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# A characterisation of the lines external to a quadric cone in $\text{PG}(3, q)$ , $q$ odd

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## Abstract

In this article, the lines not meeting a quadric cone in  $\text{PG}(3, q)$  ( $q$  odd) are characterised by their intersection properties with points and planes.

Keywords: projective space, quadric cone, lines, characterisation

MSC 2000: 51E20

## 1. Introduction

Recently, Durante and Olanda [4] and Di Gennaro, Durante and Olanda [3] have characterised the lines external to the non-singular quadrics in  $\text{PG}(3, q)$  using their combinatorial properties. These results are listed below.

**Theorem 1.1** ([4]). *Let  $\mathcal{L}$  be a set of lines in  $\text{PG}(3, q)$ ,  $q > 2$  such that:*

- (i) *Every point lies on 0 or  $\frac{1}{2}q(q+1)$  lines of  $\mathcal{L}$ ;*
- (ii) *Every plane contains  $q^2$  or  $\frac{1}{2}q(q-1)$  lines of  $\mathcal{L}$ .*

*Then  $\mathcal{L}$  is the set of external lines to an ovoid of  $\text{PG}(3, q)$ .*

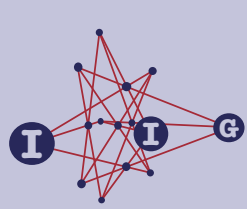
**Theorem 1.2** ([3]). *Let  $\mathcal{L}$  be a non-empty set of lines in  $\text{PG}(3, q)$ ,  $q$  odd such that:*

- (i) *Every point lies on 0 or  $\frac{1}{2}q(q-1)$  lines of  $\mathcal{L}$ ;*
- (ii) *Every plane contains 0 or  $\frac{1}{2}q(q-1)$  lines of  $\mathcal{L}$ ;*
- (iii) *In every plane there are 0,  $\frac{1}{2}(q-1)$  or  $\frac{1}{2}(q+1)$  lines of  $\mathcal{L}$  through any point.*

*Then the set of points on no lines of  $\mathcal{L}$  forms either one line, two skew lines or a hyperbolic quadric. In the last case,  $\mathcal{L}$  is precisely the set of external lines to the hyperbolic quadric.*

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**Theorem 1.3** ([3]). Let  $\mathcal{L}$  be a non-empty set of lines in  $\text{PG}(3, q)$ ,  $q$  even,  $q > 2$  such that:

- (i) In every plane there are 0 or  $\frac{1}{2}q$  lines of  $\mathcal{L}$  through any point.

Then the set of points on no lines of  $\mathcal{L}$  forms either one line, two skew lines or a hyperbolic quadric. In the last case,  $\mathcal{L}$  is precisely the set of external lines to the hyperbolic quadric.

It is also possible to characterise the external lines to the *singular* irreducible quadric in  $\text{PG}(3, q)$ . That is, the *quadric cone*. Barwick and Butler have provided this characterisation in the case when  $q$  is even:

**Theorem 1.4** ([1]). Let  $\mathcal{L}$  be a non-empty set of lines in  $\text{PG}(3, q)$ ,  $q$  even, such that:

- (i) Every point lies on 0 or  $\frac{1}{2}q^2$  lines of  $\mathcal{L}$ ;  
(ii) Every plane contains 0,  $q^2$  or  $\frac{1}{2}q(q - 1)$  lines of  $\mathcal{L}$ .

Then  $\mathcal{L}$  is the set of external lines to a hyperoval cone of  $\text{PG}(3, q)$ , and hence is the set of external lines to  $q + 2$  oval cones.

In this article, we give a characterisation of the quadric cone when  $q$  is odd. In particular, we prove the following theorem:

**Theorem 1.5.** Let  $\mathcal{L}$  be a non-empty set of lines in  $\text{PG}(3, q)$  ( $q$  odd) such that:

- (i) Every point lies on 0,  $\frac{1}{2}q(q + 1)$  or  $\frac{1}{2}q(q - 1)$  lines of  $\mathcal{L}$ ;  
(ii) Every plane contains 0,  $q^2$  or  $\frac{1}{2}q(q - 1)$  lines of  $\mathcal{L}$ ;  
(iii) For any point  $P$ , if  $P$  is on two planes which contain the same number of lines of  $\mathcal{L}$ , then  $P$  is on the same number of lines of  $\mathcal{L}$  in both planes.

Then  $\mathcal{L}$  is the set of external lines to a quadric cone.

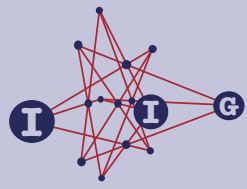
Note that a similar characterisation of the planes meeting a non-singular quadric of  $\text{PG}(4, q)$  in a conic is given in the preprint [2].

## 2. The proof of Theorem 1.5

Let  $\mathcal{L}$  be a set of lines as described in Theorem 1.5. We will prove that  $\mathcal{L}$  is the set of lines external to a quadric cone by a series of lemmas. In order to make the argument clearer, we will introduce some terminology:

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- A point on 0 lines of  $\mathcal{L}$  will be called a *black point*; all other points will be called *white points*.
- A (white) point on  $\frac{1}{2}q(q-1)$  lines of  $\mathcal{L}$  will be called an *external point* and a (white) point on  $\frac{1}{2}q(q+1)$  lines of  $\mathcal{L}$  will be called an *internal point*.
- A plane containing 0 lines of  $\mathcal{L}$  will be called a *0-plane*.
- A plane containing  $q^2$  lines of  $\mathcal{L}$  will be called a *V-plane*.
- A plane containing  $\frac{1}{2}q(q-1)$  lines of  $\mathcal{L}$  will be called a *secant plane*.

We show that the set of black points is a quadric cone  $\mathcal{C}$ , and that  $\mathcal{L}$  is precisely the set of external lines to  $\mathcal{C}$ . The 0-planes are those planes containing a generator of  $\mathcal{C}$ , the V-planes are those planes that meet  $\mathcal{C}$  in only its vertex, and the secant planes are those planes that meet  $\mathcal{C}$  in a conic.

We are now ready to state the first lemma:

**Lemma 2.1.** *For a white point  $P$ , every line of  $\mathcal{L}$  through  $P$  is on the same number of V-planes.*

*Proof.* Let  $P$  be a white point and let  $L_P$  be the number of lines of  $\mathcal{L}$  through  $P$ . By Condition (iii) of Theorem 1.5,  $P$  lies on the same number of lines of  $\mathcal{L}$  in every secant plane through  $P$ . Let this number of lines be  $L_{P_s}$ . Similarly,  $P$  lies on the same number of lines of  $\mathcal{L}$  in every V-plane through  $P$ . Let this number be  $L_{P_v}$ .

Let  $\ell$  be a line of  $\mathcal{L}$  through  $P$  and let  $v_\ell$  be the number of V-planes through  $\ell$ . Since a 0-plane contains no lines of  $\mathcal{L}$ , there are no 0-planes through  $\ell$ . So, the number of secant planes through  $\ell$  is  $(q+1-v_\ell)$ . We will count the lines of  $\mathcal{L}$  through  $P$  by considering the lines of  $\mathcal{L}$  through  $P$  in each plane about  $\ell$ .

Each V-plane through  $\ell$  contains  $L_{P_v}$  lines of  $\mathcal{L}$  through  $P$ , including  $\ell$ . Each secant plane through  $\ell$  contains  $L_{P_s}$  lines of  $\mathcal{L}$  through  $P$ , including  $\ell$ . Counting this way, we have included  $\ell$  itself  $q+1$  times. So:

$$L_P = v_\ell L_{P_v} + (q+1-v_\ell)L_{P_s} - q. \quad (1)$$

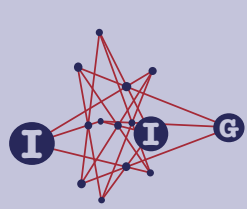
In the above equation,  $L_P$ ,  $L_{P_v}$  and  $L_{P_s}$  are constants, so  $v_\ell$  is uniquely determined by  $P$ . That is, every line of  $\mathcal{L}$  through  $P$  lies on the same number of V-planes.  $\square$

**Lemma 2.2.** *A line of  $\mathcal{L}$  lies on at most two V-planes.*

*Proof.* Let  $\ell$  be a line of  $\mathcal{L}$ . Let  $v_\ell$  be the number of V-planes through  $\ell$  and  $I_\ell$  the number of internal points on  $\ell$ . Since  $\ell$  contains no black points, there are  $(q+1-I_\ell)$  external points on  $\ell$ ; and since  $\ell$  lies on no 0-planes, there are

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$(q + 1 - v_\ell)$  secant planes through  $\ell$ . Let  $L_\ell$  be the number of lines of  $\mathcal{L}$  meeting  $\ell$  (not including  $\ell$  itself). We will count these lines in two ways.

We first count  $L_\ell$  by considering the lines of  $\mathcal{L}$  through each point on  $\ell$ . Each internal point is on  $\frac{1}{2}q(q + 1)$  lines of  $\mathcal{L}$  (including  $\ell$ ), and each external point is on  $\frac{1}{2}q(q - 1)$  lines of  $\mathcal{L}$  (including  $\ell$ ). Counting this way, we have included  $\ell$  itself  $q + 1$  times, so  $L_\ell = \frac{1}{2}q(q + 1)I_\ell + \frac{1}{2}q(q - 1)(q + 1 - I_\ell) - (q + 1)$ .

On the other hand, we may also count  $L_\ell$  by considering the lines of  $\mathcal{L}$  in each plane through  $\ell$ . Each V-plane contains  $q^2$  lines of  $\mathcal{L}$  (including  $\ell$ ), and each secant plane contains  $\frac{1}{2}q(q - 1)$  lines of  $\mathcal{L}$  (including  $\ell$ ). Again, we have included  $\ell$  itself  $(q + 1)$  times, so  $L_\ell = q^2v_\ell + \frac{1}{2}q(q - 1)(q + 1 - v_\ell) - (q + 1)$ .

Equating the above two expressions for  $L_\ell$  and simplifying gives:

$$(q + 1)v_\ell = 2I_\ell. \quad (2)$$

Now  $I_\ell \leq q + 1$ , so  $(q + 1)v_\ell \leq 2(q + 1)$ . Thus  $v_\ell \leq 2$ .  $\square$

**Lemma 2.3.** *Every point in a V-plane  $\pi$  is on 0 or  $q$  lines of  $\mathcal{L}$  in  $\pi$ .*

*Proof.* Let  $\pi$  be a V-plane. We begin by showing that every point of  $\pi$  lies on at most  $q$  lines of  $\mathcal{L}$  in  $\pi$ . Suppose that  $P$  is a point of  $\pi$  on  $q + 1$  lines of  $\mathcal{L}$  in  $\pi$ . Let  $L_P$  be the total number of lines of  $\mathcal{L}$  through  $P$  and  $v_P$  be the number of V-planes through  $P$ . By Condition (iii) of Theorem 1.5, every V-plane through  $P$  contains the same number of lines of  $\mathcal{L}$  through  $P$ . That is, every V-plane through  $P$  contains  $q + 1$  lines of  $\mathcal{L}$  through  $P$ . Also, by Lemma 2.1, every line of  $\mathcal{L}$  through  $P$  lies on the same number of V-planes. Let this number be  $v_{P\ell}$ . By Lemma 2.2,  $v_{P\ell} \leq 2$ . However, since  $P$  lies on lines of  $\mathcal{L}$  in the V-plane  $\pi$ , every line of  $\mathcal{L}$  through  $P$  is on at least one V-plane. That is,  $v_{P\ell} = 1$  or 2. We will form an equation relating  $L_P$ ,  $v_P$  and  $v_{P\ell}$  by counting a set of pairs.

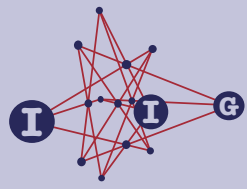
Let  $X = \{(\ell, \alpha) \mid \ell \text{ is a line of } \mathcal{L} \text{ through } P, \alpha \text{ is a V-plane through } \ell\}$ . Counting  $\ell$  then  $\alpha$ , we have  $L_P$  lines of  $\mathcal{L}$  through  $P$  and  $v_{P\ell}$  V-planes through each. So  $|X| = L_P v_{P\ell}$ . Counting  $\alpha$  then  $\ell$ , we have  $v_P$  V-planes through  $P$  and  $(q + 1)$  lines of  $\mathcal{L}$  through  $P$  in each. So  $|X| = (q + 1)v_P$ . Thus:

$$(q + 1)v_P = L_P v_{P\ell}. \quad (3)$$

Suppose  $v_{P\ell} = 1$ . That is, suppose that there is exactly one V-plane through each line of  $\mathcal{L}$  containing  $P$ . Any V-plane  $\alpha$  through  $P$  other than  $\pi$  will meet  $\pi$  in a line through  $P$ . Since all lines through  $P$  in  $\pi$  are lines of  $\mathcal{L}$ , the line  $\alpha \cap \pi$  is a line of  $\mathcal{L}$  with two V-planes through it. However, each line of  $\mathcal{L}$  through  $P$  lies on exactly one V-plane. So,  $P$  lies on no V-plane other than  $\pi$ . That is  $v_P = 1$ . Equation (3) now becomes  $q + 1 = L_P$ . Now  $L_P = \frac{1}{2}q(q - 1)$

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or  $\frac{1}{2}q(q+1)$ , and neither of these can be equal to  $q+1$  for odd integer  $q$ . Thus  $v_{P\ell} \neq 1$  and hence  $v_{P\ell} = 2$ .

Since every line of  $\mathcal{L}$  through  $P$  lies on two V-planes, the  $q+1$  lines of  $\mathcal{L}$  in  $\pi$  define  $q+1$  further V-planes. There can be no further V-planes through  $P$  as any plane through  $P$  other than  $\pi$  must meet  $\pi$  in a line through  $P$ . Thus  $v_P = q+2$ . Equation (3) now becomes  $(q+1)(q+2) = 2L_P$ . Now  $2L_P = q(q+1)$  or  $q(q-1)$ . Both of these are contradictions, so the point  $P$  cannot exist and every point of  $\pi$  lies on at most  $q$  lines of  $\mathcal{L}$  in  $\pi$ .

Let  $\ell$  be a line of  $\mathcal{L}$  in  $\pi$ . Since every line in  $\pi$  meets  $\ell$ , we may count the lines of  $\mathcal{L}$  in  $\pi$  by counting the lines of  $\mathcal{L}$  through each point on  $\ell$ . For  $i = 1, \dots, q$ , let  $a_i$  be the number of points of  $\ell$  on  $i$  lines of  $\mathcal{L}$ . (Recall that every point of  $\pi$  is on at most  $q$  lines of  $\mathcal{L}$  in  $\pi$ .) Counting this way, we have included  $\ell$  itself  $q+1$  times — once for each point on  $\ell$ . Thus:

$$a_1 \cdot 1 + \dots + a_{q-1} \cdot (q-1) + a_q \cdot q = q^2 + q. \quad (4)$$

We also have:

$$a_1 + \dots + a_{q-1} + a_q = q + 1. \quad (5)$$

Subtracting equation (4) from  $q$  times equation (5) gives:

$$(q-1) \cdot a_1 + \dots + 1 \cdot a_{q-1} = 0. \quad (6)$$

Now  $q-1, \dots, 1 > 0$  and  $a_1, \dots, a_{q-1} \geq 0$ , so equation (6) is only possible if  $a_1 = \dots = a_{q-1} = 0$ .

Hence,  $a_q = q+1$  and all points on a line of  $\mathcal{L}$  in  $\pi$  are on  $q$  lines of  $\mathcal{L}$  in  $\pi$ . That is, all points of  $\pi$  are on 0 or  $q$  lines of  $\mathcal{L}$  in  $\pi$ .  $\square$

**Lemma 2.4.** *Every line of  $\mathcal{L}$  lies on one V-plane and  $q$  secant planes. Also, every line of  $\mathcal{L}$  contains  $\frac{1}{2}(q+1)$  internal points and  $\frac{1}{2}(q+1)$  external points.*

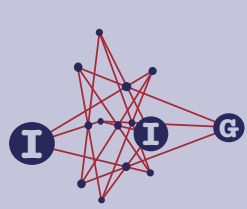
*Proof.* Let  $\ell$  be a line of  $\mathcal{L}$  lying on  $v_\ell$  V-planes and containing  $I_\ell$  internal points. Equation (2) in Lemma 2.2 states that  $2I_\ell = (q+1)v_\ell$ . Also, by Lemma 2.2,  $v_\ell \leq 2$ . We will rule out the cases of  $v_\ell = 0, 2$  by considering the lines through one point on  $\ell$ .

Let  $P$  be a point on  $\ell$  lying on  $L_P$  lines of  $\mathcal{L}$  in total and  $L_{P_s}$  lines of  $\mathcal{L}$  in each secant plane. If  $\pi$  is a V-plane through  $\ell$ , then  $P$  lies on at least one line of  $\mathcal{L}$  in  $\pi$ . Lemma 2.3 implies that  $P$  lies on  $q$  lines in  $\pi$ , so by Condition (iii) of Theorem 1.5,  $P$  lies on  $q$  lines of  $\mathcal{L}$  in each V-plane. Using equation (1) in Lemma 2.1, we have:

$$L_P = v_\ell \cdot q + (q+1 - v_\ell)L_{P_s} - q.$$

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If  $v_\ell = 0$ , then from equation (2) in Lemma 2.2,  $I_\ell = 0$ , so all points on  $\ell$  are external points. Thus  $L_P = \frac{1}{2}q(q-1)$ . Hence:

$$\begin{aligned}\frac{1}{2}q(q-1) &= (q+1)L_{Ps} - q; \\ L_{Ps} &= \frac{1}{2}q.\end{aligned}$$

But  $q$  is odd, so  $\frac{1}{2}q$  is not an integer. This is a contradiction, so  $v_\ell \neq 0$ .

If  $v_\ell = 2$ , then from equation (2) in Lemma 2.2,  $I_\ell = q+1$ , so all points on  $\ell$  are internal points. Thus  $L_P = \frac{1}{2}q(q+1)$ . Hence:

$$\begin{aligned}\frac{1}{2}q(q+1) &= 2q + (q-1)L_{Ps} - q; \\ L_{Ps} &= \frac{1}{2}q.\end{aligned}$$

This is a contradiction as before, so  $v_\ell \neq 2$ .

Hence  $v_\ell = 1$  and  $I_\ell = \frac{1}{2}(q+1) \cdot 1 = \frac{1}{2}(q+1)$ . This leaves  $q$  secant planes through  $\ell$  and  $\frac{1}{2}(q+1)$  external points on  $\ell$ .  $\square$

Note that the above lemma ensures the existence of secant planes, V-planes, internal points and external points as  $\mathcal{L}$  is non-empty.

**Lemma 2.5.** *An internal point lies on  $q$  lines of  $\mathcal{L}$  in every V-plane and  $\frac{1}{2}(q+1)$  lines of  $\mathcal{L}$  in every secant plane. An external point lies on  $q$  lines of  $\mathcal{L}$  in every V-plane and  $\frac{1}{2}(q-1)$  lines of  $\mathcal{L}$  in every secant plane.*

*Proof.* Let  $P$  be a white point and let  $\ell$  be a line of  $\mathcal{L}$  through  $P$ . By Lemma 2.4,  $\ell$  is contained in a unique V-plane. Let this plane be  $\pi$ . In the plane  $\pi$ ,  $P$  lies on at least one line of  $\mathcal{L}$ , and so by Lemma 2.3,  $P$  lies on  $q$  lines of  $\mathcal{L}$  in  $\pi$ . Condition (iii) of Theorem 1.5 implies that every V-plane through  $P$  contains the same number of lines of  $\mathcal{L}$  through  $P$ . Thus  $P$  lies on exactly  $q$  lines of  $\mathcal{L}$  in every V-plane.

Let  $L_{Ps}$  be the number of lines of  $\mathcal{L}$  through  $P$  in a secant plane and let  $L_P$  be the total number of lines of  $\mathcal{L}$  through  $P$ . We can now use equation (1) from Lemma 2.1. Through  $\ell$  there are  $q$  secant planes and one V-plane, and the V-plane contains  $q$  lines of  $\mathcal{L}$  through  $P$ . Thus  $L_P = qL_{Ps} + 1 \cdot q - q = qL_{Ps}$ . If  $P$  is an internal point, then  $L_P = \frac{1}{2}q(q+1)$ , and so  $L_{Ps} = \frac{1}{2}(q+1)$ . If  $P$  is an external point, then  $L_P = \frac{1}{2}q(q-1)$ , and so  $L_{Ps} = \frac{1}{2}(q-1)$ .  $\square$

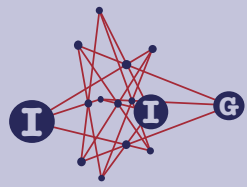
**Lemma 2.6.** *A V-plane contains exactly one black point, and the lines of  $\mathcal{L}$  in the plane are exactly those lines not through this black point.*

*Proof.* Let  $\pi$  be a V-plane and let  $W_\pi$  be the number of white points in  $\pi$ . Consider the set

$$X = \{(P, \ell) \mid P \text{ is a white point of } \pi, \ell \text{ is a line of } \mathcal{L} \text{ through } P \text{ in } \pi\}.$$

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We will count the size of  $X$  in two ways.

Each line of  $\mathcal{L}$  in  $\pi$  contains  $(q + 1)$  white points, so  $|X| = q^2(q + 1)$ . On the other hand, each white point is on  $q$  lines of  $\mathcal{L}$  in every V-plane by Lemma 2.5. So every white point in  $\pi$  lies on  $q$  lines of  $\mathcal{L}$  in  $\pi$  and  $|X| = qW_\pi$ . Thus  $qW_\pi = q^2(q + 1)$  and so  $W_\pi = q^2 + q$ . This leaves one black point  $V$  in  $\pi$ . There are  $q^2$  lines of  $\mathcal{L}$  in  $\pi$ , none of which can pass through a black point. On the other hand, there are  $q^2$  lines of  $\pi$  not through  $V$ . Thus, the lines of  $\pi$  in  $\mathcal{L}$  are exactly those lines not through  $V$ .  $\square$

Note that, since there must exist a V-plane, the above lemma ensures the existence of black points.

**Lemma 2.7.** *There exists a unique (black) point  $V$  through which all 0-planes and V-planes pass. The secant planes are precisely those planes not containing  $V$ .*

*Proof.* Let  $\pi$  be a V-plane and let its unique black point be  $V$ .

Let  $\alpha$  be another V-plane and suppose that  $\alpha$  does not pass through  $V$ . Then  $\alpha$  must meet  $\pi$  in a line  $\ell$  not through  $V$ . Since  $\ell$  is a line of  $\pi$  not through  $V$ , it is a line of  $\mathcal{L}$ . But now we have a line of  $\mathcal{L}$  on two V-planes. This is a contradiction to Lemma 2.4, so  $\alpha$  must pass through  $V$ .

Let  $\beta$  be a 0-plane and suppose that  $\beta$  does not pass through  $V$ . Then  $\beta$  must meet  $\pi$  in a line  $\ell$  not through  $V$ . Again, this line must be a line of  $\mathcal{L}$ . But now we have a line of  $\mathcal{L}$  in a 0-plane. This is a contradiction, so  $\beta$  must pass through  $V$ .

So we see that all 0-planes and all V-planes pass through  $V$ . Thus the planes not through  $V$  are all secant planes. To complete the proof we must show that there are no secant planes through  $V$ .

Let  $\gamma$  be a secant plane containing  $V$  and let  $\ell$  be a line of  $\mathcal{L}$  in  $\gamma$ . Since  $V$  is a black point,  $\ell$  does not pass through  $V$ . Now the  $q$  other planes through  $\ell$  do not contain  $V$ , and so they must all be secant planes. But now  $\ell$  is a line of  $\mathcal{L}$  on  $q + 1$  secant planes. This is a contradiction to Lemma 2.4, so  $\gamma$  cannot contain  $V$ .  $\square$

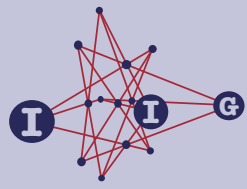
The next three lemmas will complete the proof of Theorem 1.5.

**Lemma 2.8.** *Let  $m$  be a line not in  $\mathcal{L}$ . If  $m$  passes through  $V$ , then  $m$  contains 1 or  $q + 1$  black points. If  $m$  does not pass through  $V$ , then  $m$  contains 1 or 2 black points.*

*Proof.* Suppose  $m$  passes through  $V$ , and also suppose that there exists a black point  $P$  other than  $V$  on  $m$ . Let  $\pi$  be a plane through  $m$ . Since  $\pi$  contains  $V$ ,

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it is either a 0-plane or a V-plane by Lemma 2.7. Lemma 2.6 states that every V-plane contains a single black point. However,  $\pi$  contains two black points ( $P$  and  $V$ ), so it cannot be a V-plane. Thus  $\pi$  is a 0-plane. So, every plane through  $m$  is a 0-plane. Since none of these planes has any line of  $\mathcal{L}$ , there are no lines of  $\mathcal{L}$  meeting  $m$ . Hence, there are no lines of  $\mathcal{L}$  through any point on  $m$ . That is,  $m$  consists of  $q + 1$  black points. So, if  $m$  passes through  $V$ , it has 1 or  $q + 1$  black points.

Suppose  $m$  does not pass through  $V$ . Then exactly one plane through  $m$  contains  $V$  and  $q$  planes do not. These  $q$  planes are all secant planes by Lemma 2.7. In light of this, let  $\pi$  be a secant plane through  $m$ .

Let  $B_m$  be the number of black points on  $m$ , let  $E_m$  be the number of external points on  $m$ , and let  $I_m$  be the number of internal points on  $m$ . We count the number of lines of  $\mathcal{L}$  in  $\pi$  by considering the lines of  $\mathcal{L}$  through each point on  $m$ . There are no lines of  $\mathcal{L}$  through each black point,  $\frac{1}{2}(q + 1)$  through each internal point and  $\frac{1}{2}(q - 1)$  through each external point. Thus:

$$\begin{aligned} \frac{1}{2}q(q - 1) &= \frac{1}{2}(q + 1)I_m + \frac{1}{2}(q - 1)E_m; \\ \frac{1}{2}(q - 1)(q - E_m) &= \frac{1}{2}(q + 1)I_m. \end{aligned} \quad (7)$$

Now  $\frac{1}{2}(q + 1)$  and  $\frac{1}{2}(q - 1)$  are coprime, so  $\frac{1}{2}(q + 1)$  divides  $q - E_m$ . That is,  $E_m \equiv q \equiv -1 \pmod{\frac{1}{2}(q+1)}$ . Since  $0 \leq E_m \leq q+1$ , we have that  $E_m = \frac{1}{2}(q-1)$  or  $q$ .

If  $E_m = \frac{1}{2}(q - 1)$ , then by equation (7), we have  $I_m = \frac{1}{2}(q - 1)$  and so  $B_m = q + 1 - \frac{1}{2}(q - 1) - \frac{1}{2}(q - 1) = 2$ . If  $E_m = q$ , then by equation (7),  $I_m = 0$  and so  $B_m = q + 1 - 0 - q = 1$ . Thus if  $m$  does not pass through  $V$ , it contains 1 or 2 black points.  $\square$

**Lemma 2.9.** *The set of black points in a secant plane forms a conic.*

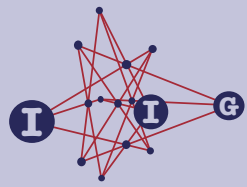
*Proof.* Let  $\pi$  be a secant plane and let  $E_\pi$  be the number of external points in  $\pi$ . Let  $X = \{(P, \ell) \mid P \text{ is an external point of } \pi, \ell \text{ is a line of } \mathcal{L} \text{ in } \pi\}$ . We will count  $X$  in two ways. Counting  $P$  first then  $\ell$ , we have  $E_\pi$  choices for an external point in  $\pi$  and  $\frac{1}{2}(q - 1)$  choices for a line of  $\mathcal{L}$  in  $\pi$  through each by Lemma 2.5. So  $|X| = E_\pi \cdot \frac{1}{2}(q - 1)$ . Counting  $\ell$  first then  $P$ , we have  $\frac{1}{2}q(q - 1)$  choices for a line of  $\mathcal{L}$  in  $\pi$  and  $\frac{1}{2}(q + 1)$  choices for an external point on each by Lemma 2.4. So  $|X| = \frac{1}{2}q(q - 1)\frac{1}{2}(q + 1)$ . Thus  $E_\pi \cdot \frac{1}{2}(q - 1) = \frac{1}{2}q(q - 1)\frac{1}{2}(q + 1)$  and so  $E_\pi = \frac{1}{2}q(q + 1)$ . A similar argument shows that there are  $\frac{1}{2}q(q - 1)$  internal points in  $\pi$ . The number of white points in  $\pi$  is thus  $\frac{1}{2}q(q - 1) + \frac{1}{2}q(q + 1) = q^2$ . This leaves  $q + 1$  black points in  $\pi$ . We will show that these  $q + 1$  points form an arc. That is, that no three are collinear.

A line of  $\mathcal{L}$  contains no black points, so let  $m$  be a line of  $\pi$  not in  $\mathcal{L}$ . No secant plane passes through  $V$  by Lemma 2.7, so the line  $m$  cannot contain  $V$ .

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By Lemma 2.8, this implies that  $m$  contains 1 or 2 black points. Thus, the lines of  $\pi$  contain at most 2 black points and the set of black points is a  $(q + 1)$ -arc. That is, the black points are an *oval*. By Segre [5], every oval in  $\text{PG}(2, q)$ ,  $q$  odd, is a conic, so the set of black points in  $\pi$  forms a conic.  $\square$

**Lemma 2.10.** *The set of black points  $\mathcal{C}$  is a quadric cone and  $\mathcal{L}$  is the set of external lines to  $\mathcal{C}$ .*

*Proof.* Let  $\pi$  be a secant plane and let  $\mathcal{O}$  the conic made by the black points in  $\pi$ . Let  $P$  be a point of  $\mathcal{O}$  and consider the line  $VP$ . This line passes through  $V$  and has more than one black point, so it has  $q + 1$  black points by Lemma 2.8. Thus, the set of black points  $\mathcal{C}$  contains the lines  $VP$  for any  $P \in \mathcal{O}$ .

On the other hand, suppose that  $Q$  is any black point other than  $V$ . Then the line  $VQ$  contains  $q + 1$  black points by the same argument as above. This line  $VQ$  meets  $\pi$  in a single point, which is a black point since  $VQ$  consists only of black points. Now the black points in  $\pi$  are precisely the points of the conic  $\mathcal{O}$ , so the line  $VQ$  is a line  $VP$  for some  $P \in \mathcal{O}$ . Thus  $\mathcal{C}$  is exactly the lines  $VP$  for  $P \in \mathcal{O}$ . That is,  $\mathcal{C}$  is a quadric cone.

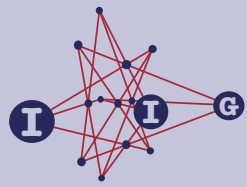
The lines of  $\mathcal{L}$  contain no black points and so are all external lines to the cone  $\mathcal{C}$ . Any line not in  $\mathcal{L}$  contains at least one black point by Lemma 2.8. So  $\mathcal{L}$  is precisely the set of external lines to the quadric cone  $\mathcal{C}$ .  $\square$

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