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Locally hermitian partial ovoids of unitary polar spaces and partial ovoids of orthogonal polar spaces

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Abstract

In order to study unitals in the projective plane $\mathsf{PG}(2,q^2)$, F. Buekenhout [5] gave a representation in $\mathsf{PG}(4,q)$ of the unitary polar space $H(2,q^2)$ as points of a quadratic cone on a $Q^-(3,q)$.

In [16], G. Lunardon used the Barlotti-Cofman representation of $\mathsf{PG}(3,q^2)$ to represent $H(3,q^2)$ in $\mathsf{PG}(6,q)$ as a cone on a $Q^+(5,q)$. He also proved that to any locally hermitian ovoid of $H(3,q^2)$ corresponds an ovoid of $Q^+(5,q)$ and conversely.

In this paper, we study the Barlotti-Cofman representation of the unitary polar space $H(n,q^2)$ for all n and we prove that to any locally hermitian partial ovoid of such spaces corresponds a partial ovoid of an orthogonal polar space, and conversely. Further the locally hermitian partial ovoid is maximal if and only if the corresponding partial ovoid of the orthogonal polar space is maximal. As a consequence of the previous connection and a result of A. Klein [14] we obtain a geometric proof to derive that the orthogonal polar space $Q^+(4n+1,q)$ has no ovoid when $n > q^3$.

Keywords: polar spaces, partial ovoids, ovoids

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1 Introduction

In the projective space $\mathsf{PG}(n,q)$ coordinatized by the finite field $\mathsf{GF}(q)$ let $\mathcal{P}$ denote a classical polar space. The *generators* of $\mathcal{P}$ are the subspaces of maximal dimension contained in it.

An *ovoid* of $\mathcal{P}$ is a set of points having exactly one common point with every generator. A *partial ovoid* of $\mathcal{P}$ is a set of points having at most one common point with every generator of $\mathcal{P}$. Equivalently, a partial ovoid is a set of points consisting of pairwise non-conjugate points of $\mathcal{P}$.

The following results on the existence and nonexistence of ovoids in finite classical polar spaces are known.

The orthogonal polar space $Q^-(2n+1,q)$ has ovoids only when $n=1$ [25] whereas $Q^+(2n+1,q)$ has no ovoid when $n \geq 4$ and $q=2,3$, see [13, 23]. For $n=1,2$ $Q^+(2n+1,q)$ admits ovoids for all values of q [20]; $Q^+(7,q)$ has ovoids at least in the following cases: q even, q an odd prime and q odd with $q \equiv 0 \text{ or } 2 \pmod 3$, see [6, 12, 13, 19, 7, 22].

The orthogonal polar space $Q(2n,q)$, q even, has ovoids if and only if $n=2$ [25]. If q is odd, ovoids of $Q(2n,q)$ do not exist if $n \geq 4$ [11] and the only known ovoids of $Q(6,q)$ are the unitary ovoid of $Q(6,3^h)$ and the Ree ovoid of $Q(6,3^{2h+1})$ [12, 25].

For n even, the unitary polar space $H(n,q^2)$ has no ovoids [25]. When n is odd, recently J. De Beule and K. Metsch [10] have proved the non-existence of ovoids in the polar space $H(5,4)$ and A. Klein [14] has proved that $H(2n+1,q^2)$ has no ovoids when $n > q^3$. On the other hand, $H(3,q^2)$ has many ovoids.

Thus, to the knowledge of the author, the existence or nonexistence of ovoids in the following cases are still open problems:

(a) $Q^+(7,q)$, q odd, $q \equiv 1 \pmod 3$ and q not a prime;

(b) $Q^+(2n+1,q)$, $n \geq 4$, $q > 3$;

(c) $Q(6,q)$, $q \neq 3^h$;

(d) $H(5,q^2)$, $q > 2$;

(e) $H(2n+1,q^2)$, $2 < n \leq q^3$.

When the classical polar space $\mathcal{P}$ does not have ovoids, the emphasis lies on studying maximal partial ovoids of it.

In this article we study in details the connection between certain type of partial ovoids (called *locally hermitian*) of the unitary polar space and partial ovoids of orthogonal polar spaces.

A line L of $\mathsf{PG}(n,q)$ not contained in the polar space $\mathcal{P}$ is said to be *external, tangent* or *hyperbolic* if L meets $\mathcal{P}$ in 0, 1 or $s > 1$ points, respectively. More precisely, if $\mathcal{P}$ is the orthogonal polar space then $s=2$ whereas if $\mathcal{P}$ is the unitary polar space $H(n,q^2)$ then $s=q+1$.

A (partial) ovoid $\mathcal{O}$ of $H(n,q^2)$ is said to be *locally hermitian* with respect to a point P of $\mathcal{O}$ if the points of $\mathcal{O}$ are contained in the union of hyperbolic lines through P.

In [8] several infinite families of locally hermitian ovoids of $H(3,q^2)$ are constructed. In particular these are translation ovoids of $H(3,q^2)$, i.e. there is a collineation group of $H(3,q^2)$ fixing all lines of $H(3,q^2)$ through P and acting regularly on points of the ovoid but not P. After [2] it is clear that there is an intimate connection between translation ovoids of $H(3,q^2)$ and semifields of dimension two over their left nucleus.

In [16], by using the Barlotti-Cofman representation of $\mathsf{PG}(3,q^2)$, G. Lunardon gave a representation of $H(3,q^2)$ in $\mathsf{PG}(6,q)$: the unitary polar space $H(3,q^2)$ is represented by a cone projecting the orthogonal polar space $Q^+(5,q)$ from its vertex at infinity. In the same paper G. Lunardon has proved that to every locally hermitian ovoid of $H(3,q^2)$ with respect to a point P lying at infinity corresponds an ovoid of $Q^+(5,q)$ and conversely.

The previous representation can be viewed as a particular case of a more general process to represent the unitary polar space $H(n,q^2)$. Actually, the case $n=2$ was realized by F. Buekenhout in [5]: the polar space $H(2,q^2)$ is represented by a quadratic cone projecting the orthogonal polar space $Q^-(3,q)$ from its vertex at infinity.

When $n=4$, F. Mazzocca, O. Polverino and L. Storme [18] have used the Barlotti-Cofman representation of $\mathsf{PG}(4,q^2)$ to construct maximal partial ovoids of $H(4,q^2)$ of size q^3+1 which are locally hermitian with respect to a point.

In this paper, by using the Barlotti-Cofman representation of $H(n,q^2)$, we prove that there is a close correspondence between local hermitian partial ovoids of $H(n,q^2)$ and partial ovoids of the corresponding orthogonal polar spaces. Further the locally hermitian partial ovoid is a maximal partial ovoid if and only if the corresponding partial ovoid of the orthogonal polar space is maximal. Finally, by using the result of A. Klein [14], we give a geometric proof that the orthogonal polar space $Q^+(4n+1,q)$ has no ovoids when $n>q^3$.

2 The geometry of $H(n,q^2)$ in the Barlotti-Cofman representation of $\mathsf{PG}(n,q^2)$

A *line-spread* of $\mathsf{PG}(2n+1,q)$, $n\geq 2$, is a set of mutually disjoint lines partitioning the point-set of $\mathsf{PG}(2n+1,q)$. A line-spread $\mathcal{N}$ of $\mathsf{PG}(2n+1,q)$ is said to be *normal* if $\mathcal{N}$ induces a spread in any subspace generated by two elements of $\mathcal{N}$, i.e. if $A,\ B\in\mathcal{N}$, then

$$\mathcal{N}_{\langle A,B\rangle}=\{X\in\mathcal{N}\mid X\cap\langle A,B\rangle\neq\emptyset\}$$

is a spread of $\langle A, B\rangle$.

It is possible to construct a normal line-spread of $\mathsf{PG}(2n+1, q^2)$ in the following way.

Denote by $(\mathbf{x}, \mathbf{y})$ be the homogeneous projective coordinates of a point of $\mathsf{PG}(2n+1, q^2)$ with $\mathbf{x} = (x_0, x_1, \ldots, x_n)$, $\mathbf{y} = (y_0, y_1, \ldots, y_n) \in \mathsf{GF}(q^2)^{n+1}$. Denote by σ the involutory collineation of $\mathsf{PG}(2n+1, q^2)$ defined by $(\mathbf{x}, \mathbf{y})^\sigma = (\mathbf{y}^q, \mathbf{x}^q)$ where $\mathbf{x}^q = (x_0^q, x_1^q, \ldots, x_n^q)$ and $\mathbf{y}^q = (y_0^q, y_1^q, \ldots, y_n^q)$.

Let $\mathsf{PG}(2n+1, q)$ be the Baer subgeometry of $\mathsf{PG}(2n+1, q^2)$ pointwise fixed by σ, i.e. the points of $\mathsf{PG}(2n+1, q)$ have homogeneous projective coordinates $(\mathbf{x}, \mathbf{x}^q)$. Let Γ be the n-dimensional subspace of $\mathsf{PG}(2n+1, q^2)$ with equation $\mathbf{y} = \mathbf{0}$. Then Γ is disjoint from $\mathsf{PG}(2n+1, q)$ and the subspace Γ^σ has equation $\mathbf{x} = \mathbf{0}$. For each point $(\mathbf{x}, \mathbf{0})$ of Γ the line joining the points $(\mathbf{x}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{x}^q)$ is fixed by σ and $\ell(\mathbf{x}) = \langle(\mathbf{x}, \mathbf{0}), (\mathbf{0}, \mathbf{x}^q)\rangle$ intersects $\mathsf{PG}(2n+1, q)$ in a line. Moreover two lines $\ell(\mathbf{x})$ and $\ell(\mathbf{y})$, with $\mathbf{x} \neq \mathbf{y}$, are disjoint otherwise the subspace $\langle(\mathbf{x}, \mathbf{0}), (\mathbf{y}, \mathbf{0}), (\mathbf{0}, \mathbf{x}^q), (\mathbf{0}, \mathbf{y}^q)\rangle$ is a plane, but this is impossible since the lines $\langle(\mathbf{x}, \mathbf{0}), (\mathbf{y}, \mathbf{0})\rangle$ of Γ and $\langle(\mathbf{0}, \mathbf{x}^q), (\mathbf{0}, \mathbf{y}^q)\rangle$ of Γ^σ are disjoint. Hence $\mathcal{N} = \{\ell(\mathbf{x}) \cap \mathsf{PG}(2n+1, q) \mid (\mathbf{x}, \mathbf{0}) \in \Gamma\}$ is a line-spread of $\mathsf{PG}(2n+1, q)$.

For each line m of Γ, let $\mathcal{N}_m = \{\ell(\mathbf{x}) \cap \mathsf{PG}(2n+1, q) \mid (\mathbf{x}, \mathbf{0}) \in m\}$. Then $\mathcal{N}_m$ is a regular spread of the 3-dimensional space $\langle m, m^\sigma\rangle \cap \mathsf{PG}(2n+1, q)$ [4]. If a 3-dimensional subspace Φ of $\mathsf{PG}(2n+1, q)$ contains two lines $\ell(\mathbf{x})$ and $\ell(\mathbf{y})$ of $\mathcal{N}$ and m is the line of Γ joining the points $(\mathbf{x}, \mathbf{0})$ and $(\mathbf{y}, \mathbf{0})$ then $\Phi = \langle m, m^\sigma\rangle \cap \mathsf{PG}(2n+1, q)$ and $\mathcal{N}_\Phi = \{n \in \mathcal{N} \mid n \cap \Phi \neq \emptyset\} = \mathcal{N}_m$ is a spread of Φ.

For further details about normal spreads see [15], [21].

The Barlotti-Cofman representation of $\mathsf{PG}(n, q^2)$ [3] can be described in the following way.

Let α be a $(n-1)$-dimensional subspace of Γ and Σ be the $(2n-1)$-dimensional subspace $\langle \alpha, \alpha^\sigma\rangle \cap \mathsf{PG}(2n+1, q)$. Then $\mathcal{N}_\Sigma = \{\ell \in \mathcal{N} \mid \ell \cap \Sigma \neq \emptyset\} = \{\ell(\mathbf{x}) \mid \mathbf{x} \in \alpha\}$ is a normal spread of Σ.

Let Σ' be a hyperplane of $\mathsf{PG}(2n+1, q)$ containing Σ. It is easily seen that each line of $\mathcal{N}$ either meets Σ' in a line of Σ or meets $\Sigma' \setminus \Sigma$ in exactly one point. Further, if m is a line of Γ not contained in α then $\mathcal{N}_m$ meets Σ' in a plane intersecting Σ in a line of $\mathcal{N}_\Sigma$.

Thus, it is possible to define an incidence structure $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ as follows: points are either points of Σ' not in Σ or lines of $\mathcal{N}_\Sigma$; lines are either planes of Σ' intersecting Σ in an element of $\mathcal{N}_\Sigma$ or the spreads $\mathcal{N}_{\langle A,B\rangle}$ where A and B are distinct elements of $\mathcal{N}_\Sigma$; the incidence relation is the natural one. As $\mathcal{N}$ is normal, $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ is isomorphic to the projective space $\mathsf{PG}(n, q^2)$. An automorphism of $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ is a 1-1 mapping of points onto points and lines onto lines preserving incidence. In [3] it was also proved that the full automorphism group of $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ is isomorphic to the stabilizer of a hyperplane in the collineation group of $\mathsf{PG}(n, q^2)$.

Now we describe in details the geometry of the unitary polar space $H(n, q^2)$ under the Barlotti-Cofman representation of $\mathsf{PG}(n, q^2)$. We will proceed by induction on n, where the link between $H(3, q^2)$ and $Q^+(5, q)$ stated in [16] is the induction basis.

Assume first that n is even. Let $Q^+(2n+1, q^2)$ be the hyperbolic quadric of $\mathsf{PG}(2n+1, q^2)$ with equation

$$x_0 y_{n-1} + x_1 y_{n-2} + \cdots + x_{n-1} y_0 + x_n y_n = 0\,.$$

Then the n-dimensional subspaces Γ and Γ^σ are contained in $Q^+(2n+1, q^2)$ and $Q^+(2n+1, q^2) \cap \mathsf{PG}(2n+1, q)$ has equation

$$x_0 x_{n-1}^q + x_1 x_{n-2}^q + \cdots + x_{n-1} x_0^q + x_n^{q+1} = 0\,, \tag{1}$$

which is quadratic over $\mathsf{GF}(q)$. In $\mathsf{GF}(q^2) \setminus \mathsf{GF}(q)$ we fix an element i whose minimal polynomial over $\mathsf{GF}(q)$ is $X^2 - \omega X + 1$, with $\omega \in \mathsf{GF}(q)$. Then we have $\bar{i} = 1/i$ and, if $x \in \mathsf{GF}(q^2)$, we uniquely can write $x = a + ib$, with $a, b \in \mathsf{GF}(q)$. Thus, through straightforward calculations, we see that equation (1) is the equation of an elliptic quadric $Q^-(2n+1, q)$ of $\mathsf{PG}(2n+1, q)$.

If a line $\ell(\mathbf{x})$ of $\mathcal{N}$ contains a point of $Q^-(2n+1, q)$, then $\ell(\mathbf{x})$ is contained in $Q^+(2n+1, q^2)$ because it is incident with three points of $Q^+(2n+1, q^2)$. This implies that $\mathcal{H} = \{\ell(\mathbf{x}) \mid \ell(\mathbf{x}) \cap Q^-(2n+1, q) \neq \emptyset\}$ is a line spread of $Q^-(2n+1, q)$. Furthermore, $H(n, q^2) = \{(\mathbf{x}, 0) \in \Gamma \mid \ell(\mathbf{x}) \in \mathcal{H}\}$ is the unitary polar space in Γ defined by the equation

$$x_0 x_{n-1}^q + x_1 x_{n-2}^q + \cdots + x_0^q x_{n-1} + x_n^{q+1} = 0\,; \tag{2}$$

see also [26].

The $(n-1)$-dimensional subspace α of Γ is either non-singular with respect to $H(n, q^2)$ or tangent to $H(n, q^2)$.

Let α be non-singular with respect to $H(n, q^2)$. Then $\alpha \cap H(n, q^2)$ is a unitary polar space $\mathcal{H}_{n-1} = H(n-1, q^2)$. Since $n-1$ is odd, by induction the set $\{\ell(\mathbf{x}) \cap \Sigma' \mid \mathbf{x} \in \mathcal{H}_{n-1}\}$ is a line spread of a orthogonal polar space $Q^+(2n-1, q)$. Further, since $H(n, q^2)$ is non-singular then Σ' is a non-singular section of $Q^-(2n+1, q)$ that is, the representation of $H(n, q^2)$ in $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ is a orthogonal polar space $Q(2n, q)$.

Now consider the case that α is tangent to $H(n, q^2)$ at $(\mathbf{x}, \mathbf{0})$. Then α intersects $H(n, q^2)$ in a cone projecting a $H(n-2, q^2)$ from $(\mathbf{x}, \mathbf{0})$. Since lines of $H(n, q^2)$ through $(\mathbf{x}, \mathbf{0})$ are represented by regular line-spread of a 3-dimensional subspace of Σ, we see that $\{\ell(\mathbf{x}) \cap \Sigma \mid \mathbf{x} \in \alpha\}$ is union of regular line-spreads which share precisely the line $\ell(\mathbf{x})$.

If P is a point of $\Sigma' \cap Q^-(2n+1,q)$ not in Σ, then $P = \ell(\mathbf{y}) \cap \Sigma'$ for some $(\mathbf{y}, \mathbf{0}) \in H(n,q^2)$ such that $\langle(\mathbf{x},\mathbf{0}),(\mathbf{y},\mathbf{0})\rangle$ is a hyperbolic line of $H(n,q^2)$ and points of $\langle(\mathbf{x},\mathbf{0}),(\mathbf{y},\mathbf{0})\rangle \cap H(n,q^2)$ are represented by points of a line of $\Sigma' \cap Q^-(2n+1,q)$ through P and intersecting $\ell(\mathbf{x})$.

It follows that Σ' intersects the quadric $Q^-(2n+1,q)$ in a cone, say $\mathcal{K}$, whose vertex is a point, say V, on the line $\ell(\mathbf{x})$; that is, Σ' is tangent to the quadric $Q^-(2n+1,q)$ at the point V. With straightforward calculations we see that $\mathcal{K}$ projects an elliptic quadric $Q^-(2n-1,q)$ from V. Furthermore Σ intersects $Q^-(2n+1,q)$ in a cone whose vertex is the line $\ell(\mathbf{x})$.

Hence in $S(\Sigma',\Sigma,\mathcal{N}_\Sigma)$ points of $H(n,q^2)$ are either points of the cone $\mathcal{K}$ not in Σ or lines of $\mathcal{N}_\Sigma$ contained in the quadratic cone $\Sigma \cap Q^-(2n+1,q) \subset \mathcal{K}$. Further lines of $H(n,q^2)$ not incident with $(\mathbf{x},\mathbf{0})$ are planes of $\mathcal{K}$ meeting Σ in a line of $\mathcal{N}_\Sigma$.

When n is odd we start with the hyperbolic quadric $Q^+(2n+1,q^2)$ with equation

$$x_0y_n + \cdots + x_ny_0 = 0\,.$$

Then $Q^+(2n+1,q^2) \cap \mathsf{PG}(2n+1,q)$ has equation

$$x_0x_n^q + \cdots + x_nx_0^q = 0\,, \tag{3}$$

which is quadratic over $\mathsf{GF}(q)$ and represents a hyperbolic quadric $Q^+(2n+1,q)$ of $\mathsf{PG}(2n+1,q)$. Again we see that $\mathcal{H} = \{\ell(\mathbf{x}) \mid \ell(\mathbf{x}) \cap Q^+(2n+1,q) \neq \emptyset\}$ is a line spread of $Q^+(2n+1,q)$. Furthermore, $H(n,q^2) = \{(\mathbf{x},0) \in \Gamma \mid \ell(\mathbf{x}) \in \mathcal{H}\}$ is the hermitian variety of Γ defined by the equation

$$x_0x_n^q + \cdots + x_0^qx_n = 0\,; \tag{4}$$

see also [26].

By arguing similarly as we have done for n even, we see that $H(n,q^2)$ is represented in $S(\Sigma',\Sigma,\mathcal{N}_\Sigma)$ either by a $Q(2n,q)$ or by a cone $\mathcal{K}$ projecting a hyperbolic quadric $Q^+(2n-1,q)$ from a point on $\ell(\mathbf{x})$.

3 Locally hermitian (partial) ovoids of unitary space and (partial) ovoids of orthogonal spaces

Consider first the case n even. Let Δ be a hyperplane of Σ' not containing V. Then $\Delta \cap Q^-(2n+1,q)$ is a orthogonal polar space $Q^-(2n-1,q)$ of Δ and $\Delta \cap \Sigma$ is the tangent hyperplane to $Q^-(2n-1,q)$ at a point, say R, of $\ell(\mathbf{x}) \cap \Delta$.

Now let $\mathcal{O}$ be a locally hermitian partial ovoid of $H(n, q^2)$ with respect to $(\mathbf{x}, \mathbf{0})$ consisting of s hyperbolic lines. In $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ the ovoid $\mathcal{O}$ is represented by the union of s lines of $\mathcal{K}$ containing V one of which is $\ell(\mathbf{x})$. Note that each of these lines but not $\ell(\mathbf{x})$ has q points in $\mathcal{K} \setminus \Sigma$. Hence there is a set $\mathbb{O}$ of $s+1$ points of $Q^-(2n-1, q)$ such that $\langle V, X \rangle$, $X \in \mathbb{O}$, are the $s+1$ lines through V representing $\mathcal{O}$.

Theorem 3.1. *The set $\mathbb{O}$ is a partial ovoid of $Q^-(2n-1, q)$ containing $R = \ell(\mathbf{x}) \cap \Delta$.*

Proof. Let X, Y be two distinct points of $\mathbb{O}$ and suppose that X and Y are collinear in $Q^-(2n-1, q)$.

Let m be the line of $\mathcal{N}_\Sigma$ incident with the point $\langle X, Y \rangle \cap \Sigma$. If $m = \ell(\mathbf{x})$ then the plane $\langle \ell(\mathbf{x}), X, Y \rangle$ is a plane of the cone $\mathcal{K}$ which represents a line of $H(n, q^2)$ through $(\mathbf{x}, \mathbf{0})$ not in α. But this is a contradiction. Hence $m \neq \ell(\mathbf{x})$. Let S be the 3-dimensional subspace $\langle \ell(\mathbf{x}), m \rangle$ and $\mathcal{S}$ the regular line-spread induced by $\mathcal{N}$ in S.

Thus the 4-dimensional subspace $\langle S, X \rangle$ represents a plane π which is tangent to $H(n, q^2)$ at a point represented by a line of $\mathcal{S}$ and containing the point $(\mathbf{x}, \mathbf{0})$ and the two points of $H(n, q^2)$ represented by X and Y. But this is impossible because the plane generated by two hyperbolic lines of $\mathcal{O}$ through $(\mathbf{x}, \mathbf{0})$ is never a tangent plane of $H(n, q^2)$. □

Theorem 3.2. *Let $\mathbb{O}$ be a partial ovoid of $Q^-(2n-1, q)$ of size $s+1$ containing $R = \ell(\mathbf{x}) \cap \Delta$. Let $\bar{\mathbb{O}}$ be the set of lines $\langle P, X \rangle$, with $X \in \mathbb{O}$. Then $\bar{\mathbb{O}}$ represents a locally hermitian partial ovoid $\mathcal{O}$ with respect to $(\mathbf{x}, \mathbf{0})$ of size $sq+1$.*

Proof. We can argue as in [16, Theorem 7]. □

When n is odd we can argue as before to prove the following result.

Theorem 3.3. *Let $\mathcal{O}$ be a locally hermitian partial ovoid of $H(n, q^2)$, n odd, with respect to $(\mathbf{x}, \mathbf{0})$ consisting of s hyperbolic lines. The set $\mathbb{O}$ defined as before is a partial ovoid of $Q^+(2n-1, q)$ of size $s+1$ containing $R = \ell(\mathbf{x}) \cap \Delta$. Conversely, if $\mathbb{O}$ is a partial ovoid of $Q^+(2n-1, q)$ of size $s+1$ containing $R = \ell(\mathbf{x}) \cap \Delta$ then there exists a locally hermitian partial ovoid $\mathcal{O}$ with respect to $(\mathbf{x}, \mathbf{0})$ of size $sq+1$.*

Corollary 3.4. *The unitary polar space $H(n, q^2)$ has no locally hermitian ovoid when n is odd, $n \geq 5$ and $q = 2, 3$.*

Proof. By way of contradiction the result follows since $Q^+(2n+1, q)$ has no ovoids when $n \geq 4$ and $q = 2, 3$ [13, 23]. □

Remark 3.5. We recall that $H(5,4)$ has no ovoid [10].

Corollary 3.6. *The orthogonal space $Q^+(4n+1,q)$ has no ovoids when $n > q^3$.*

Proof. By way of contradiction the result follows since $H(2n+1,q^2)$ has no ovoids when $n > q^3$ [14, Theorem 3]. □

A partial ovoid of a classical polar space $\mathcal{P}$ is said to be *maximal* if it is not contained in a larger partial ovoid of $\mathcal{P}$. A locally hermitian partial ovoid of $\mathcal{P}$ is said to be *maximal* if it is not contained in a larger locally hermitian partial ovoid.

Theorem 3.7. *Let $\mathcal{O}$ be a maximal locally hermitian partial ovoid of $H(n,q^2)$ with respect to a point $(\mathbf{x},\mathbf{0})$. Then $\mathcal{O}$ is a maximal partial ovoid of $H(n,q^2)$. Furthermore, $\mathcal{O}$ is maximal if and only if the associated partial ovoid $\mathbb{O}$ of the corresponding orthogonal polar space is maximal.*

Proof. Let $H(n,q^2)$ be the unitary polar space defined by equation (2) if n is even and equation (4) if n is odd. Let $\mathsf{PGU}(n+1,q^2)$ denote the group of the linear collineations of $\Gamma = \mathsf{PG}(n,q^2)$ leaving $H(n,q^2)$ invariant. Since such a group acts transitively on points of $H(n,q^2)$, we can take $\mathbf{x} = (1,0,\dots,0)$. The subgroup of $\mathsf{PGU}(n+1,q^2)$ fixing $P = (\mathbf{x},\mathbf{0})$ and leaving invariant all the lines of $H(n,q^2)$ through P contains the elementary abelian subgroup G of order q whose elements can be written in the form

$$\begin{pmatrix} 1 & 0 & \dots & a & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

if n is even and

$$\begin{pmatrix} 1 & 0 & \dots & a \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

if n is odd where $a \in \mathsf{GF}(q^2)$ and $a + a^q = 0$ (see [24] for more details).

Straightforward calculations show that each element of G fixes all hyperbolic lines through P and it has exactly one orbit on the q isotropic points distinct from P on each hyperbolic line through P. This implies that the group G is a collineation group of every locally hermitian partial ovoid of $H(n,q^2)$.

Now let $\mathcal{O}$ be a maximal locally hermitian partial ovoid of $H(n,q^2)$ and let Q be a point of $H(n,q^2) \setminus \{\mathcal{O}\}$ not collinear with any point of $\mathcal{O}$. Then the line

$\langle P, Q \rangle$ is a hyperbolic line and every isotropic point of $\langle P, Q \rangle \setminus \{P\}$ can be added to $\mathcal{O}$. In fact, if Q_1 of $\langle P, Q \rangle$ is collinear with a point P_1 of $\mathcal{O}$, take $g \in X_{P,P^\perp}$ such that $gQ_1 = Q$. Then Q and gP_1 are collinear, a contradiction. Thus, the hyperbolic line $\langle P, Q \rangle$ can be added to $\mathcal{O}$ to get a larger locally hermitian partial ovoid. But this is a contradiction because $\mathcal{O}$ is a maximal locally hermitian partial ovoid of $H(n, q^2)$.

The last statement of the theorem easily follows by way of contradiction from Theorems 3.1, 3.2 and 3.3. □

Theorem 3.8. *Let $\mathcal{O}$ be a locally hermitian maximal partial ovoid of $H(n, q^2)$, $n \geq 3$, with respect to $(\mathbf{x}, \mathbf{0})$. Then*

(i) *$\mathcal{O}$ has size at least $2q^2 - q + 1$ if $n = 3$ and $2q^2 + 1$ for $n \geq 4$;*

(ii) *$\mathcal{O}$ has size at most $q^n - q^{n/2} - q + 1$ if n is odd and $\mathcal{O}$ is not an ovoid.*

Proof. From Theorem 3.7, in $S(\Sigma', \Sigma, \mathcal{N}_\Sigma)$ $\mathcal{O}$ is represented as a maximal partial ovoid of $Q^\varepsilon(2n-1, q)$ where ε is '+' or '−' according to whether n is odd or n is even.

In [9] it was proved that a maximal partial ovoid of $Q^\varepsilon(2n-1, q)$ has at least $2q$ points if $n = 3$ and at least $2q + 1$ points if $n \geq 4$. In the same paper it was proved that a maximal partial ovoid of $Q^+(2n-1, q)$ which is not an ovoid has at most $q^{n-1} + q^{\frac{n-2}{2}}$ points.

By applying Theorems 3.1, 3.2, 3.3 and 3.7 we get the result. □

Remark 3.9. In the papers [1] and [16] it was proved that if $\mathcal{O}$ is a translation ovoid of $H(3, q^2)$ then also the corresponding ovoid $\mathbb{O}$ of $Q^+(5, q)$ is. We remark that if an analogue result is obtained for translation ovoids of $H(2n+1, q^2)$, $n \geq 2$, then by using the classification result of G. Lunardon and O. Polverino proved in [17] one gets a non-existence result for translation ovoids of such unitary polar space.

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