Innovations in Incidence Geometry
Volume 8 (2008), Pages 73–79
ISSN 1781-6475

Constructing the Tits ovoid from an elliptic quadric

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Abstract

This note details the construction of the Tits ovoid in $\mathsf{PG}(3, 2^{2e-1})$, $e \geq 2$, starting with an elliptic quadric in this space. The method employs a special type of net replacement which is here called *oval derivation* applied to a plane representation of the elliptic quadric.

Keywords: elliptic quadrics, Tits ovoid, plane representation theorem, oval derivation

MSC 2000: 51E21

1 Introduction

Let $\mathsf{PG}(3, q)$ denote the 3-dimensional projective space over the finite field $\mathsf{GF}(q)$. An *ovoid* of $\mathsf{PG}(3, q)$ is a set of $q^2 + 1$ points of $\mathsf{PG}(3, q)$, no three of which are collinear. The only known ovoids of $\mathsf{PG}(3, q)$ are the elliptic quadrics, which exist for all q, and the Tits ovoids, which exist only when $q = 2^{2e-1}$, $e \geq 2$. An *elliptic quadric* in $\mathsf{PG}(3, q)$ is the set of all points whose homogeneous coordinates satisfy a homogeneous quadratic equation such that the set contains no line. A *Tits ovoid* is the set of all absolute points of a polarity of the generalized quadrangle $W(q)$ (see [12]).

If q is odd, then it is known that every ovoid of $\mathsf{PG}(3, q)$ is an elliptic quadric [1, 11]. Furthermore, if $q = 4$ or 16, then every ovoid of $\mathsf{PG}(3, q)$ is an elliptic quadric [6, 7, 8], while if $q = 8$ or 32 then every ovoid is either an elliptic quadric or a Tits ovoid [4, 10, 14]. It is widely conjectured that these are the only families of ovoids in $\mathsf{PG}(3, q)$, but the classification for even q has only been achieved for $q < 64$.

In this note we provide a direct connection between elliptic quadrics and Tits ovoids in the spaces in which they both exist. It is hoped that this connection may be useful in shedding some light on the very difficult problem of the classification of ovoids.

2 Background material

The construction we give in the next section uses two fundamental ideas which we now review.

There is an equivalence between ovoids in $\mathsf{PG}(3, q)$ and special sets of ovals in a $\mathsf{PG}(2, q)$. This equivalence is the subject of the *Plane Equivalent Theorem* (below) independently discovered by Penttila [13] and Glynn [5].

An *oval* in a projective plane $\mathsf{PG}(2, q)$ is a set of $q+1$ points no three of which are collinear. When q is even, there is a unique point (often called the *nucleus* or *knot* of the oval) which when added to the oval gives a set of $q+2$ points, no three collinear. Such a set of $q+2$ points is called the *hyperoval* containing the given oval. Any line of the plane intersects a hyperoval in either 0 or 2 points and is called an *exterior* or *secant* line of the hyperoval respectively. Two hyperovals, $\mathcal{H}_1$ and $\mathcal{H}_2$, meeting precisely in two points N and P are said to be *compatible* at a point Q of the line $\ell = \overline{NP}$, other than N or P, if all lines through Q other than ℓ which are secant lines of $\mathcal{H}_1$ are exterior lines of $\mathcal{H}_2$ (and consequently, the exterior lines to $\mathcal{H}_1$ are secant lines of $\mathcal{H}_2$.) Consider a set of q hyperovals $\{\mathcal{H}_s\}$ indexed by the elements of $\mathsf{GF}(q)$ which mutually intersect precisely at the points $(0, 1, 0)$ and $(1, 0, 0)$ and for which $\mathcal{H}_a$ and $\mathcal{H}_b$ are compatible at $(1, a+b, 0)$ for all distinct $a, b \in \mathsf{GF}(q)$. Any set of hyperovals that are projectively equivalent to these is called a *fan* of hyperovals. Note that these definitions are usually given with respect to ovals, but that approach introduces an unnecessary distinction between the two common points (the *carrier set*) of the fan.

Theorem 2.1 (The Plane Equivalent Theorem [13, 5]). *An ovoid of* $\mathsf{PG}(3, q)$ *is equivalent to a fan of hyperovals in a plane* $\mathsf{PG}(2, q)$. □

The proof is constructive. Starting with the secant planes of an ovoid passing through a common tangent line of the ovoid and indexed by the elements of the field, homographies dependent on the index map each section of the ovoid into a single secant plane of this pencil. The resulting collection of ovals have a common nucleus and form a fan of hyperovals. The process is reversible and the ovoid can be recovered from the fan of hyperovals.

In a little more detail, in order to fix notation, we have that an ovoid Ω of $\mathsf{PG}(3, q)$ consists of the points,

$$\Omega = \{(s, t, st + f_s(t), 1) \colon s, t \in \mathsf{GF}(q)\} \cup \{(0, 0, 1, 0)\}, \qquad (1)$$

where, for each $s \in \mathsf{GF}(q)$, $f_s(t)$ is an o-permutation (that is, $f_s(0) = 0$ and the set of points $H_s = \{(t, f_s(t), 1) \colon t \in \mathsf{GF}(q)\} \cup \{(0,1,0)\} \cup \{(1,0,0)\}$ is a hyperoval in $\mathsf{PG}(2,q)$. The fan of hyperovals corresponding to Ω is given by (after suppressing the first coordinate in a $\mathsf{PG}(3,q)$ representation),

$$\mathcal{F}_s = \{(t, f_s(t), 1) \colon t \in \mathsf{GF}(q)\} \cup \{(0,1,0)\} \cup \{(1,0,0)\}.$$

The second notion that we require is that of what we call *translation oval derivation*. This is a special case of a construction of affine planes from other affine planes by a method called *derivation by net replacement* by Ostrom. The modern terminology of *partial spread replacement* does not suit our purposes so we will describe the method in very basic geometric terms in order to make the technique as transparent as possible.

The basic idea is to replace a set of parallel classes in an affine plane (a *net*) by another net which is compatible with the parallel classes that have not been replaced. For the affine plane $\mathsf{AG}(2, 2^h)$ one way to do this is to replace all except two of the parallel classes of lines by a net which is constructed from translation hyperovals. If $q = 2^h$, a *translation hyperoval* is a hyperoval of $\mathsf{PG}(2,q)$ that is projectively equivalent to the hyperoval passing through the points $(1,0,0)$ and $(0,1,0)$ and whose affine points satisfy the equation $y = x^{2^i}$, where $(i,h) = 1$. For $i = 1$ and $h - 1$ the translation hyperovals are *hyperconics* (a conic together with its nucleus, also known as a *regular hyperoval*), but other values of i give translation hyperovals which are not projectively equivalent to hyperconics. Translation hyperovals were first investigated by B. Segre [15].

For translation oval derivation the replacement net consists of the restriction to the affine plane $\mathsf{PG}(2, 2^h) \setminus \{x_2 = 0\}$ of the set $\mathcal{T}$ of $q^2 - q$ translation hyperovals (for a fixed i) given by $\{y = mx^{2^i} + k\}$, $m (\neq 0), k \in \mathsf{GF}(q)$, $(i,h) = 1$, all of which pass through $(1,0,0)$ and $(0,1,0)$. The result of performing translation oval derivation on $\mathsf{AG}(2, 2^h)$ is an isomorphic affine plane (see [2]). We shall sketch the proof of this result.

Consider the affine plane $\pi = \mathsf{PG}(2, 2^h) \setminus \{x_2 = 0\}$. We construct a new affine plane $\widetilde{\pi}$ whose points are the points of π and whose lines are the sets $T_{m,k} = \{(x,y) \colon y = mx^{2^i} + k\}$ for $m, k \in \mathsf{GF}(2^h)$ and a fixed i with $(i,h) = 1$, and the sets $T_{\infty,k} = \{(x,y) \colon x = k\}, k \in \mathsf{GF}(2^h)$. Note that for any k the sets $T_{0,k}$ and $T_{\infty,k}$ are just lines of π, while for $m \neq 0$ the sets $T_{m,k}$ are the translation hyperovals of $\mathcal{T}$ restricted to the affine plane. Verifying that $\widetilde{\pi}$ is an affine plane is straightforward using the fact that a line meeting a hyperoval (in the projective completion of the affine plane) meets the hyperoval in exactly one further point and some simple algebraic calculations. The main case of verifying that two points determine a unique line follows from the calculation that for affine points (u_1, v_1) and (u_2, v_2) with $u_1 \neq u_2$ and $v_1 \neq v_2$, the unique $T_{m,k}$ containing these points has

$$m = \frac{v_1 + v_2}{(u_1 + u_2)^{2^i}}$$

and

$$k = \frac{u_1^{2^i} v_2 + u_2^{2^i} v_1}{(u_1 + u_2)^{2^i}} \,.$$

The observation that if $m_1 \neq m_2$ and both are finite then

$$T_{m_1,k_1} \cap T_{m_2,k_2} = \left(\left(\frac{k_1 + k_2}{m_1 + m_2} \right)^{2^{-i}}, \frac{m_1 k_2 + m_2 k_1}{m_1 + m_2} \right),$$

leads to a quick verification of Playfair's Axiom and nontriviality is obvious. Thus, $\widetilde{\pi}$ is an affine plane. Finally, the point map $\tau \colon \pi \mapsto \widetilde{\pi}$ given by $\tau(x, y) = (x^{2^{-i}}, y)$ provides an isomorphism between π and $\widetilde{\pi}$, which we note is not a collineation of the affine plane for $i \neq 0$. With an obvious abuse of language we see that τ permutes the sets $T_{\infty,k}$ (lines), stabilizes the sets $T_{0,k}$ (lines) and maps the lines given by $y = mx + k$ to the sets $T_{m,k}$. The main case of showing that incidence is preserved is provided by the following calculation. Suppose the point (u, v) is incident with the line given by $y = mx + k$, then

$$\tau(u, v) = \tau(u, mu + k) = (u^{2^{-i}}, mu + k) = (z, mz^{2^i} + k) \in T_{m,k} \,.$$

Algebraically, (translation) oval derivation is carried out by applying the point map $(x, y) \mapsto (x^{\psi}, y)$ for any (maximal) automorphism ψ of $\mathsf{GF}(2^h)$. From this point of view the role that the (translation) hyperovals play is obfuscated.

3 Construction

As there is a single orbit of elliptic quadrics under $\mathsf{PGL}(4, q)$ we can take as our starting point the quadric Ω (ovoid) whose point set is

$$\Omega := \{(s, t, t^2 + st + \kappa s^2, 1) \colon s, t \in \mathsf{GF}(q)\} \cup \{(0, 0, 1, 0)\} \,,$$

where $\kappa \in \mathsf{GF}(q)$ is a fixed element of absolute trace 1. The line ℓ_∞ given by $x_0 = x_3 = 0$ is tangent to Ω at the point $Q_\infty = (0, 0, 1, 0)$. The secant planes of Ω through ℓ_∞ will be denoted by $\pi_\alpha \colon x_0 = \alpha x_3$ for $\alpha \in \mathsf{GF}(q)$. Furthermore, we define the conic sections $\mathcal{O}_\alpha := \pi_\alpha \cap \Omega$. By the Plane Equivalent Theorem we obtain a fan of conics in π_0 all passing through Q_∞ and having common nucleus $(0, 1, 0, 0)$. The q conics of this fan are represented by

$$\mathcal{F}_\alpha = \{(t, t^2 + \kappa\alpha^2, 1) \colon t \in \mathsf{GF}(q)\} \cup (0, 1, 0) \,,$$

for $\alpha \in \mathsf{GF}(q)$. Note that the o-permutations of (1) are given here by $f_\alpha(t) = t^2 + \kappa\alpha^2$.

Let σ be the automorphism of $\mathsf{GF}(2^{2e-1})$ given by $x \mapsto x^\sigma = x^{2^e}$ and note that $\sigma^2 \equiv 2 \bmod (q-1)$. We now perform translation oval derivation on the affine plane $\pi_0 \setminus \ell_\infty$ using the transformation $(x, y) \mapsto (x^\sigma, y)$. Denote the projective completion of the derived plane by π_0'. Observe that the point $(t, t^2 + \kappa\alpha^2, 1)$ of π_0 is transformed to $(u, u^\sigma + \kappa\alpha^2, 1)$ of π_0' since $u = t^\sigma$. Because $x \mapsto x^{\sigma+2}$ is a permutation of $\mathsf{GF}(q)$, to each α of $\mathsf{GF}(q)$ we can associate a $\beta \in \mathsf{GF}(q)$ such that $\beta^{\sigma+2} = \kappa\alpha^2$. We now see that this translation oval derivation transforms $\mathcal{F}_\alpha \mapsto \mathcal{F}'_\beta = \{(u, u^\sigma + \beta^{\sigma+2}, 1) \colon u \in \mathsf{GF}(q)\} \cup (0, 1, 0)$. It is known that this set of images is a fan associated to the Tits ovoid (see [9]), but for the sake of completeness we shall provide the proof that this set is a fan of translation ovals.

Proposition 3.1. *The translation ovals $\mathcal{F}'_c$ and $\mathcal{F}'_d$ ($c \neq d$), where*

$$\mathcal{F}'_\beta = \{(u, u^\sigma + \beta^{\sigma+2}, 1) \colon u \in \mathsf{GF}(q)\} \cup (0, 1, 0)\,,$$

are compatible at the point $(1, c+d, 0)$.

Proof. A line other than ℓ_∞ through $(1, c+d, 0)$ is a secant (resp. exterior) line of $\mathcal{F}'_\beta$ provided the equation

$$u^\sigma + (c+d)u + k + \beta^{\sigma+2} = 0$$

has 2 (resp. 0) solutions with $k \in \mathsf{GF}(q)$. This equation has 2 (resp. 0) solutions provided

$$\operatorname{tr}\left(\frac{k + \beta^{\sigma+2}}{(c+d)^{\frac{\sigma}{\sigma-1}}}\right)$$

is 0 (resp. 1), where tr is the absolute trace function of $\mathsf{GF}(q)$. For any k we have

$$\operatorname{tr}\left(\frac{k + c^{\sigma+2}}{(c+d)^{\frac{\sigma}{\sigma-1}}}\right) + \operatorname{tr}\left(\frac{k + d^{\sigma+2}}{(c+d)^{\frac{\sigma}{\sigma-1}}}\right) = \operatorname{tr}\left(\frac{c^{\sigma+2} + d^{\sigma+2}}{(c+d)^{\frac{\sigma}{\sigma-1}}}\right)$$
$$= \operatorname{tr}\left(\frac{c^{\sigma+1} + d^{\sigma+1}}{(c+d)^{\sigma+1}}\right) = 1.$$

The penultimate simplification uses the invariance of tr under an automorphism (σ) and the relation $\frac{1}{\sigma-1} \equiv \sigma+1 \bmod (q-1)$ and the last equation is well known (see [3]). Therefore,

$$\operatorname{tr}\left(\frac{k + c^{\sigma+2}}{(c+d)^{\frac{\sigma}{\sigma-1}}}\right) \neq \operatorname{tr}\left(\frac{k + d^{\sigma+2}}{(c+d)^{\frac{\sigma}{\sigma-1}}}\right),$$

and each line other than ℓ_∞ through $(1, c+d, 0)$ is a secant to one and exterior to the other of $\mathcal{F}'_c$ and $\mathcal{F}'_d$. □

Using the Plane Equivalent Theorem applied to the fan $\{\mathcal{F}'_\beta\}_{\beta \in \mathsf{GF}(q)}$ we obtain the Tits ovoid, thus completing the construction.

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