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Flag-transitive and almost simple orbits in finite projective planes

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Abstract

Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O}$ of length v. We investigate the triple $(\Pi, \mathcal{O}, G)$ when $\mathcal{O}$ has the structure of a non trivial 2-$(v, k, 1)$ design, G induces a flag-transitive and almost simple automorphism group on $\mathcal{O}$ and $n \leq \sum(\mathcal{O}) = b + v + r + k$.

Keywords: projective plane, collineation group, orbit

MSC 2000: 51E15, 20B25

1 Introduction and statement of the result

A classical subject in finite geometry is the investigation of a finite projective plane Π of order n admitting a collineation group G which acts doubly transitively (flag-transitively) on a point-subset $\mathcal{O}$ of size v of Π. It dates back to 1967 and it is due to Cofman [9]. It is easily seen that either

(i) the structure of a non-trivial 2-$(v, k, 1)$ design (i.e. $k \geq 3$ and at least two distinct blocks) is induced on $\mathcal{O}$, or

(ii) $\mathcal{O}$ is an arc, or

(iii) $\mathcal{O}$ is a contained in a line.

The current paper entirely focuses on the situation in which $\mathcal{O}$ is a non-trivial 2-$(v, k, 1)$ design. Several papers deal with the case $n \leq v$. Conclusive results where obtained by Lüneburg [35], in 1966, when $\mathcal{O}$ is a Ree unital, and later on by Kantor [28], by Biliotti and Korchmáros [5, 6], when $\mathcal{O}$ is a Hermitian unital. Characterization results were also provided by Ostrom-Wagner [43] in 1959 and by Lüneburg [37] in 1976 when $\mathcal{O}$ is a projective subplane of Π.

A fundamental contribution to the problem is the classification of the 2-transitive non-trivial 2-$(v,k,1)$ designs due to Kantor [30] in 1985, and some years later, the classification of the flag-transitive non-trivial 2-$(v,k,1)$ designs due to Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [2, 3, 34, 47]. Recently, Biliotti and Francot [4] investigated the general case when $n \leq v$ and G acts doubly transitively on $\mathcal{O}$, determining all possible collineation groups.

The problem of classifying the triple $(\Pi, \mathcal{O}, G)$ in the case where $\mathcal{O}$ is a non-trivial 2-$(v,k,1)$ design for $n > v$, but v close to n, has also been investigated. Indeed, Dempwolff [11] in 1985 dealt with the case $\mathcal{O} \cong \mathsf{PG}(2,q)$ and $n = q^3$. Such a case has been recently generalized by Montinaro [39] to $n \leq q^3$. Moreover, the author [40] shows that $\mathcal{O}$ cannot be a Ree unital of order q when $n \leq q^4$, generalizing the result of Lüneburg [37]. Finally, in 2005, Biliotti and Montinaro [8] showed that no solutions to the above problem involving non-trivial 2-$(v,k,1)$ designs arise for $n = v + 3$.

In this paper we investigate the triples $(\mathcal{O}, \Pi, G)$ under the assumption that $\mathcal{O}$ is a non-trivial 2-$(v,k,1)$ design, G is a collineation group of Π inducing an almost simple and flag-transitive automorphism group on $\mathcal{O}$ and $n \leq \sum(\mathcal{O})$, where $\sum(\mathcal{O}) = b + v + r + k$. Clearly b denotes the number of blocks of $\mathcal{O}$ and r the number of blocks of $\mathcal{O}$ incident with any given point of $\mathcal{O}$. In particular, our result is contained in Theorem 1.1.

Let $\mathcal{D}$ be any set of points in Π. A line l of Π is called an external line, or a tangent or a secant to $\mathcal{D}$, according to $|\mathcal{D} \cap l| = 0$, 1 or k, with $k > 1$, respectively. We may regard $\mathcal{D}$ as an incidence structure, where the blocks are the secants of $\mathcal{D}$. Any incidence structure represented in this way is said to be *embedded in* Π. In particular, any orbit $\mathcal{O}$ of a collineation group G of Π may be regarded as an embedded structure. A *flag* is an incident point-line pair of Π, so $\mathcal{O}$ is said to be *flag-transitive*, if G is transitive on the flags of $\mathcal{O}$. Finally a flag-transitive orbit is said to be *almost simple*, if the group induced by G on $\mathcal{O}$ is almost simple (i.e. G has a non-abelian simple normal subgroup L such that $L \trianglelefteq G \trianglelefteq \mathrm{Aut}(L)$).

Theorem 1.1. *Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O}$ of length v. If $\mathcal{O}$ has the structure of a non-trivial 2-$(v,k,1)$ design, the group G induces a flag-transitive and almost simple automorphism group on $\mathcal{O}$ and $n \leq \sum(\mathcal{O})$, then one of the following holds.*

(1) *G acts faithfully on $\mathcal{O}$ and one of the following occurs:*

  (a) *$v = n^2 + n + 1$, $\mathcal{O} = \Pi \cong \mathsf{PG}(2,n)$ and $\mathsf{PSL}(3,n) \leq G$;*

  (b) *$v = n + \sqrt{n} + 1$, $\mathcal{O} \cong \mathsf{PG}(2,\sqrt{n})$, Π is a Desarguesian plane or a generalized Hughes plane, and $\mathsf{PSL}(3,\sqrt{n}) \leq G$;*

(c) $v = 7$, $\mathcal{O} \cong \mathsf{PG}(2,2)$, $\Pi \cong \mathsf{PG}(2,8)$ *and* $\mathsf{PSL}(3,2) \leq G$;

(d) $v = 7$, $\mathcal{O} \cong \mathsf{PG}(2,2)$, $\Pi \cong \mathsf{PG}(2,16)$ *and* $\mathsf{PSL}(3,2) \leq G$;

(e) $v = 13$, $n = 3^3$, $\mathcal{O} \cong \mathsf{PG}(2,3)$ *and* $\mathsf{PSL}(3,3) \leq G$;

(f) $v = \sqrt{n^3} + 1$, $\mathcal{O} \cong \mathcal{H}(\sqrt{n})$, $\Pi \cong \mathsf{PG}(2,n)$ *and* $\mathsf{PSU}(3,\sqrt{n}) \leq G$;

(g) $v = \sqrt[4]{n^3}+1$, $\mathcal{O} \cong \mathcal{H}(\sqrt[4]{n})$ *is contained in a plane* $\Pi_0 \cong \mathsf{PG}(2,\sqrt{n})$ *which is left invariant by* G*, and* $\mathsf{PSU}(3,\sqrt[4]{n}) \leq G$;

(h) $v = \frac{n}{2}(n-1)$, $n = 2^r$, $\mathcal{O} \cong \mathcal{W}(n)$, $\Pi \cong \mathsf{PG}(2,n)$ *and* $\mathsf{PSL}(2,n) \leq G$. *Here* $\mathcal{W}(n)$ *denotes the Witt space associated with* $\mathsf{PSL}(2,n)$*. Furthermore, the projective extension of* $\mathcal{O}$ *is embedded in* Π *and the set* $\mathcal{C}$ *of external lines to* $\mathcal{O}$ *is a line-hyperoval extending a line-conic of* Π;

(i) $v = \frac{\sqrt{n}}{2}(\sqrt{n}-1)$, $n = 2^{2r}$, $\mathcal{O} \cong \mathcal{W}(\sqrt{n})$ *and* $\mathsf{PSL}(2,\sqrt{n}) \leq G$;

(j) $v = \frac{\sqrt{n}-1}{2}\sqrt{n}$, $n = (2^r+1)^2$, $\mathcal{O} \cong \mathcal{W}(\sqrt{n}-1)$ *and* $\mathsf{PSL}(2,\sqrt{n}-1) \leq G$.

(2) G *does not act faithfully on* $\mathcal{O}$ *and one of the following occurs:*

(a) $v = 75$, $\mathcal{O} \cong \mathsf{PG}(2,7)$, Π *is the generalized Hughes plane over the exceptional nearfield of order* 7^2 *and* $\mathsf{SL}(3,7) \leq G$;

(b) $v = \sqrt[4]{n^3}+1$, $\sqrt[4]{n^3} \equiv 2 \mod 3$, $\mathcal{O} \cong \mathcal{H}(\sqrt[4]{n})$ *is contained in a Desarguesian Baer subplane and* $\mathsf{SU}(3,\sqrt[4]{n}) \leq G$.

Furthermore, in each case, except in (1j) *and possibly in* (1i)*, the involutions in* G *are perspectivities of* Π.

Cases (1a)–(1d), (1f), (1h) and (2a) really occur. Case (1e) occurs in the Desarguesian plane and in the Hering-Figueroa plane of order 3^3 (see [14] and [19]). Furthermore, case (1g) occurs in the Desarguesian or Hughes planes, and case (1i) occurs in the Desarguesian plane. Finally, cases (1j) and (2b) are open. Some restrictions about the possible existence of an example for case (1j) are provided in Corollary 3.10.

2 Preliminaries and background

We shall use standard notation. For what concerns finite groups the reader is referred to [15] and [25]. The necessary background about finite projective planes and 2-$(v,k,1)$ designs may be found in [24] and in [49], respectively.

Let $\Pi = (\mathcal{P},\mathcal{L})$ be a finite projective plane of order n. If H is a collineation group of Π and $P \in \mathcal{P}$ ($l \in \mathcal{L}$), we denote by $H(P)$ (by $H(l)$) the subgroup of H consisting of perspectivities with center P (axis l). Also, $H(P,l) = H(P) \cap H(l)$. Furthermore, we denote by $H(P,P)$ (by $H(l,l)$) the subgroup of H consisting of elations with center P (axis l). A collineation group H of Π is said to be *irreducible* on Π, if H does not fix any point, line, or triangle of Π. An irreducible collineation group H is said to be *strongly irreducible* on Π, if H does not fix any proper subplane of Π.

Let X be a collineation group of Π and recall that $\mathrm{Fix}(X)$ denotes the subset of Π consisting of the points and of the lines of Π which are fixed by X. If X is planar, i.e. $\mathrm{Fix}(X)$ is a subplane of Π, we denote by $o(\mathrm{Fix}(X))$ the order of plane $\mathrm{Fix}(X)$.

Let $\mathcal{N}$ be a H-orbit of points in Π on which the structure of a non-trivial 2-$(v,k,1)$ design can be induced. Suppose that $\mathcal{N}$ admits a parallelism, that is the blocks of $\mathcal{N}$ are partitioned into parallel classes such that each class is a partition of the points of $\mathcal{N}$. If for each pencil Φ of parallel blocks, the blocks of Φ pass through a common point P_Φ of Π, and all P_Φ's lie in the same line $l_\mathcal{N}$ of Π, then, following [13] we say that the *projective extension* of $\mathcal{N}$ is embedded in Π.

Before starting our investigation we recall some combinatorial and group-theoretical results which are useful hereafter. The following theorem deals with the general structure of G.

Theorem 2.1 (Buekenhout et al. [2]). *Suppose that $\mathfrak{G}$ is a flag-transitive group of automorphisms of a non-trivial design $\mathcal{D}$. Then either*

(I) *$\mathfrak{G}$ is almost simple: that is, $\mathfrak{G}$ has a non-abelian simple normal subgroup N such that $N \trianglelefteq \mathfrak{G} \trianglelefteq \mathrm{Aut}(N)$, or*

(II) *$\mathfrak{G}$ is of affine type: that is, the set of points of $\mathcal{D}$ carries the structure of an affine space $\mathsf{AG}(t,p)$ which is invariant under $\mathfrak{G}$, and $\mathfrak{G}$ contains the whole translation group T of $\mathsf{AG}(t,p)$, (so $T \trianglelefteq \mathfrak{G} \leq \mathsf{AGL}(t,p)$).*

The non-trivial 2-$(v,k,1)$ designs $\mathcal{D}$ admitting a flag-transitive automorphism group $\mathfrak{G}$ are classified in [47] when $\mathfrak{G}$ is of type (I) and in [34] when $\mathfrak{G}$ is of type (II), respectively. In our analysis we focus entirely on case (I) and we have:

Theorem 2.2 (Saxl [47]). *Let $\mathcal{D}$ be a design and suppose that $\mathfrak{G}$ is a flag-transitive almost simple group of automorphisms of $\mathcal{D}$. Then one of the following occurs:*

(i) *$\mathcal{D} \cong \mathsf{PG}(d,q)$, $d \geq 2$, and either $\mathsf{PSL}(d+1) \leq \mathfrak{G} \leq \mathsf{P\Gamma L}(d+1,q)$ or $(d,q)=(3,2)$ and $\mathfrak{G} \cong \mathsf{A}_7$;*

(ii) *$\mathcal{D} \cong \mathcal{H}(q)$ is a Hermitian unital and $\mathsf{PSU}(3,q^2) \leq \mathfrak{G} \leq \mathsf{P\Gamma U}(3,q)$;*

(iii) *$\mathcal{D} \cong \mathcal{R}(q)$, $q=3^h$, h odd, $h \geq 1$, is a Ree unital associated to ${}^2\mathsf{G}_2\,(q) \leq \mathfrak{G} \leq {}^2\mathsf{G}_2\,(q)\,.Z_h$;*

(iv) *$\mathcal{D} \cong \mathcal{W}(q)$, $q=2^h \geq 8$, is a Witt space associated to $\mathsf{PSL}(2,q) \leq \mathfrak{G} \leq \mathsf{P\Gamma L}(2,q)$.*

We shall make extensive use the following definition and lemma, which are independent of the context of non-trivial 2-$(v,k,1)$ designs embedded in projective planes.

Definition 2.3. Let $\mathcal{D}$ be a non-trivial 2-$(v,k,1)$ design. Then

$$\sum(\mathcal{D}) = b + v + r + k,$$

where $r = \frac{v-1}{k-1}$ is the number of blocks which are incident with any given point of $\mathcal{D}$ and $b = \frac{v(v-1)}{k(k-1)}$ is the total number of blocks of $\mathcal{D}$.

Lemma 2.4. *The following relations hold:*

(1) $\sum\bigl(\mathsf{PG}(d,q)\bigr) = \dfrac{q^{2d+1}+q^{d+3}+q^{d+2}-2q^{d+1}-2q^{d}+q^{4}-4q^{2}+4}{(q^{2}-1)(q-1)}$*;*

(2) $\sum\bigl(\mathcal{H}(q)\bigr) = q^4+2q^2+q+2$*;*

(3) $\sum\bigl(\mathcal{R}(q)\bigr) = q^4+2q^2+q+2$*, where* $q=3^h$ *and* h *is odd,* $h\geq 1$*;*

(4) $\sum\bigl(\mathcal{W}(q)\bigr) = \dfrac{q}{2}(3q+2)$*, where* $q=2^h\geq 8$.

Proof. Omitted. □

The following theorem, which relies on the results of Ostrom-Wagner [43], Lüneburg [37] and Dempwolff [11], deals with the case $\mathcal{O}\cong \mathsf{PG}(2,q)$ and $n\leq q^3$.

Theorem 2.5 ([39])**.** *Let* Π *be a finite projective plane of order* n *and let* G *be a collineation group of* Π *inducing a group containing* $\mathsf{PSL}(3,q)$ *on a subplane* Π_0 *of order* q*. If* $n\leq q^3$*, then one of the following occurs:*

(1) $\Pi_0\cong\mathsf{PG}(2,q)$*,* $\mathsf{PSL}(3,q)\leq G$ *and one of the following occurs:*

  (a) $n=q$*, and* $\Pi=\Pi_0$*;*

  (b) $n=q^2$*,* Π *is a Desarguesian plane or a generalized Hughes plane and* Π_0 *is a Baer subplane of* Π*;*

  (c) $n=q^3$.

(2) $\Pi_0\cong\mathsf{PG}(2,7)$*,* Π *is the generalized Hughes plane over the exceptional nearfield of order* 7^2 *and* $\mathsf{SL}(3,7)\leq G$.

Based on a similar idea, the following theorem generalizes a result of Lüneburg [35].

Theorem 2.6 ([40]). *If $\mathcal{R}(q)$, $q = 3^h$, is embedded in a finite projective plane Π of order n, with $n \leq q^4$, in such a way that ${}^2\mathsf{G}_2(q)$ is induced on $\mathcal{R}(q)$ by a collineation group G of Π. Then $h = 1$, G acts faithfully on $\mathcal{R}(3)$ as $\mathsf{P\Gamma L}(2,8)$ and one of the following occurs:*

(1) *$\Pi \cong \mathsf{PG}(2,8)$, G leaves a line oval $\mathcal{C}$ of Π invariant and $\mathcal{O}$ consists of the external points of $\mathcal{C}$;*

(2) *$n = 2^6$.*

We conclude the preliminaries with the following numerical lemma that will be useful in dealing with the case when $\mathcal{O}$ is an Hermitian unital of order 2^{2h}, $h > 1$.

Lemma 2.7. *Let f, h, t and λ_1 be positive integers. Then the Diophantine equation*

$$f\frac{2^{2h}-1}{3} = 1 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1) \tag{1}$$

has no solutions for $h > 1$, $t = \lfloor (h+1)/2 \rfloor$ and $\lambda_1 \leq 2^{h-t}$.

Proof. Set $a = f/3$. Then (1) becomes

$$a(2^{2h}-1) = 1 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)\,. \tag{2}$$

As $\lambda_1 \leq 2^{h-t}$, then $1 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1) \leq 1 + 2^{2h}3$. Then $a \leq 4$ by (2).

Assume that $(f,3) = 3$. Then a is an integer. Since the second part of (2) is odd and since $a \leq 4$, then either $a = 1$ or 3. Assume that $a = 1$. Then $2^{2h} = 2 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$ by (2). A contradiction, since $2^2 \nmid 2 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$ as $t \geq 1$ while $h > 1$. So $a = 3$. Then $2^{2h}3 = 4 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$ again by (2). If $t \geq 2$, then $2^3 \nmid 4 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$. This forces $h = 1$. A contradiction by our assumption. Thus $t = 1$ and hence $h = 2$ and $\lambda_1 = 1, 2$, since $t = \lfloor (h+1)/2 \rfloor$ and $1 \leq \lambda_1 \leq 2^{h-t}$. So, substituting $t = 1$, $h = 2$ and $\lambda_1 = 1, 2$ in (2), we obtain a contradiction since a must be a positive integer as $(f,3) = 3$.

Assume that $(f,3) = 1$. Then $f \leq 12$ as $a = f/3$ and $a \leq 4$. Furthermore f is odd, since $1 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$ in (1) is so. All these informations yield $f \in \{1, 5, 7, 11\}$. Now, by managing (1), we obtain

$$2^{2h}f = (3 + f) + 2^t 3\lambda_1(2^{h+1} + 2^t\lambda_1)\,. \tag{3}$$

If $f \in \{7, 11\}$, then $2^2 \nmid 2 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$ as $t \geq 1$. A contradiction, since $h > 1$. Hence $f \in \{1, 5\}$. If $f = 1$, then (3) becomes $2^{2h} = 4 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$. If $t \geq 2$, then $2^3 \nmid 4 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$. A contradiction, since $h > 1$ by our assumptions. So $t = 1$ and hence $h = 2$ and $\lambda_1 = 1, 2$ as $h > 1$, $t = \lfloor (h+1)/2 \rfloor$

and $1 \leq \lambda_1 \leq 2^{h-t}$ by our assumptions. Now, substituting these values in (3), we obtain a contradiction since $f = 1$. Thus $f = 5$. Then (3) becomes $2^{2h}5 = 2^3 + 2^t 3\lambda_1(2^{h+1} + 2^t\lambda_1)$. If $t \geq 2$, then $2^4 \nmid 4 + 2^t\lambda_1(2^{h+1} + 2^t\lambda_1)$. This yields that $2^4 \nmid 2^{2h}5$ as $2^{2h}5 = 2^3 + 2^t 3\lambda_1(2^{h+1} + 2^t\lambda_1)$. A contradiction. So $t = 1$, $h = 2$ and $\lambda_1 = 1, 2$, as $t = \lfloor (h+1)/2 \rfloor$ and $1 \leq \lambda_1 \leq 2^{h-t}$. By substituting these values in (3), we obtain again a contradiction as $f = 5$. This completes the proof. $\square$

Now, let Π be a finite projective plane of order n and let G be a collineation group of Π with a flag-transitive and almost simple point-orbit $\mathcal{O}$. Let N be the kernel of the action of G on $\mathcal{O}$. As a consequence of the our assumptions, the pair $(\mathcal{O}, G/N)$ is listed in Theorem 2.2. So, in order to classify the triples $(\Pi, \mathcal{O}, G)$ when $n \leq \sum(\mathcal{O})$, we treat the cases $N = \langle 1 \rangle$ and $N \neq \langle 1 \rangle$ separately.

3 The faithful action

Throughout this section we assume that $N = \langle 1 \rangle$. Hence G acts faithfully on $\mathcal{O}$.

Lemma 3.1. *Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathsf{PG}(d,q)$, on which G acts faithfully and induces a flag-transitive almost simple automorphism group. If $n \leq \sum(\mathcal{O})$ then $(d,q) \neq (3,2)$.*

Proof. Assume $\mathcal{O} \cong \mathsf{PG}(3,2)$. Then either $G \cong \mathsf{A}_7$ or $\mathsf{PSL}(4,2) \leq G$ by Theorem 2.2(i). As $\mathsf{A}_7 < \mathsf{PSL}(4,2)$, we may assume $G \cong \mathsf{A}_7$.

Note that $n \leq 60$ as $\sum(\mathsf{PG}(3,2)) = 60$ by Lemma 2.4(1). On the other hand, $n > 15$ by [4]. Therefore $16 \leq n \leq 60$. Let $P \in \mathcal{O}$. Then $G_P \cong \mathsf{PSL}(2,7)$ by [1]. Now, let σ be an arbitrary involution in G_P. Then σ fixes exactly 3 of the 5 secants to $\mathcal{O}$ through P (see [42]). Thus σ is a Baer involution of Π and hence n is a square, namely $n = 16, 25$ or 49.

Assume $n = 16$. Since G_P is transitive on $\mathcal{O} - \{P\}$ and since $|\mathcal{O} - \{P\}| = 14$, we have a contradiction by [12, 4.6(a), (b), 4.8(i)].

Assume $n = 25$. Let φ be any element of order 7 in G_P. Then φ fixes at least 5 lines of $[P]$ and 2 lines of $\Pi - [P]$, as $n + 1 \equiv 5 \mod 7$ and $n^2 \equiv 2 \mod 7$. Thus φ is planar with $o(\mathrm{Fix}(\varphi)) = 4 + 7\theta$, where $\theta \geq 0$. Actually, $\theta = 0$ by [24, Theorem 3.7], since $n = 25$. Hence $o(\mathrm{Fix}(\varphi)) = 4$. Note that $N_{G_P}(\langle\varphi\rangle) = \langle\varphi, \psi\rangle$ where $o(\psi) = 3$. Also $N_{G_P}(\langle\varphi\rangle)$ is the unique maximal subgroup of $G_P \cong \mathsf{PSL}(2,7)$ containing φ. Hence for each line $u \in \mathrm{Fix}(\varphi) \cap [P]$, either $G_{P,u} = \langle\varphi\rangle$ or $G_{P,u} = \langle\varphi, \psi\rangle$ or $G_{P,u} = G_P$. Assume that $G_{P,m} = \langle\varphi\rangle$ for some line $m \in \mathrm{Fix}(\varphi) \cap [P]$. Then $\left|m^{G_P}\right| = 24$. Furthermore, we have $\left|s^{G_P}\right| = 7$ for any line $s \in [P] \cap \mathcal{O}$. Hence $\left|m^{G_P}\right| + \left|s^{G_P}\right| \geq 31$. On the other hand, $\left|m^{G_P}\right| + \left|s^{G_P}\right| \leq 26$ as $n+1 = 26$, $m^{G_P} \cup s^{G_P} \subseteq [P]$ and $m^{G_P} \cap s^{G_P} = \emptyset$. A contradiction. Thus either $G_{P,u} = \langle \varphi, \psi \rangle$ or $G_{P,u} = G_P$ for each line $u \in \mathrm{Fix}(\varphi) \cap [P]$. Thus $\mathrm{Fix}(\varphi) \cap [P] = \mathrm{Fix}(\langle \varphi, \psi \rangle) \cap [P]$ and hence $|\mathrm{Fix}(\langle \varphi, \psi \rangle) \cap [P]| = 5$. Assume that $|\mathrm{Fix}(\langle \varphi, \psi \rangle) \cap [P] - \mathrm{Fix}(G_P) \cap [P]| \geq 3$. Let u_i, $i = 1,2,3$, be some of the lines of $[P]$ fixed by the group $\langle \varphi, \psi \rangle$ but not by the group G_P. Then $\left|u_i^{G_P}\right| = 8$ for each $i = 1,2,3$. Furthermore, as $\langle \varphi, \psi \rangle$ is maximal in G_P, the line u_i is the unique line in $u_i^{G_P}$ which is fixed by $\langle \varphi, \psi \rangle$ for each $i = 1,2,3$. Thus $u_h^{G_P} \cap u_j^{G_P} = \varnothing$ for each $h, j = 1,2,3$ with $h \neq j$. Then there are at least three G_P-orbits on $[P]$ each of length 8. So these G_P-orbits involve 24 of the lines of $[P]$. Then $n+1 \geq 24+7$, since $\left|s^{G_P}\right| = 7$ for any $s \in [P] \cap \mathcal{O}$. A contradiction, since $n = 25$. Thus $|\mathrm{Fix}(\langle \varphi, \psi \rangle) \cap [P] - \mathrm{Fix}(G_P) \cap [P]| \leq 2$ and hence $|\mathrm{Fix}(G_P) \cap [P]| \geq 3$. Now, we may repeat the above argument with l in the role of $[P]$ for each line $l \in \mathrm{Fix}(G_P) \cap [P]$. This yields $|\mathrm{Fix}(G_P) \cap l| \geq 3$ for each line $l \in \mathrm{Fix}(G_P) \cap [P]$. Then G_P is planar and $\mathrm{Fix}(G_P)$ is a proper subplane of $\mathrm{Fix}(\sigma)$ since φ and σ fix exactly 0 and 3 lines in $[P] \cap \mathcal{O}$, respectively. That contradicts [24, Theorem 3.7] since $\mathrm{Fix}(\sigma)$ has order 5.

Finally, assume that $n = 49$. Then $G_P \cong \mathsf{PSL}(2,7)$ contains a unique conjugate class of involutions which are homologies of Π by [22]. A contradiction, since $\sigma \in G_P$ and σ is a Baer involution of Π. $\square$

Lemma 3.2. *Under the same assumptions as in Lemma 3.1, we have $d = 2$.*

Proof. Assume $d \geq 3$. Then $\mathsf{PSL}(d+1,q) \leq G$ by Theorem 2.2(i) and by Lemma 3.1, since G is faithful, flag-transitive and almost simple on $\mathcal{O}$. Since all the assumptions are satisfied by the subgroup of G which is isomorphic to $\mathsf{PSL}(d+1,q)$, we may assume $G \cong \mathsf{PSL}(d+1,q)$. Let T be a subgroup of G consisting of the projective transvections of $\mathsf{PSL}(d+1,q)$ fixing the same hyperplane of $\mathcal{O}$ pointwise, as $d \geq 3$. That is $\mathrm{Fix}(T) \cap \mathcal{O} \cong \mathsf{PG}(d-1,q)$. Hence T is planar. Set $\mathrm{Fix}_{\mathcal{O}}(T) = \mathrm{Fix}(T) \cap \mathcal{O}$. Clearly, $\mathrm{Fix}_{\mathcal{O}}(T) \subseteq \Psi \subseteq \mathrm{Fix}(T)$. Assume $d > 3$. Then $\mathrm{Fix}_{\mathcal{O}}(T) \subset \Psi$, since $\mathrm{Fix}_{\mathcal{O}}(T) \cong \mathsf{PG}(d-1,q)$. Let W be any given point of $\mathrm{Fix}_{\mathcal{O}}(T)$. Then there are exactly $\frac{q^{d-1}-1}{q-1}$ lines of $\mathrm{Fix}_{\mathcal{O}}(T) \cong \mathsf{PG}(d-1,q)$ through W by [49, Theorem 1.4.10]. Thus, there are at least $\frac{q^{d-1}-1}{q-1}+1$ lines of Ψ through W, since $\mathrm{Fix}_{\mathcal{O}}(T) \subset \Psi$. Therefore, $o(\Psi) \geq \frac{q^{d-1}-1}{q-1}$. If $\Psi = \mathrm{Fix}(T)$, let us consider the subgroup J of $N_G(T)$ such that $J \cong \mathsf{PSL}(d,q)$. Then J acts doubly transitively on $\mathrm{Fix}_{\mathcal{O}}(T) \cong \mathsf{PG}(d-1,q)$ and acts faithfully on $\mathrm{Fix}(T)$. If $o(\mathrm{Fix}(T)) \leq \frac{q^{d}-1}{q-1}$, we obtain a contradiction by [4, Theorem 3.13 and Proposition 3.14]. Hence, we may assume that $o(\mathrm{Fix}(T)) > \frac{q^{d}-1}{q-1}$ in this case. If

Ψ is a proper subplane of $\mathrm{Fix}(T)$, then $o(\mathrm{Fix}(T)) \geq \left(\frac{q^{d-1}-1}{q-1}\right)^2$ by [24, Theorem 3.7]. In particular, this implies $o(\mathrm{Fix}(T)) \geq \frac{q^d-1}{q-1}$ as $d > 3$. Therefore, $o(\mathrm{Fix}(T)) \geq \frac{q^d-1}{q-1}$ in any case, and

$$\left(\frac{q^d-1}{q-1}\right)^2 \leq n \leq \sum(\mathcal{O}) \tag{4}$$

by [24, Theorem 3.7]. Hence

$$\frac{(q^d-1)^2}{(q-1)^2} \leq n \leq \frac{q^{2d+1}+q^{d+3}+q^{d+2}-2q^{d+1}-2q^d+q^4-4q^2+4}{(q^2-1)\,(q-1)} \tag{5}$$

by Lemma 2.4(1). Multiplying each term of (5) by $(q^2-1)(q-1)$ we obtain

$$q^{2d}-q^{d+3}-q^{d+2}-q^4+4q^2+q-3 \leq 0\,. \tag{6}$$

That is

$$q^{d+2}\left[q(q^{d-3}-1)-1\right]+(4q^2+q-3)-q^4 \leq 0\,, \tag{7}$$

which is impossible as $d > 3$. Now, we rule out the case $d = 3$ in five steps.

(I) *q is odd.*

Assume that q is even. Then τ is a Baer involution of Π. Note that $\mathrm{Fix}(\tau) \cap \mathcal{O} \cong \mathsf{PG}(2,q)$ as $d = 3$. Let us consider the subgroup R of $C_G(\tau)$ such that $R \cong \mathsf{PSL}(3,q)$. Then R acts on $\mathrm{Fix}(\tau)$. Moreover, R acts in its natural 2-transitive permutation representation on $\mathrm{Fix}(\tau)\cap\mathcal{O} \cong \mathsf{PG}(2,q)$. Lemma 2.4(1) with $d = 3$ yields $\sum(\mathcal{O}) = q^4+2q^3+4q^2+4q+4$, and then, by using (4), that $q^4 < n < (q+2)^4$. This yields $\sqrt{n} < q^3$ for $q > 2$. Furthermore, $\sqrt{n} < q^3$ for $q = 2$ by a direct substitution in (4). Hence, we may apply Theorem 2.5, with $\mathrm{Fix}(\tau)$ in the role of Π, to assert that either $\sqrt{n} = q$ or $\sqrt{n} = q^2$. A contradiction in any case as $n > q^4$. Thus q is odd.

(II) *If G fixes a point of Π, then the involution σ in G represented by the matrix $\mathrm{diag}(-I_2, I_2)$ is a Baer collineation of Π.*

Assume that G fixes a point Q on Π. Clearly $Q \in \Pi - \mathcal{O}$. Then G acts on $[Q]$. Note that each line through Q intersecting $\mathcal{O}$ is a tangent to $\mathcal{O}$, as G is primitive on $\mathcal{O}$. As q is odd by (I), let σ be the involution in G represented by the matrix $\mathrm{diag}(-I_2, I_2)$. Then σ fixes pointwise two skew lines on $\mathcal{O}$ and hence σ fixes exactly $2(q+1)$ tangents to $\mathcal{O}$ through Q. Therefore, σ is a Baer collineation of Π, since there are exactly $\frac{q^4-1}{q-1}$ tangents to $\mathcal{O}$ through Q.

(III) *If G fixes a line of Π, then it fixes a point on this line.*

Assume that G fixes a line l of Π. Clearly l is external to $\mathcal{O}$. We assume that G does not fix points on l and this leads to a contradiction, as we are going to see.

Assume that distinct lines of $\mathcal{O}$ intersect l in distinct points. Then there exists $X \in l$ such that $\left|X^G\right| = b$, as G is transitive on the lines of $\mathcal{O}$. If $l = X^G$, then the commuting involutions σ and $\sigma' = \mathrm{diag}(1, -1, 1, -1)$ are homologies of Π, since they fix exactly two points on l. Indeed, the actions of σ and σ' on X^G and on the lines of $\mathsf{PG}(3, q)$ are the same, but they have no common fixed points on l. A contradiction by [29, Lemma 3.1]. Then there exists $Y \in l - X^G$. Then $\left|Y^G\right| > 1$ by our assumption. Therefore $\left|Y^G\right| \geq v$, where $v = \frac{q^4-1}{q-1}$ is the minimal primitive permutation representation degree of $G \cong \mathsf{PSL}(4, q)$. Note that $\left|l - (X^G \cup Y^G)\right| \leq r + k + 1$, since $\left|Y^G\right| \geq v$, $\left|X^G\right| = b$ and $n \leq \sum(\mathcal{O})$. Now observe that $v > r + k + 1$ being $v = \frac{q^4-1}{q-1}$, $r = \frac{q^3-1}{q-1}$ and $k = q + 1$. Thus, if $l \neq X^G \cup Y^G$, the group G fixes $l - (X^G \cup Y^G)$ pointwise, since any non-trivial G-orbit has length at least $v = \frac{q^4-1}{q-1}$. That contradicts our assumptions. Hence $l = X^G \cup Y^G$. Then either $\left|Y^G\right| = v$ or $\left|Y^G\right| > v$. Assume the latter occurs. Then $\left|Y^G\right| \geq qv$ by [33, Lemma 4.2 and Table II], for $|G| > 10^{12}$ and by [31] for $|G| \leq 10^{12}$. Therefore $qv \leq v + r + k + 1$ as $Y^G \subseteq l - X^G$. Again a contradiction, since $v = \frac{q^4-1}{q-1}$, $r = \frac{q^3-1}{q-1}$ and $k = q + 1$. So $l = X^G \cup Y^G$ with $\left|Y^G\right| = v$ and $\left|X^G\right| = b$. Then σ fixes exactly $q + 3$ points on l. Namely $q + 1$ points on Y^G and 2 points on X^G (this follows by the fact that the actions of σ on Y^G and on X^G are the same as those on the points and on the lines of $\mathsf{PG}(3, q)$, respectively). So σ is a Baer collineation and hence $n = (q + 2)^2$. A contradiction, since $n = b + v - 1$, with $v = \frac{q^4-1}{q-1}$ and $b = (q^2 + 1)(q^2 + q + 1)$ as $\mathcal{O} \cong \mathsf{PG}(3, q)$.

Hence there are two distinct secants to $\mathcal{O}$, say s_1 and s_2, concurring at a point P. Then $G_{s_1}, G_{s_2} \leq G_P$, as G fixes l. Therefore $\langle G_{s_1}, G_{s_2}\rangle \leq G_P$. Then $G = G_P$, since G_{s_1} and G_{s_2} are two distinct maximal parabolic subgroups of G (see [32] and [31]). That contradicts our assumptions. This completes the proof of (III).

(IV) *The group G contains involutory perspectivities.*

Let σ be the involution in G represented by the matrix $\mathrm{diag}(-I_2, I_2)$. Assume that σ is a Baer collineation of Π. Now, consider the subgroup H of $C_G(\sigma)$ consisting of the matrices $\mathrm{diag}(A, I_2)$, where $A \in \mathsf{PSL}(2, q)$. Then $H \cong \mathsf{PSL}(2, q)$ and H acts on $\mathrm{Fix}(\sigma)$. In particular there exists a secant u to $\mathcal{O}$ such that σ and H fix the $q + 1$ points of $u \cap \mathcal{O}$.

If H contains Baer collineations of $\mathrm{Fix}(\sigma)$, then n must be a fourth power. Note that, by the first inequality of (5), we have $n > q^4$. Then $\sqrt[4]{n} > q$ and hence $\sqrt[4]{n} \geq q+1$, since $\sqrt[4]{n}$ is an integer. Thus $n \geq (q+1)^4$. On the other hand, $n \leq \sum(\mathcal{O}) = q^4 + 2q^3 + 4q^2 + 4q + 4$. So $(q+1)^4 \leq q^4 + 2q^3 + 4q^2 + 4q + 4$. A contradiction. Thus each involution in H is a perspectivity of $\mathrm{Fix}(\sigma)$, since H contains a unique conjugate class of involutions. In particular each involution in H has axis $u \cap \mathrm{Fix}(\sigma)$, since σ and H fix the $q+1$ points of $u \cap \mathcal{O}$ by the above argument. Thus $H = H(u \cap \mathrm{Fix}(\sigma))$, since H is generated by its involutions.

Assume $q > 3$. Then $H = H(C, u \cap \mathrm{Fix}(\sigma))$ for some point $C \in \mathrm{Fix}(\sigma)$ by [24, Theorems 4.14 and 4.25], since H is non-abelian simple. Then either $|H| \mid \sqrt{n}$ or $|H| \mid \sqrt{n} - 1$ according to whether C does or does not lie on $u \cap \mathrm{Fix}(\sigma)$ respectively. This yields $\sqrt{n} \geq q(q^2-1)/(q-1,2)$ in any case as $|H| = q(q^2-1)/(q-1,2)$. Thus

$$\left[\frac{q(q^2-1)}{(q-1,2)}\right]^2 \leq q^4 + 2q^3 + 4q^2 + 4q + 4 \tag{8}$$

as $n \leq q^4 + 2q^3 + 4q^2 + 4q + 4$. Inequality (8) yields $q \leq 3$, contrary to our assumption.

Assume $q \leq 3$. Actually $q = 3$ by Lemma 3.1 as $d = 3$. Then $\sum(\mathcal{O}) = 187$ and hence $n \leq 13^2$ as n is a square and $n \leq \sum(\mathcal{O})$. On the other hand $n > 3^4$ by the first inequality of (5), as $q = 3$. Thus $\sqrt{n} = 10$, 11, 12 or 13, since n is a square. The case $\sqrt{n} = 10$ cannot occur by [24, Theorem 13.18]. Finally, if $\sqrt{n} = 12$, then $H \cong \mathsf{A}_4$. A contradiction by [27]. Hence either $\sqrt{n} = 11$ or $\sqrt{n} = 13$. Now, consider the group $R = H \times L$ where H is defined as above (for $q = 3$) and L consists of the matrices $\mathrm{diag}(I_2, B)$, where $B \in \mathsf{PSL}(2,3)$. Then $R \cong \mathsf{A}_4 \times \mathsf{A}_4$ and $R \leq C_G(\sigma)$. In particular R acts faithfully on $Fix(\sigma)$. As $\sqrt{n} = 11$ or 13, then R contains a subgroup R_0 of homologies of $\mathrm{Fix}(\sigma)$ isomorphic to E_{16}. Then $R_0(D, a)$ for some point $D \in \mathrm{Fix}(\sigma)$ and some line $a \in \mathrm{Fix}(\sigma)$ by [29, Lemma 3.1]. Thus $16 \mid (\sqrt{n} - 1)$. A contradiction, since $\sqrt{n} = 11$ or 13. Thus the assertion (III).

(V) *The final contradiction.*

By (IV) the group G contains involutory perspectivities of Π. In particular the proof of (IV) implies that the involution σ in G represented by the matrix $\mathrm{diag}(-I_2, I_2)$ is a perspectivity of Π. This yields that G does not fix points or lines by (II) and (III) respectively. This yields in turn that G does not fix triangles, since G is simple. Therefore the group $G \cong \mathsf{PSL}(4,q)$ is an irreducible collineation group of Π containing involutory perspectivities. A contradiction by [48].

Thus $d = 2$ and hence the assertion. $\square$

Lemma 3.3. *Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathsf{PG}(d,q)$, on which the group G acts faithfully and induces a flag-transitive almost simple automorphism group. If $n \leq \sum(\mathcal{O})$ then $d = 2$ one of the following occurs:*

(1) $n = q$, $\Pi = \mathcal{O} \cong \mathsf{PG}(2,q)$ *and* $\mathsf{PSL}(3,q) \leq G$;

(2) $n = q^2$, Π *is either a Desarguesian plane or a generalized Hughes plane,* $\mathcal{O} \cong \mathsf{PG}(2,q)$ *is a Baer subplane of* Π *and* $\mathsf{PSL}(3,q) \leq G$;

(3) $n = 8$, $q = 2$, $\Pi \cong \mathsf{PG}(2,8)$, $\mathcal{O} \cong \mathsf{PG}(2,2)$ *and* $\mathsf{PSL}(3,2) \leq G$;

(4) $n = 16$, $q = 2$, $\Pi \cong \mathsf{PG}(2,16)$, $\mathcal{O} \cong \mathsf{PG}(2,2)$ *and* $\mathsf{PSL}(3,2) \leq G$;

(5) $n = 27$, $q = 3$, $\mathcal{O} \cong \mathsf{PG}(2,3)$ *and* $\mathsf{PSL}(3,3) \leq G$.

Proof. Clearly $\mathcal{O} \cong \mathsf{PG}(2,q)$ by Lemma 3.2. Then $n \leq \sum(\mathcal{O}) = 2q^2 + 4q + 4$ by Lemma 2.4(1). If $q > 3$ then we have $2q^2 + 4q + 4 \leq q^3$. Hence if $q > 3$ or if $q \in \{2,3\}$, $n \leq q^3$ we may apply Theorem 2.5 and obtain assertions (1), (2), (3) and (5) of our statement. We assume now $q = 2$ or 3; the inequalities $q^3 < n \leq 2q^2 + 4q + 4$ yield $9 \leq n \leq 20$ or $28 \leq n \leq 34$, respectively.

Assume $q = 2$, $9 \leq n \leq 20$. If the involutions in G are Baer collineations of Π, then $n = 16$ by [22]. Since $\mathsf{PSL}(3,2) \cong \mathsf{PSL}(2,7)$ it follows from [12, 4.6(a), (b), 4.8(i)] that this cannot be the case. Hence the involutions in G are elations of Π, since they induce elations on $\mathcal{O} \cong \mathsf{PG}(2,2)$. Therefore n is even. Then $n = 10, 12, 14, 16, 18, 20$. The cases $n = 10, 14$ or 18 are ruled out by [24, Theorem 13.18]. The case $n = 12$ is ruled out by [27]. Hence either $n = 16$ or $n = 20$. Assume the latter occurs. Let $\mathcal{E}$ be the set of points of Π which do not lie on any secant to $\mathcal{O}$. Then $|\mathcal{E}| = 288$. Furthermore, G leaves $\mathcal{E}$ invariant. In particular the stabilizer in G of any point of $\mathcal{E}$ has odd order, since each involution has center in $\mathcal{O}$ and axis a secant to $\mathcal{O}$. So, for any point $P \in \mathcal{E}$ the group G_P is isomorphic either to Z_7 or to Z_3 or to $Z_7.Z_3$, as $G \cong \mathsf{PSL}(2,7)$. Thus the corresponding P^G, which clearly lies in $\mathcal{E}$, has length 24, 56 or 8, respectively. Thus $288 = |\mathcal{E}| = x_1 24 + x_2 56 + x_3 8$. Now, let $\langle\gamma\rangle$ be a subgroup of G of order 7. As $n = 20$, then either $\langle\gamma\rangle$ fixes exactly 1 point on Π or a subplane of Π of order at least 6. Actually the latter cannot occur by [24, Theorem 3.7]. Hence γ fixes exactly one point of Π. Let Q be such a point. Clearly $Q \in \mathcal{E}$, since $|\mathcal{E}| = 288$ and $o(\gamma) = 7$. On the other hand, by [41, Relation (9)], the group $\langle\gamma\rangle$ fixes 3, 0 or 1 points on P^G according to whether P^G has length 24, 56 or 8, respectively. This in conjunction with the fact that $\langle\gamma\rangle$ fixes exactly one point in $\mathcal{E}$ yields $(x_{1,} x_2, x_3) = (0,5,1)$ in the above Diophantine equation. Let $\langle\rho\rangle$ be a subgroup of G of order 3. Then $\langle\rho\rangle$ fixes exactly 2 points in each G-orbit of length 56 or 8 by [41, Relation (9)]. Thus $\langle\rho\rangle$ fixes exactly 12 points on $\mathcal{E}$, as $x_2 = 5$ and $x_3 = 1$. This yields that $\langle\rho\rangle$ fixes exactly a subplane of Π of order at least 3, since $n \equiv 2 \mod 3$. We have $o(\mathrm{Fix}(\langle\rho\rangle)) \in \{3, 4\}$ by [24, Theorem 3.7]. Moreover, as any involution in G normalizing $\langle\rho\rangle$ is an elation of Π, the group G must induce an elation on $\mathrm{Fix}(\langle\rho\rangle)$. Thus the case $o(\mathrm{Fix}(\langle\rho\rangle)) = 3$ is ruled out. Hence $o(\mathrm{Fix}(\langle\rho\rangle)) = 4$. Now, let l be any line of $\mathrm{Fix}(\langle\rho\rangle)$. Then $3 \mid (n-4)$, as $\langle\rho\rangle$ must be semiregular on $l - \mathrm{Fix}(\langle\rho\rangle)$. A contradiction, since $n = 20$. Hence $n = 16$ and part 4. of our statement follows from [12].

Assume $q = 3$, $28 \leq n \leq 34$. Thus n cannot be a square. Then the involutions in G are homologies of Π, since they induce homologies of $\mathcal{O} \cong \mathsf{PG}(2,3)$. Therefore n is odd and hence $n = 29, 31$ or 33. Actually the case $n = 33$ cannot occur by [24, Theorem 3.6]. Let $\mathcal{F}$ be the set of points of Π which do not lie on any secant to $\mathcal{O}$. Then $|\mathcal{F}| = (n-3)(n-9)$. Furthermore, G leaves $\mathcal{F}$ invariant. In particular the stabilizer in G of any point of $\mathcal{F}$ has odd order, since each involution in G has center in $\mathcal{O}$ and axis a secant to $\mathcal{O}$. Thus each G-orbit in $\mathcal{F}$ must have length divisible by 16 as $16 \mid |G|$. Therefore $16 \mid |\mathcal{E}|$. That is $16 \mid (n-3)(n-9)$. A contradiction since $n = 29$ or 31. This completes the proof. □

Lemma 3.4. *Let Π be a finite projective plane of order n and let $G \cong \mathsf{PSU}(3,q)$ be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathcal{H}(q)$, $q > 2$. If $n \leq \sum(\mathcal{O})$, then G contains involutory perspectivities of Π.*

Proof. Let σ be an involution in G and suppose that it is a Baer collineation of Π.

Assume that q is odd. Then σ fixes exactly $q+1$ points of $\mathcal{O}$ lying on a secant l. Furthermore, σ fixes a set S_σ of $q^2 - q$ secants and $S_\sigma \cup \{l\}$ is a partition of the points of $\mathcal{O}$ (see [23]). The group $C = C_G(\sigma)$ induces a transitive permutation group $\bar{C} \cong \mathsf{PGL}(2,q)$ on S_σ, and $\bar{C}_r \cong Z_{q+1}$ for any $r \in S_\sigma$. Assume there exist two distinct lines r and s of S_σ that intersect $\mathrm{Fix}(\sigma) \cap (l - \mathcal{O})$ in one and the same point X. Then all the lines of S_σ pass through X, since $\langle \bar{C}_r, \bar{C}_s \rangle \cong \mathsf{PGL}(2,q)$. Thus $o(\mathrm{Fix}(\sigma)) \geq q^2 - q$, since σ fixes $S_\sigma \cup \{l\}$.

Suppose that $o(\mathrm{Fix}(\sigma)) > q^2 - q$. Then $|\mathcal{E}| > 0$, where $\mathcal{E}$ denotes the set of external lines to $\mathcal{O}$ lying in $\mathrm{Fix}(\sigma) \cap [X]$. Clearly C acts on $\mathcal{E}$ with kernel $\langle\sigma\rangle$. In particular there exists a subgroup C_1 of C such that $\bar{C}_1 = C_1/\langle\sigma\rangle \cong \mathsf{PSL}(2,q)$. Clearly $\bar{C}_1$ acts in its natural 2-transitive permutation representation of degree $q+1$ on $l \cap \mathcal{O}$. Furthermore, either $\bar{C}_1$ fixes $\mathcal{E}$ elementwise or $\bar{C}_1$ has a non-trivial orbit on $\mathcal{E}$.

Suppose that $\bar{C}_1$ fixes $\mathcal{E}$ elementwise. Let $\bar{\rho}$ be any involution in $\bar{C}_1$. Then $\bar{\rho}$ fixes either 0 or 2 secants to $\mathcal{O}$ through X according to whether $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$ respectively, since $\bar{C}_1$ is transitive on S_σ and $\bar{C}_{1,u} \cong Z_{\frac{q+1}{2}}$ for each $u \in S_\sigma$ by [23].

Assume that $q \equiv 1 \mod 4$. Thus $\bar{\rho}$ is a Baer collineation on $\mathrm{Fix}(\sigma)$, since $\bar{\rho}$ fixes exactly 2 points lying in $l \cap \mathcal{O} \subset \mathrm{Fix}(\sigma)$ and at least the point $X \in \mathrm{Fix}(\sigma) \cap (l - \mathcal{O})$. Furthermore, $\bar{\rho}$ does not fix any secants to $\mathcal{O}$ through X, other than l. Thus $\sqrt[4]{n} + 1 = |\mathcal{E}| + 1$, since $\bar{\rho}$ fixes $\mathcal{E}$ elementwise. Since $\sqrt{n} + 1 = |\mathcal{E}| + q^2 - q + 1$, we obtain $\sqrt{n} = \sqrt[4]{n} + q^2 - q$. Thus $\sqrt{n} = q^2$. A contradiction by [41, Theorem 1.2], since $\mathcal{E} \cup \{l\} \subset \mathrm{Fix}(\bar{C}_1)$ and $|\mathcal{E}| > 0$.

Assume that $q \equiv 3 \mod 4$. Then $\bar{\rho}$ is a Baer collineation on $\mathrm{Fix}(\sigma)$, since $\bar{\rho}$ fixes exactly 2 secants to $\mathcal{O}$ through X other than l. Thus $\sqrt[4]{n}+1 = |\mathcal{E}|+3$, since $\bar{\rho}$ fixes $\mathcal{E}$ elementwise. Since $\sqrt{n} + 1 = |\mathcal{E}| + q^2 - q + 1$, we obtain $\sqrt{n} - \sqrt[4]{n} = q^2 - q - 2$. That is

$$\sqrt[4]{n}(\sqrt[4]{n} - 1) = (q + 1)(q - 2)\,. \tag{9}$$

Clearly $\sqrt[4]{n} \neq q$. Thus, either $\sqrt[4]{n}(\sqrt[4]{n} - 1) \geq q(q + 1)$ or $\sqrt[4]{n}(\sqrt[4]{n} - 1) \leq (q - 1)(q - 2)$ according to whether $\sqrt[4]{n} \geq q + 1$ or $\sqrt[4]{n} \leq q - 1$ respectively. A contradiction in any case by (9).

Suppose that $\bar{C}_1$ has a non-trivial orbit on $\mathcal{E}$. Since the length of each $\bar{C}_1$-orbit is the multiple of some non-trivial primitive permutation representation degree of $\bar{C}_1$, we have that $|\mathcal{E}| \geq d_0(\bar{C}_1)$, where $d_0(\bar{C}_1)$ denotes the minimal of such degrees of $\bar{C}_1$. If $q \notin \{3, 5, 7, 9, 11\}$, then $d_0(\bar{C}_1) = q + 1$. Hence σ fixes at least $q^2 - q + 1 + (q + 1)$ lines through X, since $\sqrt{n} + 1 = |\mathcal{E}| + q^2 - q + 1$. So $o(\mathrm{Fix}(\sigma)) \geq q^2 + 1$. That is $\sqrt{n} \geq q^2 + 1$. On the other hand $n \leq q^4 + 2q^2 + q + 2$ as $n \leq \sum(\mathcal{O})$ and $\sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$ by Lemma 2.4(2). That is $n < (q^2 + 2)^2$ and hence $\sqrt{n} < q^2 + 2$. As a consequence $\sqrt{n} = q^2 + 1$. As q is odd by our assumptions, then $\sqrt{n} \equiv 2 \mod 4$ and $\sqrt{n} > 2$. A contradiction by [24, Theorem 13.18], since $\bar{C}_1 \cong \mathsf{PSL}(2, q)$ acts faithfully on $\mathrm{Fix}(\sigma)$. Thus $q \in \{3, 5, 7, 9, 11\}$. If $q \neq 9$, then $o(\mathrm{Fix}(\sigma)) \geq q^2$ by the same argument as above, since $d_0(\bar{C}_1) = q$. The previous argument rules out the case $o(\mathrm{Fix}(\sigma)) \geq q^2 + 1$. Hence $o(\mathrm{Fix}(\sigma)) = q^2$. That contradicts [41, Theorem 1.2], since $\bar{C}_1$ fixes the flag (X, l) and $q \neq 9$. Hence $q = 9$. Then $\sqrt{n} \leq 82$, since $n \leq \sum(\mathcal{O})$ and $\sum(\mathcal{O}) = 6734$. Note that $\sqrt{n} = 82$ cannot occur by the above argument involving [24, Theorem 13.18]. Hence $\sqrt{n} \leq 81$. On the other hand, $\sqrt{n} = |\mathcal{E}| + 72$. Hence $|\mathcal{E}| < 10$. Recall that $\mathcal{E}$ contains a non-trivial $\bar{C}_1$-orbit on $\mathcal{E}$. Then either $|\mathcal{E}| = 6\lambda + \mu$ for some $\lambda \geq 1$ and $\mu \geq 0$, since the unique primitive permutation degree of $\mathsf{PSL}(2, 9)$ less than 10 is 6. Clearly $\lambda = 1$ as $|\mathcal{E}| < 10$. Note that μ denotes the number of lines of $\mathcal{E}$ fixed by the whole group $\bar{C}_1$. Moreover, each involution in $\bar{C}_1$ fixes exactly 2 lines in the $\bar{C}_1$-orbit of length 6 lying in $\mathcal{E}$. Also, each involution in $\bar{C}_1$ fixes the line l. So the involutions in $\bar{C}_1$ are Baer collineations of $\mathrm{Fix}(\sigma)$. Thus $\sqrt{n} = 81$, since $\sqrt{n}$ must be a square and $72 < \sqrt{n} \leq 81$. Then $\mu = 2$. Hence $\mathcal{E}$ consists of three $\bar{C}_1$-orbits of length 6, 1, 1. So $\mathrm{Fix}(\bar{C}_1)$ consists of more than one flag. Again a contradiction by [41, Theorem 1.2].

Now, assume that $o(\mathrm{Fix}(\sigma)) = q^2 - q$. Let $\bar{\beta}$ be any involution of $\bar{C}_1$. If $q \equiv 3$

$\bmod 4$, then $\bar{\beta}$ fixes exactly two secants to $\mathcal{O}$ through X, other than l. Thus $\bar{\beta}$ is a Baer collineation on $\mathrm{Fix}(\sigma)$ with $o(\mathrm{Fix}(\bar{\beta})) = 2$. So $q^2 - q = 4$. A contradiction. Hence $q \equiv 1 \mod 4$ as q is odd. Then $\bar{\beta}$ does not fix any secant to $\mathcal{O}$ through X, other than l. On the other hand, the collineation $\bar{\beta}$ fixes exactly 2 points lying in $l \cap \mathcal{O} \subset \mathrm{Fix}(\sigma)$ and at least the point $X \in \mathrm{Fix}(\sigma) \cap (l - \mathcal{O})$. Then $\bar{\beta}$ fixes a Baer involution of $\mathrm{Fix}(\sigma)$. A contradiction, since $\bar{\beta}$ does not fix any secant to $\mathcal{O}$ through X other than l.

Hence, we may assume that any two distinct lines of S_σ intersect l in distinct points. Then $l \cap \mathrm{Fix}(\sigma)$ contains $l \cap \mathcal{O}$ and the intersection points of l with each line of S_σ. Thus $\sqrt{n} \geq q^2$. Actually either $\sqrt{n} = q^2$ or $\sqrt{n} = q^2 + 1$, since $\sqrt{n} \leq \sqrt{\sum(\mathcal{O})}$ where $\sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$ by Lemma 2.4(2). If $\sqrt{n} = q^2$, then $\bar{C}_1$ acts transitively on $l \cap \mathrm{Fix}(\sigma) - \mathcal{O}$, since $\bar{C}_1$ is transitive on S_σ. Hence $l \cap \mathrm{Fix}(\sigma)$ consists of two $\bar{C}_1$-orbits of points of length $q+1$ and $q^2 - q$. A contradiction by [41, Theorem 1.2], since q is odd. If $\sqrt{n} = q^2 + 1$, the above argument involving Theorem 13.18 of [24] rules out this case. Thus the involutions in G are homologies of Π when q is odd.

Assume that q is even. Let Q be a Sylow 2-subgroup of G containing σ. Then $Z(Q)$ is an elementary abelian 2-group of order q fixing exactly a point Y in $\mathcal{O}$ and all the q^2 secants to $\mathcal{O}$ through Y (see [23]). Note that the involutions in $Z(Q)$ are Baer, since σ is a Baer involution and G contains a unique conjugate class of involutions. Thus $n \geq (q^2-1)^2$ by [24, Theorem 3.7]. On the other hand, we have that $n \leq \sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$ by Lemma 2.4(2), and hence $n < (q^2+2)^2$. So either $n = (q^2-1)^2$ or $n = q^4$ or $n = (q^2+1)^2$, as $n \geq (q^2-1)^2$. Thus the involutions in $Z(Q)$ fix the same $\sqrt{n}+1$ lines through Y, since they fixes the same q^2 secants to $\mathcal{O}$ through Y. Then $Z(Q)$ is semiregular on $[Y] - \mathrm{Fix}(Z(Q))$. Therefore Q is semiregular on $[Y] - \mathrm{Fix}(Z(Q))$, since each involution in Q lies in $Z(Q)$ being q even. Thus $q^3 \mid (n - \sqrt{n})$, being $|Q| = q^3$ and $|[Y] - \mathrm{Fix}(Z(Q))| = n - \sqrt{n}$. Then either $q^3 \mid \sqrt{n}$ or $q^3 \mid (\sqrt{n} - 1)$. A contradiction in any case, since $\sqrt{n} = q^2 - 1$ or $\sqrt{n} = q^2$ or $\sqrt{n} = q^2 + 1$. Thus σ is a perspectivity of Π. As a consequence each involution in G is a perspectivity of Π, since $G \cong \mathsf{PSU}(3,q)$ has a unique conjugate class of involutions. This completes the proof. $\square$

Lemma 3.5. *Let Π be a finite projective plane of order n and let $G \cong \mathsf{PSU}(3,q)$ be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathcal{H}(q)$, $q > 2$. If $n \leq \sum(\mathcal{O})$, then G acts strongly irreducibly on a G-invariant subplane $\Pi_0 \cong \mathsf{PG}(2,q^2)$ of Π containing $\mathcal{O}$. In particular, the following hold:*

(1) $\Pi = \Pi_0$ *or* $n \geq q^4$*;*

(2) *the involutions in G are either homologies or elations of Π according to whether q is odd or even, respectively.*

Proof. If $n \leq q^3+1$, the assertion follows by [4, Theorem 3.10]. Hence we may assume that $n > q^3+1$. We proceed stepwise.

(I) *G does not fix any point of* Π.

Suppose that $G = G_P$ for some point $P \in \Pi$. Clearly $P \notin \mathcal{O}$. Then $[P]$ contains q^3+1 tangents to $\mathcal{O}$, since G is primitive on $\mathcal{O}$. Then P cannot be the center of any perspectivity lying in G, since G is faithful on $\mathcal{O}$. Hence the axis of any involutory perspectivity contains P. Then q is even by [23], since any involution in G fixes $q+1$ collinear points in $\mathcal{O}$ for q odd. Furthermore, each point in $\mathcal{O}$ is the center of an involutory perspectivity by [23]. Hence each involution fixes at most 2 lines through P.

Suppose that n is even. Then each line of $[P]$ tangent to $\mathcal{O}$ is the axis of some the involutory elation in G, since q is even. As $n > q^3+1$, there exists an external line e to $\mathcal{O}$ through P. Clearly $|G_e|$ is odd. Then $n+1 \geq q^3+1+\left|e^G\right|$, since $|[P]| = n+1$, $e^G \subset [P]$ and the group G is transitive on the tangents to $\mathcal{O}$ through P. Moreover $q^3 \mid \left|e^G\right|$, since q is even and $|G_e|$ is odd. If $|G_e| \leq \frac{3(q+1)^2}{j}$, where $j = (3, q+1)$, then $\left|e^G\right| \geq q^3(q-1)(q^2-q+1)/3$. A contradiction, since $e^G \subset [P]$ and $n \leq q^4$. Hence $|G_e| > \frac{3(q+1)^2}{j}$. Then $q=2$ and $|G_e| \mid 9$ by [16], since $|G_e|$ is odd. A contradiction, since $q > 2$ by our assumption.

Now, suppose that n is odd. For each Sylow 2-subgroup Q_i of G, with $1 \leq i \leq q^3+1$, denote by U_i the center of Q_i. Then $U_i = U_i(X_i, z_i)$ for some point $X_i \in \mathcal{O}$ and some line $z_i \in [P]$, for each $1 \leq i \leq q^3+1$, since n is odd, q is even and G is transitive on $\mathcal{O}$. Clearly $X_w \neq X_f$ for each $1 \leq w, j \leq q^3+1$ with $w \neq f$. If there exist $z_h = z_t$ for some $1 \leq h, t \leq q^3+1$ with $h \neq t$, then U_i is a Frobenius complement by [24, Theorem 4.25]. A contradiction, by [44, Proposition 18.1(i)], since $q > 2$. Hence there are exactly q^3+1 external lines z_i to $\mathcal{O}$ through P such that $U_i = U_i(X_i, z_i)$. Now it easily seen that, the stabilizer of any external line, which is not a z_i, $1 \leq i \leq q^3+1$, must have odd order. So, by using similar arguments to that used above, involving [16], we may conclude that the lines z_i, with $1 \leq i \leq q^3+1$, are the unique lines of $[P]$ which are external to $\mathcal{O}$. Hence $n+1 = 2(q^3+1)$. That is $n = 2q^3+1$. Hence $[P]$ is the disjoint union of two 2-transitive G-orbits both of length q^3+1. Let $Z \cong Z_{\frac{q+1}{j}}$ be a subgroup of $G \cong \mathsf{PSU}(3, q^2)$, where j is defined as above. Clearly Z fixes exactly $2(q+1)$ lines of $[P]$. Furthermore, Z fixes two points on each of these line other than P, since $(2q^3+1, q+1) = 1$ and $(2q^3, q+1) = 1$. Hence Z is planar. In particular Z fixes exactly $q+1$ points of $\mathcal{O}$ lying on a secant s. Furthermore, Z leaves a partition $\mathcal{J}$ of q^2-q+1 secants invariant, with $s \in \mathcal{J}$, by [23]. In particular $N_G(Z)$ acts on $\mathrm{Fix}(Z)$, inducing $N_G(Z)/Z \cong \mathsf{PGL}(2, q)$. Arguing as in Lemma 3.4, either there exists a point $R \in \mathrm{Fix}(Z) \cap s$ such that

$\mathcal{J} \subset [R]$, or any two distinct lines in $\mathcal{J}$ intersect $s - \mathcal{O}$ in distinct points. This yields $o(\mathrm{Fix}(Z)) \geq q^2 - q$ in any case. A contradiction by [24, Theorem 3.7], since $n = 2q^3 + 1$, q is even and $q > 2$.

(II) *G does not fix any line of* Π.

Suppose that $G = G_l$ for some line l of Π. Clearly $l \cap \mathcal{O} = \emptyset$. If l is the axis of some involutory perspectivity in G, then q is even and hence n odd by [23]. Moreover $G(l,l) \neq \langle 1 \rangle$ by [24, Theorem 4.25], since G fixes l and acts on $\mathcal{O}$ transitively. Note that $G(l,l) \lhd G$, since G fixes l. Then $G = G(l,l)$, since G is simple as $q > 2$ and $G(l,l) \neq \langle 1 \rangle$. So G fixes l pointwise. A contradiction by (I). Hence l contains the centers of all involutory perspectivities of G. Thus q is odd by [23]. Pick two distinct involutory perspectivities σ and ϕ in G with axis a and c respectively, such that $a \neq c$ and such that $S_\sigma \cap S_\phi$ share a secant m. The secants in S_σ and in S_ϕ meet l in the center of σ and ϕ, respectively. In particular, both centers coincide with $l \cap m$, since $m \in S_\sigma \cap S_\phi$. So σ fixes $S_\sigma \cup S_\phi$. A contradiction.

(III) *G acts irreducibly on* Π.

Assume that G leaves invariant a triangle Δ. Then G fixes Δ pointwise, since G is simple as $q > 2$. A contradiction, since G does not fix points by (I). Thus G is irreducible on Π.

(IV) *G acts strongly irreducibly on a G-invariant subplane $\Pi_0 \cong \mathsf{PG}(2,q^2)$ of Π containing $\mathcal{O}$. In particular, either $\Pi = \Pi_0$ or $n \geq q^4$.*

By Lemma 3.4 and by (III), we have that G is irreducible on Π and it contains perspectivities of Π. Then by [18, Lemma 5.3] the centers and the axes of the perspectivities in G generate a subplane Π_0 containing $\mathcal{O} \cong \mathcal{H}(q)$ and on which G acts strongly irreducibly. Then $o(\Pi_0) \geq q^2$ since Π_0 contains all the $q^4+q^2-q^3$ secants of $\mathcal{O}$. Note that, again by [18], any other subplane of Π left invariant by G contains Π_0.

Assume that q is odd. Then the involutions in G are homologies of Π by Lemma 3.4, since $\Pi_0 \cong \mathsf{PG}(2,q^2)$. Therefore, by [29, Theorem C(iii)], the group G leaves invariant a plane Π_1 of order q^2 containing a unital $\mathcal{H}_1(q)$. In particular $\Pi_1 \cong \mathsf{PG}(2,q^2)$ and G acts on $\mathcal{H}_1(q)$ in its natural action by [23]. Then $\Pi_0 = \Pi_1$ and $\mathcal{H}(q) = \mathcal{H}_1(q)$, since $\Pi_0 \subset \Pi_1$ and $o(\Pi_0) \geq q^2$ by the above remark. At this point the assertion for q odd follows by [24, Theorem 3.7].

Assume that q is even. Then the involutions in G are elations of Π by [20, Proposition 6.1]. Suppose that $o(\Pi_0) > 2^{2h}$. Then it must be $o(\Pi_0) > 2^{6h}$ by

[7, Lemma 5.4]. A contradiction, since $n \leq \sum(\mathcal{O}) = 2^{4h} + 2^{2h+1} + 2^h + 2$ by Lemma 2.4 (2). Thus $\Pi_0 \cong \mathsf{PG}(2, 2^{2h})$ and again the assertion follows by [24, Theorem 3.7]. □

Lemma 3.6. *Let* Π *be a finite projective plane of order* n *and let* $G \cong \mathsf{PSU}(3,q)$*,* q *odd, be a collineation group of* Π *with a point-orbit* $\mathcal{O} \cong \mathcal{H}(q)$*. If* $n \leq \sum(\mathcal{O})$*, then* G *contains involutory perspectivities and* G *is strongly irreducible on a* G*-invariant subplane* Π_0 *of* Π *containing* $\mathcal{O}$*. In particular, one of the following occurs:*

(1) $n = q^2$*,* $\Pi_0 = \Pi \cong \mathsf{PG}(2, q^2)$*;*

(2) $n = q^4$*,* $\Pi_0 \cong \mathsf{PG}(2, q^2)$ *is a Baer subplane of* Π*.*

Proof. If $n \leq q^4$, the assertion follows from Lemma 3.5. Hence assume that $n > q^4$. We rule out this case in two steps.

(I) $n = q^4 + 2q^2 - 2q$.

Note that $n \geq q^4 + q^2$ by [24, Theorem 3.7], since $\Pi_0 \cong \mathsf{PG}(2, q^2)$ and $n > q^4$. As q is odd the involutions in G are homologies of Π by Lemma 3.5(2). Then $(q-1) \mid (n-1)$ by [29, Theorem C(iii)]. Thus $n = q^4 + q^2 + \lambda$ with $\lambda \geq 1$ and $(q-1) \mid (\lambda + 1)$. Then $\lambda = \lambda_1(q-1) - 1$, $\lambda_1 \geq 1$, and hence $n = q^4 + q^2 + \lambda_1(q-1) - 1$. Let $Z(U)$ be the center of a Sylow p-subgroup U of G. Then $Z(U)$ is an elementary abelian p-group of order q. Furthermore $Z(U)$ induces on Π_0 an elation group having the same center $P \in \mathcal{O}$ and the same axis c tangent to $\mathcal{O}$ in P (see [23]). Let s be a secant to $\mathcal{O}$ through P. Clearly s is a secant to Π_0 and $Z(U)$ is semiregular on $s \cap \Pi_0 - \{P\}$. Assume that $Z(U)_R \neq \langle 1 \rangle$ for some point R on $s - \Pi_0$. Then $Z(U)_R$ is planar in Π, since U is transitive on $[P] \cap \mathcal{O}$ and $Z(U)_R \lhd U$. Then each element in $Z(U)$ is planar, since they are conjugate in G. Let $\tau \in Z(U)$, $\tau \neq 1$. Then $o(\mathrm{Fix}(\langle\tau\rangle)) \geq q^2$, since $|\mathrm{Fix}(\tau) \cap c| \geq q^2 + 1$. Actually, either $o(\mathrm{Fix}(\langle\tau\rangle)) = q^2$ or $o(\mathrm{Fix}(\langle\tau\rangle)) = q^2 + 1$, since $n \leq \sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$ by Lemma 2.4(2).

Assume that $o(\mathrm{Fix}(\langle\tau\rangle)) = q^2 + 1$. Note that the collineation ζ in G represented by the matrix $\mathrm{diag}(-1, 1, -1)$ centralizes $Z(U)$. Hence ζ and acts on $\mathrm{Fix}(\langle\tau\rangle)$. In particular ζ acts non-trivially on $\mathrm{Fix}(\langle\tau\rangle)$, since it acts non-trivially on $\mathrm{Fix}(\langle\tau\rangle) \cap \mathcal{O}$ by [23]. This is impossible by [24, Theorem 13.18], since $o(\mathrm{Fix}(\langle\tau\rangle)) = q^2 + 1$ with $q^2 + 1 \equiv 2 \mod 4$ and $q^2 + 1 > 2$ as q is odd. Hence $o(\mathrm{Fix}(\langle\tau\rangle)) = q^2$. Then $|\mathrm{Fix}(\tau) \cap c| = q^2 + 1$ and hence $|\mathrm{Fix}(Z(U)) \cap c| = q^2 + 1$, since the non-trivial elements in $Z(U)$ are conjugate in G and since $|\mathrm{Fix}(Z(U)) \cap c| \geq q^2 + 1$. Thus $Z(U)$ is semiregular on $c - \mathrm{Fix}(Z(U))$. Hence $q \mid (n - q^2)$ as $|U| = q$ and $|c - \mathrm{Fix}(Z(U))| = n - q^2$. That is $q \mid n$. Then $q \mid (\lambda_1 + 1)$, since $n = q^4 + q^2 + \lambda_1(q-1) - 1$. Thus $\lambda_1 = \lambda_2 q - 1$, $\lambda_2 \geq 1$, and hence

$$n = q^4 + (\lambda_2 + 1)q^2 - (\lambda_2 + 1)q\,. \tag{10}$$

Since n is odd, $n \leq \sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$ by Lemma 2.4(2), it follows that $n \leq q^4 + 2q^2 + q + 1$. Hence $(\lambda_2 - 1)q^2 - (\lambda_2 + 2)q - 1 \leq 0$ by (10). This yields either $\lambda_2 = 1$, or $\lambda_2 = 2$ and $q = 3$. Thus either $n = q^4 + 2q^2 - 2q$ or $n = 99$ and $q = 3$. Assume the latter occurs. Let S be a Sylow 2-subgroup of $G \cong \mathsf{PSU}(3,3)$. Then $|S| = 2^5$ and hence $|S| \nmid 4(n+1)$ as $n = 99$. A contradiction by [17, Satz 2], since $n \equiv 3 \mod 4$. Thus the assertion (I).

(II) *The final contradiction.*

Let $\mathcal{E}$ be the set of lines of Π which are external to Π_0. Denote by $\mathcal{S}$ and $\mathcal{T}$ the sets of the secants and of tangents to Π_0. Then $\mathcal{S}$ and $\mathcal{T}$ have size q^4+q^2+1 and $(q^4 + q^2 + 1)(n - q^2)$ respectively. Now it is a straightforward computation to see that $|\mathcal{E}| = 2q^2\,(q-1)^2\,\left(q^2+q+2\right)$, since $|\mathcal{E}| = n^2 + n + 1 - (|\mathcal{S}| + |\mathcal{T}|)$ and $n = q^4 + 2q^2 - 2q$.

Since the involutions in G are perspectivities of Π and of Π_0, then G_e has odd order for each line $e \in \mathcal{E}$. Now assume there exists a line x in $\mathcal{E}$ such that $(|G_x|, \frac{q+1}{j}) = 1$, $j = (q+1, 3)$, and let γ be an element of order a prime divisor of $\frac{q+1}{j}$. Then $\langle\gamma\rangle$ induces an (A,l)-homology group on Π_0 by [23]. In particular $|\mathrm{Fix}(\gamma) \cap \Pi_0 \cap l| = q^2 + 1$. Nevertheless $\langle\gamma\rangle$ is planar on Π, since $\langle\gamma\rangle$ fixes x in $\mathcal{E}$. Note that x intersects $l - \Pi_0$ again since $x \in \mathcal{E}$. Thus $\langle\gamma\rangle$ fixes at least $q^2 + 2$ points on l and hence $o(\mathrm{Fix}(\langle\gamma\rangle)) \geq q^2 + 1$ as γ is planar on Π. Arguing as above, with $\langle\gamma\rangle$ in the role of $\langle\tau\rangle$, we have $o(\mathrm{Fix}(\langle\gamma\rangle)) = q^2 + 1$. Now, let σ be an involutions centralizing $\langle\gamma\rangle$ (we can pick σ in the same cyclic subgroup of G of order $\frac{q+1}{j}$ containing $\langle\gamma\rangle$). Then σ acts non-trivially on $\mathrm{Fix}(\langle\gamma\rangle)$, since G_x has odd order and $x \in \mathrm{Fix}(\langle\gamma\rangle) \cap \mathcal{E}$. At this point the above argument involving [24, Theorem 13.18] rules out this case. Hence $\left(|G_e|, \frac{q+1}{j}\right) = 1$, $j = (q+1, 3)$, for each line $e \in \mathcal{E}$. That is $\frac{(q+1)^2}{j} \mid \left|e^G\right|$ for each line $e \in \mathcal{E}$ (see [38]). Therefore $\frac{(q+1)^2}{j} \mid 2q^2\,(q-1)^2\,\left(q^2+q+2\right)$, since $\mathcal{E}$ consists of non-trivial G-orbits and $|\mathcal{E}| = 2q^2\,(q-1)^2\,\left(q^2+q+2\right)$. Actually $\frac{(q+1)^2}{j} \mid 16$, as $(\frac{q+1}{j}, q-1) = (\frac{q+1}{j}, q^2+q+2) = 2$. Hence $(q+1)^2 = 2^i j$, where $1 \leq i \leq 4$ and $j = (q+1, 3)$. Thus $j = 1$ and hence $q \equiv 1 \mod 3$. At this point, easy computations show that no value of q is admissible. This completes the proof. $\square$

Lemma 3.7. *Let Π be a finite projective plane of order n and let $G \cong \mathsf{PSU}(3,q)$ be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathcal{H}(q)$, $q > 2$. If $n \leq \sum(\mathcal{O})$, then G contains involutory perspectivities and G is strongly irreducible on a G-invariant subplane Π_0 of Π containing $\mathcal{O}$. In particular, one of the following occurs:*

(1) $n = q^2$, $\Pi_0 = \Pi \cong \mathsf{PG}(2, q^2)$;

(2) $n = q^4$, $\Pi_0 \cong \mathsf{PG}(2, q^2)$ *is a Baer subplane of* Π.

Proof. In order to obtain the assertion we have to investigate only the case $q = 2^{2h}$, since the assertion is true for q odd by Lemma 3.6. Hence, assume that $q = 2^{2h}$. Recall that the involutions in G are elations of Π by [20, Proposition 6.1].

Assume $n \geq 2^{4h} + 2^{2h}$. Recall that $n \leq \sum(\mathcal{O}) = 2^{4h} + 2^{2h+1} + 2^h + 2$. If $n = 2^{4h} + 2^{2h+1} + 2^h + 2$, then $n \equiv 2 \mod 4$ as $h > 1$. A contradiction by [24, Theorem 13.18]. Then $n \leq 2^{4h} + 2^{2h+1} + 2^h$, since n is even. We prove that this leads to a contradiction in five steps.

(I) *Let* $C \cong Z_{(2^h+1)/j}$, $j = (2^h + 1, 3)$. *If there exists* $\delta \in C$, $\delta \neq 1$, *which is planar on* Π, *then* $o(\mathrm{Fix}(\delta)) = 2^{2h}$ *and hence* $|\mathrm{Fix}(\delta) \cap \Pi_0 \cap c| = 2^{2h} + 1$.

Let $C \cong Z_{(2^h+1)/j}$, where $j = (2^h + 1, 3)$. Then $N_G(C)/C = \mathsf{PSL}(2, 2^h)$ by [23]. Clearly $N_G(C)$ is the minimal such that $N_G(C)/C \cong \mathsf{PSL}(2, 2^h)$. Since C is abelian and $C \lhd N_G(C)$, then $C \trianglelefteq C_{N_G(C)}(C) \trianglelefteq N_G(C)$. As $N_G(C)/C \cong \mathsf{PSL}(2, 2^h)$, then either $C_{N_G(C)}(C) = C$ or $C_{N_G(C)}(C) = N_G(C)$. If the former occurs, then $N_G(C)/C = N_G(C)/C_{N_G(C)}(C) \leq \mathrm{Aut}(C)$. A contradiction, since C is cyclic while $N_G(C)/C = \mathsf{PSL}(2, 2^h)$. So $C_{N_G(C)}(C) = N_G(C)$. That is $C \leq Z(N_G(C))$. Furthermore $N_G(C)' = N_G(C)$ by the minimality of this one. Hence $N_G(C)$ is perfect central extension of $\mathsf{PSL}(2, 2^h)$ by C. Then $N_G(C) = C \times L$, where $L \cong \mathsf{PSL}(2, 2^h)$ by [32, Theorem 5.1.4].

Assume there exists $\delta \in C$, $\delta \neq 1$, which is planar on Π. Then $o(\mathrm{Fix}(\delta)) \geq 2^{2h}$, since C induces a group of homologies on $\Pi_0 \cong \mathsf{PG}(2, 2^{2h})$ having the same center X and the same axis l by [23]. Clearly L acts on $\mathrm{Fix}(\delta)$, since L centralizes C. In particular $L \cong \mathsf{PSL}(2, 2^h)$ has two orbits on $\mathrm{Fix}(\delta) \cap \Pi_0 \cap l$ of length $2^{2h} + 1$ and $2^{2h} - 2^h$ respectively (see [23]). If $o(\mathrm{Fix}(\delta)) > 2^{2h}$, then $o(\mathrm{Fix}(\delta)) = 2^{2h} + 1$ by [24, Theorem 3.7], since $n \leq 2^{4h} + 2^{2h+1} + 2^h$. Then L fixes the point D, where $\{D\} = \mathrm{Fix}(\delta) \cap l - \Pi_0$. Thus any Sylow 2-subgroup of L fixes exactly 2 points on $\mathrm{Fix}(\delta) \cap l$, namely D and a point lying in the L-orbit of length $2^h + 1$. Therefore any Sylow 2-subgroup of L, which is elementary abelian of order 2^h, induces a group of homologies of $\mathrm{Fix}(\delta)$. Note that L fixes X, since L centralizes C and C induces a group of homologies on $\Pi_0 \cong \mathsf{PG}(2, 2^{2h})$ having the same center X and the same axis l. Hence L fixes the lines l and XD and the points X and D on $\mathrm{Fix}(\delta)$. Thus distinct Sylow 2-subgroups of L are homology groups having either the same center and distinct axes, or distinct centers and the same axis. Indeed, distinct Sylow 2-subgroups of L fixes distinct points on the L-orbit on $l \cap \Pi_0$ of length $q + 1$ (exactly one per each Sylow 2-subgroup of L). Then the Sylow 2-subgroups of L are Frobenius complements by [24, Theorem 4.25]. Then $h = 1$ and $q = 2$, since they must be cyclic by [44, Proposition 18.1.(i)]. A contradiction, since $q > 2$ by our assumptions. Thus $o(\mathrm{Fix}(\delta)) = 2^{2h}$ and hence assertion (I).

(II) *We have*

$$n = 2^{4h} + 2^{2h} + 2^{h+t}\lambda_1$$

where $0 \leq \lambda_1 \leq 2^{h-t}$ *and* $t = \lfloor (h+1)/2 \rfloor$.

Let Z be the center of a Sylow 2-subgroup Q of G. Then Z is an elementary abelian group of order 2^h and $Z = Z(A, c)$ for some point $A \in \mathcal{O}$ and some line c of Π_0 which is tangent to $\mathcal{O}$ in A by Lemma 3.4 and by [23]. Thus $2^h \mid n$. Then $n = 2^h\lambda$ for some positive integer $\lambda \geq 2^{3h} + 2^h$ as $n \geq 2^{4h} + 2^{2h}$. Thus $\lambda = 2^{3h} + 2^h + \lambda_0$ with $\lambda_0 \geq 0$. Hence $n = 2^{4h} + 2^{2h} + 2^h\lambda_0$, $\lambda_0 \geq 0$.

Assume $\lambda_0 \geq 1$. Since Z is semiregular on $\Pi - c$, and since each involution in Q lies in Z as $q = 2^h$, then also Q is semiregular on $\Pi - c$. So $2^{3h} \mid n^2$, since $|Q| = 2^{3h}$. Then $2^{3h} \mid 2^{2h}\lambda_0^2$ and hence $2^h \mid \lambda_0^2$, since $n = 2^{4h} + 2^{2h} + 2^h\lambda_0$. Thus $\lambda_0 = 2^t\lambda_1$, where $\lambda_1 \geq 1$ and $t = \lfloor (h+1)/2 \rfloor$ (note that λ_1 even number is also admissible). Hence $n = 2^{4h}+2^{2h}+2^{h+t}\lambda_1$, $\lambda_1 \geq 1$. As $n \leq 2^{4h}+2^{2h+1}+2^h$ by the above argument, and being $n = 2^{4h} + 2^{2h} + 2^{h+t}\lambda_1$, we have $2^{h+t}\lambda_1 \leq 2^{2h} + 2^h$. That is $2^t\lambda_1 \leq 2^h + 1$ and hence $1 \leq \lambda_1 \leq 2^{h-t}$, since λ_1 is a positive integer and $t = \lfloor (h+1)/2 \rfloor$.

Assume that $\lambda_0 = 0$. Then it is easy to determine that $n = 2^{4h} + 2^{2h}$, which is assertion (II) for $\lambda_1 = 0$.

(III) *Let* $\mathcal{E}$ *be the set of lines of* Π *which are external to* Π_0. *Then*

$$|\mathcal{E}| = 2^{6h} + 2^{5h+t}\lambda_1 + 2^{3h+t}\lambda_1 + 2^{2h+2t}\lambda_1^2\,,$$

where $0 \leq \lambda_1 \leq 2^{h-t}$ *and* $t = \lfloor (h+1)/2 \rfloor$.

Let $\mathcal{E}$ be the set of lines of Π which are external to Π_0. Denote by $\mathcal{S}$ and $\mathcal{T}$ the sets of the secants and of tangents to Π_0, respectively. Then $\mathcal{S}$ and $\mathcal{T}$ have size $2^{4h} + 2^{2h} + 1$ and $(2^{4h} + 2^{2h} + 1)(n - 2^{2h})$, respectively. Therefore $|\mathcal{S}| + |\mathcal{T}| = (2^{4h} + 2^{2h} + 1)(n + 1 - 2^{2h})$ and we obtain

$$|\mathcal{S}| + |\mathcal{T}| = (2^{4h} + 2^{2h})n + (n+1) - 2^{6h}\,. \tag{11}$$

As $|\mathcal{E}| = n^2+n+1-(|\mathcal{S}|+|\mathcal{T}|)$ and by bearing in mind that $n = 2^{4h}+2^{2h}+2^{h+t}\lambda_1$, $\lambda_1 \geq 0$, we have

$$|\mathcal{E}| = (2^{4h} + 2^{2h})n + 2^{h+t}\lambda_1 n + (n+1) - (|\mathcal{S}| + |\mathcal{T}|)\,. \tag{12}$$

Now, by combining (11) and (12), we have

$$|\mathcal{E}| = 2^{h+t}\lambda_1 n + 2^{6h}\,. \tag{13}$$

As $n = 2^{4h} + 2^{2h} + 2^{h+t}\lambda_1$, by (13) we obtain

$$|\mathcal{E}| = 2^{6h} + 2^{5h+t}\lambda_1 + 2^{3h+t}\lambda_1 + 2^{2h+2t}\lambda_1^2\,. \tag{14}$$

(IV) *Each G-orbit in $\mathcal{E}$ has length $\frac{2^{3h}(2^h+1)(2^{2h}-1)}{\mu}$, $\mu = 1$ or 3.*

Observe that $\mathcal{E} = \cup_{i=1}^{y} e_i^G$, $y \geq 1$ and $\left|e_i^G\right| > 0$, since G_{e_i} has odd order for each $1 \leq i \leq y$. Hence

$$|\mathcal{E}| = \sum_{i=1}^{y} \left|e_i^G\right|\,. \tag{15}$$

Set $\left|e^G\right| = \min\left\{\left|e_i^G\right| : 1 \leq i \leq y\right\}$. Then $|\mathcal{E}| \geq y\left|e^G\right|$ by (15). Hence

$$|G_e| \geq \frac{y2^{3h}(2^{3h}+1)(2^{2h}-1)}{j(2^{6h} + 2^{5h+t}\lambda_1 + 2^{3h+t}\lambda_1 + 2^{2h+2t}\lambda_1^2)}\,, \tag{16}$$

by (II) and since $|G| = 2^{3h}(2^{3h}+1)(2^{2h}-1)/j$, $j = (2^h+1, 3)$.

As $\lambda_1 \leq 2^{h-t}$, we have

$$\frac{y2^{3h}(2^{3h}+1)(2^{2h}-1)}{j(2^{6h} + 2^{5h+t}\lambda_1 + 2^{3h+t}\lambda_1 + 2^{2h+2t}\lambda_1^2)} \geq \frac{y2^{3h}(2^{3h}+1)(2^{2h}-1)}{j(2^{6h+1} + 2^{4h+1})}$$

and hence

$$|G_e| \geq \frac{y(2^{3h}+1)(2^{2h}-1)}{2^{h+1}j(2^{2h}+1)}\,. \tag{17}$$

It is easily seen that $|G_e| > \max(9, 2^h - 1)$ as $h > 1$. Now, assume that $\left(|G_e|\,, \frac{2^h+1}{j}\right) \neq 1$. Then there exists a non-trivial element ψ in G_e of order a divisor of $\frac{2^h+1}{j}$. Since the cyclic subgroups of G of order $\frac{2^h+1}{j}$ are conjugate in G (indeed G is transitive on the lines on $\mathcal{O}$), then we may assume that $\psi \in C$. Then ψ is planar with $o(\mathrm{Fix}(\psi)) = 2^{2h}$ and $|\mathrm{Fix}(\psi) \cap \Pi_0 \cap c| = 2^{2h} + 1$ by (I). Nevertheless ψ fixes the external e to Π_0, the secant c to Π_0 and hence the point $e \cap c$. As $e \in \mathcal{E}$ then $e \cap c$ lies in $c - \Pi_0$. Thus ψ fixes at least $q^2 + 2$ points on C contradicting (I). Thus $|G_e|$ is odd, $|G_e| > \max(9, 2^h - 1)$ and $(|G_e|\,, \frac{2^h+1}{j}) = 1$. Then $G_e \leq Z_{\frac{2^{2h}-2^h+1}{j}}.Z_3$ by [16] (note that $(\frac{2^{2h}-2^h+1}{j}, 3) = 1$ by [18, Lemma 3.9]). Now, by using (17), it is plain to see that $|G_e| > \frac{2^{2h}-2^h+1}{3j}$ as $h > 1$. Hence $\frac{2^{2h}-2^h+1}{j} \mid |G_e|$ as $G_e \leq Z_{\frac{2^{2h}-2^h+1}{j}}.Z_3$. Actually, the previous proof can be repeated for each G_{e_i}, $1 \leq i \leq y$, in order to show that $\frac{2^{2h}-2^h+1}{j} \mid |G_{e_i}|$ and $G_{e_i} \leq Z_{\frac{2^{2h}-2^h+1}{j}}.Z_3$. Hence $|G_{e_i}| = \mu_i \frac{2^{2h}-2^h+1}{j}$, $\mu_i = 1$ or 3, for each $1 \leq i \leq y$. Therefore $\left|e_i^G\right| = \frac{2^{3h}(2^h+1)(2^{2h}-1)}{\mu_i}$, $\mu_i = 1$ or 3, for each $1 \leq i \leq y$.

(V) *The final contradiction.*

By (IV), we have $\frac{2^{2h}-1}{3} \mid \left|e_i^G\right|$ for each $1 \leq i \leq y$, since

$$|G| = 2^{3h}(2^{3h}+1)(2^{2h}-1)/j,\ j = (2^h+1, 3)\,.$$

Then $\frac{2^{2h}-1}{3} \mid |\mathcal{E}|$ as $|\mathcal{E}| = \sum_{i=1}^{y} \left|e_i^G\right|$. Then

$$\frac{2^{2h}-1}{3} \;\middle|\; \left(2^{6h} + 2^t\lambda_1(2^{5h} + 2^{3h} + 2^{2h+t}\lambda_1)\right)$$

by (III). Easy computations yield $\frac{2^{2h}-1}{3} \mid (1+2^t\lambda_1(2^{h+1}+2^t\lambda_1))$. Hence $f\frac{2^{2h}-1}{3} = 1 + 2^t\lambda_1(2^{h+1}+2^t\lambda_1)$ with $f \geq 1$, $h > 1$, $t = \lfloor (h+1)/2 \rfloor$ and $1 \leq \lambda_1 \leq 2^{h-t}$. A contradiction by Lemma 2.7.

We conclude that $n = q^4$. □

Lemma 3.8. *Let Π be a finite projective plane of order n and let $G \cong {}^2\mathsf{G}_2(q)$, $q = 3^h$, h odd, $h \geq 1$, be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathcal{R}(q)$. If $n \leq \sum(\mathcal{O})$, then $h = 1$, $G \cong \mathsf{P\Gamma L}(2,8)$, $\mathcal{O} \cong \mathcal{R}(3)$ and one of the following occurs:*

(1) $\Pi \cong \mathsf{PG}(2,8)$*, G leaves a line oval $\mathcal{C}$ of Π invariant and $\mathcal{O}$ consists of the external points of $\mathcal{C}$;*

(2) $n = 2^6$.

Proof. Note that $n \leq \sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$, where $q = 3^h$ and h is odd, $h \geq 1$ by Lemma 2.4(3). Assume that $n > q^4$. Then $q^4 < n < (q^2+2)^2$, since $\sum(\mathcal{O}) < (q^2+2)^2$. Let ψ be any involution of $G \cong {}^2\mathsf{G}_2(q)$. If ψ is a Baer involution of Π, then n is square. Then $n = (q^2+1)^2$, since $q^4 < n < (q^2+2)^2$. Note that $C_G(\psi) = \langle\psi\rangle \times \mathsf{PSL}(2,q)$ by [35]. Furthermore $\mathsf{PSL}(2,q)$ acts non-trivially on $\mathrm{Fix}(\psi)$, since $\mathrm{Fix}(\psi) \cap \mathcal{O}$ contains a line $s \cap \mathcal{O}$, s line of Π, and $\mathsf{PSL}(2,q)$ acts on $s \cap \mathcal{O}$ in its natural 2-transitive permutation representation of degree $q+1$ (see [35]). So $\mathsf{PSL}(2,q)$ acts faithfully on $\mathrm{Fix}(\psi)$. A contradiction by [24, Theorem 13.18], since $\sqrt{n} \equiv 2 \mod 4$ and $\sqrt{n} > 2$, being $\sqrt{n} = q^2 + 1$ and $q = 3^h$, $h \geq 1$. Thus ψ is a perspectivity of Π.

Assume $h > 1$. Then $G = G(Q,l)$ by [20, Lemma 4.3] and $|G| \mid n$. That is $q^3(q^3+1)(q-1) \mid n$. A contradiction, since $n \leq (q^2+1)^2 + (q+1)$.

Assume $h = 1$. Then $G \cong \mathsf{P\Gamma L}(2,8)$. Let S be any Sylow 2-subgroup of G. If n is odd, then each involution in S is a homology of Π. Then $S = S(P,l)$ by [29, Lemma 3.1], since $S \cong E_8$, the elementary abelian group of order 8. A contradiction, since distinct involutions in G fix distinct blocks of $\mathcal{R}(3)$ pointwise by [35]. Thus n is even and either $S = S(X,X)$ for some point $X \in \Pi - \mathcal{R}(3)$, or $S = S(a, a)$ for some external line a to $\mathcal{R}(3)$, since $S \cong E_8$. Actually, the latter is ruled out by the above argument. Hence $S = S(X, X)$. In particular the axis of each elation in S is a secant of $\mathcal{R}(3)$ by [35]. Let U be another Sylow 2-subgroup of G. Then $U = U(Y, Y)$ for some point Y of $\Pi - \mathcal{R}_1$ by the above argument with U in the role of S. Clearly $G' = \langle S, U\rangle \cong \mathsf{PSL}(2, 8)$. If $X = Y$, then $G' = G'(X, X)$ which is abelian by [24, Theorem 4.14], since distinct involutions in G have distinct axes which are secants of $\mathcal{R}(3)$. A contradiction, since $G' \cong \mathsf{PSL}(2, 8)$. Hence $X \neq Y$. Then S and U fix the external line $e = XY$ to $\mathcal{R}_1$. Hence $G' = \langle S, U\rangle$ fixes e. Clearly $X^{G'} \subseteq e$ and $\left|X^{G'}\right| = 9$, since $G'_X \cong E_8.Z_7$. As $n > 81$, we have $X^{G'} \subsetneq e$. Then G'_Q has odd order for each $Q \in e - X^{G'}$, since each involution in G' is an elation of center in $X^{G'}$ and axis a secant to $\mathcal{R}_1$. Hence $e - X^{G'}$ is union of non-trivial G'-orbits of length divisible by 2^3, since $2^3 \mid |G'|$. Then each of these orbits must have length divisible by either 56 or 72 by [1], since $\left|e - X^{G'}\right| \leq 72$. Consequently, since $81 < n < 104$, we have that $n - 8 = n + 1 - 9$ must be divisible by 56 or by 72, a contradiction. Hence $n \leq q^4$ and the assertion follows from Theorem 2.6. □

Lemma 3.9. *Let Π be a finite projective plane of order n and let $G \cong \mathsf{PSL}(2, 2^h)$, $h \geq 3$, be a collineation group of Π with a point-orbit $\mathcal{O} \cong \mathcal{W}(2^h)$. If $n \leq \sum(\mathcal{O})$, then*

(1) $\Pi \cong \mathsf{PG}(2, 2^h)$, *the projective extension of $\mathcal{O}$ is embedded in Π and the set $\mathcal{C}$ of the external lines to $\mathcal{O}$ is a line-hyperoval extending a line conic, or*

(2) $n = 2^{2h}$, *or*

(3) $n = (2^h + 1)^2$, $h > 3$, *and the involutions in G are Baer collineations of Π.*

Proof. If $h = 3$, then $\mathcal{R}(3) \cong \mathcal{W}(8)$ by [47, Example 1.4], and hence assertions (1) and (2) follow from Lemma 3.8 in this case. Hence we may assume $h > 3$. Let S be a Sylow 2-subgroup of G. Then S is elementary abelian of order 2^h fixing a point P on Π, since $n^2 + n + 1$ is odd. Actually $P \in \Pi - \mathcal{O}$ by [47, Lemma 7.1].

Assume that each involution in S is a Baer collineation of Π (recall that there exists a unique conjugate class of involutions in G). Then each involution in S fixes exactly $\sqrt{n} + 1$ lines through P. Then

$$2^h \mid \left[n + 1 + (2^h - 1)(\sqrt{n} + 1)\right]$$

by [24, Result 1.4]. Hence $2^h \mid (n - \sqrt{n})$. Thus either $2^h \mid (\sqrt{n} - 1)$ or $2^h \mid \sqrt{n}$. Then either $\sqrt{n} = 2^h$ or $\sqrt{n} = 2^h + 1$ or $\sqrt{n} \geq 2^{h+1}$. On the other hand, $n \leq \sum(\mathcal{O}) = 2^h(2^{h-1}3 + 1)$ by Lemma 2.4(4). Thus the case $\sqrt{n} \geq 2^{h+1}$ is ruled out. Hence either $\sqrt{n} = 2^h$ or $\sqrt{n} = 2^h + 1$ and we have assertions (2) and (3), respectively.

Now, assume that each involution in S is a perspectivity of Π. By [10], the group S fixes a pencil Φ of parallel lines in $\mathcal{O}$ elementwise. Furthermore, for each line s in Φ there exists a unique non-trivial element in S fixing $s \cap \mathcal{O}$ pointwise. Hence $|\Phi| = 2^h - 1$. Moreover $S = S(Q,Q)$ for some Q on $\Pi - \mathcal{O}$ by [29, Lemma 3.1], since S is abelian of order at least 8. Since $n > 2^h - 1$, there exists a line $a \in [Q]$ which is external to $\mathcal{O}$. Then S is semiregular on $a - \{Q\}$, since the axis of each involution in S is a secant to $\mathcal{O}$. Thus $2^h \mid n$. Now, let R be another Sylow 2-subgroup of G. Then $R = R(C,C)$ by arguing as above with R in the role of S. Then $G = \langle R, S\rangle$ fixes the line QC, since both S and R fix QC. Then the projective extension of $\mathcal{O}$ is embedded in Π, since $Q^G \subseteq QC$, the point Q is the center of a pencil of parallel lines in $\mathcal{O}$ and G is flag-transitive on $\mathcal{O}$. Then either $n+1 = 2^h+1$, or $n = 2^{h-1}(2^h-1)$, or $n \geq 2^{2h-1}$ by [13, Proposition 2.1], since $v = 2^{h-1}(2^h-1)$. Actually the case $n = 2^{h-1}(2^h-1)$ is ruled out, since $2^h \mid n$. Hence either $n = 2^h$ or $n \geq 2^{2h-1}$ and $2^h \mid n$.

Assume that $n = 2^h$. Then $G \cong \mathsf{PSL}(2,2^h)$ is 2-transitive on QC, since $Q^G = QC$ and $\left|Q^G\right| = 2^h+1$. Then $\Pi \cong \mathsf{PG}(2,2^h)$, by [36, Satz 3]. It is easily seen that there is exactly one external line to $\mathcal{O}$ through each point of QC, other than QC, since each Sylow 2-subgroup of G fixes a pencil of parallel lines in $\mathcal{O}$ each of size 2^h-1 by [10]. Let $\mathcal{C}$ be the set of these external lines to $\mathcal{O}$. Then $|\mathcal{C}| = 2^h+1$, since G is 2-transitive on $\mathcal{C}$. Thus either there exists a point $X \in \Pi - QC$ such that $\mathcal{C} \subset [X]$, or $\mathcal{C}$ is a line conic of Π. If $\mathcal{C} \subset [X]$, then G fixes X and hence $S = S(Q,QX)$. A contradiction, since S consists of elations having the same center Q but distinct axes which are secants to $\mathcal{O}$. Hence $\mathcal{C}$ is a line conic and $\mathcal{C} \cup \{QC\}$ is a line-hyperoval of Π extending $\mathcal{C}$. That is assertion (1).

Assume that $n \geq 2^{2h-1}$ and $2^h \mid n$. Then $Q^G \subsetneq QC$. Let B be a point on QC which is not the center of any pencil of parallel lines to $\mathcal{O}$. Then G_B must have odd order. Hence $G_B \leq Z_{2^h \pm 1}$ by direct inspection of the list of subgroups of G given in [25, Haupsatz 8.27]. Furthermore $\left|B^G\right| \leq 2^{2h-1}3$, since $n \leq 2^h(2^{h-1}3+1)$ and $\left|Q^G\right| = 2^h+1$. Thus $|G_B| \geq (2^{2h}-1)/2^{h-1}3$ and hence $G_B \cong Z_{2^h \pm 1}$, as $G_B \leq Z_{2^h \pm 1}$. That is $\left|B^G\right| = 2^h(2^h \mp 1)$. Let x_- and x_+ be the number of G-orbits on $QC - Q^G$ of length $2^h(2^h-1)$ and $2^h(2^h+1)$ respectively. Then

$$2^h(2^h-1)x_- + 2^h(2^h+1)x_+ = n - 2^h\,, \tag{18}$$

since $\left|QC - Q^G\right| = n - 2^h$. In particular $(x_- + x_+)2^h(2^h-1) \leq 2^{2h-1}3$ by (18), since $n \leq 2^h(2^{h-1}3+1)$. If $x_- + x_+ \geq 2$, then $2^{h+1}(2^h-1) \leq 2^{h-1}3$. A contradiction. Thus $x_- + x_+ = 1$ and hence either $(x_-, x_+) = (1,0)$ or $(x_-, x_+) = (0,1)$. Assume the latter occurs. Then $2^h(2^h+1) = n - 2^h$ by (18) and hence $n = 2^{2h} + 2^{h+1}$. So, QC consists of two G orbits of length 2^h+1 and $2^h(2^h+1)$. Then the group $G_B \cong Z_{2^h-1}$ fixes exactly 4 points on QC, namely two in each G-orbit on QC. Furthermore there exists a subgroup G_B^* of G_B, such that $[G_B : G_B^*] \leq 3$, fixing at least 4 lines, including QC, through B, since $(n, 2^h - 1) = 3$ and being $n = 2^{2h} + 2^{h+1}$. Let η be any involution in G normalizing G_B. Then η normalizes G_B^*, since $G_B \cong Z_{2^h-1}$. In particular η moves B as $\left|B^G\right| = 2^h(2^h + 1)$. Thus G_B^* is planar and $o(\mathrm{Fix}(G_B^*)) = 3$, since G_B (and hence G_B^*) fixes exactly 4 points on QC. Clearly η acts non-trivially on $\mathrm{Fix}(G_B^*)$. Thus η induces a homology on $\mathrm{Fix}(G_B^*)$ as $o(\mathrm{Fix}(G_B^*)) = 3$. A contradiction, since η is an elation of Π. Thus $(x_-, x_+) = (1, 0)$ and hence $n = 2^{2h}$. That is assertion (2). $\square$

Corollary 3.10. *If case (2) or (3) of Lemma 3.9 occurs and if the involutions in G are Baer collineations of Π when case (2) occurs, then the following hold:*

(1) *Each Sylow 2-subgroup of G induces a group of perspectivities having the same center and the same axis on the Baer subplane fixed by any one of its involution;*

(2) *If $n = (2^h + 1)^2$ is a prime power, then n is actually the square of a Fermat prime.*

Proof. Assume either $n = 2^{2h}$ or $n = (2^h + 1)^2$, and that the involutions in G are Baer collineations of Π. We may also assume $h > 3$ by Lemmas 3.8 and 3.9. Let S be a Sylow 2-subgroup of G. Then S is elementary abelian of order 2^h. As $G_P \cong D_{2(2^h+1)}$ for each $P \in \mathcal{O}$, then $\mathrm{Fix}(\alpha) \neq \mathrm{Fix}(\beta)$ for $\alpha, \beta \in S - \{1\}$, with $\alpha \neq \beta$. In particular $\mathrm{Fix}(S) \subset \mathrm{Fix}(\sigma)$ for each $\sigma \in S - \{1\}$. Furthermore, since $b = 2^{2h} - 1$ and since G is transitive on the lines of $\mathcal{O}$, the group S fixes exactly $2^h - 1$ lines of $\mathcal{O}$.

Assume that β induces a Baer involution on $\mathrm{Fix}(\alpha)$. Then $\sqrt{n}$ must be a square. If $\sqrt{n} = 2^h + 1$, then $(\sqrt[4]{n} - 1)(\sqrt[4]{n} + 1) = 2^h$. As $(\sqrt[4]{n} - 1, \sqrt[4]{n} + 1) = 2$, then $\sqrt[4]{n} - 1 = 2$ and $\sqrt[4]{n} + 1 = 2^{h-1}$. A contradiction since $h > 3$. Hence $\sqrt{n} = 2^h$ and h is even. Note that S acts on the plane $\mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$ fixing at least $\sqrt{n} - 1 = 2^h - 1$ lines on it. Clearly among these lines there are $2^h - 1$ secants to $\mathcal{O}$. As $\sqrt{n} - 1 > \sqrt[4]{n} + \sqrt[8]{n} + 1$, and since the order of $\mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$ is $\sqrt[4]{n}$ (recall that $\mathrm{Fix}(\alpha) \neq \mathrm{Fix}(\beta)$), we have $\mathrm{Fix}(S) = \mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$. Let l be any line of $\mathrm{Fix}(S)$. Then each non-trivial involution in S fixes exactly $\sqrt{n} - \sqrt[4]{n}$ points on l which are not fixed by any other involution in S. Then S must be semiregular on the non-empty point-set $l - \cup_{\sigma \in S - \{1\}} (\mathrm{Fix}(\sigma) \cap l)$. Hence

$$2^h \mid (n + 1 - (2^h - 1)(\sqrt{n} - \sqrt[4]{n}) - (\sqrt[4]{n} + 1)).$$

Hence $2^h \mid (n + \sqrt{n} - 2\sqrt[4]{n})$. As $\sqrt{n} = 2^h$ and h is even, we have $2^h \mid 2^{\frac{h}{2}+1}$. This yields $h = 2$. A contradiction since $h > 3$. Thus each non-trivial element in S induces a perspectivity on $\mathrm{Fix}(S)$. Actually, as S fixes exactly $2^h - 1 > 3$ lines of $\mathcal{O}$ and since S is a abelian, the group S induces the group $S/\langle\alpha\rangle$ on $\mathrm{Fix}(\alpha)$ consisting of perspectivities having the same center C and the same axis a. We have thus assertion (1).

Now, assume that $n = (2^h + 1)^2$ is a prime power, then $2^h + 1 = u^j$ for some prime u and some $j \geq 1$. Then $j = 1$ and u is a Fermat prime by [45, Result (B1.1)], as $h > 3$. We have thus assertion (2). □

Example 3.11. *Let $\Pi_0 \cong \mathsf{PG}(2, 2^h)$ be a subplane of $\Pi \cong \mathsf{PG}(2, 2^{2h})$ and let $\mathcal{C}$ be a line hyperoval of Π_0 consisting of a line-conic $\mathcal{C}_0$ and an additional line l. Clearly $H_{\mathcal{C}} \cong \mathsf{PSL}(2, 2^h)$ fixes l and acts 2-transitively on $\mathcal{C}_0$. By [10], the external lines to $\mathcal{C}$ and the points of $\Pi - \mathcal{C}$ define an incidence structure $\mathcal{O}$ which is left invariant by $H_{\mathcal{C}}$. In particular $\mathcal{O} \cong W(2^h)$ and $H_{\mathcal{C}}$ acts flag-transitively on $\mathcal{O}$.*

The first part of Theorem 1.1 is now a consequence of the results in this section.

4 The non-faithful action

Throughout this section we assume that $N \neq \langle 1 \rangle$. Then N is planar on Π, since $\mathcal{O} \subseteq \mathrm{Fix}(N)$ and $\mathcal{O}$ is a non-trivial 2-$(v, k, 1)$ design. We may also assume that G is minimal with respect to the property that G/N is flag transitive on $\mathcal{O}$.

Lemma 4.1. *Then $N = \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G.*

Proof. Let S be a Sylow t-subgroup of N. Then $S \lhd G$ by the minimality of G, since $G = N_G(S)N$ by the Frattini's argument. Thus N is nilpotent. Suppose that $N \nleq \Phi(G)$. Then there exists a maximal subgroup H of G such that $G = NH$ by [25, Satz 3.2(b)]. Clearly H is flag-transitive on $\mathcal{O}$. Furthermore $H < G$ and $\frac{H}{H \cap N} \cong \frac{G}{N}$. A contradiction by the minimality of G. Hence $N \leq \Phi(G)$. Note that G_P is maximal in G for each point $P \in \mathcal{O}$, since $N \lhd G_P$ and since G/N is primitive on $\mathcal{O}$ by [21]. Hence $\Phi(G) \lhd G_P$ for each point $P \in \mathcal{O}$. Thus $N = \Phi(G)$. □

Lemma 4.2. *If G/N is non-abelian simple, then one of the following holds:*

(1) *G is a perfect central extension of G/N, or*

(2) *there exists a Sylow t-subgroup S of N such that $G/N \leq \mathsf{SL}(W)$, where $W = S/\Phi(S)$ and $\Phi(S)$ is the Frattini subgroup of S.*

Proof. If $N \leq Z(G)$ we have the assertion (1), since $G = G'$ by the minimality of G. Hence we may assume that $N \nleq Z(G)$. Then there exists a Sylow t-subgroup S of N such that $S \nleq Z(G)$, since N is nilpotent. Set $W = S/\Phi(S)$, where

$\Phi(S)$ is the Frattini subgroup of S. Clearly G acts on W. Let R be the kernel of the action of G on V. If U is the Sylow u-subgroup of N, where u is a prime, $u \neq t$, then $[S,U] = \langle 1 \rangle$, since N is nilpotent. This yields $N \trianglelefteq R \trianglelefteq G$, since $S' \leq \Phi(S)$, S being a t-group. If $R = G$, then each Sylow j-subgroup of G, with $j \neq t$, centralizes S by [15, Theorem 5.1.4]. That is $C_G(S) \nleq N$. Furthermore, $C_G(S) \lhd G$ as $S \lhd G$. Then $N \lhd C_G(S)N \trianglelefteq G$. Hence $G = C_G(S)N$, since G/N is non-abelian simple and since $C_G(S) \nleq N$. Actually, $G = C_G(S)$ since $N = \Phi(G)$ by Lemma 4.1. A contradiction, since $S \nleq Z(G)$. Hence $R < G$. Then $R = N$ as G/N is non-abelian simple. Then $G/N \leq \Gamma\mathsf{L}(W)$, since W is a vector space over $\mathsf{GF}(t)$. Actually $G/N \leq \mathsf{SL}(W)$, since G/N is non-abelian simple. Thus we have assertion (2). $\square$

Corollary 4.3. *In case (2) of Lemma 4.2, we have $|S| \geq 1 + d_0(G/N)$, where $d_0(G/N)$ is the minimal primitive permutation representation of G/N.*

Proof. By Lemma 4.2(2) there exists a Sylow t-subgroup S of N such that $G/N \leq \mathsf{SL}(W)$, where $W = S/\Phi(S)$ and $\Phi(S)$ is the Frattini subgroup of S. In particular $G/N \leq \mathsf{PSL}(W)$ as G/N is non-abelian simple. Then $|\mathsf{PG}(W)| \geq d_0(G/N)$, where $d_0(G/N)$ denotes the minimal primitive permutation representation of G/N. This yields $|W| \geq 1+d_0(G/N)$ and hence $|S| \geq 1+d_0(G/N)$. $\square$

It should be pointed out that the lower bound for $|S|$ given in the previous corollary is not the best one. Indeed, stronger inequalities are given in Theorem 5.3.9 and Proposition 5.4.11 of [32].

Lemma 4.4. *Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O}$ of length v. Assume that $\mathcal{O}$ has the structure of a non-trivial 2-$(v,k,1)$ design, the group G induces a flag-transitive and almost simple automorphism group on $\mathcal{O}$ and $n \leq \sum(\mathcal{O})$. If G does not act faithfully on $\mathcal{O}$, then one of the following holds:*

(1) $n \geq q^4$, $\mathcal{O} \cong \mathcal{H}(q)$, $q > 2$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ *and* $G/N \cong \mathsf{PSU}(3,q)$*;*

(2) $n \geq q^2$, $\mathcal{O} \cong \mathcal{W}(q)$, $q = 2^{2h}$, $h \geq 3$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$ *and* $G/N \cong \mathsf{PSL}(2,q)$*;*

(3) Π *is the generalized Hughes plane over the exceptional nearfield of order* 7^2*,* $\mathcal{O} \cong \mathsf{PG}(2,7)$ *and* $\mathsf{SL}(3,7) \leq G$*;*

(4) $n > q^3$, $q = 2$ *or* 3, $\mathcal{O} = \mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$ *and* $G/N \cong \mathsf{PSL}(3,q)$.

Proof. Assume that $\mathcal{O} \subsetneq \mathrm{Fix}(N)$. Clearly $\mathrm{Fix}(N)$ is a proper subplane of Π, since $N \neq \langle 1 \rangle$. Clearly G/N acts faithfully on $\mathrm{Fix}(N)$. If m is the order of $\mathrm{Fix}(N)$, then from [24, Theorem 3.7] and our assumption we obtain

$$m \leq \sqrt{n} \leq \sqrt{\sum(\mathcal{O})}\,. \tag{19}$$

If $\mathcal{O} \cong \mathsf{PG}(d,q)$, $d \geq 2$, then $q < m \leq \sqrt{\sum(\mathcal{O})}$ (note that the first inequality follows from the fact that $\mathcal{O} \subsetneq \mathrm{Fix}(N)$). Now, we may use the same argument of Lemma 3.2 involving transvections and Theorem 3.7 of [24] in order to obtain

$$\left(\frac{q^d-1}{q-1}-1\right)^2 \leq m \leq \sqrt{\sum(\mathcal{O})}\,. \tag{20}$$

Using Lemma 2.4(1), we obtain

$$\frac{(q^d-q)^4}{(q-1)^4} \leq \frac{q^{2d+1}+q^{d+3}+q^{d+2}-2q^{d+1}-2q^d+q^4-4q^2+4}{(q^2-1)\,(q-1)}\,. \tag{21}$$

Now, arguing as in the first part of Lemma 3.2, we reduce to the case $d=2$ or 3. These values are ruled out, since they do not satisfy (21) as $(d,q) \neq (2,2),(2,3)$.

Assume either $\mathcal{O} \cong \mathcal{H}(q)$, $q>2$, or $\mathcal{O} \cong \mathcal{R}(q)$, $q=3^{2h+1}$. Then $m \leq \sqrt{q^4+2q^2+q+2}$ by (19) and by Lemma 2.4(2)–(3). This yields $m < q^2+2$. Then, by the first part of Theorem 1.1, with $\mathrm{Fix}(N)$ in the role of Π, we have $\mathcal{O} \cong \mathcal{H}(q)$, $q>2$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ and by $G/N \cong \mathsf{PSU}(3,q)$, since $N \neq \langle 1 \rangle$ and $m < q^2+2$. That is assertion (1).

Assume $\mathcal{O} \cong \mathcal{W}(q)$, $q=2^h$, $h \geq 3$. Then $m \leq \sqrt{q(3q+1)/2}$ by (19) and Lemma 2.4(2). This yields $m < q^2$. Then $m=q$ and hence $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$ by the first part of Theorem 1.1. Thus we obtain assertion (2).

Finally, assume $\mathcal{O} = \mathrm{Fix}(N)$. Then $\mathcal{O}$ is symmetric and hence $\mathcal{O} = \mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$ by Theorem 2.2. Therefore $G/N \cong \mathsf{PSL}(3,q)$, since G/N is flag-transitive on $\mathcal{O}$ and G is almost simple. If $n \leq q^3$, then $\mathcal{O} \cong \mathsf{PG}(2,7)$, Π is the generalized Hughes plane over the exceptional nearfield of order 7^2 and $\mathsf{SL}(3,7) \leq G$ by Theorem 2.5. That is assertion (3). If $n > q^3$, then $q=2$ or 3, since $n \leq \sum(\mathcal{O}) = 2q^2+4q+4$ by our assumption and Lemma 2.4(1). Therefore we have assertion (4). $\square$

Lemma 4.5. *The case $n > q^3$, $q=2$ or 3, $\mathcal{O} = \mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$ and $G/N \cong \mathsf{PSL}(3,q)$ does not occur.*

Proof. Assume $\mathcal{O} = \mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$, $q=2$ or 3, $G/N \cong \mathsf{PSL}(3,q)$ and $n > q^3$. If $q=3$, then $28 \leq n \leq 34$, since $q^3 < n < \sum(\mathcal{O}) = 2q^2+4q+4$ by Lemma 2.4(1). Thus n cannot be a square and hence N has odd order. Furthermore, the involutions in G are homologies of Π, since they induce homologies on $\mathcal{O} \cong \mathsf{PG}(2,3)$. At this point we may use the same argument of Lemma 3.3 to rule out this case. Hence $q=2$. Then $9 \leq n \leq 20$.

Assume there exists $\delta \in N$, $\delta \neq 1$, such that $\mathrm{Fix}(N) \subsetneq \mathrm{Fix}(\delta)$. Then either $o(\mathrm{Fix}(\delta)) = 4$ or $o(\mathrm{Fix}(\delta)) \geq 6$ by [24, Theorem 3.7]. If the latter occurs, then $n \geq 36$ again by [24, Theorem 3.7]. A contradiction, since $n \leq 20$. Hence $o(\mathrm{Fix}(\delta)) = 4$, then $n=16$ by [46], since $n \leq 20$.

Now consider $N(\delta) = \{\alpha \in N\colon \operatorname{Fix}(\alpha) = \operatorname{Fix}(\delta)\}$. Then $N(\delta)$ is a non-trivial subgroup of N, since $\operatorname{Fix}(\delta)$ is a Baer subplane of Π. Clearly $N(\delta) = N(\alpha)$ for each $\alpha \in N(\delta)$. Thus we may set $N_0 = N(\delta)$. Clearly $\operatorname{Fix}(N_0) = \operatorname{Fix}(\delta)$ and hence $\operatorname{Fix}(N_0)$ is a Baer subplane of Π. Assume there exists $g \in G$ such that $N_0^g \cap N_0 \neq \langle 1\rangle$. Thus $\operatorname{Fix}(N_0) = \operatorname{Fix}(N_0^g)$, since $\operatorname{Fix}(N_0)$ and hence $\operatorname{Fix}(N_0^g)$ are Baer subplanes of Π. Then $N_0 = N_0^g$. Hence distinct conjugates of N_0 in G have trivial intersection and the corresponding Baer subplanes intersect exactly in $\operatorname{Fix}(N)$.

Now, consider N_0^G and N_0^N. Then $\left|N_0^N\right| \mid \left|N_0^G\right|$, since $N \lhd G$. In particular, it is easy to see that

$$\frac{\left|N_0^G\right|}{\left|N_0^N\right|} = \frac{|G|\,/\,|N_G(N_0)|}{|N|\,/\,|N_N(N_0)|} = \frac{|G/N|}{|N/(N_G(N_0)\cap N)|}\,.$$

Since $N/(N_G(N_0)\cap N) \cong N_G(N_0)N/N$, then

$$\left|N_0^G\right| / \left|N_0^N\right| = [G/N : N_G(N_0)N/N]\,. \tag{22}$$

If $[G/N : N_G(N_0)N/N] = 1$, then $\left|N_0^G\right| = \left|N_0^N\right|$ by (22). Hence $N_0 \lhd G$. In particular $N_0 \lhd N$ and hence N acts as N/N_0 on $\operatorname{Fix}(N_0)$. Since N/N_0 must be semiregular on $u \cap (\operatorname{Fix}(N_0) - \operatorname{Fix}(N_0))$, where u is any secant of $\operatorname{Fix}(N)$, we have $[N : N_0] = 2$ and hence $G/N_0 \cong Z_2.\mathsf{PSL}(2,7)$. That is $G/N_0 \cong \mathsf{SL}(2,7)$. A contradiction by [1], since $G/N_0 \leq \mathsf{P\Gamma L}(3,4)$ being $\operatorname{Fix}(N_0) \cong \mathsf{PG}(2,4)$. Thus $[G/N : N_G(N_0)N/N] > 1$ and hence $[G/N : N_G(N_0)N/N] \geq 7$, since the minimal primitive permutation representation degree of $G/N \cong \mathsf{PSL}(2,7)$ is 7. Then $\left|N_0^G\right| \geq 7\left|N_0^N\right|$ by (22). Let l be a secant line of $\operatorname{Fix}(N)$. Note that, by the above argument different conjugates of N_0 in G cover different pairs of points on $l - \operatorname{Fix}(N)$, since $|l \cap (\operatorname{Fix}(N_0) - \operatorname{Fix}(N))| = 2$. Hence $2\left|N_0^G\right| \leq |l - \operatorname{Fix}(N)|$. That is $14\left|N_0^N\right| \leq 14$, since $\left|N_0^G\right| \geq 7\left|N_0^N\right|$ and $|l - \operatorname{Fix}(N)| = 14$. This yields $\left|N_0^G\right| = 7$, $\left|N_0^N\right| = 1$ and hence $N \lhd N_0$. Then $[N : N_0] = 2$ arguing as above. Suppose there exists a distinct conjugate N_0^* of N_0 in G. By the above argument with N_0^* in the role of N_0, we have $[N : N_0^*] = 2$. This yields that $N \cong E_4$, since $[N : N_0] = 2$ and $N \cap N_0^* = \langle 1\rangle$. Thus $N \leq Z(G)$ and hence G is a perfect central extension of $\mathsf{PSL}(2,7)$ by Lemma 4.2. Then $N \cong Z_2$ by [32, Theorem 5.1.4], since $N \neq \langle 1\rangle$ by our assumptions. Therefore N_0 must be trivial. A contradiction, since $\delta \in N_0$ and $\delta \neq 1$. Thus $N_0 \lhd G$ and hence $N_0 \lhd N$. Then the above argument rules out this case. Hence, we may assume that N is semiregular on $l - \operatorname{Fix}(N)$, where l is the above secant. If N has even order, then N contains Baer involutions of Π, since $\operatorname{Fix}(N) = \mathcal{O} \cong \mathsf{PG}(2,2)$. Then $n = 4$, since N is semiregular on $l - \operatorname{Fix}(N)$. A contradiction, since $n > 2^3$ by our assumption. Hence N has odd order. Thus each involution of G acts faithfully on $\mathcal{O}$. Then each such involution is either a Baer collineation or an elation of Π, since $\mathcal{O} \cong \mathsf{PG}(2,2)$. Thus either n is a square or n is even. Then $n = 9, 12, 14, 16, 18, 20$, since $9 \leq n \leq 20$. The cases $n = 10, 14$ or 18 are ruled out by [24, Theorem 13.18]. The case $n = 12$ is ruled out by [26]. Hence $n = 9, 16$ or 20. Then either $N \cong Z_7$ for $n = 9$ or 16, or $N \cong Z_5$ for $n = 9$, since $|N| \mid |l - \mathrm{Fix}(N)|$ again by the semiregularity of N, since $|l - \mathrm{Fix}(N)| = n - 2$ and since N is non-trivial of odd order. Then $N \leq Z(G)$ and hence G is a perfect central extension of $\mathsf{PSL}(2,7)$ by Lemma 4.2. Then $G = G_0 \times N$, where $G_0 \cong \mathsf{PSL}(2,7)$ by [32, Theorem 5.1.4], as N has odd order. Therefore $G = G_0 \cong \mathsf{PSL}(2,7)$, since $N = \Phi(G)$ by Lemma 4.1. A contradiction, since $N \neq \langle 1 \rangle$ by our assumptions. $\square$

Lemma 4.6. *Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O}$. Assume that $\mathcal{O}$ has the structure of a non-trivial 2-$(v,k,1)$ design, the group G induces a flag-transitive and almost simple automorphism group on $\mathcal{O}$ and $n \leq \sum(\mathcal{O})$. If G does not act faithfully on $\mathcal{O}$, then one of the following holds:*

(1) *$\mathcal{O} \cong \mathsf{PG}(2,7)$, Π is the generalized Hughes plane over the exceptional nearfield of order 7^2 and $\mathsf{SL}(3,7) \leq G$;*

(2) *we have $n \geq q^4$, $q \equiv 2 \mod 3$, $q > 2$, $\mathcal{O} \cong \mathcal{H}(q)$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ and $\mathsf{SU}(3,q) \leq G$.*

Proof. By Lemmas 4.4 and 4.5, one of the following occurs:

(1) $\mathcal{O} \cong \mathsf{PG}(2,7)$, Π is the generalized Hughes plane over the exceptional nearfield of order 7^2 and $\mathsf{SL}(3,7) \leq G$;

(2) $\mathcal{O} \cong \mathcal{H}(q)$, $q > 2$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ is a Baer subplane of Π and $G/N \cong \mathsf{PSU}(3,q)$;

(3) $\mathcal{O} \cong \mathcal{W}(q)$, $q = 2^h$, $h \geq 3$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q)$ and $G/N \cong \mathsf{PSL}(2,q)$.

As case (1) does indeed occur, we shall focus on the remaining two cases.

Suppose that $N \leq Z(G)$. Then G is a perfect central extension of G/N by Lemma 4.2(1). Therefore $G \cong \mathsf{SU}(3,q)$, $N = Z_3$ and $q \equiv 2 \mod 3$ by [32, Theorem 5.1.4], since $N \neq \langle 1 \rangle$ by our assumption. Thus we obtain assertion (2).

Suppose that $N \not\leq Z(G)$. Then, by Corollary 4.3, there exists a Sylow t-subgroup S of N such that $|S| \geq 1 + d_0(G/N)$, where $d_0(G/N)$ is the minimal primitive permutation representation of G/N.

Assume that $\mathcal{O} \cong \mathcal{H}(q)$, $q > 2$. As $d_0(G/N) = q^3 + 1$ for $q \neq 5$ and $d_0(G/N) = 51$ for $q = 5$ by [32], we have that $|S| \geq q^3 + 2$ for $q \neq 5$ and $|S| \geq 51$ for $q = 5$, respectively. Moreover, it is easy to see that $\mathrm{Fix}(N) = \mathrm{Fix}(\alpha)$ for each non-trivial $\alpha \in N$ by [24, Theorem 3.7]. Thus S is semiregular on $s - s \cap \mathrm{Fix}(N)$

for each secants s to $\mathcal{O}$. Therefore $|S| \mid q^2(q^2-1)$. A contradiction, since S is a t-group with either $|S| \geq q^3+2$ for $q \neq 5$ or $|S| \geq 51$ for $q=5$.

Assume that $\mathcal{O} \cong \mathcal{W}(2^h)$, $h \geq 3$. Let $\alpha \in N$, $\alpha \neq 1$. Then α is planar, since $\mathrm{Fix}(N) \cong \mathsf{PG}(2,2^h)$. If $\mathcal{O} \subsetneq \mathrm{Fix}(\alpha)$, then $o(\mathrm{Fix}(\alpha)) \geq 2^{2h}$ and hence $n \geq 2^{4h}$ by [24, Theorem 3.7]. A contradiction since $n \leq \sum(\mathcal{O}) = 2^h(2^{h-1}3+1)$ by Lemma 2.4(4). Thus $\mathrm{Fix}(N) = \mathrm{Fix}(\beta)$ for each $\beta \in N$, $\beta \neq 1$. Then $|N| \mid (n-2^h)$, since N is semiregular on $l-(l\cap\mathcal{O})$ for any secant line l to $\mathcal{O}$. In particular $|S| \mid (n-2^h)$. Then $|S| \leq 2^{2h-1}3$ as $n \leq 2^h(2^{h-1}3+1)$. If $t \neq 2$, instead of using the inequality $|S| \geq 1 + d_0(G/N)$, we use the inequality $|S| \geq 2^{2^h-1}$, as $|S/\Phi(S)| \geq 2^{2^h-1}$ by [32, Theorem 5.3.9], since $G/N \leq \mathsf{PSL}(S/\Phi(S))$ and $h \geq 3$. Then $|S| \geq 2^{2^h-1}$ and hence $2^{2^h-1} \leq 2^{2h-1}3$, as $|S| \leq 2^{2h-1}3$ by the above argument. It is a straightforward computation to show that the inequality $2^{2^h-1} \leq 2^{2h-1}3$ is impossible, as $h \geq 3$. So $t=2$. Thus N contains Baer involutions. This yields $n = 2^{2h}$, since $\mathrm{Fix}(N) \cong \mathsf{PG}(2,2^h)$ and since $\mathrm{Fix}(N) = \mathrm{Fix}(\beta)$ for each $\beta \in N$, $\beta \neq 1$. Moreover $|N| \mid 2^h(2^h-1)$, since N is semiregular on $l-(l\cap\mathcal{O})$ for any secant line l to $\mathcal{O}$. In particular $|S| \mid 2^h(2^h-1)$. A contradiction, since S is a 2-group and since $|S| \geq 2^h+2$ by Corollary 4.3, being $d_0(G/N) = 2^h+1$. This completes the proof. □

Theorem 4.7. *Let Π be a projective plane of order n and let G be a collineation group of Π with a point-orbit $\mathcal{O}$ of length v. Assume that $\mathcal{O}$ has the structure of a non-trivial 2-$(v,k,1)$ design, the group G induces a flag-transitive and almost simple automorphism group on $\mathcal{O}$ and $n \leq \sum(\mathcal{O})$. If G does not act faithfully on $\mathcal{O}$, then one of the following holds:*

(1) *Π is the generalized Hughes plane over the exceptional nearfield of order 7^2, $\mathcal{O} \cong \mathsf{PG}(2,7)$ and $\mathsf{SL}(3,7) \leq G$;*

(2) *we have $n = q^4$, $q \equiv 2 \mod 3$, $q > 2$, $\mathcal{O} \cong \mathcal{H}(q)$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ and $\mathsf{SU}(3,q) \leq G$.*

Furthermore, the involutions in G are perspectivities of Π.

Proof. As a consequence of Lemma 4.6, in order to complete the proof of this theorem, we need to investigate only the case $n \geq q^4$, $q \equiv 2 \mod 3$, $q > 2$, $\mathcal{O} \cong \mathcal{H}(q)$, $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ and $\mathsf{SU}(3,q) \leq G$.

We proceed stepwise.

(I) *The involutions in G are either homologies or elations of Π according to whether q is odd or even respectively.*

Note that the group $G \cong \mathsf{SU}(3,q)$ has a unique conjugate class of involutions, since $N \cong Z_3$ is central in G. Hence assume that the involutions in G are Baer collineations of Π. Then either $n = q^4$ or $(q^2+1)^2$, since $q^4 \leq n \leq \sum(\mathcal{O}) = q^4 + 2q^2 + q + 2$ by Lemma 2.4(2).

Assume that q is odd. The case $n = (q^2+1)^2$ cannot occur by Theorem 13.18 of [24], since for any involution in γ in G the group $C_G(\gamma)$ induces a group containing $\mathsf{PGL}(2,q)$ on $\mathrm{Fix}(\gamma)$. Thus $n = q^4$. Then the involutions in G are homologies of Π by [41, Proposition 2.6], and since G contains a unique conjugate class of involutions. A contradiction by our assumption. Therefore q is even. Now, let Z be the center of a Sylow 2-subgroup Q of G. Then Z is an elementary abelian 2-group inducing on $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$ an elation group having the same center $A \in \mathcal{O}$ and the same axis c which is tangent to $\mathcal{O}$ in A (see [23]). Thus $|\mathrm{Fix}(Z) \cap \mathrm{Fix}(N) \cap c| = q^2+1$. Then $\sqrt{n}+1 = |\mathrm{Fix}(Z) \cap c| = q^2+1$ or q^2+2 according to whether $n = q^4$ or $(q^2+1)^2$ respectively, since the involutions in G are Baer collineations of Π. Therefore Z is semiregular on $c - \mathrm{Fix}(Z)$. Then Q is semiregular on $c - \mathrm{Fix}(Z)$, since each involution in Q lies in Z and q is even. So $q^3 \mid (n - \sqrt{n})$, since $|Q| = q^3$ and $|c - \mathrm{Fix}(Z)|$. Thus either $q^3 \mid \sqrt{n}$ or $q^3 \mid (\sqrt{n}-1)$. A contradiction in any case, since $\sqrt{n} = q^2$ or q^2+1. Thus we have assertion (I), since $\mathrm{Fix}(N) \cong \mathsf{PG}(2,q^2)$.

(II) $n = q^4$.

As $n \geq q^4$ we have either $n = q^4$ or $n \geq q^4 + q$ by [24, Theorem 3.7]. If the latter occurs, then the set $\mathcal{E}$ of the external lines to $\mathrm{Fix}(N)$ is non-empty. Now, it is a plain argument to see that the proofs of Lemmas 3.6 and 3.7 still work, with $\mathrm{Fix}(N)$ in the role of Π_0, leading to a contradiction. Indeed, $N \cong Z_3$ is semiregular on $\mathcal{E}$ and N is disjoint from any Sylow p-subgroup of G, p a prime divisor of q, as $q \equiv 2 \mod 3$. Thus $n = q^4$ and hence the assertion. □

The proof of the second part of Theorem 1.1 is now complete.

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