG}(n, q)$

First we prove a lower bound on the size of a minimal weighted  $t$ -fold 1-blocking set.

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**Theorem 5.1.** Let  $B$  be a minimal weighted  $t$ -fold 1-blocking set in  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ . Assume that  $|B| = tq + t + k$ , where  $t + k \leq (q - 1)/2$ . Let  $e$  be the largest integer for which each hyperplane intersects  $B$  in  $t \pmod{p^e}$  points. Then

$$|B| \geq tq + \frac{q^{n-1} + p^e}{q^{n-2}(p^e + 1)} - \frac{t}{q^{n-2}}.$$

For simplicity we note that this bound implies the slightly weaker bound

$$|B| \geq tq + \frac{q}{p^e + 1} - 1.$$

*Proof.* By the maximality of  $e$ , there exists a hyperplane  $H$  such that  $|B \cap H| \not\equiv t \pmod{p^{e+1}}$ . Embed  $\text{PG}(n, q)$  into  $\text{PG}(2n - 2, q)$  and as in the proof of Theorem 4.1, construct the cone  $\mathcal{C}$ . In the corresponding plane  $\pi^W$ ,  $\mathcal{C}'$  is a weighted  $t$ -fold blocking set of size  $|\mathcal{C}'| = |B|q^{n-2} + t$ . The blocking set  $\mathcal{C}'$  is not necessarily minimal, but due to our construction, the subspace  $H$  corresponds to a line  $h$  of  $\pi^W$  so that all the points of  $h \cap \mathcal{C}'$  are essential to  $\mathcal{C}'$ . If there are non-essential points in  $\mathcal{C}'$ , delete them one-by-one until a minimal  $t$ -fold blocking set  $B'$  of  $\pi^W$  is obtained. By Theorem 4.1,  $B'$  intersects each line of  $\pi^W$  in  $t \pmod{p^{e^*}}$  points for some  $e^* \leq e$ . Since the lower bound in Theorem 3.1 is decreasing in  $e$ ,  $|\mathcal{C}'| \geq |B'| \geq tq^{n-1} + \frac{q^{n-1} + p^{e^*}}{p^{e^*+1}} + 1$  holds, from which the bound on  $|B|$  follows.  $\square$

Theorem 5.1 immediately yields a lower bound on the size of minimal  $t$ -fold  $(n-k)$ -blocking sets in  $\text{PG}(n, q)$ . As in the proof of Theorem 4.2, embed  $\text{PG}(n, q)$  in  $\text{PG}(n, q^{n-k})$  as a subgeometry and note again that  $B$  is a  $t$ -fold blocking set with respect to hyperplanes of  $\text{PG}(n, q^{n-k})$ .

**Corollary 5.2.** Let  $B$  be a minimal weighted  $t$ -fold  $(n-k)$ -blocking set in  $\text{PG}(n, q)$ ,  $q = p^h$ ,  $p$  prime,  $h \geq 1$ . Assume that  $|B| = tq^{n-k} + t + k'$ , where  $t + k' \leq (q^{n-k} - 1)/2$ . Let  $e$  be the largest integer for which each  $k$ -space intersects  $B$  in  $t \pmod{p^e}$  points. Then

$$|B| \geq tq^{n-k} + \frac{q^{n-k}}{p^e + 1} - 1. \quad \square$$

**Warning.** From now on, we consider point sets *without* weights.

**Theorem 5.3.** Let  $B$  be a minimal  $t$ -fold  $(n-k)$ -blocking set in  $\text{PG}(n, q)$ ,  $n \geq 2$ ,  $|B| = tq^{n-k} + t + k'$ , with  $t + k' \leq (q^{n-k} - 1)/2$ . Assume that  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , and that  $B$  intersects every  $k$ -dimensional space in  $t \pmod{E}$  points, with  $E = p^e$ . If  $2t < E$ , then

$$tq^{n-k} + \frac{q^{n-k}}{p^e + 1} - 1 \leq |B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E}.$$

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*Proof.* Let  $\tau_{t+iE}$  be the number of  $k$ -dimensional spaces intersecting  $B$  in  $t+iE$  points. We count the number of  $k$ -dimensional spaces, the number of incident pairs  $(R, \pi)$ , with  $R \in B$  and with  $\pi$  a  $k$ -dimensional space through  $R$ , and the number of triples  $(R, R', \pi)$ , with  $R$  and  $R'$  distinct points of  $B$  and  $\pi$  a  $k$ -dimensional space passing through  $R$  and  $R'$ .

Then the following formulas are valid.

$$\sum_{i \geq 0} \tau_{t+iE} = \frac{(q^{n+1} - 1)(q^n - 1)}{(q^{k+1} - 1)(q^k - 1)} \cdot C,$$

$$\sum_{i \geq 0} (t + iE) \tau_{t+iE} = |B| \left( \frac{q^n - 1}{q^k - 1} \right) \cdot C,$$

$$\sum_{i \geq 0} (t + iE)(t + iE - 1) \tau_{t+iE} = |B|(|B| - 1) \cdot C,$$

where

$$C = \frac{(q^{n-1} - 1) \dots (q^{n+1-k} - 1)}{(q^{k-1} - 1) \dots (q - 1)}.$$

Then  $\sum_{i \geq 0} i(i-1)E^2 \tau_{t+iE} \geq 0$  implies that

$$\begin{aligned} & |B|(|B| - 1) - (2t - 1)|B| \left( \frac{q^n - 1}{q^k - 1} \right) + t^2 \left( \frac{(q^{n+1} - 1)(q^n - 1)}{(q^{k+1} - 1)(q^k - 1)} \right) \\ & - |B|E \left( \frac{q^n - 1}{q^k - 1} \right) + tE \left( \frac{(q^{n+1} - 1)(q^n - 1)}{(q^{k+1} - 1)(q^k - 1)} \right) \geq 0. \end{aligned}$$

Under the condition  $2t < E$ , this implies that

$$|B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E} \quad \text{or} \quad |B| \geq Eq^{n-k} + t.$$

The lower bound on the size of  $B$  was proved earlier. □

The preceding proof also leads to the following corollary.

**Corollary 5.4.** *Let  $B$  be a minimal  $t$ -fold  $(n-k)$ -blocking set in  $\text{PG}(n, q)$ . Assume that  $q = p^h$ ,  $p$  prime,  $h \geq 1$ , and that  $B$  intersects every  $k$ -dimensional space in  $t \pmod{E}$  points, with  $E = p^e$ . If  $\max\{2t, 4\} < E$ , then*

$$|B| \leq tq^{n-k} + \frac{2tq^{n-k}}{E} \quad \text{or} \quad |B| \geq Eq^{n-k} + t. \quad \square$$

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## 6. A characterization result which follows from the $t \pmod{p}$ result

In [5], the preceding  $t \pmod{p}$  results are used to characterize minimal  $t$ -fold  $(n - k)$ -blocking sets in  $\text{PG}(n, q)$ ,  $q$  square, of small cardinality.

**Theorem 6.1.** *Let  $B$  be a minimal  $t$ -fold  $(n - k)$ -blocking set in  $\text{PG}(n, q)$ ,  $q$  square, of size at most  $|B| \leq tq^{n-k} + 2tq^{n-k-1}\sqrt{q} < tq^{n-k} + q^{n-k-1/3}$ . Then  $B$  is a union of  $t$  pairwise disjoint cones  $\langle \pi_{m_i}, \text{PG}(2(n - k - m_i - 1), \sqrt{q}) \rangle$ ,  $-1 \leq m_i \leq n - k - 1$ , with vertex an  $m_i$ -dimensional space  $\pi_{m_i}$  and base  $\text{PG}(2(n - k - m_i - 1), \sqrt{q})$ ,  $i = 1, \dots, t$ .*

*If  $t \geq 2$ , then  $k > n/2$  if  $B$  contains at least one  $(n - k)$ -dimensional space  $\text{PG}(n - k, q)$  and  $k \geq n/2$  in the other cases.  $\square$*

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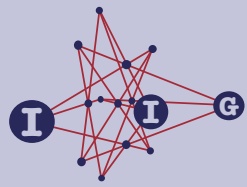
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