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A characterisation of the lines external to a quadric cone in $\mathsf{PG}(3,q)$, q odd

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Abstract

In this article, the lines not meeting a quadric cone in $\mathsf{PG}(3,q)$ (q odd) are characterised by their intersection properties with points and planes.

Keywords: projective space, quadric cone, lines, characterisation

MSC 2000: 51E20

1 Introduction

Recently, Durante and Olanda [4] and Di Gennaro, Durante and Olanda [3] have characterised the lines external to the non-singular quadrics in $\mathsf{PG}(3,q)$ using their combinatorial properties. These results are listed below.

Theorem 1.1 ([4])**.** *Let $\mathscr{L}$ be a set of lines in $\mathsf{PG}(3,q)$, $q>2$ such that:*

(i) *Every point lies on 0 or $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$;*

(ii) *Every plane contains q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$.*

Then $\mathscr{L}$ is the set of external lines to an ovoid of $\mathsf{PG}(3,q)$.

Theorem 1.2 ([3])**.** *Let $\mathscr{L}$ be a non-empty set of lines in $\mathsf{PG}(3,q)$, q odd such that:*

(i) *Every point lies on 0 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$;*

(ii) *Every plane contains 0 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$;*

(iii) *In every plane there are 0, $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$ lines of $\mathscr{L}$ through any point.*

Then the set of points on no lines of $\mathscr{L}$ forms either one line, two skew lines or a hyperbolic quadric. In the last case, $\mathscr{L}$ is precisely the set of external lines to the hyperbolic quadric.

Theorem 1.3 ([3]). *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathsf{PG}(3,q)$*, q even, $q > 2$ such that:*

(i) *In every plane there are 0 or $\frac{1}{2}q$ lines of $\mathscr{L}$ through any point.*

Then the set of points on no lines of $\mathscr{L}$ forms either one line, two skew lines or a hyperbolic quadric. In the last case, $\mathscr{L}$ is precisely the set of external lines to the hyperbolic quadric.

It is also possible to characterise the external lines to the *singular* irreducible quadric in $\mathsf{PG}(3,q)$. That is, the *quadric cone*. Barwick and Butler have provided this characterisation in the case when q is even:

Theorem 1.4 ([1]). *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathsf{PG}(3,q)$*, q even, such that:*

(i) *Every point lies on 0 or $\frac{1}{2}q^2$ lines of $\mathscr{L}$;*
(ii) *Every plane contains 0, q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$.*

Then $\mathscr{L}$ is the set of external lines to a hyperoval cone of $\mathsf{PG}(3,q)$*, and hence is the set of external lines to $q+2$ oval cones.*

In this article, we give a characterisation of the quadric cone when q is odd. In particular, we prove the following theorem:

Theorem 1.5. *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathsf{PG}(3,q)$ *(q odd) such that:*

(i) *Every point lies on 0, $\frac{1}{2}q(q+1)$ or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$;*
(ii) *Every plane contains 0, q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$;*
(iii) *For any point P, if P is on two planes which contain the same number of lines of $\mathscr{L}$, then P is on the same number of lines of $\mathscr{L}$ in both planes.*

Then $\mathscr{L}$ is the set of external lines to a quadric cone.

Note that a similar characterisation of the planes meeting a non-singular quadric of $\mathsf{PG}(4,q)$ in a conic is given in the preprint [2].

2 The proof of Theorem 1.5

Let $\mathscr{L}$ be a set of lines as described in Theorem 1.5. We will prove that $\mathscr{L}$ is the set of lines external to a quadric cone by a series of lemmas. In order to make the argument clearer, we will introduce some terminology:

- A point on 0 lines of $\mathscr{L}$ will be called a *black point*; all other points will be called *white points*.
- A (white) point on $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ will be called an *external point* and a (white) point on $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$ will be called an *internal point*.
- A plane containing 0 lines of $\mathscr{L}$ will be called a *0-plane*.
- A plane containing q^2 lines of $\mathscr{L}$ will be called a *V-plane*.
- A plane containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ will be called a *secant plane*.

We show that the set of black points is a quadric cone $\mathscr{C}$, and that $\mathscr{L}$ is precisely the set of external lines to $\mathscr{C}$. The 0-planes are those planes containing a generator of $\mathscr{C}$, the V-planes are those planes that meet $\mathscr{C}$ in only its vertex, and the secant planes are those planes that meet $\mathscr{C}$ in a conic.

We are now ready to state the first lemma:

Lemma 2.1. *For a white point P, every line of $\mathscr{L}$ through P is on the same number of V-planes.*

Proof. Let P be a white point and let L_P be the number of lines of $\mathscr{L}$ through P. By Condition (iii) of Theorem 1.5, P lies on the same number of lines of $\mathscr{L}$ in every secant plane through P. Let this number of lines be L_{Ps}. Similarly, P lies on the same number of lines of $\mathscr{L}$ in every V-plane through P. Let this number be L_{Pv}.

Let ℓ be a line of $\mathscr{L}$ through P and let v_ℓ be the number of V-planes through ℓ. Since a 0-plane contains no lines of $\mathscr{L}$, there are no 0-planes through ℓ. So, the number of secant planes through ℓ is $(q+1-v_\ell)$. We will count the lines of $\mathscr{L}$ through P by considering the lines of $\mathscr{L}$ through P in each plane about ℓ.

Each V-plane through ℓ contains L_{Pv} lines of $\mathscr{L}$ through P, including ℓ. Each secant plane through ℓ contains L_{Ps} lines of $\mathscr{L}$ through P, including ℓ. Counting this way, we have included ℓ itself $q+1$ times. So:

$$L_P = v_\ell L_{Pv} + (q+1-v_\ell)L_{Ps} - q\,. \tag{1}$$

In the above equation, L_P, L_{Pv} and L_{Ps} are constants, so v_ℓ is uniquely determined by P. That is, every line of $\mathscr{L}$ through P lies on the same number of V-planes. □

Lemma 2.2. *A line of $\mathscr{L}$ lies on at most two V-planes.*

Proof. Let ℓ be a line of $\mathscr{L}$. Let v_ℓ be the number of V-planes through ℓ and I_ℓ the number of internal points on ℓ. Since ℓ contains no black points, there are $(q+1-I_\ell)$ external points on ℓ; and since ℓ lies on no 0-planes, there are

$(q+1-v_\ell)$ secant planes through ℓ. Let L_ℓ be the number of lines of $\mathscr{L}$ meeting ℓ (not including ℓ itself). We will count these lines in two ways.

We first count L_ℓ by considering the lines of $\mathscr{L}$ through each point on ℓ. Each internal point is on $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$ (including ℓ), and each external point is on $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ (including ℓ). Counting this way, we have included ℓ itself $q+1$ times, so $L_\ell = \frac{1}{2}q(q+1)I_\ell + \frac{1}{2}q(q-1)(q+1-I_\ell) - (q+1)$.

On the other hand, we may also count L_ℓ by considering the lines of $\mathscr{L}$ in each plane through ℓ. Each V-plane contains q^2 lines of $\mathscr{L}$ (including ℓ), and each secant plane contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ (including ℓ). Again, we have included ℓ itself $(q+1)$ times, so $L_\ell = q^2 v_\ell + \frac{1}{2}q(q-1)(q+1-v_\ell) - (q+1)$.

Equating the above two expressions for L_ℓ and simplifying gives:

$$(q+1)v_\ell = 2I_\ell\,. \tag{2}$$

Now $I_\ell \le q+1$, so $(q+1)v_\ell \le 2(q+1)$. Thus $v_\ell \le 2$. □

Lemma 2.3. *Every point in a V-plane π is on 0 or q lines of $\mathscr{L}$ in π.*

Proof. Let π be a V-plane. We begin by showing that every point of π lies on at most q lines of $\mathscr{L}$ in π. Suppose that P is a point of π on $q+1$ lines of $\mathscr{L}$ in π. Let L_P be the total number of lines of $\mathscr{L}$ through P and v_P be the number of V-planes through P. By Condition (iii) of Theorem 1.5, every V-plane through P contains the same number of lines of $\mathscr{L}$ through P. That is, every V-plane through P contains $q+1$ lines of $\mathscr{L}$ through P. Also, by Lemma 2.1, every line of $\mathscr{L}$ through P lies on the same number of V-planes. Let this number be $v_{P\ell}$. By Lemma 2.2, $v_{P\ell} \le 2$. However, since P lies on lines of $\mathscr{L}$ in the V-plane π, every line of $\mathscr{L}$ through P is on at least one V-plane. That is, $v_{P\ell} = 1$ or 2. We will form an equation relating L_P, v_P and $v_{P\ell}$ by counting a set of pairs.

Let $X = \{(\ell,\alpha) \mid \ell$ is a line of $\mathscr{L}$ through P, α is a V-plane through $\ell\}$. Counting ℓ then α, we have L_P lines of $\mathscr{L}$ through P and $v_{P\ell}$ V-planes through each. So $|X| = L_P v_{P\ell}$. Counting α then ℓ, we have v_P V-planes through P and $(q+1)$ lines of $\mathscr{L}$ through P in each. So $|X| = (q+1)v_P$. Thus:

$$(q+1)v_P = L_P v_{P\ell}\,. \tag{3}$$

Suppose $v_{P\ell} = 1$. That is, suppose that there is exactly one V-plane through each line of $\mathscr{L}$ containing P. Any V-plane α through P other than π will meet π in a line through P. Since all lines through P in π are lines of $\mathscr{L}$, the line $\alpha \cap \pi$ is a line of $\mathscr{L}$ with two V-planes through it. However, each line of $\mathscr{L}$ through P lies on exactly one V-plane. So, P lies on no V-plane other than π. That is $v_P = 1$. Equation (3) now becomes $q+1 = L_P$. Now $L_P = \frac{1}{2}q(q-1)$

or $\frac{1}{2}q(q+1)$, and neither of these can be equal to $q+1$ for odd integer q. Thus $v_{P\ell} \neq 1$ and hence $v_{P\ell} = 2$.

Since every line of $\mathscr{L}$ through P lies on two V-planes, the $q+1$ lines of $\mathscr{L}$ in π define $q+1$ further V-planes. There can be no further V-planes through P as any plane through P other than π must meet π in a line through P. Thus $v_P = q+2$. Equation (3) now becomes $(q+1)(q+2) = 2L_P$. Now $2L_P = q(q+1)$ or $q(q-1)$. Both of these are contradictions, so the point P cannot exist and every point of π lies on at most q lines of $\mathscr{L}$ in π.

Let ℓ be a line of $\mathscr{L}$ in π. Since every line in π meets ℓ, we may count the lines of $\mathscr{L}$ in π by counting the lines of $\mathscr{L}$ through each point on ℓ. For $i = 1, \dots, q$, let a_i be the number of points of ℓ on i lines of $\mathscr{L}$. (Recall that every point of π is on at most q lines of $\mathscr{L}$ in π.) Counting this way, we have included ℓ itself $q+1$ times — once for each point on ℓ. Thus:

$$a_1 \cdot 1 + \cdots + a_{q-1} \cdot (q-1) + a_q \cdot q = q^2 + q\,. \tag{4}$$

We also have:

$$a_1 + \cdots + a_{q-1} + a_q = q + 1\,. \tag{5}$$

Subtracting equation (4) from q times equation (5) gives:

$$(q-1) \cdot a_1 + \cdots + 1 \cdot a_{q-1} = 0\,. \tag{6}$$

Now $q-1, \dots, 1 > 0$ and $a_1, \dots, a_{q-1} \geq 0$, so equation (6) is only possible if $a_1 = \cdots = a_{q-1} = 0$.

Hence, $a_q = q+1$ and all points on a line of $\mathscr{L}$ in π are on q lines of $\mathscr{L}$ in π. That is, all points of π are on 0 or q lines of $\mathscr{L}$ in π. □

Lemma 2.4. *Every line of $\mathscr{L}$ lies on one V-plane and q secant planes. Also, every line of $\mathscr{L}$ contains $\frac{1}{2}(q+1)$ internal points and $\frac{1}{2}(q+1)$ external points.*

Proof. Let ℓ be a line of $\mathscr{L}$ lying on v_ℓ V-planes and containing I_ℓ internal points. Equation (2) in Lemma 2.2 states that $2I_\ell = (q+1)v_\ell$. Also, by Lemma 2.2, $v_\ell \leq 2$. We will rule out the cases of $v_\ell = 0$, 2 by considering the lines through one point on ℓ.

Let P be a point on ℓ lying on L_P lines of $\mathscr{L}$ in total and L_{Ps} lines of $\mathscr{L}$ in each secant plane. If π is a V-plane through ℓ, then P lies on at least one line of $\mathscr{L}$ in π. Lemma 2.3 implies that P lies on q lines in π, so by Condition (iii) of Theorem 1.5, P lies on q lines of $\mathscr{L}$ in each V-plane. Using equation (1) in Lemma 2.1, we have:

$$L_P = v_\ell \cdot q + (q + 1 - v_\ell)L_{Ps} - q\,.$$

If $v_\ell = 0$, then from equation (2) in Lemma 2.2, $I_\ell = 0$, so all points on ℓ are external points. Thus $L_P = \frac{1}{2}q(q-1)$. Hence:

$$\frac{1}{2}q(q-1) = (q+1)L_{Ps} - q\,;$$
$$L_{Ps} = \frac{1}{2}q\,.$$

But q is odd, so $\frac{1}{2}q$ is not an integer. This is a contradiction, so $v_\ell \neq 0$.

If $v_\ell = 2$, then from equation (2) in Lemma 2.2, $I_\ell = q+1$, so all points on ℓ are internal points. Thus $L_P = \frac{1}{2}q(q+1)$. Hence:

$$\frac{1}{2}q(q+1) = 2q + (q-1)L_{Ps} - q\,;$$
$$L_{Ps} = \frac{1}{2}q\,.$$

This is a contradiction as before, so $v_\ell \neq 2$.

Hence $v_\ell = 1$ and $I_\ell = \frac{1}{2}(q+1)\cdot 1 = \frac{1}{2}(q+1)$. This leaves q secant planes through ℓ and $\frac{1}{2}(q+1)$ external points on ℓ. □

Note that the above lemma ensures the existence of secant planes, V-planes, internal points and external points as $\mathscr{L}$ is non-empty.

Lemma 2.5. *An internal point lies on q lines of $\mathscr{L}$ in every V-plane and $\frac{1}{2}(q+1)$ lines of $\mathscr{L}$ in every secant plane. An external point lies on q lines of $\mathscr{L}$ in every V-plane and $\frac{1}{2}(q-1)$ lines of $\mathscr{L}$ in every secant plane.*

Proof. Let P be a white point and let ℓ be a line of $\mathscr{L}$ through P. By Lemma 2.4, ℓ is contained in a unique V-plane. Let this plane be π. In the plane π, P lies on at least one line of $\mathscr{L}$, and so by Lemma 2.3, P lies on q lines of $\mathscr{L}$ in π. Condition (iii) of Theorem 1.5 implies that every V-plane through P contains the same number of lines of $\mathscr{L}$ through P. Thus P lies on exactly q lines of $\mathscr{L}$ in every V-plane.

Let L_{Ps} be the number of lines of $\mathscr{L}$ through P in a secant plane and let L_P be the total number of lines of $\mathscr{L}$ through P. We can now use equation (1) from Lemma 2.1. Through ℓ there are q secant planes and one V-plane, and the V-plane contains q lines of $\mathscr{L}$ through P. Thus $L_P = qL_{Ps} + 1\cdot q - q = qL_{Ps}$. If P is an internal point, then $L_P = \frac{1}{2}q(q+1)$, and so $L_{Ps} = \frac{1}{2}(q+1)$. If P is an external point, then $L_P = \frac{1}{2}q(q-1)$, and so $L_{Ps} = \frac{1}{2}(q-1)$. □

Lemma 2.6. *A V-plane contains exactly one black point, and the lines of $\mathscr{L}$ in the plane are exactly those lines not through this black point.*

Proof. Let π be a V-plane and let W_π be the number of white points in π. Consider the set

$$X = \{(P,\ell) \mid P \text{ is a white point of } \pi,\ \ell \text{ is a line of } \mathscr{L} \text{ through } P \text{ in } \pi\}\,.$$

We will count the size of X in two ways.

Each line of $\mathscr{L}$ in π contains $(q+1)$ white points, so $|X| = q^2(q+1)$. On the other hand, each white point is on q lines of $\mathscr{L}$ in every V-plane by Lemma 2.5. So every white point in π lies on q lines of $\mathscr{L}$ in π and $|X| = qW_\pi$. Thus $qW_\pi = q^2(q+1)$ and so $W_\pi = q^2 + q$. This leaves one black point V in π. There are q^2 lines of $\mathscr{L}$ in π, none of which can pass through a black point. On the other hand, there are q^2 lines of π not through V. Thus, the lines of π in $\mathscr{L}$ are exactly those lines not through V. □

Note that, since there must exist a V-plane, the above lemma ensures the existence of black points.

Lemma 2.7. *There exists a unique (black) point V through which all 0-planes and V-planes pass. The secant planes are precisely those planes not containing V.*

Proof. Let π be a V-plane and let its unique black point be V.

Let α be another V-plane and suppose that α does not pass through V. Then α must meet π in a line ℓ not through V. Since ℓ is a line of π not through V, it is a line of $\mathscr{L}$. But now we have a line of $\mathscr{L}$ on two V-planes. This is a contradiction to Lemma 2.4, so α must pass through V.

Let β be a 0-plane and suppose that β does not pass through V. Then β must meet π in a line ℓ not through V. Again, this line must be a line of $\mathscr{L}$. But now we have a line of $\mathscr{L}$ in a 0-plane. This is a contradiction, so β must pass through V.

So we see that all 0-planes and all V-planes pass through V. Thus the planes not through V are all secant planes. To complete the proof we must show that there are no secant planes through V.

Let γ be a secant plane containing V and let ℓ be a line of $\mathscr{L}$ in γ. Since V is a black point, ℓ does not pass through V. Now the q other planes through ℓ do not contain V, and so they must all be secant planes. But now ℓ is a line of $\mathscr{L}$ on $q+1$ secant planes. This is a contradiction to Lemma 2.4, so γ cannot contain V. □

The next three lemmas will complete the proof of Theorem 1.5.

Lemma 2.8. *Let m be a line not in $\mathscr{L}$. If m passes through V, then m contains 1 or $q+1$ black points. If m does not pass through V, then m contains 1 or 2 black points.*

Proof. Suppose m passes through V, and also suppose that there exists a black point P other than V on m. Let π be a plane through m. Since π contains V, it is either a 0-plane or a V-plane by Lemma 2.7. Lemma 2.6 states that every V-plane contains a single black point. However, π contains two black points (P and V), so it cannot be a V-plane. Thus π is a 0-plane. So, every plane through m is a 0-plane. Since none of these planes has any line of $\mathscr{L}$, there are no lines of $\mathscr{L}$ meeting m. Hence, there are no lines of $\mathscr{L}$ through any point on m. That is, m consists of $q+1$ black points. So, if m passes through V, it has 1 or $q+1$ black points.

Suppose m does not pass through V. Then exactly one plane through m contains V and q planes do not. These q planes are all secant planes by Lemma 2.7. In light of this, let π be a secant plane through m.

Let B_m be the number of black points on m, let E_m be the number of external points on m, and let I_m be the number of internal points on m. We count the number of lines of $\mathscr{L}$ in π by considering the lines of $\mathscr{L}$ through each point on m. There are no lines of $\mathscr{L}$ through each black point, $\frac{1}{2}(q+1)$ through each internal point and $\frac{1}{2}(q-1)$ through each external point. Thus:

$$\begin{aligned} \tfrac{1}{2}q(q-1) &= \tfrac{1}{2}(q+1)I_m + \tfrac{1}{2}(q-1)E_m\,; \\ \tfrac{1}{2}(q-1)(q-E_m) &= \tfrac{1}{2}(q+1)I_m\,. \end{aligned} \tag{7}$$

Now $\frac{1}{2}(q+1)$ and $\frac{1}{2}(q-1)$ are coprime, so $\frac{1}{2}(q+1)$ divides $q-E_m$. That is, $E_m \equiv q \equiv -1 \pmod{\frac{1}{2}(q+1)}$. Since $0 \le E_m \le q+1$, we have that $E_m = \frac{1}{2}(q-1)$ or q.

If $E_m = \frac{1}{2}(q-1)$, then by equation (7), we have $I_m = \frac{1}{2}(q-1)$ and so $B_m = q+1-\frac{1}{2}(q-1)-\frac{1}{2}(q-1) = 2$. If $E_m = q$, then by equation (7), $I_m = 0$ and so $B_m = q+1-0-q = 1$. Thus if m does not pass through V, it contains 1 or 2 black points. □

Lemma 2.9. *The set of black points in a secant plane forms a conic.*

Proof. Let π be a secant plane and let E_π be the number of external points in π. Let $X = \{(P,\ell) \mid P \text{ is an external point of } \pi,\ \ell \text{ is a line of } \mathscr{L} \text{ in } \pi\}$. We will count X in two ways. Counting P first then ℓ, we have E_π choices for an external point in π and $\frac{1}{2}(q-1)$ choices for a line of $\mathscr{L}$ in π through each by Lemma 2.5. So $|X| = E_\pi \cdot \frac{1}{2}(q-1)$. Counting ℓ first then P, we have $\frac{1}{2}q(q-1)$ choices for a line of $\mathscr{L}$ in π and $\frac{1}{2}(q+1)$ choices for an external point on each by Lemma 2.4. So $|X| = \frac{1}{2}q(q-1)\frac{1}{2}(q+1)$. Thus $E_\pi \cdot \frac{1}{2}(q-1) = \frac{1}{2}q(q-1)\frac{1}{2}(q+1)$ and so $E_\pi = \frac{1}{2}q(q+1)$. A similar argument shows that there are $\frac{1}{2}q(q-1)$ internal points in π. The number of white points in π is thus $\frac{1}{2}q(q-1)+\frac{1}{2}q(q+1) = q^2$. This leaves $q+1$ black points in π. We will show that these $q+1$ points form an arc. That is, that no three are collinear.

A line of $\mathscr{L}$ contains no black points, so let m be a line of π not in $\mathscr{L}$. No secant plane passes through V by Lemma 2.7, so the line m cannot contain V.

By Lemma 2.8, this implies that m contains 1 or 2 black points. Thus, the lines of π contain at most 2 black points and the set of black points is a $(q+1)$-arc. That is, the black points are an *oval*. By Segre [5], every oval in PG$(2, q)$, q odd, is a conic, so the set of black points in π forms a conic. □

Lemma 2.10. *The set of black points $\mathscr{C}$ is a quadric cone and $\mathscr{L}$ is the set of external lines to $\mathscr{C}$.*

Proof. Let π be a secant plane and let $\mathscr{O}$ the conic made by the black points in π. Let P be a point of $\mathscr{O}$ and consider the line VP. This line passes through V and has more than one black point, so it has $q+1$ black points by Lemma 2.8. Thus, the set of black points $\mathscr{C}$ contains the lines VP for any $P \in \mathscr{O}$.

On the other hand, suppose that Q is any black point other than V. Then the line VQ contains $q+1$ black points by the same argument as above. This line VQ meets π in a single point, which is a black point since VQ consists only of black points. Now the black points in π are precisely the points of the conic $\mathscr{O}$, so the line VQ is a line VP for some $P \in \mathscr{O}$. Thus $\mathscr{C}$ is exactly the lines VP for $P \in \mathscr{O}$. That is, $\mathscr{C}$ is a quadric cone.

The lines of $\mathscr{L}$ contain no black points and so are all external lines to the cone $\mathscr{C}$. Any line not in $\mathscr{L}$ contains at least one black point by Lemma 2.8. So $\mathscr{L}$ is precisely the set of external lines to the quadric cone $\mathscr{C}$. □

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