A note on the group of projectivities of finite projective planes

Peter Müller Gábor P. Nagy*

Dedicated to Professor Gábor Korchmáros on the occasion of his sixtieth birthday.

Abstract

In this short note we show that the group of projectivities of a projective plane of order 23 cannot be isomorphic to the Mathieu group M_{24}. By a result of T. Grundhöfer [6], this implies that the group of projectivities of a non-desarguesian projective plane of finite order n is isomorphic either to the alternating group A_{n+1} or to the symmetric group S_{n+1}.

Keywords: projective planes, projectivities, loops

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1. Introduction

Any projective plane Π can be coordinatized by a planar ternary ring (R, T), see [7]. There is a natural bijection between the set of points of an arbitrary line ℓ and the set $R \cup \{\infty\}$. Let P denote the group of projectivities of Π; then P acts 3-transitively on the point set of ℓ. Equivalently, we can consider the group P of projectivities as a permutation group acting on $R \cup \{\infty\}$.

The fundamental theorem of projective planes says that Π is pappian if and only if P is sharply 3-transitive. In [6], T. Grundhöfer has shown that the group of projectivities of a non-desarguesian projective plane Π of finite order n is either the alternating group A_{n+1}, or the symmetric group S_{n+1}, or $n = 23$ and

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P is the Mathieu group M_{24}. In this paper, we show that the latter case cannot occur. Our proof uses computer calculations.

2. Coordinate loops and their multiplication groups

For a loop $(L, \cdot, 1)$, we denote by L_x and R_x the left and right translation maps by x, respectively. These maps generate the multiplication group $\mathrm{Mlt}(L)$ of L. The stabilizer of the unit element $1 \in L$ is the inner mapping group $\mathrm{Inn}(L)$ of L. The left (or right) translations form a sharply transitive set of permutations. Moreover, for any $x, y \in L$, $L_x R_y L_x^{-1} R_y^{-1} \in \mathrm{Inn}(L)$.

The next result was already noticed by A. Drápal [3] in a slightly weaker form.

Lemma 2.1. *The Mathieu group M_{22} of degree 22 does not contain the multiplication group of a loop of order 22.*

Proof. Let $G = \mathsf{M}_{22}$ act on $\{1, 2, \ldots, 22\}$. Let e be the unit element of G, and $H = G_1$ be the stabilizer in G of 1. Assume that $\mathrm{Mlt}(L) \leq G$. Then G contains two sharply transitive subsets U, V of order 22 such that $e \in U, V$ and $uvu^{-1}v^{-1} \in H$ for all $u \in U, v \in V$. For any $c \in N_{\mathsf{S}_{22}}(G)$ there is an element $w \in V$ such that $H^{wc} = H$. Thus the pair $c^{-1}Uc, c^{-1}w^{-1}Vc$ has the same properties as U, V: the commutator element

$$c^{-1}(uw^{-1}vu^{-1}(w^{-1}v)^{-1})c = c^{-1}w^{-1}(wuw^{-1}u^{-1})(uvu^{-1}v^{-1})wc$$

is indeed contained in $H^{wc} = H$. Thus, U can be replaced by some conjugate under $\mathrm{Aut}(\mathsf{M}_{22}) = N_{\mathsf{S}_{22}}(G)$. Up to conjugacy by $\mathrm{Aut}(\mathsf{M}_{22})$ there are only 3 fixed point free elements in G represented by

$$(1\ 2\ 15\ 14\ 17\ 11)\,(3\ 8\ 19\ 22\ 9\ 13)\,(4\ 10)\,(5\ 7\ 18)\,(6\ 16\ 12)\,(20\ 21),$$
$$(1\ 2\ 20\ 3\ 18\ 21\ 9\ 22)\,(4\ 6\ 19\ 8\ 5\ 11\ 7\ 17)\,(10\ 15\ 16\ 14)\,(12\ 13), \text{ and}$$
$$(1\ 2\ 9\ 16\ 18\ 22\ 8\ 15\ 10\ 11\ 6)\,(3\ 7\ 5\ 19\ 17\ 14\ 12\ 21\ 4\ 20\ 13).$$

These three elements generate G, so they describe the action of G we work with. Pick $e \neq a \in U$. By the previous remark we may assume that a is one of the given 3 elements. Note that $1^a = 2$. By transitivity of U there are $b, c \in U$ with $1^b = 3$ and $1^c = 4$.

Let F denote the set of fixed point free elements of G and for $X \subseteq G$ define the set $S_X = \{g \in F \mid xgx^{-1}g^{-1} \in H\ \forall x \in X\}$. Note that if X is a subset of U, then S_X contains V. In particular, S_X is transitive on $\{1, \ldots, 22\}$. However, a straightforward computer calculation (see the remark below) shows that for any a as above and $b, c \in F$ with $ab^{-1}, bc^{-1}, ca^{-1} \in F$, $1^b = 3$, $1^c = 4$, the set $S_{\{a,b,c\}}$ is intransitive on $\{1, \ldots, 22\}$. This proves the lemma. □

With a given planar ternary ring (R, T), one can introduce two binary operations $x + y = T(1, x, y)$ and $x \cdot y = T(x, y, 0)$ in such a way that $(R, +, 0)$ and $(R^* = R \setminus \{0\}, \cdot, 1)$ are loops.

Lemma 2.2. *Let P be the group of projectivities of the projective plane Π. Then the 2-point stabilizer $P_{0,\infty}$ contains the multiplication group $\mathrm{Mlt}(R^*, \cdot)$ of the multiplicative loop $(R^*, \cdot)$.*

Proof. Easy calculation shows that for any $a \in R^*$, the projectivities

$$\alpha = \big([1]\ (0)\ [1,0]\ (\infty)\ [a,0]\ (0)\ [1]\big),$$
$$\beta = \big([1]\ (0,0)\ [a]\ (0)\ [1]\big)$$

map the point $(1, y)$ of $[1]$ to $(1, a \cdot y)$ and $(1, y \cdot a)$, respectively. Moreover, α and β leave the points $(1, 0)$ and (∞) fixed. □

Our main result completes the solution of the conjecture in [2, p. 160].

Theorem 2.3. *The group of projectivities of a non-desarguesian projective plane of finite order n contains the alternating group A_{n+1}.*

Proof. By [6], we only have to exclude the case $n = 23$ and $P = \mathsf{M}_{24}$. However, if this case would exist, then by Lemma 2.2, M_{22} would contain the multiplication group of a loop, which contradicts Lemma 2.1. □

We conclude this note with two remarks. First, we notice that both the alternating and the symmetric group can be the group of projectivities of a non-desarguesian finite projective plane, see [5] and the references therein. The second remark concerns the computer calculation in the proof of Lemma 2.1. Let a be one of the 3 possibilities from above, then the number of possibilities for $b \in F$ with $ab^{-1} \in F$ and $1^b = 3$ is 3214, 3290, or 3318, respectively. The sizes of the sets $S_{\{a,b\}}$ are between 355 and 538. In the majority of the cases $S_{\{a,b\}}$ is intransitive on $\{1, \ldots, 22\}$. In the remaining cases one determines the possibilities for c, and shows that $S_{\{a,b,c\}}$ is intransitive again.

The computation takes about 40 minutes on an average home PC. The algorithm was implemented twice independently in the computer algebra systems GAP [4] and Magma [1].

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Peter Müller

Institut für Mathematik, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany

e-mail: `peter.mueller@mathematik.uni-wuerzburg.de`

website: `http://mathematik.uni-wuerzburg.de/~mueller`

Gábor P. Nagy

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

e-mail: `nagyg@math.u-szeged.hu`

website: `http://www.math.u-szeged.hu/~nagyg`