Double-Baer groups

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Abstract

We show that a double-Baer group implies the existence of a double-retraction in a translation plane with kernel containing a field $K = \mathsf{GF}(q)$. If the associated spread is in $\mathsf{PG}(3,q)$ then a lifted spread in $\mathsf{PG}(3,q^2)$ admits a double-Baer group. The double-retraction group produces a maximal partial mixed partition of $\mathsf{PG}(3,q^2)$ of lines and $\mathsf{PG}(3,q)$. This result is generalized and new examples of translation planes admitting double-Baer groups are given.

Keywords: Baer groups, subgeometry partitions, lifting

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We both met Gabor Korchmáros in 1985 when the "triangle" of Potenza-Bari-Lecce met for a series of seminars at Lecce and we were visiting the university. Many finite geometers have been greatly influenced by the tremendous depth and range of the geometry of Gabor. In particular, we gratefully acknowledge that our foundations' book on translation planes with Mauro Biliotti grew out of a summer school organized by Gabor and his group at Potenza during the summer of 1996. We are privileged to dedicate this article to Gabor on the occasion of his sixtieth birthday.

1. Introduction

This article somewhat concerns the concept of 'geometric lifting', which is described in the text by Hirschfeld and Thas [4]. In particular, from a subgeometry partition of a finite projective space, there is a construction that produces a corresponding finite spread, and hence, from which we obtain a finite translation plane. The subgeometry partition may consist entirely of Baer subgeometries or be a so-called 'mixed subgeometry partition' where the partition has at least two non-isomorphic subgeometries comprising the partition.

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In Johnson [10], the question was considered as how to recognize spreads that have been geometrically lifted from Baer subgeometry or mixed subgeometry partitions of a finite projective space. It was determined that the intrinsic character is that the translation plane have order q^t with subkernel K isomorphic to $\mathsf{GF}(q)$ and admit a collineation group (on the nonzero vectors) which contains the scalar group K^*, written as GK^*, such that GK^* union the zero mapping is a field isomorphic to $\mathsf{GF}(q^2)$ (see Johnson [10] and Johnson-Mellinger [11]). With such a recognition theorem on collineation groups, it is then possible to 'retract' such a translation plane or spread to construct a variety of Baer subgeometry or mixed subgeometry partitions of an associated projective space written over GK as a quadratic field extension of K. Since subgeometry partitions often first arise directly from translation planes admitting a retraction group, it is therefore of common interest to study situations that force translation planes to admit such a group. Furthermore, mixed partitions produce translation planes of square order and often implicit in the translation plane is a combination of Baer groups (the group fixes a Baer subplane pointwise). The main thrust of this article is that such a combination of Baer groups produces, in turn, a retraction group that produces a mixed subgeometry partition. Of course, the question of what sorts of translation planes admit such Baer groups becomes of major interest. In this regard, there is a very unlikely connection with what are called 'algebraically lifted planes'.

There is also an algebraic construction procedure for spreads which is called 'algebraic lifting' (or more simply 'lifting' in Johnson [9]) by which a spread in $\mathsf{PG}(3, q)$ may be lifted to a spread in $\mathsf{PG}(3, q^2)$. More precisely, this construction is a construction on the associated quasifields for the spread and different quasifields may produce different algebraically lifted spreads. This material is explicated in Biliotti, Jha and Johnson and the reader is referred to this text for additional details and information (see [1, p. 437]).

Recently, in Jha and Johnson [5] the following surprising result is obtained.

Theorem 1.1 (Jha and Johnson [5]). *Let S be any spread in $\mathsf{PG}(3, q)$. Then there is a mixed subgeometry partition of $\mathsf{PG}(3, q^2)$, which geometrically lifts to a spread in $\mathsf{PG}(3, q^2)$ that algebraically contracts to S.*

Corollary 1.2 (Jha and Johnson [5]). *The set of mixed subgeometry partitions of a 3-dimensional projective space $\mathsf{PG}(3, k^2)$, constructs all spreads of $\mathsf{PG}(3, k)$.*

In this article, we provide further fundamental connections between these two constructions. In particular, we establish connections between what are called 'double-Baer groups' and 'double-retraction' and basically show that the existence of one is equivalent to the existence of the other. (In this context, double-retraction refers to the possibility that there are two collineation groups of a given translation plane with the above properties allowing retraction using either one of these fields to produce subgeometry partitions of isomorphic projective spaces.) Furthermore, in Johnson [10] and using Jha and Johnson [5], it is shown that spreads in $\mathsf{PG}(3,q^2)$ lifted from spreads in $\mathsf{PG}(3,q)$ always admit double-Baer groups of the type above and hence there is an associated double-retraction. In this case, one retraction provides a mixed subgeometry partition of q^2+1 lines and $q^2(q-1)$ $\mathsf{PG}(3,q)$'s of a $\mathsf{PG}(3,q^2)$.

Considered vectorially, retraction may be considered more generally over infinite vector spaces, even infinite vector spaces of infinite dimension over their kernels. Hence, in essence, geometric lifting may be more generally considered, even though in this article, we restrict ourselves to the finite situation.

To be clear on what we mean by 'retraction', we end this section with the second author's theorem involving groups of order q^2-1 and our specific definitions. The following theorem deals with the situation considered in this article, although the theorem can be stated in a more general form.

Theorem 1.3 (Johnson [10])**.** *Let π be a translation plane of order $q^{2^a r}$ and kernel containing $\mathsf{GF}(q)$. Let G be a collineation group of order q^2-1 containing the kernel homology group of order $q-1$ and assume that $G \cup \{0\}$ (the zero mapping) is a field K.*

Then the component orbit lengths of G are either 1 or $q+1$. Forming the projective space $\mathsf{PG}(2^a r-1,q^2)$, the orbits of length 1 become projective subgeometries isomorphic to $\mathsf{PG}(2^{a-1}r-1,q^2)$ and the orbits of length $q+1$ become projective subgeometries isomorphic to $\mathsf{PG}(2^{a-1}r-1,q)$.

The set of subgeometries partition the points of the projective space providing a 'mixed subgeometry partition'.

Furthermore, as mentioned, there is a 'geometric lifting' process that constructs translation planes of order $q^{2^a r}$ and kernel containing $\mathsf{GF}(q)$ from any mixed subgeometry partition.

There is also a corresponding result in Johnson [10] for translation planes of order q^t, for t odd but admitting collineation groups of order q^2-1 having the properties of the group in the previous theorem. In this setting, all component orbits will have length $q+1$ and the associated subgeometry partition is said to be a 'Baer subgeometry partition'.

In general, we define the process of 'retraction' as follows.

Definition 1.4. Assume that a finite translation plane with kernel containing $\mathsf{GF}(q)$ admits a collineation group of order q^2-1 that contains the kernel homology group of order $q-1$ such that together with the zero mapping, a field isomorphic to $\mathsf{GF}(q^2)$ is obtained. Then the group is said to be a 'retraction group'.

Definition 1.5. A retraction group is said to be 'trivial' in the projective space $\mathsf{PG}(2n-1, q^2)$ associated with the spread, if the subgeometries are all isomorphic to $\mathsf{PG}(n-1, q^2)$.

Remark 1.6. It is immediate that a retraction group is trivial if and only if it is a kernel homology group of the translation plane.

Proof. The construction process of 'geometric lifting' constructs a translation plane with kernel isomorphic to $\mathsf{GF}(q)$, such that each subgeometry produces exactly one component of the translation plane, which is fixed by a group of order $q^2 - 1$. Hence, the retraction group is a kernel homology group of order $q^2 - 1$. □

Definition 1.7. The associated process of constructing the subgeometry partition from a non-trivial retraction group is said to be 'retraction' (or 'spread retraction'). If there are two distinct retraction groups G_1 and G_2 that centralize each other, we shall say that we have 'double-retraction'. More generally, if there are k distinct retraction groups that centralize each other, we say that we have 'k-retraction'. We note that we are allowing 'double-retraction' to include the possibility that one the associated fields is a kernel field of the translation plane.

In this article, we show fundamental connections between double-retraction and what are called 'double-Baer groups'. Furthermore, new constructions of Dempwolff [2] provide several new double-Baer groups and hence will construct some new subgeometry partitions. In addition, the concept of 'multiple-retraction' is considered within the context of double-Baer groups and double-homology groups. In a sense made clear in the article, double-retraction is equivalent to the existence of either a double-Baer group or a double-homology group.

2. Double-Baer groups

We recall that a 'Baer group' in a translation plane of order h^2 is a group in the translation complement that fixes a Baer subgroup pointwise. Indeed, such Baer groups have orders that divide $h(h-1)$. Here we let $h = q^{2^a r}$ and consider the possibility of having two Baer groups of order $q + 1$. The reason for consideration of such Baer groups may be seen from algebraic lifting, where a spread in $\mathsf{PG}(3, q)$ algebraically lifts to a spread in $\mathsf{PG}(3, q^2)$ admitting a Baer group of order $q+1$. In this setting, the order of the translation plane has order q^4. The following theorem recalls the main ideas. The reader is directed to Biliotti, Jha, and Johnson [1] for details on algebraic lifting. Actually, this concept originated in Hiramine-Matsumoto-Oyama [3] for the odd order case and here we provide the details for arbitrary order planes.

Theorem 2.1. *Let π be a translation plane with spread S in $\mathsf{PG}(3,q)$. Let F denote the associated field of order q and let K be a quadratic extension field with basis $\{1,\theta\}$ such that $\theta^2=\theta\alpha+\beta$ for $\alpha,\beta\in F$. Choose any quasifield and write the spread as follows:*

$$x=0, y=x\begin{bmatrix} g(t,u) & h(t,u)-\alpha g(t,u)=f(t,u) \\ t & u \end{bmatrix} \forall t,u\in F\,,$$

where g,f are unique functions on $F\times F$ and h is defined as noted in the matrix, using the term α.

Define $F(\theta t+u)=-g(t,u)\theta+h(t,u)$. Then

$$x=0, y=x\begin{bmatrix} \theta t+u & F(\theta s+v) \\ \theta s+v & (\theta t+u)^q \end{bmatrix} \forall t,u,s,v\in F$$

is a spread S^L in $\mathsf{PG}(3,q^2)$ called the spread 'algebraically lifted' from S. We note that there is a derivable net

$$x=0, y=x\begin{bmatrix} w & 0 \\ 0 & w^q \end{bmatrix} \forall w\in K\simeq \mathsf{GF}(q^2)$$

with the property that the derived net (replaceable net) contains exactly two Baer subplanes which are $\mathsf{GF}(q^2)$-subspaces and the remaining q^2-1 Baer subplanes form $(q-1)$ orbits of length $q+1$ under the kernel homology group.

Hence, we obtain a mixed partition of $(q-1)$ $\mathsf{PG}(3,q)$'s and q^4-q lines of $\mathsf{PG}(3,q^2)$.

So, from any quasifield, we obtain a spread in $\mathsf{PG}(3,q)$ which lifts and derives to a spread permitting retraction which produces a mixed partition of $(q-1)$ $\mathsf{PG}(3,q)$'s and q^4-q lines of $\mathsf{PG}(3,q^2)$.

In the previous result, we used the derived plane of the algebraically lifted plane to construct the subgeometry partitions. However, there is also a more direct construction which proves Theorem 1.1. We revisit that proof here as it may be directly generalized.

We note that in the above situation,

$$B_1=\left\langle \mathrm{diag}(e^{-1},1,1,e^{-1}) \mid e \text{ has order } q+1\right\rangle$$

maps

$$y = x \begin{bmatrix} u & F(z) \\ z & u^q \end{bmatrix}$$

onto

$$y = x \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & F(z) \\ z & u^q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-1} \end{bmatrix} = x \begin{bmatrix} ue & F(z) \\ z & (ue)^q \end{bmatrix}$$

and so is a collineation of the algebraically lifted translation plane. Furthermore, B_1 fixes the vector subspace

$$\operatorname{Fix} B_1 = \big\langle (0, x_2, y_1, 0) \mid x_2, y_1 \in \mathsf{GF}(q^2) \big\rangle$$

pointwise, which implies that B_1 is a Baer group of order $q+1$. Since, we have kernel $\mathsf{GF}(q^2)$ in the lifted plane, it follows that

$$B_2 = \langle \operatorname{diag}(1, e, e, 1) \mid e \text{ has order } q+1 \rangle$$

is also a Baer group fixing the vector subspace

$$\big\langle (x_1, 0, 0, y_2) \mid x_1, y_2 \in \mathsf{GF}(q^2) \big\rangle$$

pointwise. Furthermore, B_1 and B_2 commute and note that B_1 is contained in the group $B_2 K^{2*}$, where K^{2*} denotes the kernel homology group of order $q^2 - 1$. Now let K^2 be the kernel homology group of order $q^2 - 1$ and form $\langle B_1, B_2 \rangle K^{2*}$, which we claim is

$$D = \big\langle \operatorname{diag}(a, b, b, a) \mid a, b \in \mathsf{GF}(q^2)^* \text{ such that } a^{q+1} = b^{q+1} \big\rangle$$

of order $(q+1)(q^2-1)$. Note that the order of $\langle B_1, B_2 \rangle K^{2*}$ is $(q+1)^2(q^2-1)/I$, where I is the intersection with B_1B_2 and K^{2*}. This group leaves invariant the subplane $\operatorname{Fix} B_1$, B_1 fixes it pointwise, B_2 induces a kernel group on it as does K^{2*}. Hence, the group induced on $\operatorname{Fix} B_1$ is isomorphic to K^{2*}. But, since B_1 of order $q+1$ fixes $\operatorname{Fix} B_1$ pointwise, it follows that the group has order $(q+1)(q^2-1)$.

When $a = b$, D contains K^{2*} and when $a = 1$, D contains B_1, similarly, D contains B_2. Since D has a K^{2*} as a normal subgroup of index $q+1$, we see that $B_1 B_2 K^{2*} = D$.

Let $q = p^r$, for p a prime and form the following set of fields K_{p^i} of order q^2, for $\sigma = p^i$ for i dividing $2r$.

$$L_\sigma = \big\langle \operatorname{diag}(a, a^\sigma, a^\sigma, a) \mid a \in \mathsf{GF}(q^2)^* \big\rangle \cup \{0\}\,.$$

To obtain L_σ^* in our group, then $a^{\sigma(q+1)} = a^{q+1}$, which implies that if $\sigma = p^i$, then q divides p^i, so let $\sigma = q$ or 1.

Now each of the fields L_σ satisfies the hypothesis of the retraction theorem so that we obtain a double-retraction. However, when $\sigma = 1$, in this particular setting, the subgeometry partition is essentially trivial as K_1 is the kernel homology group of the associated translation plane so that we do not actually obtain a mixed partition as all subgeometries are isomorphic to $\mathsf{PG}(1, q^2)$'s. In a more general setting, we often do obtain true double-retraction from such pairs of Baer groups.

We generalize this concept below. In the following setting, however, we make no assumption on the kernel of the translation plane, so π is a translation plane of order $q^{2^a r}$, with unspecified kernel.

There are two principal situations where the use of Baer groups provides double-retraction. First, two Baer groups of order $q+1$ in translation planes that admit a subkernel isomorphic to $\mathsf{GF}(q^2)$ and two Baer groups of order $(2, q-1)(q+1)$ in translation planes that admit a subkernel isomorphic to $\mathsf{GF}(q)$. Of course, when q is even, we may use either of these conditions for our results.

Definition 2.2. Let B_1 and $\mathrm{co}B_1$ be distinct commuting Baer groups of the same order and in the same net of degree $q^{2^{a-1}r} + 1$ in a translation plane π of order $q^{2^a r}$ and kernel containing K for $(r, 2) = 1$. If either K is isomorphic to $\mathsf{GF}(q^2)$ and B_1 is divisible by $q+1$ or if K is isomorphic to $\mathsf{GF}(q)$ and B_1 is divisible by $(2, q-1)(q+1)$, the pair $(B_1, \mathrm{co}B_1)$ shall be called a 'double-Baer group'. The 'order' of the double-Baer group is the order of either Baer group in the set.

Definition 2.3. Any Baer group whose order is that of a double-Baer group is said to be 'critical'.

2.1. $(q+1, q^2)$-double-Baer groups

To fix the situation, we assume that we have a double-Baer group of order divisible by $q+1$ and a translation plane of order $q^{2^a r}$, where there is a subkernel of K_{q^2} isomorphic to $\mathsf{GF}(q^2)$.

Lemma 2.4. *We may choose coordinates so that B_1 and $\mathrm{co}B_1$ have the following representation:*

$$B_1 = \left\langle \mathrm{diag}(I, A, I, A) \mid A \in K_1^+ \textit{ of order dividing } q+1 \right\rangle$$

and

$$\mathrm{co}B_1 = \left\langle \mathrm{diag}(C, I, C, I) \mid C \in K_2^+ \textit{ of order dividing } q+1 \right\rangle$$

where both K_1^+ and K_2^+ are fields isomorphic to $\mathsf{GF}(q^2)$, and K_i^+ is the K_q-module generated over K_q by A or C respectively for $i = 1, 2$.

Proof. To see this, we note that any Baer group of order $q+1$ fixes both of the Baer subplanes indicated and induces a kernel homology group on one subplane and the identity on the other. Hence, the elements A are contained in the kernel of a translation plane of order $q^{2^a r}$. So, it follows that there exist sub-kernels of the Baer subplanes that are isomorphic to $\mathsf{GF}(q^2)$. Note also that the Baer groups are linear over K_q so it must be that both of the Baer subplanes are fixed by the kernel of the super translation plane. Hence, we see that K_q may be assumed to be a subfield of K_1^+ and K_2^+. □

Then, we have the following representation for $B_1\,\mathrm{co}B_1K_{q^2}^*$:

$$B_1\,\mathrm{co}B_1K_{q^2}^* = \big\langle \mathrm{diag}(C\alpha, A\alpha, C\alpha, A\alpha) \mid A \in K_1^+, C \in K_2^+ \text{ of orders dividing } (q+1) \text{ and } \alpha \in K_{q^2} - \{0\}\big\rangle\,.$$

We note our previous argument in the dimension 2 situation applies directly here to show that the order of this group is $(q+1)(q^2-1)$.

$$\begin{aligned} B_1\,\mathrm{co}B_1K_{q^2}^* &= \langle \mathrm{diag}(C\alpha, A\alpha, C\alpha, A\alpha)\rangle \\ &= \big\langle \mathrm{diag}(D, E, D, E) \mid D^{q+1} = E^{q+1} \text{ for } D \in K_1^+, E \in K_2^+\big\rangle\,. \end{aligned}$$

Since K_1^+ and K_2^+ are isomorphic fields of matrices containing K_q, they are conjugate by an K_q-matrix H. So, let $K_2^{+H} = K_1^+$ and consider the following fields K^σ

$$L_\sigma = \big\langle \mathrm{diag}(D, D^{\sigma H}, D, D^{\sigma H}) \mid D \in K_1^+, \sigma \text{ automorphism of } K_1^+\big\rangle\,.$$

By our previous argument L_σ^* belongs to our group if and only if $\sigma = 1$ or q. We now have the conditions for double-retraction as given in Theorem 1.3. Of course, in this setting, A and C are in the kernel group K_{q^2} but in a related setting to follow, this will not be the case.

Hence, we have the following theorem.

Theorem 2.5. *Let π be a translation plane of order $q^{2^a r}$ for $(r,2)=1$, admitting a double-Baer group of order divisible by $q+1$ with kernel containing K_{q^2} isomorphic to* $\mathsf{GF}(q^2)$.

From the double-Baer group, there are exactly two retraction fields L_σ, for $\sigma = 1$ or q, for the translation plane so we have double-retraction.

2.2. $((2, q-1)(q+1), q)$-double-Baer groups

Now we will consider a translation plane of order $q^{2^a r}$ with kernel containing K_q isomorphic to $\mathsf{GF}(q)$ that admits a double-Baer group of order divisible by

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$(2, q-1)(q+1)$. Now if we reread the argument given in the previous subsection, we see that we can still create two fields L_σ, for $\sigma = 1$ or q. In particular, we have

$$B_1 \operatorname{co}B_1 K_q^* = \langle \operatorname{diag}(C\alpha, A\alpha, C\alpha, A\alpha) \mid C \text{ and } A \text{ have orders divisible by} (2, q-1)(q+1)z \text{ and } \alpha \in K_q^* \rangle.$$

The order of this group is $((2, q-1)(q+1))^2(q-1)/I$, where I is the intersection of $B_1 \operatorname{co}B_1$ and K_q^*. In order that an element of $g \in B_1 \operatorname{co}B_1$ be also in K_q^*, then $C = A$ has order dividing $(q-1)$. If q is even, then I is clearly $\langle 1 \rangle$, so assume that q is odd. Then, we have a group of order $4(q+1)^2(q-1)/I$. First assume that $(q-1)/2$ is odd. Then clearly we have a group of order $2(q+1)(q^2-1)$. If $(q-1)/2$ is even then there is a cyclic group of order 4 in both B_1, $\operatorname{co}B_1$ and K_q^*, so that the group has order $(q+1)(q^2-1)$.

Therefore, in all situations, we have the subgroup of $B_1 \operatorname{co}B_1 K_q^*$

$$\langle \operatorname{diag}(D, E, D, E) \mid D^{q+1} = E^{q+1}, \text{ for } D \in K_1^+, E \in K_2^* \rangle$$

of order $(q+1)(q^2-1)$ and we obtain the fields L_σ, for $\sigma = 1$ or q, exactly as in the previous situation.

However, in this setting, the field L_1 need not be a kernel homology group. Therefore, we obtain the following theorem.

Theorem 2.6. *Let π be a translation plane of order $q^{2^a r}$ for $(r, 2) = 1$, admitting a double-Baer group of of order divisible by $(2, q-1)(q+1)$ with kernel containing K_q isomorphic to* $\mathsf{GF}(q)$.

(1) *From the double-Baer group, we may construct two retraction fields L_σ, for $\sigma = 1$ or q, for the translation plane so we have double-retraction.*

(2) *If the plane has kernel* $\mathsf{GF}(q^2)$ *but L_1 is not a kernel homology group, we obtain triple-retraction.*

3. The fusion of Baer groups

One important procedure for the construction of double-Baer groups is the following theorem.

Theorem 3.1. *Let π be a semifield plane of order $q^{2^a r}$ and kernel containing K isomorphic to* $\mathsf{GF}(q)$, *for $(r, 2) = 1$. Assume that the right nucleus and middle nucleus of the associated semifield contain fields isomorphic to* $\mathsf{GF}(q^2)$. *If there is a derivable net that contains a middle nucleus net invariant under the associated affine homology group then the derived plane admits a double-Baer group.*

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Proof. In Jha and Johnson [7], it is possible to 'fuse' isomorphic sub- right and middle nuclei in semifields (identify the fields defining the nuclei). This means that there are affine homology groups of order $q^2 - 1$ that leave invariant the derivable net. Under derivation, such groups are transformed into Baer groups that form a double-Baer group. □

So, it would appear that a natural development to the study of Baer groups is to consider a 'fusion' of Baer groups. Note in the case of the $(q+1, q^2)$-Baer groups, we will have a natural fusion.

Definition 3.2. Let π be a translation plane of order $q^{2^a r}$ admitting a double-Baer group. Then both associated Baer groups are represented by fields K_1^+ and K_2^+ isomorphic to $\mathsf{GF}(q^2)$. If these fields can be identified, we shall say that the Baer groups are 'fused'.

Let B_1 and B_2 denote the two Baer groups of the double-Baer group. If there is a field L^+ isomorphic to $\mathsf{GF}(q^2)$ whose multiplicative group is a collineation group such that $B_2 \subseteq B_1 L^{+*}$, we shall say that L^+ 'intertwines' the double-Baer group, or that L^+ is an 'intertwining field' for the double-Baer group.

We note that when the fields corresponding to the Baer groups in a double-Baer group are fused, we still might not have the situation that one of the fields constructed above L_σ, for $\sigma = 1$ or q is a kernel homology group.

We now connect 'spread-retracton' and the fusion of Baer groups.

We observe that fusion arises when a double-Baer group actually commutes with a retraction field and although this result is less general than when fusion does not occur, it conceivably is a more common occurrence.

Theorem 3.3. *Let π of order $q^{2^a r}$ and kernel containing K isomorphic to $\mathsf{GF}(q)$ admit spread-retraction relative to the group $G_1 K^*$. Assume that a double-Baer group $B_1 B_2$ commutes with $G_1 K^*$. Let N denote the net containing the axes of the Baer groups B_i, for $i = 1, 2$.*

(1) *Then $G_1 K^* B_1 B_2$ fixes at least two components of the net N containing the axes of B_i, $i = 1, 2$.*

(2) *Assume that $G_1 K^*$ fixes all components of the net N.*

   (a) *Then the kernel of the Baer subplane $\operatorname{Fix} B_i$ contains $G_1 K^* \cup \{0\} = L^+$.*

   (b) *Furthermore, the double-Baer group may be fused and L^+ is an intertwining field for the double-Baer group.*

Proof. The double-Baer group fixes a set of $q^{2^{a-1} r} + 1$ components which are necessarily permuted by $G_1 K^*$, a group of order $q^2 - 1$. The component orbit lengths of G_1K^* are 1 and $q+1$. Since $q+1$ divides $q^{2^{a-1}r}-1$, it follows that there must be a common set of at least two fixed components, say L and M. Decompose both L and M relative to K_1^+ and apply the previous arguments which show that the groups B_1 and B_2, since both cyclic, must diagonalize relative to K_1^+ in the same manner on both L and M. Furthermore, the group element acting on a 1-dimensional K_1^+-subspace on L or M must act on that 1-space as a K_1^+-scalar mapping. Assume that on L, B_1 has the general form:

$$(x_1,\dots,x_{q^{2^{a-1}r}}) \mapsto (a_1x_1, a_2x_2, a_3x_3, a_4x_4, \dots)$$

for $a_i \in K_1^+$ of orders dividing $q+1$. Since we know that the vectors on $\mathrm{Fix}\, B_1$ and the vectors on $\mathrm{Fix}\, B_2$ direct sum on L to the entire space, this is also true when considering the vectors written over K_1^+. Since B_2 must also have the same general form, it follows that we may rearrange the K_1^+-basis elements on L so that the nontrivial elements of the group B_1 have $a_{2j+1}=1$ and $a_{2k}\neq 1$ and the nontrivial elements of the group B_2 have $a_{2k}=1$ and $a_{2j+1}\neq 1$. Furthermore, since G_1K^* has orbits of length 1 or $q+1$, we have the kernel group of order $q-1$ acting on the two Baer subplanes, since they are pointwise fixed by collineation groups of π. On $\mathrm{Fix}\, B_1=\pi_1$, we also have the K-scalar group fixing all components relative to K_1^+. This means that relative to π_1, we have a subgroup of order at least $(q^2-1)/(2,q-1)$ of K_1^+ acting on each K_1^+-component on L. It follows easily that there is a subkernel on π_1 which may be identified with K_1^+. The same is true for $\mathrm{Fix}\, B_2$. Relative to π_1, the kernel of π_1 is a K_1^+-group which acts like a scalar group on each component. Furthermore, the group is cyclic and if g is a generator then g^{q-1} is the K-kernel homology group restricted to π_1. Let $x_1\mapsto ax_1$ and $x_{2j+1}\mapsto a^{i_{2j+1}}x_{2j+1}$ on L. Since we have a field, it follows that $i_{2j+1}=p^{\lambda(2j+1)}$ and since g^{q-1}, it follows that $i_{2j+1}=1$ or q for all components.

It then follows that a generator h_1 for B_1maps on L

$$(x_1,\dots,x_{q^{2^{a-1}r}}) \mapsto (cx_1, x_2, c^{q^{\lambda(3)}}x_3, x_4, c^{q^{\lambda(5)}}x_5,\dots),$$

where c has order $q+1$. The analogous decomposition may be assumed on M.

If a generator h_2 is chosen so that $x_2\mapsto c^qx_2$, then the product of h_1h_2 maps the K_1^+-components x_i or y_i to $c^{q^{\lambda(i)}}x_i$ or $c_i^{q^{\rho(i)}}y_i$, respectively. Since we have G_1K^+ as a collineation group, it follows that h_1h_2 cannot be in G_1K^*. Now assume that there are more c's than $c^{q\prime}$s in the decomposition of h_1h_2. Then multiplication by $c^{-1}I$ in G_1K^* implies that we have a collineation which violates the Baer condition on numbers of fixed points. Hence, there are exactly half c's and exactly half $c^{q\prime}$s in the decomposition of h_1h_2. Now suppose on L there are more than half c's then $c^{q\prime}$s. If there are at least some c's on M, we obtain again by multiplication of $c^{-1}I$, a planar collineation which violates the

Baer condition. We choose the collineations h_1 and h_2 so that we would have at least one c and at least one c^q on L. If there are no c's on M then multiplication by $c^{-q}I$ provides a contradiction. Hence, there are half c's and half $c^{q\prime}$s on L and half c's and half $c^{q\prime}$s on M. We now assert that relative to B_1, we still have essentially the same parity on elements which are not 1. If we have more c's than $c^{q\prime}$s, again multiplication by $c^{-1}I$ gives a collineation which must be Baer, thus implying that we have all c's. *Hence, either we have all c's, all $c^{q\prime}$s or half c's and half $c^{q\prime}$s relative to B_1.*

However, we have assumed that the group G_1K^* fixes all components on the net N defined by π_1 and π_2. Since G_1K^* leaves π_1 and π_2 invariant and fixes all components on N, it must induce a kernel homology group of order q^2-1 on both subplanes.

In this case, then choosing a third component of N as $y=x$, the representation already given for G_1K^* is basically valid over the translation plane, since now $x=0, y=0, y=x$ are fixed. However, the representation of B_1 must then have all c's or all $c^{q\prime}$s. In other words, we may represent the group B_1 as previous

$$B_1 = \left\langle \mathrm{diag}(I,A,I,A) \mid A \in K_1^+ \text{ of order dividing } q+1 \right\rangle .$$

Now since the kernel acting on π_2 is $\mathrm{diag}(A,A)$, it follows that L^+ acting on π_1 is K_1^+. It now follows that $A = a^{q-1}I_{2^a-2_r}$ (recalling that $a \in K_1^+$ isomorphic to $\mathsf{GF}(q^2)$). Furthermore, we now have that $a^{1-q}I_{2^a r}B_1$ is B_2. Hence, the double-Baer group may be fused and L^+ is an intertwining group. □

Corollary 3.4. *Let π be a translation plane of order $q^{2^a r}$ and kernel containing K isomorphic to* GF(q). *If π admits spread-retraction with a group G_1K^* and admits a Baer group B_1 or order $q+1$ such that B_1 and G_1K^* commute and G_1K^* fixes all components of the net N containing $\operatorname{Fix} B_1$ then there is a double-Baer group of order $q+1$ and two intertwining fields.*

Hence, the plane admits double-retraction.

Proof. By the above theorem, it remains to check that there is a second intertwining field which gives rise to retraction. We have the following group:

$$B_1\,\mathrm{co}B_1G_1K^* = \big\langle \mathrm{diag}(CD,AD,CD,AD) \mid A,C \in K^+ \supseteq K \text{ of orders dividing } q+1 \text{ and } D \in K^+ - \{0\} \big\rangle .$$

Letting $A = a^{q-1}I_{2^a-2_r}$, $C = I_{2^a-2_r}$ and $D = aI_{2^a-2_r}$, we obtain:

$$\left\langle \mathrm{diag}(aI_{2^a-2_r}, a^qI_{2^a-2_r}, aI_{2^a-2_r}, a^qI_{2^a-2_r}) \mid a \in K^+ - \{0\} \right\rangle .$$

We clearly obtain a double-Baer group and this group is fixed-point-free and the union with the zero forms a field containing the K-kernel homology group. Hence, by Johnson and Mellinger [11], it follows that this is a G_2K^* group providing a second retraction. □

Corollary 3.5. *Let π be a translation plane of order $q^{2^a r}$ and kernel containing F isomorphic to* $\mathsf{GF}(q^2)$.

(1) *If π admits a Baer group of order $q+1$ then a double-Baer group exists with intertwining field F.*

(2) *Furthermore, then the three fields, kernel and two associated Baer-fields may be fused.*

(3) *Double-retraction exists.*

Proof. Since there is a subkernel isomorphic to $\mathsf{GF}(q^2)$, we may apply the previous theorem since now all components of the net N are fixed and the Baer subplanes are fixed pointwise by groups of order $q+1$, the subplanes are kernel subspaces and so commutes with the kernel homology group. □

We now consider the possibility of multiple-retraction under the assumption that there is a kernel isomorphic to $\mathsf{GF}(q^2)$, and two commuting double-Baer groups.

Theorem 3.6. *Let π be a translation plane of order $q^{2^a r}$ and kernel containing K isomorphic to* $\mathsf{GF}(q)$. *Furthermore, assume that there is a kernel subfield K^+ isomorphic to* $\mathsf{GF}(q^2)$ *with associated group G_0K^*. Assume that there exist k mutually distinct commuting double-Baer groups $B_{1i}B_{2i}$ for $i = 1, 2, \ldots, k$. Then the plane admits $(k+1)$-retraction.*

Proof. Each of the double-Baer groups produces a retraction group G_iK^* in addition to G_0K^*. If G_0K^* is generated by aI then the associated fixed-point-free groups whose union with the zero are fields may be written as aIB_{1i}. If two such retraction groups are identical then $aIB_{1j} = aIB_{1i}$ for $i \neq j$, a contradiction. Hence, we obtain a set of k retraction groups all distinct from each other which together with G_0K^* gives $(k+1)$-retraction. □

Thus, we have the following converse to our main result of the previous section on multiple-retraction.

Theorem 3.7. *Let π be a translation plane of order $q^{2^a r}$, $(r, 2) = 1$ with subkernel F isomorphic to* $\mathsf{GF}(q^2)$ *containing K isomorphic to* $\mathsf{GF}(q)$.

(1) *If π admits t distinct commuting Baer subgroups of order $q+1$ then π admits $(t+1)$-retraction and has kernel a subfield of* $\mathsf{GF}(q^{2r})$ *where r is odd.*

(2) *Moreover, for $2^{a-1}r = n$ and if $t = 2^n - 1$, then π admits symmetric homology groups of orders $q+1$.*

Actually, part (2) of the previous theorem requires some explanation. Suppose that we have a translation plane of order q^6, with kernel containing K isomorphic to $\mathsf{GF}(q)$ so $n = 3$ in the above notation. Let F be a field isomorphic to $\mathsf{GF}(q^2)$ and assume the associated vector space V is an F-space. Now consider the following $2^3 - 1$ fields in the associated vector space of dimension 12 over $\mathsf{GF}(q)$

$$F_{(i_1,\ldots,i_6)} = \left\langle \mathrm{diag}(a^{q^{i_1}}, a^{q^{i_2}}, \ldots, a^{q^{i_6}}) \mid a \in \mathsf{GF}(q^2) \right\rangle ,$$

where exactly three of the integers $(i_1, \ldots, i_6)$ are equal to 1 and the remaining three are q. Then each of these fields satisfies the basic conditions to be a retraction field. Conversely, suppose that have 7-retraction in the translation plane. Choose a representation for that $F = F_1$ appears as the scalar field for V as a 6-dimensional F-space. By assumption, the remaining retraction fields commute with F_1. In this setting, the remaining fields F_i, for $i = 2, \ldots, 6$ each permute the orbits of length 1 of F_1^*. Suppose that F_1^* has a orbits of components of length 1 and b orbits of length $q+1$, so that

$$a + b(q+1) = q^6 + 1 .$$

Hence, a is at least 2 since $q+1$ divides $q^6 - 1$. Consider the action of F_i^*, for $i > 1$ on the components of length 1 Λ_1 of F_1^*. If F_i^* does not fix a component of Λ_1 then $q+1$ divides Λ_1, a contradiction. Hence, $F_1^* F_i^*$ fixes two component for each $i > 1$. Since $a \equiv 2 \pmod{q+1}$, then $F_1^* F_i^*$ fixes a_i components, where also $a_i \equiv 2 \pmod{q+1}$. Hence, F_j^* for $j > 1$ and $j \neq i$ fixes two components. Therefore, the group in the product of the set of 6 fields fixes at least two components, say L and M. Let F_2^* have an irreducible component N on L. Then by Schur's lemma, the centralizer of F_2^* is a field. Since both fields F_1 and F_2 have the same order, this means that the two fields act identically on an irreducible component and the dimension of N is 1 over F. Using Maschke's theorem, this implies that we may diagonalize each of the fields F_i and since these are retraction groups, our previous analysis shows that the fields are those of $F_{(i_1,\ldots,i_6)}$.

However, not all of the associated groups can act as a collineation group of a translation plane as can be seen as follows. Note that we obtain a $q+1$-homology group from the product fields $F_{(1,1,1,1,1,1)}F_{(1,1,1,q,q,q)}$.

Consider $F_{(1,1,1,1,1,1)}F_{(1,q,1,q,q,1)}$ and $F_{(1,1,1,1,1,1)}F_{(1,q,1,q,1,q)}$. In the first group, we obtain the Baer group

$$B_1 = \left\langle \mathrm{diag}(1, a^{1-q}, 1, a^{1-q}, a^{1-q}, 1) \mid a \in \mathsf{GF}(q^2)^* \right\rangle ,$$

with fixed point space

$$\mathrm{Fix}\, B_1 = \left\{(x_1, 0, x_3, 0, 0, x_6) \mid x_1, x_3, x_6 \in \mathsf{GF}(q^2)\right\}.$$

In the second group, we have the Baer group

$$B_2 = \left\langle \mathrm{diag}(1, a^{q-1}, 1, a^{1-q}, 1, a^{1-q}) \mid a \in \mathsf{GF}(q^2)^* \right\rangle$$

with fixed point space

$$\mathrm{Fix}\, B_2 = \left\{(x_1, 0, , x_3, 0, x_5, 0) \mid x_1, x_3, x_5 \in \mathsf{GF}(q^2)\right\}.$$

Hence, $B_1 B_2$ fixes

$$\mathrm{Fix}\, B_1 \cap \mathrm{Fix}\, B_2 = \left\{(x_1, 0, x_3, 0, 0, 0) \mid x_1, x_3, x_5 \in \mathsf{GF}(q^2)\right\},$$

which cannot be a subplane or line or either, which is a contradiction. In a similar manner and in general, it follows fairly direct that if we have $(2^n - 1)$-retraction in a translation plane of order q^{2n}, then $n \leq 2$ and since retraction makes sense only when $n \geq 2$, we have the following theorem.

Theorem 3.8. *Let π be a translation plane of order q^{2n} and kernel containing $\mathsf{GF}(q)$ that admits $(2^n - 1)$-retraction. Then, $n = 2$ and we obtain triple-retraction.*

4. Double-homology groups

We revisit part of our previous results and show connections to what are called 'double-homology groups'.

Theorem 4.1. *Let π be a translation plane of order $q^{2^a r}$ and kernel containing K isomorphic to $\mathsf{GF}(q)$.*

(1) *When q is even, a double-Baer group of order $q + 1$ implies double-retraction.*

(2) *For arbitrary order, a double-Baer group of order $q + 1$ with intertwining field implies double-retraction.*

(3) *A double-Baer group of order $(2, q - 1)(q + 1)$ implies double-retraction.*

(4) *Double-retraction implies a double-Baer group of order $q+1$ or two commuting homology groups of order $q + 1$–a 'double-homology group' of order $q + 1$.*

(5) *A double-homology group of order $q + 1$ in a semifield plane implies double-retraction.*

Proof. It remains to prove (4) and (5). Assume that we have double-retraction. Write one of the fixed-point-free groups as $G_1 K^*$ and decompose the space so that this group looks like a scalar group of order q^2-1 generated by $z\mapsto az$. Then, we know that the second group G_2K^* must look either like $x_i\mapsto ax_i$ or $x_j\mapsto a^qx_j$ and there are exactly half a's and half $a^q{}'$s. It follows that by multiplication, we may obtain a commuting pair of homology groups of order $q+1$ or a double-Baer group of order $q+1$, which proves (3).

Now assume that we have a semifield plane and we have a double-homology group of order $q+1$. Hence, we must have a double-homology group of order q^2-1. Moreover, we can 'fuse' the nuclei. The possible products produce two fixed-point-free fields of order q^2-1. We need that these fields contain the K-kernel homology group of order $q-1$. Let a subgroup of the product of the two homology groups be given by

$$\mathcal{F}^{q^\lambda}:(x,y)\mapsto(xA,yA^{q^\lambda});A\in F^{+*},$$

where F^{+*} is a field of order q^2. However, since the homologies are K-linear groups, it follows that F^{+*} commutes with the kernel homology group K^* so that by Schur's lemma, $\langle F^{+*},K^*\rangle$ are contained in a field. It follows by uniqueness of cyclic groups that $K\subseteq F^+$.

Hence, both $\mathcal{F}^{q^\lambda}$ for $\lambda=0$ or 1 produce retraction. Hence, double-retraction exists. □

Definition 4.2. Let π be a translation plane of order q^{2n} and kernel containing K isomorphic to $\mathsf{GF}(q)$. A 'double-homology group' is a pair of commuting homology groups of orders divisible by a critical Baer order. A 'double-generalized central group' is either a double-homology group or a double-Baer group.

Corollary 4.3. *Let π be a semifield plane of even order q^{2n} and kernel containing K isomorphic to* $\mathsf{GF}(q)$.

Then double-retraction is equivalent to a double-generalized central group of order $(q+1)$, when the kernel contains $\mathsf{GF}(q^2)$.

We now assume that we have a translation plane of order q^{2^ar}, which has kernel isomorphic to $\mathsf{GF}(q^2)$ and also assume that we have right and middle subnuclei isomorphic to this subkernel F. Assume that we can fuse these nuclei (this is possible, for example when the plane is a semifield by Jha and Johnson [7]). Let Q denote a quasifield coordinatizing the plane π. Let K be a subkernel subfield of F and isomorphic to $\mathsf{GF}(q)$. If K commutes with Q then there is an associated K-regulus in the spread. Furthermore, we assume that F does not commute over the associated quasifield. Then, we take the subkernel F acting as $z\mapsto az$ for all vectors z, writing the vector space over F. Moreover, if the axis and coaxis are taken as $x=0$ and $y=0$, we have the action of either homology group as $z\mapsto a^tz$ where $t=0,1$ or q, assuming that K commutes with the quasifield.

We claim that from the two homology groups, we obtain two distinct retraction-groups not equal to the kernel group. Each of these groups provides in turn a double-Baer group of order $q+1$. Hence, we obtain:

Theorem 4.4. *Let π be a translation plane of order $q^{2^a r}$, $(r,2)=1$ with subkernel F isomorphic to $\mathsf{GF}(q^2)$. Assume that we have right and middle subnuclei both fields and field-isomorphic to this subkernel F. Let K denote the subfield isomorphic to $\mathsf{GF}(q)$.*

If the associated quasifield commutes over K but does not commute over F then we obtain commuting double-Baer groups providing double-retraction, as well as double-homology groups that provide double-retraction.

Now applying this to semifield planes, we obtain:

Theorem 4.5. *Let π be a semifield plane of order $q^{2^a r}$ for $(r,2)=1$ admit subnuclei, left, right and middle all isomorphic to $\mathsf{GF}(q^2)$. Assume also that there is a double-Baer group of order $q+1$ that is not in the group generated by the three associated homology groups.*

Then π admits triple-retraction.

Proof. Since we have a semifield plane, we know that we can fuse the nuclei. By the above theorem, a double-Baer group must be commensurate with a kernel group of order q^2-1. □

Actually, the results stated above can be made somewhat more general. For example, assume that a right and middle nucleus of a semifield plane of square order are isomorphic to $\mathsf{GF}(q^2)$. In the product of order $(q^2-1)^2$ of the two affine homology groups of order q^2-1 union the zero vector, there are a number of fields of order q^2-1. Represent the groups as follows:

$$\left\langle \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \mid M \in L^* \right\rangle, \text{where } L \text{ is a field of order } q^2, \text{ and}$$

$$\left\langle \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \mid N \in J^* \right\rangle, \text{where } J \text{ is a field of order } q^2.$$

In general, we would know that it is possible to 'split' the fields so that $L \cap J$ share any subfield of order p^z, for $q = p^r$, where $p^z - 1$ divides q^2-1. Let $L^H = J$, where H is a suitable matrix, considering J and L are matrix fields. Then in

$$\left\langle \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} \mid N \in J, M \in L \right\rangle,$$

we may form the fields

$$\left\langle \begin{bmatrix} M^{\sigma H} & 0 \\ 0 & M \end{bmatrix} \mid M \in L \right\rangle ,$$

where σ is a fixed automorphism of M. This has all of the requirements for a retraction field except that the field may not contain the kernel homology field isomorphic to $\mathsf{GF}(q)$. But, if $\sigma = 1$ or q, we do have proper retraction and when σ is not one of these two, we simply retract with a smaller field. For example, if $\sigma = p$ then we have a retraction field of order p^2. In this way, there are a tremendous variety of subgeometry partitions available to us in various projective spaces.

In the settings where $\sigma = 1$ or q and we have kernel $\mathsf{GF}(q^2)$, and with the manner of splitting, we obtain triple-retraction. Hence, the previous theorem can be improved as follows.

Theorem 4.6. *Let π be a semifield plane of order $q^{2^a r}$ for $(r, 2) = 1$ admit subnuclei, left, right and middle all isomorphic to $\mathsf{GF}(q^2)$.*

(1) *Then π admits triple-retraction.*

(2) *Assume that $q = p^r$ and r is odd then π admits triple-retraction with fields of order $p^{2t} - 1$, for any t dividing r. (Note that the size of the projective space remains the same while the size of the subgeometries decreases.)*

5. New examples of double-Baer groups

As we mentioned previously, one of the main sources of translation planes or quasifibrations (see [6]) admitting double-Baer groups comes from the structures that are algebraically lifted. When the original geometry is a semifield plane, then the lifted structure is a semifield plane.

Furthermore, Johnson [8] has shown that a semifield plane of order q^4 that admits a Baer collineation of order a prime p-primitive divisor of $q^2 - 1$ is necessarily a plane that may be algebraically lifted. More generally, recently, Dempwolff [2] has extended the general theory of Johnson and determined a variety of semifield planes that admit Baer groups.

In general, let π be a translation plane of order q^4, where $q = p^r$, for p a prime. And, let B be a Baer group of order dividing $q^2 - 1$. Then, in the net defined by the $\mathrm{Fix}\, B$, there is a second Baer subplane, called $\mathrm{coFix}\, B$, upon which B acts faithfully. If B acts irreducibly as a linear $\mathsf{GF}(p)$-subgroup on any component of $\mathrm{coFix}\, B$, then Dempwolff terms the group B an 'irreducible planar Baer group'. Note that B necessarily acts on $\mathrm{coFix}\, B$ as a cyclic kernel homology group so if the order of B contains a p-primitive divisor of q^2-1 then B is irreducible.

Dempwolff [2] provides three classes of semifield planes of order q^4 that admit irreducible planar Baer groups. We shall begin with the last of these classes (Dempwolff's class (4.3)). Consider the following matrices:

$$\left\{ \begin{bmatrix} u & 0 & at+bt^q & (ct+dt^q)^q \\ 0 & u^q & ct+dt^q & (at+bt^q)^q \\ t & 0 & u^q & 0 \\ 0 & t^q & 0 & u \end{bmatrix} \mid u,t \in \mathsf{GF}(q^2) \right\},$$

a,b,c,d constants. For certain choices of a,b,c,d, this gives a semifield of order q^4, and if two of a,c,d are non-zero then it has right $=$ middle $=$ kernel $=$ $\mathsf{GF}(q)$ $=$ center. So, it is a semifield of dimension 4 over the center. Furthermore,

- ⋆ $c=d=0$ if and only if we have the left nucleus as $\mathsf{GF}(q^2)$;
- ⋆ $a=d=0$ if and only if the middle nucleus is $\mathsf{GF}(q^2)$;
- ⋆ $a=c=0$ if and only if the right nucleus is $\mathsf{GF}(q^2)$;
- ⋆ and if $ac \neq 0$ then right $=$ middle $=$ $\mathsf{GF}(q)$.

Dempwolff shows that it is always possible to find elements so that $acd \neq 0$. This set forms a new class of semifields.

Note that the group $\langle \mathrm{diag}(I,P,P,I)$ for $P=\mathrm{diag}(e,e^q)$ such that $e^q=e^{-1}\rangle$ is a Baer group of order $q+1$, although we do not actually obtain a double-Baer group in this case. Although, it is easy to see that we do obtain a double-Baer group when $c=d=0$.

In any case, there is a way to obtain a double-Baer group situation.

Furthermore, the semifield plane is derivable, and the right $=$ middle homology groups of order $q-1$ become Baer groups in the derived plane. So, the derived plane of order q^4 admits two Baer groups of order $q-1$. Now suppose that $q=p^{2^a r}$, where $(2,r)=1$ so we have a plane of order $p^{2^{a+2}r}$ with two Baer groups of order $p^{2^a r}-1$. So, assume that $q=h^2$ so we have two Baer groups of order $h+1$ in a translation plane of order h^8 with kernel $\mathsf{GF}(h^2)$. Therefore, we obtain double-Baer groups that are fused and hence we have double-retraction.

Theorem 5.1. *The class of Dempwolff semifield planes π of order q^4, for $acd \neq 0$, listed above are derivable.*

(1) *When q is a square, let π^* denote the translation plane obtained by derivation of the net*

$$x=0, y=x\,\mathrm{diag}(u,u^q,u^q,u)\,;u\in \mathsf{GF}(q^2)\,.$$

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There is a collineation group of order $q = h^2$ that contains a kernel subgroup of order $h - 1$ and is fixed-point-free and we obtain a double-Baer group and double-retraction.

(2) *Hence, there is a retraction to a mixed subgeometry partition of* $\mathsf{PG}(7, h^2)$ *by subgeometries isomorphic to* $\mathsf{PG}(7, h)$ *or* $\mathsf{PG}(3, h^2)$.

Dempwolff also points out that there are known classes of semifield planes of order q^4 that admit a Baer group of order $q + 1$, as well as double-Baer groups in some situations.

Dempwolff's class (4.1) is defined as follows: Let $q = p^k$, for $m = 2k$ and define multiplication as follows:

$$(u, v) * (x, y) = (ux + gv^{p^a} y^{p^b}, y + vx^{p^k}), \text{ for } u, v, x, y \in \mathsf{GF}(q^2 = p^m)\,,$$

where $a, b \in \{0, 1, \ldots, m-1\}$ and such that $p^m \neq p^{m(p^b+1+p^{|a-b|}+1+p^{k+b}-1)}$ and g in $\mathsf{GF}(p^m)$ and of order not $p^{m(p^b+1+p^{|a-b|}+1+p^{k+b}-1)}$. Then the above multiplication defines a semifield (actually, a generalized Knuth semifield). The kernel is $\mathsf{GF}(p^{(a,m)})$, middle nucleus $\mathsf{GF}(p^{(k+a-b,m)})$ and right nucleus $\mathsf{GF}(p^{(k-b,m)})$. In all cases, there is a Baer group of order $q + 1 = p^k + 1$. Moreover, the semifield plane is derivable with a net of the form

$$x = 0, y = x\,\mathrm{diag}(u, u^q, u^q, u)\,; u \in \mathsf{GF}(q^2)\,.$$

First assume that a is even so that we have a subkernel $\mathsf{GF}(p^{2(a/2,k)})$. Now consider $\mathsf{GF}(p^{2(a/2,k)}) \cap \mathsf{GF}(q^2) = \mathsf{GF}(p^{2(a/2,k,2k)})$. Now assume that k is odd so as $(a/2, k, 2k)$ divides k, we see that we have a sub-Baer group of order $p^{(a/2,k,2k)} + 1 = p^{(a/2,k)} + 1$. This means we have a double-Baer group of order $p^{(a/2,k)} + 1$, and hence by Corollary 3.5, we have double-retraction. Hence, we have the following theorem.

Theorem 5.2. *Consider the semifield*

$$(u, v) * (x, y) = (ux + gv^{p^a} y^{p^b}, y + vx^{p^k}), \textit{ for } u, v, x, y \in \mathsf{GF}(q^2 = p^m)\,.$$

Assume that a is even and k is odd ($a \leq 2k - 1$). Then there is a double-Baer group of order $p^{(a/2,k)} + 1$, which may be fused so that double-retraction occurs.

When we derive the net, we obtain Baer groups of orders $p^{(k+a-b,m)} - 1$ and $p^{(k-b,m)} - 1$, and we may fuse the obvious intersection Baer groups of order $p^{((k-a-b,2k),(k-b,2k))} - 1$. Hence, we have double-Baer groups of this order. The kernel of the derived plane is

$$\mathsf{GF}(q) \cap \mathsf{GF}(p^{(a,2k)}) = \mathsf{GF}(p^{(a,2k,k)}) = \mathsf{GF}(p^{(a,k)}).$$

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Furthermore, we may fuse all subnuclei isomorphic to

$$\mathsf{GF}(p^{(a,2k)}) \cap \mathsf{GF}(p^{(k+a-b,2k)}) \cap \mathsf{GF}(p^{(k-b,2k)}) = \mathsf{GF}(p^{((a,2k),(k+a-b,2k),(k-b,2k))}).$$

And if the subfield is a square $\mathsf{GF}(h^2)$, then we obtain double-retraction.

Theorem 5.3. *The derivation of the semifield plane listed in the previous theorem admits two fused Baer groups of order*

$$p^{((k-a-b,2k),(k-b,2k))} - 1\,.$$

If $((a,2k),(k+a-b,2k),(k-b,2k))$ *is even, then we obtain double-Baer groups and double-retraction.*

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