



# A note on the group of projectivities of finite projective planes

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*Dedicated to Professor Gábor Korchmáros on the occasion of his sixtieth birthday.*

## Abstract

In this short note we show that the group of projectivities of a projective plane of order 23 cannot be isomorphic to the Mathieu group  $M_{24}$ . By a result of T. Grundhöfer [6], this implies that the group of projectivities of a non-desarguesian projective plane of finite order  $n$  is isomorphic either to the alternating group  $A_{n+1}$  or to the symmetric group  $S_{n+1}$ .

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## 1 Introduction

Any projective plane  $\Pi$  can be coordinatized by a planar ternary ring  $(R, T)$ , see [7]. There is a natural bijection between the set of points of an arbitrary line  $\ell$  and the set  $R \cup \{\infty\}$ . Let  $P$  denote the group of projectivities of  $\Pi$ ; then  $P$  acts 3-transitively on the point set of  $\ell$ . Equivalently, we can consider the group  $P$  of projectivities as a permutation group acting on  $R \cup \{\infty\}$ .

The fundamental theorem of projective planes says that  $\Pi$  is pappian if and only if  $P$  is sharply 3-transitive. In [6], T. Grundhöfer has shown that the group of projectivities of a non-desarguesian projective plane  $\Pi$  of finite order  $n$  is either the alternating group  $A_{n+1}$ , or the symmetric group  $S_{n+1}$ , or  $n = 23$  and

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$P$  is the Mathieu group  $M_{24}$ . In this paper, we show that the latter case cannot occur. Our proof uses computer calculations.

## 2 Coordinate loops and their multiplication groups

For a loop  $(L, \cdot, 1)$ , we denote by  $L_x$  and  $R_x$  the left and right translation maps by  $x$ , respectively. These maps generate the multiplication group  $\text{Mlt}(L)$  of  $L$ . The stabilizer of the unit element  $1 \in L$  is the inner mapping group  $\text{Inn}(L)$  of  $L$ . The left (or right) translations form a sharply transitive set of permutations. Moreover, for any  $x, y \in L$ ,  $L_x R_y L_x^{-1} R_y^{-1} \in \text{Inn}(L)$ .

The next result was already noticed by A. Drápal [3] in a slightly weaker form.

**Lemma 2.1.** *The Mathieu group  $M_{22}$  of degree 22 does not contain the multiplication group of a loop of order 22.*

*Proof.* Let  $G = M_{22}$  act on  $\{1, 2, \dots, 22\}$ . Let  $e$  be the unit element of  $G$ , and  $H = G_1$  be the stabilizer in  $G$  of 1. Assume that  $\text{Mlt}(L) \leq G$ . Then  $G$  contains two sharply transitive subsets  $U, V$  of order 22 such that  $e \in U, V$  and  $wvu^{-1}v^{-1} \in H$  for all  $u \in U, v \in V$ . For any  $c \in N_{S_{22}}(G)$  there is an element  $w \in V$  such that  $H^{wc} = H$ . Thus the pair  $c^{-1}Uc, c^{-1}w^{-1}Vc$  has the same properties as  $U, V$ : the commutator element

$$c^{-1}(ww^{-1}vu^{-1}(w^{-1}v)^{-1})c = c^{-1}w^{-1}(wuw^{-1}u^{-1})(wvu^{-1}v^{-1})wc$$

is indeed contained in  $H^{wc} = H$ . Thus,  $U$  can be replaced by some conjugate under  $\text{Aut}(M_{22}) = N_{S_{22}}(G)$ . Up to conjugacy by  $\text{Aut}(M_{22})$  there are only 3 fixed point free elements in  $G$  represented by

$$\begin{aligned} &(1\ 2\ 15\ 14\ 17\ 11)\ (3\ 8\ 19\ 22\ 9\ 13)\ (4\ 10)\ (5\ 7\ 18)\ (6\ 16\ 12)\ (20\ 21), \\ &(1\ 2\ 20\ 3\ 18\ 21\ 9\ 22)\ (4\ 6\ 19\ 8\ 5\ 11\ 7\ 17)\ (10\ 15\ 16\ 14)\ (12\ 13), \text{ and} \\ &(1\ 2\ 9\ 16\ 18\ 22\ 8\ 15\ 10\ 11\ 6)\ (3\ 7\ 5\ 19\ 17\ 14\ 12\ 21\ 4\ 20\ 13). \end{aligned}$$

These three elements generate  $G$ , so they describe the action of  $G$  we work with. Pick  $e \neq a \in U$ . By the previous remark we may assume that  $a$  is one of the given 3 elements. Note that  $1^a = 2$ . By transitivity of  $U$  there are  $b, c \in U$  with  $1^b = 3$  and  $1^c = 4$ .

Let  $F$  denote the set of fixed point free elements of  $G$  and for  $X \subseteq G$  define the set  $S_X = \{g \in F \mid xgx^{-1}g^{-1} \in H \ \forall x \in X\}$ . Note that if  $X$  is a subset of  $U$ , then  $S_X$  contains  $V$ . In particular,  $S_X$  is transitive on  $\{1, \dots, 22\}$ . However, a straightforward computer calculation (see the remark below) shows that for

any  $a$  as above and  $b, c \in F$  with  $ab^{-1}, bc^{-1}, ca^{-1} \in F$ ,  $1^b = 3$ ,  $1^c = 4$ , the set  $S_{\{a,b,c\}}$  is intransitive on  $\{1, \dots, 22\}$ . This proves the lemma.  $\square$

With a given planar ternary ring  $(R, T)$ , one can introduce two binary operations  $x + y = T(1, x, y)$  and  $x \cdot y = T(x, y, 0)$  in such a way that  $(R, +, 0)$  and  $(R^* = R \setminus \{0\}, \cdot, 1)$  are loops.

**Lemma 2.2.** *Let  $P$  be the group of projectivities of the projective plane  $\Pi$ . Then the 2-point stabilizer  $P_{0,\infty}$  contains the multiplication group  $\text{Mlt}(R^*, \cdot)$  of the multiplicative loop  $(R^*, \cdot)$ .*

*Proof.* Easy calculation shows that for any  $a \in R^*$ , the projectivities

$$\begin{aligned}\alpha &= ([1] (0) [1, 0] (\infty) [a, 0] (0) [1]), \\ \beta &= ([1] (0, 0) [a] (0) [1])\end{aligned}$$

map the point  $(1, y)$  of  $[1]$  to  $(1, a \cdot y)$  and  $(1, y \cdot a)$ , respectively. Moreover,  $\alpha$  and  $\beta$  leave the points  $(1, 0)$  and  $(\infty)$  fixed.  $\square$

Our main result completes the solution of the conjecture in [2, p. 160].

**Theorem 2.3.** *The group of projectivities of a non-desarguesian projective plane of finite order  $n$  contains the alternating group  $A_{n+1}$ .*

*Proof.* By [6], we only have to exclude the case  $n = 23$  and  $P = M_{24}$ . However, if this case would exist, then by Lemma 2.2,  $M_{22}$  would contain the multiplication group of a loop, which contradicts Lemma 2.1.  $\square$

We conclude this note with two remarks. First, we notice that both the alternating and the symmetric group can be the group of projectivities of a non-desarguesian finite projective plane, see [5] and the references therein. The second remark concerns the computer calculation in the proof of Lemma 2.1. Let  $a$  be one of the 3 possibilities from above, then the number of possibilities for  $b \in F$  with  $ab^{-1} \in F$  and  $1^b = 3$  is 3214, 3290, or 3318, respectively. The sizes of the sets  $S_{\{a,b\}}$  are between 355 and 538. In the majority of the cases  $S_{\{a,b\}}$  is intransitive on  $\{1, \dots, 22\}$ . In the remaining cases one determines the possibilities for  $c$ , and shows that  $S_{\{a,b,c\}}$  is intransitive again.

The computation takes about 40 minutes on an average home PC. The algorithm was implemented twice independently in the computer algebra systems GAP [4] and Magma [1].

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