A characterization of the geometry of large maximal cliques of the alternating forms graph

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Abstract

We prove that the geometry of vertices, edges and q^n-cliques of the graph $\mathsf{Alt}(n+1, q)$ of $(n+1)$-dimensional alternating forms over $\mathsf{GF}(q)$, $n \geq 4$, is the unique flag-transitive geometry of rank 3 where planes are isomorphic to the point-line system of $\mathsf{AG}(n, q)$ and the star of a point is dually isomorphic to a projective space.

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MSC 2000: 51E24, 05E30, 05E20

1. Introduction

We recall that the alternating forms graph $\mathsf{Alt}(n+1, q)$ is the graph the vertices of which are the alternating bilinear forms on $V(n+1, q)$, two such forms α and β being adjacent precisely when $\mathrm{rk}(\alpha - \beta) = 2$. The following is known:

Theorem 1.1 (Munemasa and Shpectorov [15]). *For a prime power $q > 2$ and an integer $n \geq 3$, let Γ be a graph with the following properties:*

(1) *Γ is locally the $(q-1)$-clique extension of the collinearity graph of the grassmannian of lines of $\mathsf{PG}(n, q)$;*

(2) *any two vertices of Γ at distance 2 have exactly $q^2(q^2+1)$ common neighbours.*

Then Γ is covered by the alternating forms graph $\mathsf{Alt}(n+1, q)$.

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Needless to say, $\mathsf{Alt}(n+1,q)$ satisfies both (1) and (2). In view of (1), the maximal cliques of $\mathsf{Alt}(n+1,q)$ have size q^n and q^3. When $n>3$ the q^n-cliques are the largest ones. The intersection of two of them is either empty or a singular line, namely a q-clique of the form $\{a,b\}^{\perp\perp}$ (notation as in Brouwer, Cohen and Neumaier [3, Appendix A.8]). Let $\mathcal{G}(\mathsf{Alt}(n+1,q))$ be the geometry of rank 3 with the vertices of $\mathsf{Alt}(n+1,q)$ as points, the singular lines as lines and the q^n-cliques as planes. In this paper we will exploit Theorem 1.1 to prove that $\mathcal{G}(\mathsf{Alt}(n+1,q))$ is the unique flag-transitive L.PG*-geometry with n-dimensional affine spaces of order q as planes. That result, combined with the reduction theorem by Huybrechts [9, Theorem 5.5.9] and the characterization theorems of Huybrechts and Pasini [11] and Cardinali and Pasini [5], completes the classification of flag-transitive L.PG*-geometries.

We will state our main result in a slightly different and more complete way at the end of this introduction, after having translated Theorem 1.1 into the languague of diagram geometry. In view of that translation, we need to recall the definition and a few properties of certain subgeometries of D_{n+1}-buildings, which we call 'affine half-spin geometries'.

1.1. Notation and terminology

We follow [17] for diagram geometries but for a few minor changes in notation, as using the symbol Tr_J for Tr_J^+ (see below), the symbol Res for residues and denoting geometries by capital italics instead of greek letters. We only recall a few definitions here, which we need as soon as in the next subsection.

Given a geometry $\mathcal{G}$ over the type-set I and a nonempty subset J of I, we denote by $\mathrm{Tr}_J(\mathcal{G})$ the geometry induced by $\mathcal{G}$ on the set of elements of type $j\in J$ and we call it the *J-truncation* of $\mathcal{G}$. Suppose $0,1\in I$ are such that 1 is the unique type joined to 0 in the diagram graph of $\mathcal{G}$. The $\{0,1\}$-truncation of $\mathcal{G}$ is usually regarded as a point-line geometry, with the 0-elements as points and the 1-elements as lines and two 0-elements are said to be *collinear* when they are incident with the same 1-element. We call the collinearity graph of $\mathrm{Tr}_{\{0,1\}}(\mathcal{G})$ the *0-graph* of $\mathcal{G}$ and we denote it by $\Gamma_0(\mathcal{G})$. We use the symbol $\perp$ for the collinearity relation. Accordingly, given a 0-element x, the set $x^\perp$ consists of x and its neighbours in $\Gamma_0(\mathcal{G})$.

1.2. Affine half-spin geometries

For an integer $n\geq 3$ and a prime power q, we denote by $\mathcal{D}_{n+1}(q)$ the building of type D_{n+1} defined over $\mathsf{GF}(q)$ (see [21]) and we take the integers $0,1,\ldots,n-1$ and the symbol 0^* as types, as follows:

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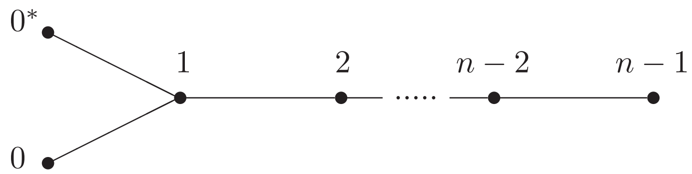
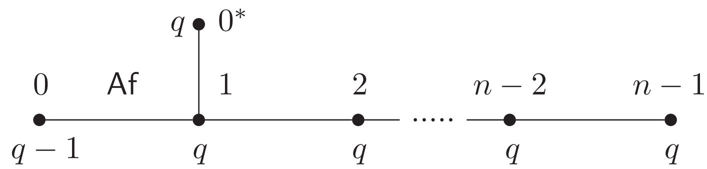


Given an ε-element S_0 of $\mathcal{D}_{n+1}(q)$, where ε stands for 0 or 0^* according to whether n is odd or even, let $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ be the subgeometry of $\mathcal{D}_{n+1}(q)$ far from S_0, namely the geometry formed by the elements of $\mathcal{D}_{n+1}(q)$ that, compatibly with their type, have maximal distance from S_0, with the incidence relation inherited from $\mathcal{D}_{n+1}(q)$ but rectified as follows: a 0^*-element S and an element X of $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ of type $i \in \{2, 3, \ldots, n-1\}$ are incident in $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ if and only if they are incident in $\mathcal{D}_{n+1}(q)$ and the flag $\{S, X\}$ is as far as possible from S_0 (see Blok and Brouwer [2]).

It is easy to see that $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ belongs to the following diagram, which we call $\mathsf{D}^{\mathsf{Af}}_{n+1}$, where $\bullet\!\overset{\mathsf{Af}}{\rule[2pt]{2.5em}{0.4pt}}\!\bullet$ stands for the class of affine planes and the integers $q-1$ and q are finite orders, $q = p^h$ for a prime p and a positive integer h:

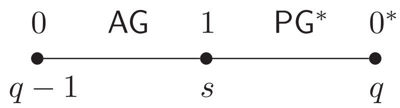
$(\mathsf{D}^{\mathsf{Af}}_{n+1})$



The $\{0, 1, 0^*\}$-truncation $\mathrm{Tr}_{\{0,1,0^*\}}(\mathrm{Far}(\mathcal{D}_{n+1}(q)))$ of $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ will be denoted by the symbol $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ and called *affine half-spin geometry* of *Far-type* and *affine rank* n. (Recall that the point-line geometry $\mathrm{Tr}_{\{0,1\}}(\mathcal{D}_{n+1}(q))$ is commonly called 'half-spin geometry'; note also that $\mathrm{Tr}_{\{0,1\}}(\mathrm{Far}(\mathcal{D}_{n+1}(q)))$ admits a family of maximal singular subspaces isomorphic to the n-dimensional affine space $\mathsf{AG}(n, q)$ over $\mathsf{GF}(q)$.)

Clearly, $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ belongs to the following diagram, where the labels AG and PG* denote the class of affine spaces and, respectively, the class of dual projective spaces, both being regarded as point-line geometries:

(AG.PG*)



$(s = q^{n-1} + \cdots + q^2 + q)$

The geometry $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ inherits the following properties from $\mathcal{D}_{n+1}(q)$:

(LL) no two distinct 0-elements are incident with the same pair of distinct 1-elements;

(T) every 3-clique of the 0-graph is incident to a 0^*-element.

Furthermore, the 0-graph of $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ is isomorphic to $\mathsf{Alt}(n+1,q)$ (see Brouwer, Cohen and Neumaier [3, 9.5.11(ii)]). So, the following also holds in $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ (compare condition (2) of Theorem 1.1):

(μ) if two 0-elements a, b have distance 2 in the 0-graph, then $|a^\perp \cap b^\perp| = q^2(q^2+1)$.

The set $H(S_0)$ of 0-elements of $\mathcal{D}_{n+1}(q)$ at non-maximal distance from S_0 is a hyperplane of the half-spin geometry $\mathrm{Tr}_{\{0,1\}}(\mathcal{D}_{n+1}(q))$ (Shult [20]). Thus, $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ can be described as the geometry obtained from $\mathcal{D}_{n+1}(q)$ by removing the hyperplane $H(S_0)$. More generally, a geometry $\mathcal{D} \setminus H$ belonging to diagram $\mathrm{D}^{\mathsf{Af}}_{n+1}$ can be obtained from $\mathcal{D} = \mathcal{D}_{n+1}(q)$ by removing an arbitrary hyperplane H of $\mathrm{Tr}_{\{0,1\}}(\mathcal{D})$. Properties (LL) and (T) hold in $\mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{D} \setminus H)$, no matter which hyperplane we choose as H. On the other hand, $\mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{D} \setminus H)$ satisfies (μ) if and only if it is of Far-type, namely $H = H(S_0)$ for an ε-element S_0. Moreover, as it follows from Proposition 4.1 of this paper (section 4), when $n > 3$ then $\mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{D} \setminus H)$ is flag-transitive if and only it is of Far-type. So, in this paper, we are not going to discuss $\mathcal{D} \setminus H$ for an arbitrary $H \neq H(S_0)$ when $n > 3$. The case of $H \neq H(S_0)$ will be considered only for $n = 3$.

Assume $n = 3$. When we regard the elements of $\mathcal{D} := \mathcal{D}_4(q)$ of type 0 and 1 as singular points and totally singular lines of $\mathsf{PG}(7,q)$ for a given non-singular quadratic form f, the elements of $\mathcal{D}$ of type 0^* and 2 are the planes of $\mathsf{PG}(7,q)$ that are totally singular for f. It is known that every geometric hyperplane of $\mathrm{Tr}_{\{0,1\}}(\mathcal{D})$ is the intersection of the set of 0-elements of $\mathcal{D}$ with a hyperplane of $\mathsf{PG}(7,q)$ (Cohen and Shult [7]). Given a hyperplane H of $\mathsf{PG}(7,q)$, the *complement* $\mathcal{D} \setminus H$ of H is the geometry formed by the elements $\mathcal{D}$ that are not contained in H, with the incidence relation inherited from $\mathcal{D}$ but rectified as follows: a $\{0^*,2\}$-flag $\{X,Y\}$ of $\mathcal{D}$ is a flag of $\mathcal{D} \setminus H$ if and only if $X \cap Y \not\subseteq H$.

If H is *tangent* to $\mathcal{D}$, namely the form f_H induced by f on H is singular, then $H = H(S_0)$, where S_0 is the radical point of f_H. In this case $\mathcal{D} \setminus H$ is the subgeometry of $\mathcal{D}$ far from S_0, whence $\mathcal{D} \setminus H \cong \mathrm{Far}(\mathcal{D}_4(q))$. On the other hand, let H be *secant*, namely f_H is non-singular. Then the 0-graph of $\mathcal{D} \setminus H$ does not satisfy (μ). In fact, we have $|a^\perp \cap b^\perp| = q^2(q^2+1)$ only for some pairs $\{a, b\}$ of points at distance 2, whereas $|a^\perp \cap b^\perp| = q^4$ for the remaining pairs.

When H is secant, we put $\mathrm{TSec}(\mathcal{D}_4(q)) := \mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{D} \setminus H)$ and we call it *affine half-spin geometry* of *Secant-type* and *affine rank* 3.

1.3. Theorem 1.1 revisited

As remarked in the previous subsection, the 0-graph $\Gamma := \Gamma_0(\mathrm{TFar}(\mathcal{D}_{n+1}(q)))$ of $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ is isomorphic to $\mathsf{Alt}(n+1,q)$. On the other hand, the maximal cliques of Γ correspond to the elements of $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ of type 0^* and 2. More explicitly, if X is a 0^*-element, then the set of 0-elements incident to X is a maximal clique of Γ of size q^n whereas, if X has type 2, then the 0-elements incident to it form a maximal clique of size q^3. Furthermore, the set of 0-elements incident to a given 1-element is a singular line of Γ. Thus, when $n > 3$, $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ is isomorphic to the geometry $\mathcal{G}(\mathsf{Alt}(n+1,q))$ defined at the very beginning of this introduction.

Suppose $n = 3$. The 1-elements of $\mathrm{TFar}(\mathcal{D}_4(q))$ still correspond to the singular lines of $\mathsf{Alt}(4,q)$. However, all maximal cliques of $\mathsf{Alt}(4,q)$ have size q^3, so we cannot recognize those corresponding to 0^*-elements as the largest ones. Nevertheless, we can still recover a partition of the maximal cliques in two classes, as follows: Define a graph Λ on the set of maximal cliques of $\mathsf{Alt}(4,q)$ by declaring two of them to be adjacent when they meet in a 4-clique. The graph Λ is connected and bipartite. The two classes of its bipartition correspond to the types 0^* and 2, except that we don't know which class correspond to 0^* and which to 2. However, there is no need to know that. Indeed, as $\mathcal{D}_4(q)$ admits a non-type-preserving automorphism permuting the types 0^* and 2, we may pick any of those two classes, claiming it corresponds to the type 0^*. So, we can still recover a copy $\mathcal{G}(\mathsf{Alt}(4,q))$ of $\mathrm{TFar}(\mathcal{D}_4(q))$ inside $\mathsf{Alt}(4,q)$.

We are now ready to give Theorem 1.1 an openly geometric formulation.

Corollary 1.2. *Let $\mathcal{G}$ be a geometry belonging to diagram* AG.PG* *with $n \geq 3$ and $q > 2$ and suppose that properties* (LL), (T) *and* (μ) *of subsection 1.2 hold in it. Then $\mathcal{G}$ is a quotient of* $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$.

Proof. Properties (LL) and (T) allow us to recover $\mathcal{G}$ inside its 0-graph $\Gamma_0(\mathcal{G})$ just in the same way as we have recovered $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ as $\mathcal{G}(\mathsf{Alt}(n+1,q))$ in $\mathsf{Alt}(n+1,q)$. As $\mathcal{G}$ belongs to AG.PG*, $\Gamma_0(\mathcal{G})$ satisfies condition (1) of Theorem 1.1. Condition (2) of that theorem is property (μ), which holds by assumption. So, by Theorem 1.1, there exists a covering $f : \mathsf{Alt}(n+1,q) \to \Gamma_0(\mathcal{G})$. However, $\mathcal{G}$ can be recovered from $\Gamma_0(\mathcal{G})$, as remarked above. Hence f induces a covering from $\mathcal{G}(\mathsf{Alt}(n+1,q))$ ($\cong \mathrm{TFar}(\mathcal{D}_{n+1}(q))$) to $\mathcal{G}$. □

1.4. Main results

The automorphism group of $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ is induced by the stabilizer of S_0 in $\mathrm{Aut}(\mathcal{D}_{n+1}(q))$ and acts flag-transitively on $\mathrm{Far}(\mathcal{D}_{n+1}(q))$. The complement of a secant hyperplane of $\mathcal{D}_4(q)$ is also flag-transitive, with automorphism group isomorphic to $\Omega_7(q).C$, where $C = \mathrm{Aut}(\mathsf{GF}(q))$. Thus, putting aside properties (LL), (T) and (μ), we focus on flag-transitivity.

Theorem 1.3. *Let $\mathcal{G}$ be a flag-transitive geometry belonging to diagram* AG.PG* *with $n \geq 3$ and finite orders $q-1, s, q$, where q is a prime power and where $s = q^{n-1} + \cdots + q^2 + q$. Then either $\mathcal{G} \cong \mathrm{TFar}(\mathcal{D}_{n+1}(q))$ or $n = 3$ and $\mathcal{G} \cong \mathrm{TSec}(\mathcal{D}_4(q))$.*

In particular:

Corollary 1.4. *Suppose $n > 3$. Then $\mathrm{TFar}(\mathcal{D}_{n+1}(q))$ is the unique flag-transitive geometry belonging to* AG.PG* *with orders $q-1, s, q$ as in Theorem 1.3.*

We recall that, as proved by Kantor [12], if a subgroup X of $\mathsf{P\Gamma L}_{n+1}(q)$ acts line-transitively on $\mathsf{PG}(n, q)$ $(n \geq 3)$, then either X is flag-transitive or $n = 4$, $q = 2$ and $X =$ Frob$(31 \cdot 5)$, regular on the set of lines of $\mathsf{PG}(4, 2)$. Therefore, if for an AG.PG*-geometry $\mathcal{G}$ of rank $n \geq 3$ its automorphism group is transitive on the set of $\{0, 1\}$-flags, then either $\mathcal{G}$ is flag-transitive and the conclusions of Theorem 1.3 hold, or $n = 4$, $q = 2$ and the stabilizer in $\mathrm{Aut}(\mathcal{G})$ of a 0-element p acts faithfully and regularly as $\mathrm{Frob}(31 \cdot 5)$ on the set of 1-elements incident to p. (The faithfulness of that action follows from Lemma 2.8 of Huybrechts and Pasini [10].) A graph-theoretic translation of the above sounds as follows:

Corollary 1.5. *Let Γ be a graph satisfying condition (1) of Theorem 1.1, for a given integer $n \geq 3$ and a prime power q, but allowing $q = 2$. Suppose that $\mathrm{Aut}(\Gamma)$ acts transitively on the set of pairs (v, L), where v is a vertex and L a singular line of Γ containing v. Then one of the following occurs:*

(1) $\Gamma = \mathsf{Alt}(n+1, q)$;

(2) $n = 3$ *and Γ is the 0-graph of* $\mathrm{TSec}(\mathcal{D}_4(q))$;

(3) $n = 4$, $q = 2$ *and the stabilizer in $\mathrm{Aut}(\Gamma)$ of a vertex v of Γ is isomorphic* $\mathrm{Frob}(31 \cdot 5)$*, acting regularly on the neighbourhood of v.*

We are not aware of any example as in case (3) of the above corollary. Nevertheless, we have not been able to rule out that case.

We finish this section with a generalization of Theorem 1.3. By combining it with the characterization theorems of Huybrechts and Pasini [11] and Cardinali and Pasini [5] and the reduction theorem of Huybrechts [9, Theorem 5.5.9] (see also Cardinali and Pasini [5, Proposition 2.1]), we immediately obtain the following:

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Theorem 1.6. *Let $\mathcal{G}$ be a flag-transitive geometry belonging to the diagram*

$$(\mathsf{L.PG}^*) \qquad \overset{0}{\underset{r}{\bullet}} \overset{\mathsf{L}}{\rule{3cm}{0.4pt}} \overset{1}{\underset{s}{\bullet}} \overset{\mathsf{PG}^*}{\rule{3cm}{0.4pt}} \overset{0^*}{\underset{q}{\bullet}}$$

where the label L *denotes the class of linear spaces, q is a prime power, $s = q^{n-1} + \cdots + q^2 + q$ for an integer $n \geq 3$ and $1 \leq r \leq s$. Then one of the following occurs:*

(1) $r = q$ *and* $\mathcal{G} \cong \mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{D}_{n+1}(q))$*;*

(2) $r = q - 1$ *and* $\mathcal{G} \cong \mathrm{TFar}(\mathcal{D}_{n+1}(q))$*;*

(3) $r = q - 1$*,* $n = 3$ *and* $\mathcal{G} \cong \mathrm{TSec}(\mathcal{D}_4(q))$*.*

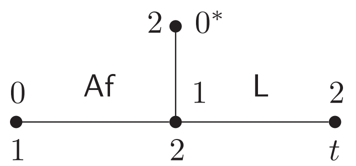
The proof of Theorem 1.3 will take the rest of this paper. We outline its main steps here. We shall firstly consider $\mathrm{D}^{\mathsf{Af}}_{n+1}$-geometries. In section 2, after a few elementary lemmas, we exploit Corollary 1.2 to obtain a sufficent condition for a $\mathrm{D}^{\mathsf{Af}}_{n+1}$-geometry to be covered by $\mathrm{Far}(\mathcal{D}_{n+1}(q))$. In section 3 we prove that, when $n > 3$, the $\{0, 0^*, 1, 2\}$-residues of a flag-transitive $\mathrm{D}^{\mathsf{Af}}_{n+1}$-geometry are isomorphic to $\mathrm{Far}(\mathcal{D}_4(q))$. At that stage we can prove that $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ is the unique flag-transitive $\mathrm{D}^{\mathsf{Af}}_{n+1}$-geometry when $n > 3$ (section 4). We turn to diagram $\mathsf{AG.PG}^*$ in sections 5 and 6, proving that every flag-transitive geometry belonging to that diagram extends to a flag-transitive $\mathrm{D}^{\mathsf{Af}}_{n+1}$-geometry. In view of the result of section 4, this finishes the proof of Theorem 1.3.

We warn the reader that our proof exploits the classification of finite 2-transitive groups (see the proof of Proposition 3.6), which in its turn depends on the classification of finite simple groups.

Note also that $q = 2$ is allowed in Theorem 1.3. However, as that case has been settled by Huybrechts and Pasini [11] a few years ago, we will not spend much time on it in this paper. On the other hand, the assumption $q > 2$ is essential in Theorem 1.1 and Corollary 1.2, as we will explain in the next remark.

1.4.0.1. Remark. We recall that the quadratic forms graph $\mathrm{Quad}(n, q)$ has the quadratic forms on $V(n, q)$ as vertices and adjacency relation defined as follows: two quadratic forms f and g are adjacent when $1 \leq \mathrm{rk}(f - g) \leq 2$. The graph $\mathrm{Quad}(n, 2)$ is distance-regular with the same local structure and the same intersection numbers as $\mathsf{Alt}(n + 1, 2)$ (see Brouwer, Cohen and Neumaier [3, 9.6.3] for the intersection array and Munemasa, Pasechnik and Shpectorov [16, Proposition 3.1] for the local structure). The following is known (Munemasa, Pasechnik and Shpectorov [16]): Given a graph Γ satisfying hypotheses (1) and (2) of Theorem 1.1 but with $q = 2$, suppose that, for any two vertices a, b at distance 2, precisely $15 \cdot 2^n - 105$ of the vertices adjacent to b have distance 2 from a; then Γ is covered by either $\mathsf{Alt}(n+1, 2)$ or $\mathrm{Quad}(n, 2)$. Accordingly, in Corollary 1.2, the geometry $\mathcal{G}(\mathrm{Quad}(n, 2))$ of vertices, edges and 2^n-cliques of $\mathrm{Quad}(n, 2)$ should be allowed as a possible cover of $\mathcal{G}$ when $q = 2$ and $n > 3$. (As $\mathrm{Quad}(n, 2)$ and $\mathsf{Alt}(n+1, 2)$ have the same local structure, the maximal cliques of $\mathrm{Quad}(n, 2)$ have size 2^n and 8, as in $\mathsf{Alt}(n+1, 2)$.)

As $\mathrm{Quad}(3, 2) \cong \mathsf{Alt}(4, 2)$, nothing new arises when $n = 3$. Suppose $n > 3$. Then, in view of Theorem 1.3 and since $\mathrm{Quad}(n, 2) \not\cong \mathsf{Alt}(n+1, 2)$ (Brouwer, Cohen and Neumaier [3, 9.6.4]), the geometry $\mathcal{G}(\mathrm{Quad}(n, 2))$ is not flag-transitive. We can enrich $\mathcal{G}(\mathrm{Quad}(n, 2))$ by taking the maximal cliques of size 8 as 2-elements, stating that a 2^n-clique and a maximal 8-clique are incident precisely when they meet in a 4-clique. Thus, we obtain a geometry $\overline{\mathcal{G}}(\mathrm{Quad}(n, 2))$ for the following diagram:



$(t = 2^{n-2} + \cdots + 2^2 + 2)$

Using [6], one can prove that $\overline{\mathcal{G}}(\mathrm{Quad}(n, 2))$ is a truncation of a chamber system belonging to $\mathrm{D}^{\mathsf{Af}}_{n+1}$, but we do not know if that chamber system arises from a geometry.

2. Elementary properties of $\mathrm{D}^{\mathsf{Af}}_{n+1}$-geometries

Throughout this section $\mathcal{G}$ is a given geometry belonging to diagram $\mathrm{D}^{\mathsf{Af}}_{n+1}$ with finite orders $q-1, q, \ldots, q$. We will denote the incidence relation of $\mathcal{G}$ by $*$ and, for an element x of $\mathcal{G}$, we denote by $\sigma(x)$ the set of 0-elements incident to x. We also call the 0- and 1-elements *points* and *lines*. The *distance* between two points is their distance in the 0-graph $\Gamma_0(\mathcal{G})$.

If x is an element of type $i = 2, 3, \ldots, n-2$, its residue is the direct sum $\mathrm{Res}(x) = \mathrm{Res}^-(x) \oplus \mathrm{Res}^+(x)$ of an $(n-1-i)$-dimensional projective geometry $\mathrm{Res}^+(x)$ and a geometry $\mathrm{Res}^-(x)$ over the set of types $\{0, 0^*, 1, \ldots, i-1\}$. Clearly, $\mathrm{Res}^-(x) \cong \mathsf{AG}(3, q)$ when $i = 2$. If $i > 2$, then $\mathrm{Res}^-(x)$ belongs to $\mathrm{D}^{\mathsf{Af}}_{i+1}$. Sometimes, in the sequel, we will also take the liberty of writing $\mathrm{Res}^-(x)$ for $\mathrm{Res}(x)$ for x of type $n-1$.

We firstly recall a well known result [17, Theorem 7.57], which settles the case of $n = 3$.

Proposition 2.1. *If $n = 3$ then $\mathcal{G} = \mathcal{D}\backslash H$ where $\mathcal{D} = \mathcal{D}_4(q)$ and H is a hyperplane H of $\mathcal{D}$.*

We now turn to the general case.

Lemma 2.2. *Given a line l and an element x of type $i \neq 0$, if $|\sigma(l) \cap \sigma(x)| > 1$, then $l * x$.*

Proof. By induction on n. It is easy to see that the lemma holds true for complements of hyperplanes of D_4-buildings. So, in view of Proposition 2.1, we may assume $n > 3$. Suppose first $i = n - 1$. Given $a \in \sigma(l) \cap \sigma(x)$, the elements l and x appear as a line and a hyperplane of the projective geometry $\mathrm{Res}(a) \cong \mathsf{PG}(n, q)$. Hence $\mathrm{Res}(a)$ contains a 0^*-element A incident to both l and x. In the affine geometry $\mathrm{Res}(A) \cong \mathsf{AG}(n, q)$ we see l as a line and x as a hyperplane. Furthermore, $\sigma(l) \subseteq \sigma(A)$. Hence $l * x$, as $|\sigma(l) \cap \sigma(x)| > 1$.

Assume now $1 \leq i < n - 1$. Pick an $(n-1)$-element $y * x$. Then $\sigma(x) \subseteq \sigma(y)$. Therefore $|\sigma(l) \cap \sigma(y)| > 1$, whence $l * y$ by the above. Then $l * x$, by induction hypothesis on the $\mathsf{D}_n^{\mathsf{Af}}$-geometry $\mathrm{Res}(y)$.

Finally, suppose x is a 0^*-element. Let m be the line of $\mathrm{Res}(x) \cong \mathsf{AG}(n, q)$ through two given points of $\sigma(l) \cap \sigma(x)$. Thus, $|\sigma(l) \cap \sigma(m)| > 1$. However, we have already proved the statement of the lemma for all types $i = 1, 2, \ldots, n-1$. In particular, that statement holds for $i = 1$. Hence $l = m$, implying $l * x$. □

Note that property (LL) is contained in Lemma 2.2. So, given two collinear points a, b, the line through them is unique. We will denote it by ab. The next lemma is a stronger version of property (T).

Lemma 2.3. *Let a, b, c be distinct points forming a 3-clique in the 0-graph of $\mathcal{G}$, but not in the same line. Then there exists a unique $\{0^*, 2\}$-flag incident to all of a, b, c, ab, bc and ca.*

Proof. Given an $(n-1)$-element $x * bc$, $\mathrm{Res}(b)$ contains a 0^*-element A incident to bc and ab. Thus, $a, c \in \sigma(A) \cap \sigma(ac)$. Hence $ac * A$ by Lemma 2.3. So, $\mathrm{Res}(A)$ contains each of the lines ab, bc and ca. A 2-element α incident to all of ab, bc and cb can be found in $\mathrm{Res}(A)$. Clearly, A and α are uniquely determined, as the lines ab, bc and ca are mutually distinct. □

Lemma 2.4. *Suppose $n = 3$. Then, for any two points a, b at distance 2, the 0-graph of $\mathcal{G}$ induces a connected graph on $a^\perp \cap b^\perp$.*

Proof. Easy, by Proposition 2.1 and well known properties of complements of hyperplanes in D_4-buildings. □

Lemma 2.5. *Suppose $n > 3$ and, for two points a, b at distance 2, let $\mathcal{S}_{a,b}$ be the set of 3-elements incident to both of them. Then the intersections $a^\perp \cap b^\perp \cap \sigma(S)$ ($S \in \mathcal{S}_{a,b}$) are the connected components of the graph induced by $\Gamma_0(\mathcal{G})$ on $a^\perp \cap b^\perp$.*

Proof. Given $c \in a^\perp \cap b^\perp$, the lines ac and cb are non-coplanar in the projective geometry $\mathrm{Res}(c)$. Indeed, if otherwise, they are incident to the same 0^*-element A and in $\mathrm{Res}(A)$ we see that $a \perp b$, contrary to our assumptions.

As ac and cb are skew lines of $\mathrm{Res}(c)$, they are contained in a unique 3-space S of the projective geometry $\mathrm{Res}(c)$. Clearly, S is a 3-element of $\mathcal{G}$. By Lemma 2.4, $a^\perp \cap b^\perp \cap S$ is contained in a connected component of $a^\perp \cap b^\perp$. Let d be a point of $a^\perp \cap b^\perp \cap c^\perp$. By Lemma 2.3, there exists a 2-element α incident to ac, ad and cd and a 2-element β incident to bc, bd and cd. Regarded as planes of the projective geometry $\mathrm{Res}(c)$, α and β meet in the line cd. Hence they are contained in a 3-space S'. However, S is the unique 3-space of $\mathrm{Res}(c)$ containing both ac and bc. Therefore, $S' = S$, that is: $d \in \sigma(S)$. So, $\sigma(S)$ contains the neighbourhood of c in $a^\perp \cap b^\perp$. It is now clear that $\sigma(S) \cap a^\perp \cap b^\perp$ is the connected component of c in $a^\perp \cap b^\perp$. □

Lemma 2.6. *Assume $n > 3$ and suppose that $\mathrm{Res}^-(S) \cong \mathrm{Far}(\mathcal{D}_4(q))$ for every 3-element S of $\mathcal{G}$. Then, for any two points a, b at distance 2, the 0-graph induces a connected graph on $a^\perp \cap b^\perp$.*

Proof. Given two distinct non-collinear points $c, d \in a^\perp \cap b^\perp$, for $x = c, d$ let S_x be the 3-element of $\mathrm{Res}(x)$ incident to xa and xb. We shall prove that $S_c = S_d$. Then, by Lemma 2.4 in $\mathrm{Res}^-(S_c)$, the points c and d belong to the same connected component of $a^\perp \cap b^\perp$ and we are done.

Suppose $n = 4$. Then S_c and S_d are hyperplanes of the 4-dimensional projective geometry $\mathrm{Res}(a)$. So, $\mathrm{Res}(a)$ contains a plane, namely a 2-element, incident to both S_c and S_d. Suppose firstly that $x^\perp \cap b^\perp \cap \sigma(X) \neq \emptyset$ for $x = c$ or $x = d$. For instance, $c^\perp \cap b^\perp \cap \sigma(X) \neq \emptyset$. Pick a point $z \in c^\perp \cap b^\perp \cap \sigma(X)$. As $X * S_d$, $z \in \sigma(S_d)$. Hence $zb * S_d$ by Lemma 2.2. By Lemma 2.5, $c \in \sigma(S_d)$. Therefore, by Lemma 2.2, S_d is incident to each of the lines ac and bc. Consequently, $S_d = S_c$ by the uniqueness of the 3-space of $\mathrm{Res}(c)$ on the skew lines ac and bc.

Assume now that $c^\perp \cap b^\perp \cap \sigma(X) = d^\perp \cap b^\perp \cap \sigma(X) = \emptyset$. By assumption, $\mathrm{Res}(S_c) = \mathcal{D} \setminus H$ for $\mathcal{D} = \mathcal{D}_4(q)$ and a tangent hyperplane H. So, $H \cap \mathcal{D} = p^\perp$ for a unique point p of $\mathcal{D}$. (Warning: p is not a point of $\mathcal{G}$ and we are using the symbol $\perp$ for the collinearity relation of $\mathcal{D}$, too.) Regarded X as a singular 3-space of $\mathcal{D}$, $b^\perp \cap X$ is a plane of X. However, $b^\perp \cap X \not\subseteq H$. Indeed, only two 3-spaces of $\mathcal{D}$ exist that contain $X \cap H$; the space X is one of them, the other one is contained in H and is spanned by $X \cap H$ and p, but none of these two spaces contains b. Thus, turning to $\mathcal{G}$, both $\beta := b^\perp \cap \sigma(X)$ and $\gamma := c^\perp \cap \sigma(X)$

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are planes of the affine geometry $\mathrm{Res}^{-}(X) \cong \mathsf{AG}(3, q)$. Similarly, $\delta := d^{\perp} \cap \sigma(X)$ is a plane of that affine geometry. Furthermore, $a \in \gamma \cap \delta$. Hence γ and δ are non-parallel in $\mathrm{Res}^{-}(X)$. Consequently, β meets one of them, contrary to the assumption that both intersections $c^{\perp} \cap b^{\perp} \cap \sigma(X)$ and $d^{\perp} \cap b^{\perp} \cap \sigma(X)$ are empty. This proves the equality $S_c = S_d$ when $n = 4$.

Suppose $n = 5$. The elements S_c and S_d are 3-dimensional subspaces of the 5-dimensional projective geometry $\mathrm{Res}(a)$. So, there exists a line l_a of $\mathrm{Res}(a)$ incident to both S_c and S_d. Similarly, a line l_b incident to S_c and S_d exists in $\mathrm{Res}(b)$. As $a \not\perp b$, $l_a \neq l_b$. If $a^{\perp} \cap \sigma(l_b) \neq \emptyset$, pick a point $x \in a^{\perp} \cap \sigma(l_b) \neq \emptyset$. Then $x \in \sigma(S_c) \cap \sigma(S_d)$. By Lemma 2.2, both ax and xb are incident with both S_c and S_d. However, the lines ax and xb are skew in $\mathrm{Res}(x)$ and both S_c and S_d are 3-spaces of $\mathrm{Res}(x)$. Hence $S_c = S_d$.

Similarly, if $b^{\perp} \cap \sigma(l_a) \neq \emptyset$, then $S_c = S_d$. Suppose now $a^{\perp} \cap \sigma(l_b) = b^{\perp} \cap \sigma(l_a) = \emptyset$. By assumption, $\mathrm{Res}(S_c) = \mathcal{D} \setminus H$ for $\mathcal{D} = \mathcal{D}_4(q)$ and a tangent hyperplane H. Regard l_a and l_b as lines of $\mathcal{D}$. Then the hypothesis that $a^{\perp} \cap \sigma(l_b) = b^{\perp} \cap \sigma(l_a) = \emptyset$ in $\mathcal{G}$ implies that the point of l_a collinear with b in $\mathcal{D}$ belongs to H. Similarly, the point of l_b collinear with a in $\mathcal{D}$ belongs to H. Therefore, every point x of $\mathcal{G}$ in $\sigma(l_a) \setminus \{a\}$ is collinear in $\mathcal{G}$ with a unique point $y \in \sigma(b) \setminus \{b\}$. Put $l := xy$, for x, y as above. As both l_a and l_b are incident to both S_c and S_d, we have $\{x, y\} \subseteq \sigma(S_c) \cap \sigma(S_d)$ and Lemma 2.2 implies that $S_c * l * S_d$. As $l_a \neq l_b$, either $l \neq l_a$ or $l \neq l_b$. Suppose $l \neq l_a$, to fix ideas. The lines l_a and l are skew in $\mathrm{Res}(x)$. Indeed $a \not\perp y$, as $a^{\perp} \cap \sigma(l_b) = \emptyset$ by assumption. However, both l and l_a are incident to both S_c and S_d. Hence $S_c = S_d$.

Finally, let $n > 5$. Then $\mathrm{Res}(a)$ contains a 3-element S incident with both ac and ad. In $\mathrm{Res}(c)$ we see S and S_c as 3-spaces incident to the same line ac. Therefore, there exists a 5-element X incident to both S and S_c. All lines ac, bd and ad are incident to X. Hence $bd * X$ too, by Lemma 2.2. However, $\mathrm{Res}^{-}(X)$ ($= \mathrm{Res}(X)$ when $n = 6$) is a $\mathsf{D}_6^{\mathsf{Af}}$-geometry (case of $n = 5$) and we have already proved that the statement of the lemma holds for $\mathsf{D}_6^{\mathsf{Af}}$-geometries. So, a path of $a^{\perp} \cap b^{\perp}$ going from c to d can be found inside $\mathrm{Res}(X)$. □

Corollary 2.7. *Assume the hypotheses of Lemma 2.6. Then $\mathcal{G}$ satisfies property (μ) of subsection 1.2.*

Proof. The 0-graph of $\mathrm{Far}(\mathcal{D}_4(q))$ is isomorphic to $\mathsf{Alt}(4, q)$. Hence property (μ) holds in $\mathrm{Far}(\mathcal{D}_4(q))$. The conclusion follows from this remark and Lemmas 2.5 and 2.6. □

Proposition 2.8. *Assume the hypotheses of Lemma 2.6. Then $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ is the universal 2-cover of $\mathcal{G}$.*

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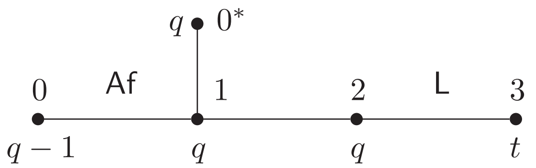
Proof. By Lemmas 2.2 and 2.3 and Corollary 2.7, properties (LL), (T) and (μ) hold in $\mathcal{G}$. By Corollary 1.2, there exists a covering $f\colon \mathrm{TFar}(\mathcal{D}_{n+1}(q)) \to \mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{G})$. The elements of $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ of type 2 can be recovered in the 0-graph as maximal cliques of size q^3. Those of type $i = 3, 4, \ldots, n-1$ can be recovered in the 0-graph as distinguished subgraphs isomorphic to $\mathsf{Alt}(i+1, q)$, but we can also regard them as i-dimensional subspaces in residues of points. It is now clear that f induces a 2-covering from $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ to $\mathcal{G}$. Furthermore, $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ is 2-simply connected (see [18, Corollary 1.7], or use Proposition 6.1 of Munemasa and Shpectorov [15], combining it with [17, Theorem 12.64] and recalling that $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ satisfies (T)). The conclusion follows. $\square$

3. Residues of type $\{0, 0^*, 1, 2\}$

In this section $\mathcal{G}$ is a geometry belonging to the following diagram:



$(t = q^{n-3} + \cdots + q^2 + q)$

As in the previous section, we call the elements of type 0 and 1 *points* and *lines* and, for an element x of type $t(x) \neq 0$, we denote by $\sigma(x)$ the set of points incident to it. Note that, when $n > 4$, we cannot claim that the statement of Lemma 2.2 holds for $i = 3$. Actually that statement holds for $i = 0^*, 1$ and 2 (in particular, property (LL) holds), as one can easily prove, but we will not make any use of this fact in this section.

Henceforth we assume that the automorphism group of $\mathcal{G}$ is flag-transitive. Given an element x of $\mathcal{G}$, we denote by G_x the stabilizer of x in $G := \mathrm{Aut}(\mathcal{G})$ and by K_x the elementwise stabilizer of $\mathrm{Res}(x)$ in G_x. That is, $\overline{G}_x := G_x/K_x$ is the group induced by G_x on $\mathrm{Res}(x)$. Clearly, if x, y are incident elements, then K_x and K_y normalize each other. The next lemma follows from a well known theorem of Higman [8]:

Lemma 3.1. *$\mathsf{PSL}_{n+1}(q) \leq \overline{G}_a \leq \mathsf{P\Gamma L}_{n+1}(q)$ for every point a.*

Lemma 3.2. *$|K_a K_S/K_S| \leq 2$ for every 3-element S and every point $a \in \sigma(S)$. If $|K_a K_S/K_S| = 2$, then q is odd and one of the following occurs:*

(1) *$\mathrm{Res}(S)$ is isomorphic to the complement of a tangent hyperplane of $\mathcal{D}_4(q)$ and a is the unique point of $\sigma(S)$ fixed by K_a;*

(2) $\mathrm{Res}(S)$ *is isomorphic to the complement of a secant hyperplane of* $\mathcal{D}_4(q)$ *and* K_a *has exactly two fixed-points in* $\sigma(S)$*, namely the point* a *and the unique point of* $\mathrm{Res}(S)$ *at distance* 3 *from* a.

Proof. By Proposition 2.1, $\mathrm{Res}(S) \cong \mathcal{A}$ where $\mathcal{A}$ is the complement of a hyperplane of $\mathcal{D} = \mathcal{D}_4(q)$.

Given a point a of $\mathcal{A}$, let H_a be the elementwise stabilizer in $\mathrm{Aut}(\mathcal{A})$ of the residue of a in $\mathcal{A}$. It is straightforward to check that $H_a = 1$ if q is even and $|H_a| = 2$ if q is odd. In the latter case, H_a acts semi-regularly on the set of points of $\mathcal{A}$ at distance 1 or 2 from a. When $\mathcal{A}$ is the complement of a tangent hyperplane, then its 0-graph has diameter 2. In that case, a is the unique fixed-point of H_a. On the other hand, if $\mathcal{A}$ is the complement of a secant hyperplane, then $\mathcal{A}$ contains exactly one point a' at distance 3 from a. In this case H_a fixes a and a' and displaces all remaining points. As $\mathcal{A} = \mathrm{Res}(S)$, we have $K_aK_S/K_S \leq H_a$. □

Lemma 3.3. *We have* $K_a \cap G_b = 1$ *for any two distinct collinear points* a, b.

Proof. Let l be a line through a and b and m a line on b skew with $l := ab$ in $\mathrm{Res}(b)$. (As noticed at the beginning of this section, (LL) holds, hence l is unique; but this fact is irrelevant for the sequel.)

In $\mathrm{Res}(b) \cong \mathsf{PG}(n, q)$, we find a 3-element S incident to both l and m. By Lemma 3.2, $K_a \cap G_b \leq K_S$. Hence $K_a \cap G_b$ fixes m, all points of l and all points of m. However, m is any of the lines of $\mathrm{Res}(b)$ skew with l. Therefore $K_a \cap G_b$ is contained in K_b and fixes all points collinear with b. So, $K_a \cap G_b = K_a \cap K_b = K_b \cap G_a$ (by symmetry) and $K_a \cap K_b$ fixes all points collinear with either a or b. So, given a point $c \perp b$, $K_a \cap K_b \leq K_b \cap G_c = K_b \cap K_c$. By symmetry, $K_a \cap K_b = K_b \cap K_c$. As the 0-graph is connected, we have $K_a \cap K_b = K_x \cap K_y$ for any two collinear points x, y. This forces $K_a \cap K_b = 1$. □

Corollary 3.4. *We have* $K_a \cap K_S = 1$ *and* $[K_a, K_S] = 1$ *for every* $\{0, 3\}$*-flag* $\{a, S\}$ *of* $\mathcal{G}$.

Proof. Clearly, $K_a \cap K_S \leq K_a \cap G_b$ for any $b \in \sigma(S) \cap a^{\perp}$. Hence $K_a \cap K_S = 1$, by Lemma 3.3. So, $[K_a, K_S] = 1$, as K_a and K_S normalize each other. □

Henceforth, we denote by p the prime of which q is a power. Given a $\mathsf{GF}(q)$-vector space V, we will make no distinction between V and its additive group, thus writing $X \cong V$ for an elementary abelian p- group X when X is isomorphic to the additive group of V.

Lemma 3.5. *For every 3-element S of $\mathcal{G}$, K_S is a split extension $V : L$ where $V := O_p(K_S) = V_1 \otimes V_2$ for $V_1 = V(4,q)$ and $V_2 = V(n-3,q)$, and L is a subgroup of $\mathsf{GL}_{n-3}(q)$ acting on $V = V_1 \otimes V_2$ via its natural action on V_2, namely every $g \in L$ maps $v_1 \otimes v_2$ onto $v_1 \otimes v_2^g$. Furthermore,*

(1) *if $n > 4$ and $q > 2$, then $L \geq \mathsf{SL}_{n-3}(q)$;*

(2) *if $n = 5$ and $q = 2$, then $|L| \geq 3$;*

(3) *if $n = 4$, then $V = V_1$ (as V_2 is 1-dimensional), L is a subgroup of the multiplicative group $\mathsf{GF}(q)^*$ of $\mathsf{GF}(q)$ and acts on V by scalar multiplication; the index $|\mathsf{GF}(q)^* : L|$ is a common divisor of $q-1$ and 10.*

Proof. Given a point $a \in \sigma(S)$, $K_S K_a / K_a$ is a subgroup of $\overline{G}_a$. The latter is described in Lemma 3.2. On the other hand, $K_S K_a / K_a \cong K_S/(K_S \cap K_a) = K_S$ (as $K_S \cap K_a = 1$ by Corollary 3.4). So, K_S is recognizable inside $\overline{G}_a$ as a subgroup of the elementwise stabilizer of $\mathrm{Res}(a,S)$, normalized by the stabilizer $\overline{G}_{a,S}$ of S in $\overline{G}_S$. In view of Lemma 3.2, $\overline{G}_{a,S}$ contains a split extension $X := V : (L_1 \times L_2)$ of $V := V_1 \otimes V_2$ by subgroups $L_1 \leq \mathsf{GL}_4(q)$ and $L_2 \leq \mathsf{GL}_{n-3}(q)$ where L_1 consists of all (4×4)-matrices of determinant t^{n-2} for $t \in \mathsf{GF}(q)^*$ and L_1 is formed by the $(n-3) \times (n-3)$-matrices of determinant t^5 for $t \in \mathsf{GF}(q)^*$. Furthermore, for $i = 1, 2$ the group L_i acts on V via its natural action on V_i. Explicitly, an element $g \in L_1$ (resp. L_2) sends $v_1 \otimes v_2$ to $v_1^g \otimes v_2$ (resp. $v_1 \otimes v_2^g$). Clearly, $V = O_p(X)$ and the group $H := VL_2$ is the elementwise stabilizer of $\mathrm{Res}(a,S)$ in X.

If $X \leq K_S$, then we are done. Suppose that $X \not\leq K_S$ and let $\widehat{X}$ be the preimage of X in G_a. So, $X = \widehat{X}/K_a$. As $X \not\leq K_S$, $\widehat{X}$ has a non-trivial action in $\mathrm{Res}(S)$. However, as X is contained in the elementwise stabilizer of $\mathrm{Res}(a,S)$ in $\overline{G}_a$, $\widehat{X}$ stabilizes all 0^*-elements of $\mathrm{Res}(S)$ on a. By the proof of Lemma 3.2, $|\widehat{X}K_S/K_S| = 2$. Hence $|X/(X \cap K_S)| = 2$, namely: $K_S \cap X$ has index 2 in X. It is straightforward to check that the subgroups of X of index 2 and normalized by $\overline{G}_{a,S}$ are as in (1), (2) and (3) of the lemma. □

Proposition 3.6. *For every 3-element S of $\mathcal{G}$, $\mathrm{Res}(S)$ is isomorphic to the complement of a tangent hyperplane of $\mathcal{D}_4(q)$.*

Proof. When $q = 2$, the statement follows from Huybrechts and Pasini [11]. So, we assume $q > 2$.

Suppose that $\mathrm{Res}(S) = \mathcal{D} \setminus H$ for a secant hypeplane H of $\mathcal{D} = \mathcal{D}_4(q)$. Then the intersection $H \cap \mathcal{D}$ is isomorphic to the non-singular quadric of $\mathsf{PG}(7,q)$ and $\overline{G}_S$ induces a flag-transitive action on $H \cap \mathcal{D}$. Furthermore, that action is faithful. Indeed, the pointwise stabilizer of $H \cap \mathcal{D}$ in $\mathrm{Aut}(\mathcal{D})$ has order 2

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and permutes the two families of maximal singular subspaces of $\mathcal{D}$, namely it permutes 0^*-elements with 2-elements. So, it cannot be involved in $\overline{G}_S$.

By a celebrated theorem of Seitz [19], $\overline{G}_S$ contains a normal subgroup $O \cong \mathsf{P}\Omega_7(q)$. Clearly, O normalizes $V = O_p(K_S)$. However, $K_S = VL$ acts as L on $V = V_1 \otimes V_2$ (notation as in Lemma 3.5). Assume first $n > 4$. Then, as $q > 2$ by assumption, we are in case (1) of Lemma 3.5. In that case, the orbits of L on V are the subspaces $\langle v_1 \rangle \otimes V_2$, for $v_1 \in V_1$. As O normalizes K_S and K_S acts as L on V, O permutes the orbits of L. So, O acts on the set $\mathcal{P}$ of 1-dimensional subspaces of V_1.

We warn the reader that we cannot claim that O preserves a projective structure on the set $\mathcal{P}$, as we have not proved that O acts $\mathsf{GF}(q)$-linearly on V_1. The center $Z(L)$ of L indeed acts on V_1 by scalar multiplication and it is not difficult to prove that O preserves multiplication by scalars corresponding to elements of $Z(L)$, however $Z(L)$ might be smaller than $\mathsf{GF}(q)^*$. (One can only prove that $|Z(L)| = (q-1)/d$ for a common divisor d of $q-1$ and $n+1$.)

Denoted by O_a the stabilizer in O of a point $a \in \sigma(S)$, the family $\{O_a\}_{a\in\sigma(S)}$ generates O. Denoted by $\widetilde{O}_a$ the preimage of O_a in G_S, the group $\widetilde{O}_a K_a/K_a$ contains the commutator subgroup $L_1' \cong \mathsf{SL}_4(q)$ of L_1 (notation as in the proof of Lemma 3.5) and L_1' acts on V according to its natural action on V_1. Thus the preimage of L_1' in $\overline{G}_a$ induces on $\mathcal{P}$ the natural action of $\mathsf{PSL}_4(q)$ on a suitable copy $\mathcal{S}_a$ of $\mathsf{PG}(3, q)$. That action is 2-transitive. Hence O, which is isomorphic to $\mathsf{P}\Omega_7(q)$, acts 2-transitively on $\mathcal{P}$, which has size $q^3 + q^2 + q + 1$. (Warning: we cannot claim that $O \leq \mathsf{P\Gamma L}_4(q)$, as the projective structure $\mathcal{S}_a$ preserved by O_a might depend on the choice of a.) However, $\mathsf{P}\Omega_7(q)$ does not admit any 2-transitive action of degree q^3+q^2+q+1 (see Cameron [4]). We have reached a contradiction.

The case of $n = 4$ remains to be examined. Regarded L as a subgroup of $\mathsf{GF}(q)^*$, let R be the subring of $\mathsf{GF}(q)$ generated by L and let M be the module defined on $V = V_1$ by taking R as the ring of scalars. Clearly, the 1-dimensional submodules of M are minimal among the L-invariant submodules of M, and O permutes them. So, O acts on the set $\mathcal{P}_0$ of 1-dimensional subspaces of M. If $R = \mathsf{GF}(q)$, then $\mathcal{P}_0$ is the set of 1-dimensional subspaces of V_1 and a contradiction is reached as in the case of $n > 4$.

Suppose $R < \mathsf{GF}(q)$. Then $|\mathsf{GF}(q) : R| = p^r$ for a positive integer $r < h$, where h is such that $p^h = q$. As L is a group, its cosets in the multiplicative semigroup R^* of R form a partition of R^*. So, denoted by d_1 the number of cosets of L in R^* and by d the index of L in $\mathsf{GF}(q)^*$, we have

$$|L| = \frac{p^{h-r}-1}{d_1} = \frac{p^h-1}{d}\,. \tag{1}$$

However, according to Lemma 3.5, d is a common divisor of $q-1$ and 10. Hence

$$\begin{cases} p^h - 1 \leq 10(p^{h-r} - 1) & \text{if } p > 2\,, \\ 2^h - 1 \leq \ 5(2^{h-r} - 1) & \text{when } p = 2\,. \end{cases} \tag{2}$$

The following are the only cases that pass through (2): $r = 1$ and $p = 2, 3, 5$ or 7, or $r = 2$ and $p = 2$ or 3. On the other hand, (1) also forces $|L|$ to be a common divisor of $p^h - 1$ and $p^{h-r} - 1$. When $r = 1$, the above amounts to say that $|L|$ divides $p - 1$. So, $(p^h - 1)/d$ $(= |L|)$ divides $p - 1$, namely $1 + p + \cdots + p^{h-1}$ divides d. However, this does not fit with the fact that $h > 1$ and d divides 10 (actually 5, when $p = 2$). So, $r = 2$ and $p = 2$ or 3. The greatest common divisor of $p^h - 1$ and $p^{h-2} - 1$ divides $p^2 - 1$. Hence $p^h - 1$ divides $d(p^2 - 1)$. This forces $h = 4$ and either $p = 2$ and $d = 5$ or $p = 3$ and $d = 10$. In any case $d_1 = 1$, hence $L = R^*$. In fact, we have either $q = 16$ and $R = \mathsf{GF}(4)$ or $q = 81$ with $R = \mathsf{GF}(9)$. Therefore, $q = q_0^2$ for $q_0 = 4$ or 9, $M \cong V(8, q_0)$ and $\mathcal{P}_0$ is the set of 1-dimensional subspaces of M.

As before, denoted by O_a the stabilizer in O of a point $a \in \sigma(S)$ and by $\widetilde{O}_a$ its preimage in G_S, the group $\widetilde{O}_a K_a / K_a$ contains the commutator subgroup of L_1. As L_1 is linear, we have $(\lambda v)^g = \lambda v^g$ for any $v \in M$, $\lambda \in L$ and $g \in L_1$. In other words, the actions of L_1 and L on M mutually commute. However, M, regarded as a group, is nothing but $O_p(K_S)$. So, O_a contains a non-trivial subgroup which commutes with the action of L on $O_p(K_S)$. Consequently, the elements of O that commute with L in their action on $O_p(K_S)$ form a non-trivial subgroup of O. Clearly, that subgroup is normal in O. However, O is simple. Hence the action of O on $O_p(K_S)$ commutes with the action of L. That is, $O = \mathsf{P}\Omega_7(q)$ is a group of linear transformations of $M = V(8, q_0)$, namely a subgroup of $\Gamma\mathsf{L}_8(q_0)$. However this is false, as one can see by comparing orders (recall that $q = q_0^2$). We have reached a final contradiction. □

4. The flag-transitive $\mathsf{D}^{\mathsf{Af}}_{n+1}$-geometries

In the next proposition $\mathcal{G}$ belongs to diagram $\mathsf{D}^{\mathsf{Af}}_{n+1}$, with orders $q-1$ and q.

Proposition 4.1. *Suppose $\mathcal{G}$ is flag-transitive. Then either $\mathcal{G} \cong \mathrm{Far}(\mathcal{D}_{n+1}(q))$ or $n = 3$ and $\mathcal{G}$ is isomorphic to the complement of a secant hyperplane of $\mathcal{D}_4(q)$.*

Proof. As the case of $n = 3$ is classified in Proposition 2.1, we may assume $n > 3$. By Propositions 3.6 and 2.8, $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ is the universal 2-cover of $\mathcal{G}$. By [17, Theorem 12.59], $\mathcal{G} = \mathrm{Far}(\mathcal{D}_{n+1}(q))/D$ for a suitable subgroup D of $G := \mathrm{Aut}(\mathrm{Far}(\mathcal{D}_{n+1}(q)))$ such that $N_G(D)$ is flag-transitive on $\mathrm{Far}(\mathcal{D}_{n+1}(q))$

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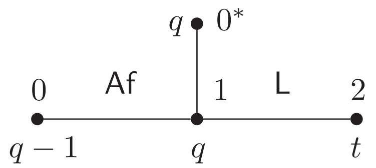
and D acts semi-regularly on the set of elements of $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ of type 0 and 0^*. (The latter condition follows from the fact that, in view of [17, Proposition 12.45], the residues of the 0- and 0^*-elements of $\mathcal{G}$ are 2-simply connected.) We recall that $\mathrm{Far}(\mathcal{D}_{n+1}(q))$ is the subgeometry of $\mathcal{D}_{n+1}(q)$ far from an element S_0 of type 0 (when n is odd) or 0^* (if n is even). So, $\mathrm{Aut}(\mathrm{Far}(\mathcal{D}_{n+1}(q)))$ is the stabilizer of S_0 in the automorphism group $\mathsf{P\Gamma O}^+_{2n+2}(q)$ of the polar spaces associated to $\mathcal{D}_{n+1}(q)$. Computing that stabilizer is a routine exercise. It is easy to see that it does not contain any non-trivial subgroup D satisfying the above conditions. Hence $D = 1$. $\square$

5. From AG.PG* to $\mathsf{D}^{\mathsf{Af}}_{n+1}$ — first step

In this section $\mathcal{G}$ is a flag-transitive geometry belonging to diagram AG.PG* with $q > 2$. We shall prove that $\mathcal{G}$ is the $\{0, 1, 0^*\}$-truncation of a geometry belonging to the following diagram, which we call $\mathsf{TD}^{\mathsf{Af}}_{n+1,2}$:

$(\mathsf{TD}^{\mathsf{Af}}_{n+1,2})$



$(t = q^{n-2} + \cdots + q^2 + q)$

As in sections 2 and 3, we call the elements of type 0 and 1 *points* and *lines*. Those of type 0^* will be called *dual points*. If x is a line or a dual point, then $\sigma(x)$ is the set of points incident to x. If x is a line or a point, we denote by $\sigma^*(x)$ the set of dual points incident to x. Also, the set of lines incident to a given point or dual point x will be denoted by $\sigma_1(x)$.

5.1. Property (LL) and the Intersection Property

As noticed by Cardinali and Pasini [5, Remark 3.3], the flag-transitivity of $\mathcal{G}$ implies property (LL). The latter in its turn implies the Intersection Property [17, Lemma 7.25]. In particular, the dual of (LL) holds, too: $|\sigma^*(l) \cap \sigma^*(m)| \leq 1$ for any two distinct lines l, m. Also: If $|\sigma(A) \cap \sigma(l)| > 1$ for a dual point A and a line l, then $l * A$; if A, B are distinct dual points, then either $\sigma(A) \cap \sigma(B) = \emptyset$ or $\sigma(A) \cap \sigma(B) = \sigma(l)$ for a unique line $l \in \sigma_1(A) \cap \sigma_1(B)$. Dually: If $|\sigma^*(a) \cap \sigma^*(l)| > 1$ for a point a and a line l, then $a * l$; if a, b are distinct points, then either $\sigma^*(a) \cap \sigma^*(b) = \emptyset$ or $\sigma^*(a) \cap \sigma^*(b) = \sigma^*(l)$ for a unique line $l \in \sigma_1(a) \cap \sigma_1(b)$.

5.2. A few lemmas on stabilizers

We keep for the symbols G_x, K_x and $\overline{G}_x$ the meaning stated in section 3, extending that notation to flags. Thus, given a flag F, G_F is its stabilizer in $G := \mathrm{Aut}(\mathcal{G})$, K_F is the elementwise stabilizer of $\mathrm{Res}(F)$ and $\overline{G}_F := G_F/K_F$ is the group induced by G_F on $\mathrm{Res}(F)$. Furthermore, given a line l, we denote by K_l^- the pointwise stabilizer of $\sigma(l)$ and by K_l^+ the elementwise stabilizer of $\sigma^*(l)$. So, $K_l = K_l^+ \cap K_l^-$ and the quotients G_l/K_l^-, G_l/K_l^+ are the groups induced by G_l on $\sigma(l)$ and $\sigma^*(l)$ respectively.

Lemma 5.1. *All the following hold:*

(1) $\mathsf{PSL}_{n+1}(q) \le \overline{G}_a \le \mathsf{P\Gamma L}_{n+1}(q)$ *for every point* a*;*

(2) $\mathsf{PGL}_n(q) \le \overline{G}_{a,A} \le \mathsf{P\Gamma L}_n(q)$ *for every* $\{0, 0^*\}$*-flag* $\{a, A\}$*;*

(3) $\mathsf{ASL}_n(q) \le \overline{G}_A \le \mathsf{A\Gamma L}_n(q)$ *for every dual point* A*.*

Proof. Claim (1) follows from Higman [8] (compare Lemma 3.1). Note that, according to [8], if $q = 2$, one more case should be considered, where $n = 3$ and $\overline{G}_a$ is isomorphic to the alternating group of degree 7; but we have assumed $q > 2$, so we need not trouble about that exceptional case. Claim (1) implies (2), which implies (3). (Recall that, by Wagner [22], every flag-transitive automorphism group of a finite affine space contains all translations.) □

Lemma 5.2. *We have* $K_a \cap K_A = 1$ *for every* $\{0, 0^*\}$*-flag* $\{a, A\}$*.*

Proof. Given a dual point $B \in \sigma^*(a) \setminus \{A\}$, let l be the line of $\mathrm{Res}(a)$ incident to both A and B. The group $(K_a \cap K_A)K_B/K_B$ fixes all points of $\sigma(l)$ and all lines of $\mathrm{Res}(a, B)$. Hence $(K_a \cap K_A)K_B/K_B = 1$, that is $K_a \cap K_A \le K_B$, namely $K_a \cap K_A \le K_a \cap K_B$. By symmetry, $K_a \cap K_A = K_a \cap K_B$.

Given a point $b \in \sigma(A) \setminus \{a\}$, let m be the line of $\mathrm{Res}(A)$ through a and b. By the above, $(K_a \cap K_A)K_b/K_b \le K_B K_b/K_b$ for every dual point $B \in \sigma^*(m)$. Hence $(K_a \cap K_A)K_b/K_b$ fixes all lines of $\mathrm{Res}(b)$ incident to a dual point of $\sigma^*(m)$. Consequently, $(K_a \cap K_A)K_b/K_b = 1$, namely $K_a \cap K_A \le K_b$. This forces $K_a \cap K_A = K_b \cap K_A$. However, according to the above, $K_b \cap K_A = K_b \cap K_B$ for any dual point $B \in \sigma^*(b)$. Hence $K_a \cap K_A = K_b \cap K_B$. By connectedness, $K_a \cap K_A = K_x \cap K_X$ for any $\{0, 0^*\}$-flag $\{x, X\}$, that is: $K_a \cap K_A = 1$. □

Corollary 5.3. *The group* K_a *is cyclic of order* d *for a given divisor* d *of* $q - 1$*. Furthermore, for every line* l *on* a*,* K_a *acts semi-regularly on* $\sigma(l) \setminus \{a\}$*.*

Proof. Given a dual point $A \in \sigma^*(a)$, we have $K_a K_A/K_A \cong K_a/(K_a \cap K_A) = K_a$ (as $K_a \cap K_A = 1$ by Lemma 5.2). Thus, K_a has the same structure and the same action as $K_a K_A/K_A$. □

5.2.0.2. Comment. The reader might wonder why we have not put Lemma 5.2 and Corollary 5.3 in section 3. We have not done so because we needed Lemma 3.2 there, which is stronger than Lemma 5.2. Note that Lemmas 3.2, 3.3 and Corollary 3.4 imply that $|K_a| \leq 2$. The statement of Corollary 5.3 is not so sharp, but it is the best we can obtain now.

5.3. Twin pairs

Given a point a, the lines of $\mathrm{Res}(a)$ contained in a given plane of the projective space $\mathrm{Res}(a) \cong \mathsf{PG}(n,q)$ are said to form a $(+)$-*plane*. The point a is called the *pole* of that $(+)$-plane. (Note that, by property (LL), a $(+)$-plane admits a unique pole.) Similarly, for a dual point A, we say that the lines of $\mathrm{Res}(A)$ incident to a given plane of the affine space $\mathrm{Res}(A) \cong \mathsf{AG}(n,q)$ form a $(-)$-*plane* with A as its *pole* (uniquely determined in view of the dual of property (LL); see subsection 5.1). For a $(+)$-plane α^+, we put $\sigma^*(\alpha^+) := \cup_{l\in\alpha^+}\sigma^*(l)$ and we say that a dual point X is *incident* to α^+ if $X \in \sigma^*(\alpha^+)$. Similarly, for a $(-)$-plane α^-, $\sigma(\alpha^-) := \cup_{l\in\alpha^-}\sigma(l)$ is the set of points *incident* to α^-.

For a $\{0, 0^*\}$-flag $\{a, A\}$, we say that a $(+)$-plane $\alpha^+ \in \mathrm{Res}(a)$ with $A \in \sigma^*(\alpha^+)$ and a $(-)$-plane $\alpha^- \in \mathrm{Res}(A)$ with $a \in \sigma(\alpha^-)$ are *twinned* if $\sigma_1(A) \cap \alpha^+ = \sigma_1(a) \cap \alpha^-$. We call the set of lines $\sigma_1(A) \cap \alpha^+ = \sigma_1(a) \cap \alpha^-$ the *pencil* of the twin pair (α^+, α^-). The pole a of α^+ and the pole A of α^- are the $(+)$-*pole* and $(-)$-*pole* of (α^+, α^-).

Denoted by G_{a,α^+} the setwise stabilizer of α^+ in G_a, we put $G_{a,l,\alpha^+} := G_{a,\alpha^+} \cap G_l$ for $l \in \sigma_1(a) \cap \alpha^+$ and $G_{a,A,\alpha^+} := G_{a,\alpha^+} \cap G_A$ for $A \in \sigma^*(\alpha^+)$. The groups G_{A,α^-}, G_{A,l,α^-} and G_{a,A,α^-} are defined in a similar way.

Lemma 5.4. *Given a $\{0, 0^*\}$-flag $\{a, A\}$, the twinning relation induces a bijection between the set of $(+)$-planes α^+ of $\mathrm{Res}(a)$ incident to A and the set of $(-)$-planes α^- of $\mathrm{Res}(A)$ incident to a. Furthermore, a $(+)$-plane α^+ and a $(-)$-plane α^- are twinned if and only if $G_{a,A,\alpha^+} = G_{a,A,\alpha^-}$.*

Proof. The setwise stabilizer X of $\mathcal{L} := \sigma_1(A) \cap \alpha^+$ in $G_{a,A}$ contains K_aK_A and is the stabilizer of a bundle of lines of $\mathrm{Res}(a) \cong \mathsf{PG}(n,q)$ with A as the center and α^+ as the support. The quotient group X/K_a is maximal in $G_{a,A}/K_a$ and, by Lemma 5.1(1), it induces on $\mathcal{L}$ a group containing $\mathsf{PGL}_2(q)$ and contained in $\mathsf{P\Gamma L}_2(q)$. As X/K_a is maximal in $G_{a,A}/K_a$, X is maximal in $G_{a,A}$, whence X/K_A is maximal in $G_{a,A}/K_A$. On the other hand, $\mathcal{L}$ is also a set of $q+1$ lines of $\mathrm{Res}(A)$ on a. It is stabilized by X/K_A, which is maximal in $G_{a,A}/K_A$ and induces on $\mathcal{L}$ a group contained between $\mathsf{PGL}_2(q)$ and $\mathsf{P\Gamma L}_2(q)$. The group $G_{a,A}/K_A$ is contained between $\mathsf{SL}_n(q)$ and $\mathsf{\Gamma L}_n(q)$ (Lemma 5.1(3)). All maximal subgroups of such groups are known (Aschbacher [1]; also Kleidman and

Liebeck [13]). In particular, X/K_A either belongs to the class $\mathcal{C}(Y)$ of natural subgroups of $Y = G_{a,A}/K_A$ (see Table 3.5 of [13]) or it is almost simple. In the latter case, K_aK_A/K_A must be trivial, namely $K_a \leq K_A$. However, this is a contradiction with Lemma 5.2 and Corollary 5.3. Therefore $X/K_A \in \mathcal{C}(Y)$ and, by checking the various cases listed in Table 3.5 of [13], one can see that X/K_A is the stabilizer of a plane α^- of the affine space $\mathrm{Res}(A)$. Accordingly, $\mathcal{L}$ is the pencil of lines of that plane with a as the center. Clearly, α^+ is the only plane of the projective space $\mathrm{Res}(a)$ stabilized by X. □

Corollary 5.5. *For every line l, the pointwise stabilizer K_l^- of $\sigma(l)$ contains a subgroup L that acts faithfully and 2-transitively as $\mathsf{PSL}_2(q)$ on $\sigma^*(l)$ and stabilizes all $(+)$-planes of $\mathrm{Res}(x)$ on l, for every point $x \in \sigma(l)$.*

Proof. Given $a \in \sigma(l)$, it follows from Lemma 5.1(1) that $G_{a,l}$ contains a subgroup X such that $K_a \leq X$, X/K_a acts faithfully as $\mathsf{PSL}_2(q)$ on $\sigma^*(l)$ and X stabilizes all $(+)$-planes of $\mathrm{Res}(a)$ containing l. Let U_A be a Sylow p-subgroup of the stabilizer X_A of A in X, where p is the prime of which q is a power. Then $U_A \cap K_a = 1$, as $|K_a|$ divides $q-1$ (Corollary 5.3). Hence K_aU_A is a semidirect product of K_a and U_A.

According to the conditions assumed on X/K_a, the group X/K_a is contained in the commutator subgroup of $\overline{G}_a$. Hence $X \subseteq G'_aK_a$, where G'_a is the commutator subgroup of G_a, and K_aU_A is contained in the stabilizer $(G'_aK_a)_A$ of A in G'_aK_a. The group induced by $(G'_aK_a)_A$ on the set of lines incident to the flag $\{a, A\}$ sits between $\mathsf{PSL}_n(q)$ and $\mathsf{PGL}_n(q)$. Therefore, $(G'_aK_a)_AK_A/K_A$ acts in $\mathrm{Res}(A)$ as a subgroup of the stabilizer $\mathsf{GL}(n,q)$ of a in $\mathsf{AGL}(n,q)$. In particular, $(G'_aK_a)_{A,l}K_A/K_A$ induces on $\sigma(l)\setminus\{a\}$ a subgroup of the cyclic $(q-1)$-subgroup C of $\mathsf{AGL}_1(q)$. However, $K_aU_A \leq (G'_aK_a)_{A,l}$ and U_A, being a p-group, cannot be involved in C. Therefore $U_A \leq K_l^-$. On the other hand, $K_a \cap K_l^- = 1$, as K_a acts semi-regularly on $\sigma(l)$ (Corollary 5.3). Hence $\langle K_a, K_l^-\rangle = K_a \times K_l^-$, as K_a and K_l^- normalize each other. It follows that $K_aU_A = K_a \times U_A$ and $U_A = (K_aU_A) \cap K_l^-$.

Similarly, given another dual point $B \neq A$ on l and a Sylow p-subgroup U_B of X_B, we have $K_aU_B = K_a \times U_B$ and $U_B \leq K_l^-$. Thus, $X \cap K_l^- \geq L := \langle U_A, U_B\rangle$ and, as $K_l^- \cap K_a = 1$, $(X \cap K_l^-)K_a/K_a \cong X \cap K_l^- \geq L$. The canonical projection of X onto X/K_a maps U_A and U_B onto two distinct Sylow p-subgroups $\overline{U}_A$ and $\overline{U}_B$ of $\mathsf{PSL}_2(q) \cong X/K_a$ and $\langle \overline{U}_A, \overline{U}_B\rangle = \mathsf{PSL}_2(q)$. Therefore, $LK_a = X$ and L acts as $\mathsf{PSL}_2(q)$ on $\sigma^*(l)$. As $L \leq K_l^-$ (in fact, $L = X \cap K_l^-$), that action is faithful.

The group L, being contained in X, stabilizes all $(+)$-planes of $\mathrm{Res}(a)$ on l. It remains to prove that L also stabilizes all $(+)$-planes of $\mathrm{Res}(b)$ on l for every $b \in \sigma(l) \setminus \{a\}$. The group U_A, being a subgroup of L, stabilizes all $(+)$-planes of $\mathrm{Res}(a)$ on l. Hence, by Lemma 5.4, it also stabilizes all $(-)$-planes of $\mathrm{Res}(A)$ on l. Therefore, again by Lemma 5.4, but applied backward from $(-)$-planes to $(+)$-planes, U_A stabilizes all $(+)$-planes of $\mathrm{Res}(b)$ on l for every $b \in \sigma(l)$. The same holds true for U_B, for any other dual point $B \in \sigma^*(l)$. As $L = \langle U_A, U_B \rangle$, L stabilizes all $(+)$-planes on l. □

5.4. The structure $\mathrm{Ext}(\mathcal{G})$

Let Π be the bipartite graph with the $(+)$- and $(-)$-planes as vertices and the twin pairs as edges. We form an incidence structure $\mathrm{Ext}(\mathcal{G})$ of rank 4 with $\{0, 0^*, 1, 2\}$ as the type-set, the 0-, 0^*- and 1-elements of $\mathcal{G}$ as elements of type 0, 0^* and 1 respectively and the connected components of Π as 2-elements. The incidence relation of $\mathcal{G}$ induces on the set of elements of type 0, 0^* and 1 the incidence relation of $\mathrm{Ext}(\mathcal{G})$. A line l and a 2-element S are said to be incident when l belongs to some $(+)$- or $(-)$-plane of S; an element x of $\mathcal{G}$ of type 0 or 0^* is declared to be incident to S when $\sigma_1(x)$ contains a line incident to S. We shall prove the following:

Proposition 5.6. *The structure* $\mathrm{Ext}(\mathcal{G})$ *is a flag-transitive geometry for diagram* $\mathrm{TD}^{\mathrm{Af}}_{n+1,2}$.

The proof of this Proposition will take the rest of this section. We shall consider the parabolic system naturally associated to a given chamber of $\mathrm{Ext}(\mathcal{G})$ and, after having studied some of its properties, we recognize three possible cases. Proposition 5.6 holds in one of them. We shall show that the other two cases are impossible, thus finishing the proof.

5.5. The parabolic system $(\{P_0, P_0^*, P_1, P_2\}, B)$

Given a twin pair $\alpha := (\alpha^+, \alpha^-)$, let a and A be its $(+)$- and $(-)$-pole, l a line in the pencil of α and S the 2-element of $\mathrm{Ext}(\mathcal{G})$ containing $\alpha^+ \cup \alpha^-$. We have $G_{a,A,\alpha^+} = G_{a,A,\alpha^-}$ (Lemma 5.4). So, we may write $G_{a,A,\alpha}$ for G_{a,A,α^+} or G_{a,A,α^-}. With that notation, we define the *minimal parabolics* P_0, P_0^*, P_1, P_2 and the *Borel subgroup* B as follows:

$$P_0 := G_{A,l,\alpha^-},\ P_0^* := G_{a,l,\alpha^+},\ P_1 := G_{a,A,\alpha},\ P_2 := G_{a,A,l},\ B := G_{a,A,l,\alpha}.$$

Denoted by G_S the stabilizer of S in $G = \mathrm{Aut}(\mathcal{G})$, we put $G_{a,S} := G_a \cap G_S$, $G_{A,S} := G_A \cap G_S$, and so on. Clearly, $G_{a,\alpha^+} \leq G_{a,S}$ and $G_{A,\alpha^-} \leq G_{A,S}$. The claims gathered in the following lemma are obvious:

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Lemma 5.7. *The Borel subgroup B is the intersection of any two minimal parabolics and all the following hold:*

$$\begin{aligned}
&G_{a,\alpha^+} = \langle P_0^*, P_1\rangle, \quad G_{A,\alpha^-} = \langle P_0, P_1\rangle;\\
&G_{a,A} = \langle P_1, P_2\rangle;\\
&P_0^* P_2 = P_2 P_0^*, \quad P_0 P_2 = P_2 P_0;\\
&G_a = \langle P_0^*, P_1, P_2\rangle, \quad G_A = \langle P_0, P_1, P_2\rangle;\\
&G = \langle P_0, P_0^*, P_1, P_2\rangle\,.
\end{aligned}$$

It remains to describe $\langle P_0, P_0^*\rangle$ and $\langle P_0, P_0^*, P_1\rangle$. We focus on $\langle P_0, P_0^*\rangle$ first. We put $Q := \langle P_0, P_0^*\rangle$ $(\leq G_l)$, $Q_0 := Q \cap G_A$ $(\leq G_{A,l})$ and $Q_0^* = Q \cap G_a$ $(\leq G_{a,l})$.

Lemma 5.8. *We have $P_0^* Q_0 = Q_0 P_0^*$ and $P_0 Q_0^* = Q_0^* P_0$.*

Proof. As the $\{0, 0^*\}$-residues of $\mathcal{G}$ are generalized digons, $G_{a,l}G_{A,l} = G_{A,l}G_{a,l}$. So, given $g \in P_0^*$ and $f \in Q_0$, we have $gf = f_1 g_1$ for suitable elements $f_1 \in G_{A,l}$ and $g_1 \in G_{a,l}$. Note that $g_1(\alpha^+)$ might be different from α^+. However, $G_{a,A,l}$ is transitive on the set of $(+)$-planes of $\mathrm{Res}(a)$ incident to A (compare Lemma 5.1(1)). Hence we can pick an element $g_0 \in G_{a,A,l}$ sending $g_1(\alpha^+)$ back to α^+. So, $g_2 := g_0 g_1 \in P_0^*$ and $f_2 := f_1 g_0^{-1} \in G_A$. However, $f_2 = gfg_2^{-1} \in Q$. Therefore $f_2 \in Q \cap G_A = Q_0$ and $gf = f_2 g_2 \in Q_0 P_0^*$. The equality $P_0^* Q_0 = Q_0 P_0^*$ is now evident. The equality $P_0 Q_0^* = Q_0^* P_0$ can be proved in a similar way. □

Corollary 5.9. $Q = P_0^* Q_0 = P_0 Q_0^* = Q_0 Q_0^* = Q_0^* Q_0$.

Proof. As $Q = \langle P_0, P_0^*\rangle$, Lemma 5.8 implies that $Q = P_0^* Q_0 = P_0 Q_0^*$. Also, $Q \supseteq Q_0 Q_0^* \supseteq P_0 Q_0^* = Q$. Hence $Q_0 Q_0^* = Q$. Similarly, $Q_0^* Q_0 = Q$. □

Lemma 5.10. *Either $P_0 P_0^* = P_0^* P_0$ or $P_0 < Q_0 = G_{A,l}$, $P_0^* < Q_0^* = G_{a,l}$ and $Q = G_l$.*

Proof. In view of Lemma 5.1(1), the group $P_0^* = G_{a,l,\alpha^+}$ is maximal in $G_{a,l}$. Therefore, either $Q_0^* = P_0^*$ or $Q_0^* = G_{a,l}$. In the first case, $P_0 P_0^* = P_0^* P_0$ by the equality $P_0 Q_0^* = Q_0^* P_0$ of Lemma 5.8. Suppose $Q_0^* = G_{a,l}$. Then $Q_0^* \cap G_A = G_{a,A,l}$. So, $Q_0 = Q \cap G_A \geq \langle P_0, G_{a,A,l}\rangle = \langle G_{A,l,\alpha^-}, G_{a,A,l}\rangle = G_{A,l}$. Therefore $Q_0 = G_{A,l}$ and $Q = G_l$. Clearly, $G_{A,l,\alpha^-} < G_{A,l}$, namely $P_0 < Q_0$. □

Lemma 5.11. $P_0 P_0^* = P_0^* P_0$.

Proof. Suppose the contrary. Then, by Lemma 5.10, $Q = G_l$ and the subgroups Q_0 and Q_0^* of Q form the parabolic system associated to the generalized digon $\mathrm{Res}(l)$. However, P_0 and P_0^* also form a parabolic system in Q, hence they also define a geometry $\mathcal{R}$ of rank 2 admitting Q as a flag-transitive automorphism group. Taken $\{0, 0^*\}$ as the type-set of $\mathcal{R}$, the 0-elements of $\mathcal{R}$ correspond to the cosets gP_0^* of P_0^* in Q and may be regarded as the pairs (x, ξ^+) for $x \in \sigma(l)$ and ξ^+ a $(+)$-plane of $\mathrm{Res}(x)$ containing l. The 0^*-elements of $\mathcal{R}$ are pairs (X, ξ^-) for $X \in \sigma^*(l)$ and ξ^- a $(-)$-plane of $\mathrm{Res}(X)$ containing l, and correspond to the cosets gP_0. Two pairs (x, ξ^+) and (X, ξ^-) are incident in $\mathcal{R}$ precisely when ξ^+ and ξ^- are twinned. (This happens precisely when the corresponding cosets of P_0^* and P_0 meet non-trivially.) We emphasize that $\mathcal{R}$, being a geometry, is connected. (Indeed, every incidence structure arising from a parabolic system is connected, whence it is a geometry if that system has rank 2.)

Given two 0-elements (b, β^+) and (c, γ^+) of $\mathcal{R}$ incident with the same 0^*-element, let L be the group considered in Corollary 5.5. As $Q = G_l$, we have $L \leq Q$. Thus, L is a subgroup of $\mathrm{Aut}(\mathcal{R})$. Therefore, as L fixes both (b, β^+) and (c, γ^+), it permutes the 0^*-elements of $\mathcal{R}$ incident to (b, β^+) and (c, γ^+). However, L acts transitively on $\sigma^*(l)$. So, we get $(q + 1)$ 0^*-elements of $\mathcal{R}$ incident to both (b, β^+) and (c, γ^+). On the other hand, as $|P_0^* : B|$ is equal to $|G_{a,l,\alpha^+} : G_{a,A,l,\alpha}| = q + 1$, every 0-element of $\mathcal{R}$ is incident to precisely $(q + 1)$ 0^*-elements of $\mathcal{R}$. Thus, we have proved that, if two 0-elements of $\mathcal{R}$ are incident to the same 0^*-element, then they are incident with just the same 0^*-elements of $\mathcal{R}$. As $\mathcal{R}$ is connected, the above forces $\mathcal{R}$ to be a generalized digon. Hence P_0 and P_0^* commute, contrary to our initial assumption. □

Lemma 5.12. *The group G_S acts transitively on the set of edges of Π contained in S and we have $G_S = \langle P_0, P_0^*, P_1 \rangle$.*

Proof. Clearly, $G_S \geq X := \langle P_0, P_0^*, P_1 \rangle$. We shall prove the following first:

$(*)$ for every twin pair (ξ^+, ξ^-) with $\xi^+, \xi^- \in S$, there exists an element $g \in X$ sending α^+ to ξ^+ and α^- to ξ^-.

Let Σ be the graph defined on the set of twin pairs of S by stating that two twin pairs (β^+, β^-) and (γ^+, γ^-) of S are adjacent when either $\beta^+ = \gamma^+$ or $\beta^- = \gamma^-$. As S is connected as an iduced subgraph of Π, Σ is connected, too. So, we can prove $(*)$ by induction on the distance d of (ξ^+, ξ^-) from (α^+, α^-).

When $d = 0$, there is nothing to prove. Suppose $d > 0$ and let (υ^+, υ^-) be a twin pair of S adjacent with (ξ^+, ξ^-) and at distance $d - 1$ from (α^+, α^-). Then $(\upsilon^+_{d-1}, \upsilon^-_{d-1}) = (g(\alpha^+), g(\alpha^-))$ for some $g \in X$, by the inductive hypothesis. Consequently, $(g^{-1}(\xi^+), g^{-1}(\xi^-))$ is adjacent with (α^+, α^-) in Σ. So, either $g(\xi^+) = \alpha^+$ or $g(\xi^-) = \alpha^-$. Suppose $g(\xi^+) = \alpha^+$, to fix ideas. (If $g(\xi^-) = \alpha^-$ an argument quite similar to the following can be used.) The point a is the pole of $g(\xi^+)$. Let B be the pole of $g(\xi^-)$. Then $B \in \sigma^*(m)$ for a suitable line $m \in \alpha^+$.

However, $\langle P_0^*, P_1\rangle = G_{a,\alpha^+}$ by Lemma 5.7 and G_{a,α^+} is transitive on the point-line flags of the plane α^+ of the projective space $\mathrm{Res}(a)$. So, $f(B) = A$ and $f(m) = l$ for some $f \in \langle P_0^*, P_1\rangle$. However, $fg^{-1}(\xi^+) = g^{-1}(\xi^+) = \alpha^+$, because $f \in G_{a,\alpha^+}$. Hence $fg^{-1}(\xi^-)$ is twinned with α^+. Therefore $fg^{-1}(\xi^-) = \alpha^-$, as $A = f(B)$ is the pole of $fg^{-1}(\xi^-)$ and α^- is the unique $(-)$-plane of $\mathrm{Res}(A)$ twinned with α^+, by Lemma 5.4. Thus, gf^{-1} maps (α^+, α^-) onto (ξ^+, ξ^-). Claim $(*)$ is proved.

The transitivity of G_S on the set of twin pairs of S follows from $(*)$. It remains to prove the equality $G_S = X$. Given $f \in G_S$, the twin pair $(f(\alpha^+), f(\alpha^-))$ belongs to S. By $(*)$, $(f(\alpha^+), f(\alpha^-)) = (g(\alpha^+), g(\alpha^-))$ for an element $g \in \langle P_0, P_0^*, P_1\rangle$. Hence $g^{-1}f$ stabilizes both α^+ and α^-, namely it belongs to $G_{a,A,\alpha} = P_1$. Therefore, $f \in X$. □

In the next lemma $\sigma(S)$ and $\sigma^*(S)$ are the set of points and the set of dual points that are incident to S.

Lemma 5.13. *Every point of $\sigma(S)$ is the pole of a $(+)$-plane of S and every dual point of $\sigma^*(S)$ is the pole of a $(-)$-plane of S.*

Proof. We will only prove the part dealing with points. The dual claim can be proved in the same way.

By definition, if $x \in \sigma(S)$ then $x * l$ for a line l of a $(+)$- or $(-)$-plane $\xi \in S$. If ξ is a $(-)$-plane, then the conclusion follows from Lemma 5.4 applied to the flag $\{x, X\}$, where X is the pole of ξ. Suppose ξ is a $(+)$-plane and let y be its pole. If $x = y$, there is nothing to prove. Suppose $x \neq y$ and, given $X \in \sigma^*(l)$, let ξ^- be the $(-)$-plane of $\mathrm{Res}(A)$ twinned with ξ (Lemma 5.4 applied to the flag $\{y, X\}$). Then x is the pole of the $(+)$-plane of $\mathrm{Res}(x)$ twinned with ξ^-. □

Lemma 5.14. *The group G_S acts transitively on $\sigma(S)$ and $\sigma^*(S)$.*

Proof. This is obvious, by Lemma 5.13 and the first claim of Lemma 5.12. □

Lemma 5.15. *One of the following holds:*

(1) *$G_{a,\alpha^+} = G_{a,S}$ and $G_{A,\alpha^-} = G_{A,S}$;*

(2) *$G_{a,S} = G_a$ and $G_{A,S} = G_A$. In this case $\langle P_0, P_0^*, P_1\rangle = \mathrm{Aut}(\mathcal{G})$;*

(3) *$G_{a,S} = G_{a,\alpha^+}$, $|G_{A,S} : G_{A,\alpha^-}| = q^{n-2}$ and $G_{A,S}$ is the stabilizer in G_A of the parallel class of the plane α^- in the affine space $\mathrm{Res}(A)$.*

Proof. As G_{a,α^+} is maximal in G_a and $G_{a,\alpha^+} \leq G_{a,S} \leq G_a$, either $G_{a,\alpha^+} = G_{a,S}$ or $G_{a,S} = G_a$. Suppose the latter. Then $G_{a,A,S} = G_{a,A}$. Hence $G_{A,S} \geq G_{a,A}$. However, $G_{a,A}$ is maximal in G_A. Therefore, either $G_{A,S} = G_A$ or $G_{A,S} = G_{a,A}$.

On the other hand, $G_{A,S}$ contains G_{A,α^-}, which transitively permutes the points of α^-. So, $G_{A,S} \neq G_{a,A}$. Consequently, $G_{A,S} = G_A$, as in case (2). The equality $\langle P_0, P_0^*, P_1\rangle = \mathrm{Aut}(\mathcal{G})$ follows from the fact that G_a and G_A generate $G = \mathrm{Aut}(\mathcal{G})$, by the flag-transitivity of G and the residual connectedness of $\mathcal{G}$.

Assume that $G_{a,S} = G_{a,\alpha^+}$. If $G_{A,S} = G_{A,\alpha^-}$, then we have (1). Suppose that $G_{A,S} > G_{A,\alpha^-}$. The stabilizer in G_A of the parallel class of α^- in $\mathrm{Res}(A)$ is the unique group between G_{A,α^-} and G_A. So, either we have case (3) or $G_{A,S} = G_A$. In the latter case, $G_{a,A,S} = G_{a,A}$. However, $G_{a,A} > G_{a,A,\alpha^+} = G_{a,A,S}$ (as $G_{a,S} = G_{a,\alpha^+}$ by assumption), and we get a contradiction. □

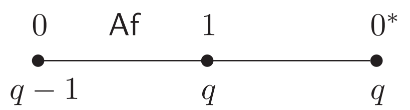
5.6. End of the proof of Proposition 5.6

Every parabolic system defines a chamber system [17, 12.4]. Let $\mathcal{C}$ be the chamber system arising from the parabolic system $(\{P_0, P_0^*, P_1, P_2\}, B)$ of $G = \mathrm{Aut}(\mathcal{G})$. By Lemmas 5.7 and 5.11 we obtain the following:

Lemma 5.16. *The chamber system $\mathcal{C}$ belongs to diagram* $\mathrm{TD}^{\mathrm{Af}}_{n+1,2}$.

If we have case (1) of Lemma 5.15, then Lemmas 5.7 and 5.12 imply that $\mathrm{Ext}(\mathcal{G})$ is the geometry associated to $\mathcal{C}$ and Proposition 5.6 follows.

Suppose we have case (2) or (3) of Lemma 5.15 and let $\mathcal{C}_\alpha$ be the $\{0, 0^*, 1\}$-residue of $\mathcal{C}$ containing the chamber B. That is, $\mathcal{C}_\alpha$ is the chamber system associated to the parabolic system $(\{P_0, P_0^*, P_1\}, B)$ of G_S $(= \langle P_0, P_0^*, P_1\rangle$, by Lemma 5.12). As $\mathcal{C}$ belongs to $\mathrm{TD}^{\mathrm{Af}}_{n+1,2}$, $\mathcal{C}_\alpha$ belongs to the same diagram as $\mathrm{AG}(3, q)$:



Let $V_0(\mathcal{C}_\alpha)$ be the set of 0-vertices of $\mathcal{C}_\alpha$, namely the set of cells of $\mathcal{C}_\alpha$ of type $\{1, 0^*\}$.

Lemma 5.17. $|V_0(\mathcal{C}_\alpha)| \leq q^3$.

Proof. If the chamber system $\mathcal{C}_\alpha$ arises from a geometry, then that geometry is necessarily a copy of $\mathrm{AG}(3, q)$, the 0-vertices of $\mathcal{C}_\alpha$ correspond to the points of $\mathrm{AG}(3, q)$ and we get $|V_0(\mathcal{C}_\alpha)| = q^3$. However, $\mathcal{C}_\alpha$ might be non-geometric. So, we must argue differently.

We define a graph Γ_0 on $V_0(\mathcal{C}_\alpha)$ by stating that two 0-vertices V, W of $\mathcal{C}_\alpha$ are adjacent in Γ_0 when $c \sim_0 d$ for some $c \in V$ and some $d \in W$. (Needless to say, the symbol $\sim_i$ means i-adjacency, for $i = 0, 1, 0^*$.) Let (V_1, W, V_2) be a path of Γ_0 of length 2. Then $c_1 \sim_0 d_1$ and $d_2 \sim_0 c_2$ for suitable chambers $c_1 \in V_1$, $d_1, d_2 \in W$ and $c_2 \in V_2$. As W, regarded as a $\{1, 0^*\}$-residue, is isomorphic to the chamber system of $\mathsf{PG}(2, q)$, it contains two chambers x_1 and x_2 such that $d_1 \sim_{0^*} x_1 \sim_1 x_2 \sim_{0^*} d_2$. As $c_i \sim_0 d_i$ for $i = 1, 2$ and, in view of Lemma 5.11, 0- and 0^*-adjacencies commute, we also have $c_1 \sim_{0^*} y_1 \sim_0 x_1$ and $x_2 \sim_0 y_2 \sim_{0^*} c_2$ for suitable chambers y_1, y_2. Clearly, $y_1 \in V_1$ and $y_2 \in V_2$. Furthermore, as $y_1 \sim_0 x_1 \sim_1 x_2 \sim_0 y_2$, the chambers y_1 and y_2 belong to the same $\{0, 1\}$-residue W' of $\mathcal{C}_\alpha$. The residues of $\mathcal{C}_\alpha$ of that type are isomorphic to the chamber systems of $\mathsf{AG}(2, q)$. Therefore there exist chambers $z_1, z_2 \in W'$ such that $y_1 \sim_1 z_1 \sim_0 z_2 \sim_1 y_1$. As $z_i \sim_1 y_i \in V_i$, $z_i \in V_i$ for $i = 1, 2$. As $z_1 \sim_0 z_2$, V_1 and V_2 are adjacent in Γ_0.

So far, we have proved that Γ_0 is a complete graph. A vertex V_0 of Γ_0, being isomorphic to the chamber system of $\mathsf{PG}(2, q)$, contains $(q^2 + q + 1)(q + 1)$ chambers. Each of them is 0-adjacent to $q - 1$ more chambers. Furthermore, given a 0-vertex V_1 and a 1-vertex W with $V_0 \cap W \neq \emptyset \neq W \cap V_1$, both $V_0 \cap W$ and $W \cap V_1$ contain at least $q + 1$ chambers and every chamber of $V_0 \cap W$ is 0-adjacent to a chamber of $W \cap V_1$. (Recall that W, regarded as a $\{0, 0^*\}$-residue, is the chamber system of a generalized digon with q elements of type 0 and $q+1$ elements of type 0^*). It follows that V_0 has at most

$$(q^2 + q + 1)(q + 1)(q - 1)/(q + 1) = q^3 - 1$$

neighbours in Γ_0. As Γ_0 is a complete graph, $|V_0(\mathcal{C}_\alpha)| \leq q^3$. □

The next corollary finishes the proof of Proposition 5.6.

Corollary 5.18. *Cases* (2) *and* (3) *of Lemma* 5.15 *are impossible.*

Proof. The 0-vertices of $\mathcal{C}_\alpha$ are the (right) cosets of $\langle P_1, P_0^* \rangle$ in $\langle P_0, P_0^*, P_1 \rangle$. However, $\langle P_0^*, P_1 \rangle = G_{a,\alpha^+}$ and $\langle P_0, P_0^*, P_1 \rangle = G_S$, by Lemma 5.7 and 5.12. Therefore,

$$|G_S : G_{a,\alpha^+}| \leq q^3 \tag{i}$$

by Lemma 5.17. This inequality immediately rules out case (2), as in that case $G_S = G$ whereas

$$|G : G_{a,\alpha^+}| \geq \frac{q^n(q^{n+1} - 1)(q^n - 1)(q^{n-1} - 1)}{(q^3 - 1)(q^2 - 1)(q - 1)} .$$

Indeed $\mathcal{G}$ has at least q^n points and the number of $(+)$-planes in the residue of a point is $[(q^{n+1} - 1)(q^n - 1)(q^{n-1} - 1)]/[(q^3 - 1)(q^2 - 1)(q - 1)]$.

In case (3), $|G_{A,S} : G_{A,\alpha^-}| = q^{n-2}$. As $|G_{A,\alpha^-} : G_{a,A,\alpha}| = q^2$ (which is the number of points of $\mathrm{Res}(A)$ in α^-) and

$$|G_S : G_{a,A,\alpha}| = |G_S : G_{A,S}| \cdot |G_{A,S} : G_{A,\alpha^-}| \cdot |G_{A,\alpha^-} : G_{a,A,\alpha}| ,$$

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we obtain $|G_S : G_{a,A,\alpha}| = q^n \cdot |G_S : G_{A,S}|$. We also have $G_{a,S} = G_{a,\alpha^+}$ and $|G_{a,\alpha^+} : G_{a,A,\alpha}| = q^2+q+1$. Therefore,

$$|G_S : G_{a,A,\alpha}| = |G_S : G_{a,S}| \cdot |G_{a,\alpha^+} : G_{a,A,\alpha}| = (q^2+q+1) \cdot |G_S : G_{a,S}|\,.$$

So far, $|G_S : G_{a,A,\alpha}| = q^n \cdot |G_S : G_{A,S}| = (q^2+q+1) \cdot |G_S : G_{a,S}|$. Comparing this with (i) and recalling that $G_{a,S} = G_{a,\alpha^+}$, we get

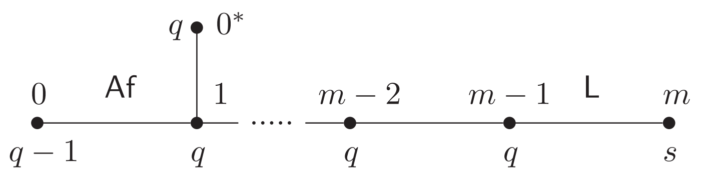
$$q^3(q^2+q+1) \geq (q^2+q+1)|G_S : G_{a,S}| = q^n \cdot |G_S : G_{A,S}|\,. \tag{ii}$$

By Lemma 5.14, G_S is transitive on $\sigma^*(S)$. Hence $|\sigma^*(S)| = |G_S : G_{A,S}|$. By Lemma 5.4, every dual point of $\sigma^*(\alpha^+)$ is the pole of a $(-)$-plane twinned with α^+. So, $|\sigma^*(S)| \geq q^2+q+1$. Therefore (ii) forces $n = 3$ (recall that $n \geq 3$ by assumption) and $|\sigma^*(S)| = q^2+q+1$. Hence $\sigma^*(x) \supseteq \sigma^*(S)$ for every $x \in \sigma(S)$, because all q^2+q+1 dual points of the $(+)$-plane of S having x as its pole belong to $\sigma^*(S)$. Consequently, $|\sigma^*(x) \cap \sigma^*(y)| \geq q^2+q+1$ for any two distinct points $x, y \in \sigma(S)$. This is in contradiction with the Intersection Property, which holds in $\mathcal{G}$, as noticed in subsection 5.1. □

6. From AG.PG* to D$^{\mathsf{Af}}_{n+1}$ — continuation

In this section $\mathcal{G}$ is a flag-transitive geometry belonging to the following diagram of rank $m+2$, where $q > 2$, $2 \leq m < n-1$ and $s = q^{n-m} + \cdots + q^2 + q$:

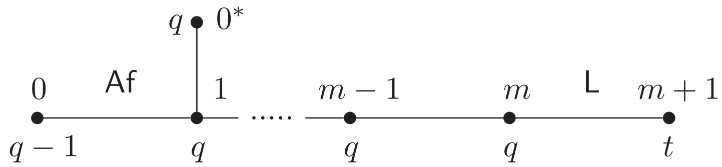
(TD$^{\mathsf{Af}}_{n+1,m}$)



The integer $n+1$, uniquely determined by m and the orders q and s, will be called the *virtual rank* of $\mathcal{G}$. Accordingly, we call $m+2$ the *actual* rank of $\mathcal{G}$.

Note that the residues of the points and the residues of the dual points of $\mathcal{G}$ are isomorphic to truncations of PG(n,q) and AG(n,q), respectively [17, Corollaries 7.11, 7.13, 7.15 and Exercise 7.1]. We shall prove that $\mathcal{G}$ is the $\{0, 0^*, 1, \ldots, m\}$-truncation of a geometry belonging to a diagram like the above, with the same virtual rank as $\mathcal{G}$ but of actual rank $m+3$:

(TD$^{\mathsf{Af}}_{n+1,m+1}$)



Clearly, $t = q^{n-m-1} + \cdots + q^2 + q = (s-q)/q$. We keep for 0-, 1- and 0^*-elements the terminology and the notation of section 5. Furthermore, for a flag F of type $\{1, 2, \ldots, m\}$, we denote by $\sigma(F)$ (resp. $\sigma^*(F)$) the set of (dual) points incident to F and, given a point or a dual point x, we denote by $\sigma_{[1,m]}(x)$ the set of $\{1, 2, \ldots, m\}$-flags incident to x. The symbols G_x, K_x and $\overline{G}_x$ keep the meaning stated in the previous sections. Clearly, Lemma 5.1 remains valid. We will freely use it in the sequel with no explicit reference.

6.1. Twin pairs and the structure $Ext(\mathcal{G})$

Given a point a, the flags of $\mathrm{Res}(a)$ of type $\{1, 2, \ldots, m\}$ contained in a given $(m+1)$-space of the projective space $\mathrm{Res}(a)$ are said to form an *$(m+1)^+$-space*. The point a is called the *pole* of that $(m+1)^+$-space. Similarly, for a dual point A, we say that the $\{1, 2, \ldots, m\}$-flags of $\mathrm{Res}(A)$ incident to a given $(m+1)$-space of the affine space $\mathrm{Res}(A)$ form an *$(m+1)^-$-space* with A as the *pole*. For an $(m+1)^+$-space α^+, we put $\sigma^*(\alpha^+) := \cup_{F\in\alpha^+}\sigma^*(F)$ and we say that a dual point *belongs* to α^+ if it belongs to $\sigma^*(\alpha^+)$. If an element x of $\mathcal{G}$ of type $1, 2, \ldots$ or m belongs to a flag $F \in \alpha^+$, then we say that x *belongs* to α^+. Similarly, for an $(m+1)^-$-space α^-, $\sigma(\alpha^-) := \cup_{F\in\alpha^-}\sigma(F)$ is the set of points *contained* in α^- and $\cup_{F\in\alpha^-}F$ is the set of elements of type $i \in \{1, 2, \ldots, m\}$ that *belong* to α^-.

For a $\{0, 0^*\}$-flag $\{a, A\}$, we say that an $(m+1)^+$-space $\alpha^+ \in \mathrm{Res}(a)$ with $A \in \sigma^*(\alpha^+)$ and an $(m+1)^-$-space $\alpha^- \in \mathrm{Res}(A)$ with $a \in \sigma(\alpha^-)$ are *twinned* if $\sigma_{[1,m]}(A) \cap \alpha^+ = \sigma_{[1,m]}(a) \cap \alpha^-$. We call the set of flags $\sigma_{[1,m]}(A) \cap \alpha^+ = \sigma_{[1,m]}(a) \cap \alpha^-$ the *$\{1, 2, \ldots, m\}$-pencil* of the twin pair (α^+, α^-). The pole a of α^+ and the pole A of α^- are the *$(+)$-pole* and *$(-)$-pole* of (α^+, α^-).

Denoted by G_{a,α^+} the setwise stabilizer of α^+ in G_a, we put $G_{a,F,\alpha^+} := G_{a,\alpha^+} \cap G_F$ for $F \in \sigma_{[1,m]}(a) \cap \alpha^+$ and $G_{a,A,\alpha^+} := G_{a,\alpha^+} \cap G_A$ for $A \in \sigma^*(\alpha^+)$. Similarly for α^-. The following can be proved in the same way as Lemma 5.4.

Lemma 6.1. *Given a $\{0, 0^*\}$-flag $\{a, A\}$, the twinning relation induces a bijection between the set of $(m+1)^+$-spaces α^+ of $\mathrm{Res}(a)$ containing A and the set of $(m+1)^-$-spaces α^- of $\mathrm{Res}(A)$ containing a. An $(m+1)^+$-space α^+ and an $(m+1)^-$-space α^- are twinned if and only if $G_{a,A,\alpha^+} = G_{a,A,\alpha^-}$.*

Let Π be the bipartite graph with the $(m+1)^+$- and $(m+1)^-$-spaces as vertices and the twin pairs as edges. We form an incidence structure $\mathrm{Ext}(\mathcal{G})$ of rank $m+3$ with $\{0, 0^*, 1, \ldots, m, m+1\}$ as the type-set, the 0-, 0^*-, 1-, ..., m-elements of $\mathcal{G}$ as elements of type 0, $0^*, 1, \ldots, m$ and the connected components of Π as elements of type $m+1$. The incidence relation of $\mathcal{G}$ induces on the set of elements of type 0, $0^*, 1, \ldots, m$ the incidence relation of $\mathrm{Ext}(\mathcal{G})$. An element x of type $0, 0^*, 1, 2, \ldots, m$ and an $(m+1)$-element S of $\mathrm{Ext}(\mathcal{G})$ are said to be incident when x belongs to an $(m+1)^-$- or $(m+1)^-$-space of S. We shall prove the following:

Proposition 6.2. *The structure* $\mathrm{Ext}(\mathcal{G})$ *is a flag-transitive geometry for diagram* $\mathrm{TD}^{\mathsf{Af}}_{n+1,m+1}$.

We will exploit induction on the virtual rank $n+1$. We have already proved this proposition when $n=3$. Indeed, in that case, $m=1$ and Proposition 6.2 reduces to Proposition 5.6. So, we assume the following induction hypothesis:

(I1) *For every flag-transitive geometry* $\mathcal{G}_0$ *belonging to diagram* $\mathrm{TD}^{\mathsf{Af}}_{k,h}$ *with* $k \leq n$ *and* $1 \leq h < k-2$, *the structure* $\mathrm{Ext}(\mathcal{G}_0)$ *is a flag-transitive geometry and belongs to diagram* $\mathrm{TD}^{\mathsf{Af}}_{k,h+1}$.

By repeatedly applying (I1) we obtain that, if $\mathcal{G}_0$ is as in (I1), then it is a truncation of a flag-transitive geometry of rank k belonging to $\mathrm{D}^{\mathsf{Af}}_k$. In view of Proposition 4.1, we can rephrase our induction hypothesis as follows:

(I2) *Let* $\mathcal{G}_0$ *be a flag-transitive geometry for* $\mathrm{TD}^{\mathsf{Af}}_{k,h}$ *with* $k \leq n$ *and* $1 \leq h < k-2$. *Then either* $\mathcal{G}_0$ *is the* $\{0, 0^*, 1, \ldots, h\}$*-truncation of* $\mathrm{Far}(\mathcal{D}_k(q))$ *or* $k=4$ *and* $\mathcal{G}_0 \cong \mathrm{TSec}(\mathcal{D}_4(q))$.

6.2. The parabolic system $(\{P_0, P_0^*, P_1, \ldots, P_{m+1}\}, B)$

Given a twin pair $\alpha := (\alpha^+, \alpha^-)$, let a and A be its $(+)$- and $(-)$-pole, F a flag in the $\{1, 2, \ldots, m\}$-pencil of α and S the $(m+1)$-element of $\mathrm{Ext}(\mathcal{G})$ containing $\alpha^+ \cup \alpha^-$. For $i = 1, 2, \ldots, m$, we denote by F_i the subflag of F formed by the elements of type different from i and, for $i, j = 1, 2, \ldots, m$, we put $F_{i,j} := F_i \cap F_j$.

We have $G_{a,A,\alpha^+} = G_{a,A,\alpha^-}$ (Lemma 6.1). So, we may write $G_{a,A,\alpha}$ for G_{a,A,α^+} or G_{a,A,α^-}. With that notation, we define the *minimal parabolics* P_0, P_0^*, $P_1, \ldots, P_{m+1}$ and the *Borel subgroup* B as follows:

$$\begin{array}{lll} P_0 := G_{A,F,\alpha^-}, & P_0^* := G_{a,F,\alpha^+}, & P_{m+1} := G_{a,A,F}, \\ P_i := G_{a,A,\alpha} \cap G_{F_i} \text{ for } i = 1, 2, \ldots, m, & & B := G_{a,A,F,\alpha}. \end{array}$$

As in section 5, we denote by G_S the stabilizer of S in $G = \mathrm{Aut}(\mathcal{G})$ and we put $G_{a,S} = G_a \cap G_S$, $G_{A,S} := G_A \cap G_S$, and so on. The following is the analogous of Lemma 5.7:

Lemma 6.3. *The Borel subgroup B is the intersection of any two minimal parabolics and all the following hold:*

$$
\begin{aligned}
&G_{a,F_i,\alpha^+} = \langle P_0^*, P_i\rangle, \quad G_{A,F_i,\alpha^-} = \langle P_0, P_i\rangle;\\
&G_{a,A,F_i} = \langle P_i, P_{m+1}\rangle \text{ for } i = 1, 2, \ldots, m;\\
&G_{a,A,F_{i,j},\alpha} = \langle P_i, P_j\rangle \text{ for } 1 \le i < j \le m;\\
&G_{a,F} = \langle P_0^*, P_{m+1}\rangle, \quad G_{A,F} = \langle P_0, P_{m+1}\rangle;\\
&G_a = \langle P_0^*, P_1, \ldots, P_m, P_{m+1}\rangle, \quad G_A = \langle P_0^*, P_1, \ldots, P_m, P_{m+1}\rangle;\\
&G = \langle P_0, P_0^*, P_1, \ldots, P_m, P_{m+1}\rangle\,.
\end{aligned}
$$

Lemma 6.4. $P_0P_0^* = P_0^*P_0$.

Proof. As the $\{0, 0^*\}$-residues of $\mathcal{G}$ are generalized digons, we have $G_{a,F}G_{A,F} = G_{A,F}G_{a,F}$. So, given $g \in P_0^*$ and $f \in P_0$, $gf = f_1g_1$ for suitable elements $f_1 \in G_{A,F}$ and $g_1 \in G_{a,F}$. Furtermore, $G_{a,A,F}$ is transitive on the set of $(m+1)^+$-spaces of $\mathrm{Res}(a)$ containing A. Let $g_0 \in G_{a,A,F}$ map $g_1(\alpha^+)$ back to α^+. Then $g_2 := g_0g_1 \in P_0^*$ and $f_2 := f_1g_0^{-1} \in G_{A,F} \le G_{A,l}$, where l is the 1-element (line) of F. It remains to prove that $f_2 \in P_0$.

Let $\mathcal{F}(l)$ be the set of $\{1, 2, \ldots, m\}$-flags containing l. Then both g and g_2^{-1} stabilize $\mathcal{F}(l) \cap \alpha^+$ and f stabilizes $\mathcal{F}(l) \cap \alpha^-$. However, all flags of $\mathcal{F}(l)$ are incident to both a and A. So, as α^+ and α^- are twinned, $\mathcal{F}(l) \cap \alpha^+ = \mathcal{F}(l) \cap \alpha^-$. Consequently f_2, which is equal to gfg_2^{-1}, stabilizes $\mathcal{F}(l) \cap \alpha^+$. Hence f_2 stabilizes α^-, as α^- is the unique $(m+1)^-$-space of $\mathrm{Res}(A)$ that contains $\mathcal{F}(l) \cap \alpha^-$. So, $f_2 \in P_0$. (Note that the assumption $m > 1$ is essential for this argument.) □

The proofs of the next four statements are quite similar to those of Lemmas 5.12, 5.13, 5.14 and 5.15. We leave them for the reader.

Lemma 6.5. *The group G_S acts transitively on the set of edges of Π contained in S and we have $G_S = \langle P_0, P_0^*, P_1, \ldots, P_m\rangle$.*

Lemma 6.6. *Denoted by $\sigma(S)$ and $\sigma^*(S)$ the set of points and the set of dual points incident to S, every point of $\sigma(S)$ is the pole of an $(m+1)^+$-space of S and every dual point of $\sigma^*(S)$ is the pole of an $(m+1)^-$-space of S.*

Lemma 6.7. *The group G_S acts transitively on $\sigma(S)$ and $\sigma^*(S)$.*

Lemma 6.8. *One of the following holds:*

(1) $G_{a,\alpha^+} = G_{a,S}$ *and* $G_{A,\alpha^-} = G_{A,S}$;

(2) $G_{a,S} = G_a$ *and* $G_{A,S} = G_A$. *In this case,* $\langle P_0, P_0^*, P_1, \ldots, P_m\rangle = \mathrm{Aut}(\mathcal{G})$;

(3) $G_{a,S} = G_{a,\alpha^+}$, $|G_{A,S} : G_{A,\alpha^-}| = q^{n-m-1}$ *and* $G_{A,S}$ *is the stabilizer in* G_A *of the parallel class of the* $(m+1)$*-space* α^- *in the affine geometry* $\mathrm{Res}(A)$.

6.3. Proof of Proposition 6.2

In this subsection, $\mathcal{C}$ is the chamber system arising from the parabolic system $(\{P_0, P_0^*, P_1, \ldots, P_{m+1}\}, B)$ of $G = \mathrm{Aut}(\mathcal{G})$. Lemmas 6.3 and 6.4 imply the following:

Lemma 6.9. *The chamber system* $\mathcal{C}$ *belongs to diagram* $\mathrm{TD}^{\mathsf{Af}}_{n+1,m+1}$.

In case (1) of Lemma 6.8, Lemmas 6.3 and 6.5 imply that $\mathrm{Ext}(\mathcal{G})$ is the geometry associated to $\mathcal{C}$ and Proposition 6.2 follows from Lemma 6.9.

Suppose now that we are in case (2) or (3) of Lemma 5.15. Let $\mathcal{C}_\alpha$ be the $\{0, 0^*, 1, \ldots, m\}$-residue of $\mathcal{C}$ containing the chamber B. That is, $\mathcal{C}_\alpha$ is the chamber system associated to the parabolic system $(\{P_0, P_0^*, P_1, \ldots, P_m\}, B)$ of G_S. (We recall that $G_S = \langle P_0, P_0^*, P_1, \ldots, P_m\rangle$, by Lemma 6.5.) As $\mathcal{C}$ belongs to $\mathrm{TD}^{\mathsf{Af}}_{n+1,m}$, $\mathcal{C}_\alpha$ belongs to $\mathrm{D}^{\mathsf{Af}}_{m+2}$. We will obtain a contradiction as in the proof of Proposition 5.6, but firstly we need to show that $\mathcal{C}_\alpha$ is geometric, namely it is the chamber system of a geometry. We consider the $\{0, 1, 0^*\}$-truncation of $\mathcal{C}_\alpha$ first, that is the chamber system $\mathrm{Tr}(\mathcal{G}_\alpha)$ associated to the parabolic system $(\{Q_0, Q_1, Q_0^*\}, \overline{B})$ where

$$Q_0 := G_{A,l,\alpha^-}\,, \quad Q_0^* := G_{a,l,\alpha^+}\,, \quad Q_1 := G_{a,A,\alpha}\,, \quad \overline{B} := G_{a,A,l,\alpha}\,,$$

l being the 1-element of the flag F. Note that $Q_0 \cap Q_1 = Q_1 \cap Q_0^* = Q_0^* \cap Q_0 = \overline{B}$. So, $\mathrm{Tr}(\mathcal{C}_\alpha)$ is well defined.

Lemma 6.10. *The chamber system* $\mathrm{Tr}(\mathcal{C}_\alpha)$ *belongs to diagram* $\mathsf{AG.PG}^*$ *with order* $q-1$, s, q *where* $s = q^m + \cdots + q^2 + q$.

Proof. The definition of Q_0, Q_1 and $\overline{B}$ makes it clear that the residues of $\mathrm{Tr}(\mathcal{G}_\alpha)$ of type $\{0, 1\}$ are chamber systems of $(m+1)$-dimensional affine spaces of order q. Similarly, the $\{0^*, 1\}$-residues are chamber systems of $(m+1)$-dimensional projective spaces of order q. It remains to prove that the residues of type $\{0, 0^*\}$ are generalized digons, namely $Q_0 Q_0^* = Q_0^* Q_0$. In view of the diagram of $\mathcal{C}_\alpha$ and the definition of Q_0 and Q_0^*, we have $Q_0 = P_0\overline{B} = \overline{B}P_0$ and $Q^* = P_0^*\overline{B} = \overline{B}P_0^*$. These equalities and the equality $P_0 P_0^* = P_0^* P_0$ of Lemma 6.4 imply that $Q_0 Q_0^* = Q_0^* Q_0$. □

Lemma 6.11. *The chamber system* $\mathrm{Tr}(\mathcal{G}_\alpha)$ *is geometric.*

Proof. According to Meixner and Timmesfeld [14], $\mathrm{Tr}(\mathcal{C}_\alpha)$ is geometric if and only if all the following hold:

$$\begin{array}{llll}
(1) & \langle Q_0^*, Q_1\rangle \cap Q_0 = \overline{B}\,, & (4) & \langle Q_0, Q_0^*\rangle \cap \langle Q_0, Q_1\rangle = Q_0\,, \\
(2) & \langle Q_0, Q_1\rangle \cap Q_0^* = \overline{B}\,, & (5) & \langle Q_0^*, Q_0\rangle \cap \langle Q_0^*, Q_1\rangle = Q_0^*\,, \\
(3) & \langle Q_0, Q_0^*\rangle \cap Q_1 = \overline{B}\,, & (6) & \langle Q_1, Q_0\rangle \cap \langle Q_1, Q_0^*\rangle = Q_1\,.
\end{array}$$

Properties (1), (2), (3) and (6) easily follow from the definitions of Q_0, Q_0^*, Q_1 and $\overline{B}$. Turning to (4) and (5), we recall that $\langle Q_0, Q^*\rangle = Q_0Q_0^* = Q_0Q^*$ by Lemma 6.11 and, as remarked in the proof of that lemma, $Q_0 = P_0\overline{B} = \overline{B}P_0$ and $Q^* = P_0^*\overline{B} = \overline{B}P_0^*$. So, we can rewrite (4) and (5) as follows:

$$(4')\quad P_0^* \cap \langle P_0, Q_1\rangle \subseteq P_0\overline{B}, \qquad (5')\quad P_0 \cap \langle P_0^*, Q_1\rangle \subseteq P_0^*\overline{B}.$$

However it is clear that $P_0^* \cap \langle P_0, Q_1\rangle = \overline{B}$ and $P_0 \cap \langle P_0^*, Q_1\rangle = \overline{B}$. Properties (4) and (5) are proved. □

Lemma 6.12. *The chamber system $\mathcal{C}_\alpha$ is geometric and, denoted by $\mathcal{G}_\alpha$ its underlying geometry, either $\mathcal{G}_\alpha \cong \mathrm{Far}(\mathcal{D}_{m+2}(q))$ or $m = 2$ and $\mathcal{G}_\alpha$ is isomorphic to the complement of a secant hyperplane of $\mathcal{D}_4(q)$.*

Proof. Let $\mathcal{T}$ be the underlying geometry of $\mathrm{Tr}(\mathcal{C}_\alpha)$ (Lemma 6.11). Then $\mathcal{T}$ has the same diagram and orders as $\mathrm{Tr}(\mathcal{C}_\alpha)$. By the Inductive Hypothesis (I2) of section 6.1, $\mathcal{T} = \mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{G}_\alpha)$ where $\mathcal{G}_\alpha$ is either $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ or the complement of a secant hyperplane of $\mathcal{D}_4(q)$. The chamber system $\mathcal{C}(\mathcal{G}_\alpha)$ of $\mathcal{G}_\alpha$ is associated to the the same parabolic system as $\mathcal{C}_\alpha$. Hence $\mathcal{C}_\alpha \cong \mathcal{C}(\mathcal{G}_\alpha)$. □

Lemma 6.13. *The number of 0-vertices of $\mathcal{C}_\alpha$ is either $q^{(m+2)(m+1)/2}$ or $q^3(q^3+1)$, with $m = 2$ in the latter case.*

Proof. The 0-vertices of $\mathcal{C}_\alpha$ are the 0-elements of its underlying geometry $\mathcal{G}_\alpha$ and, by Lemma 6.12, the latter is either $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ or the complement of a secant hyperplane of $\mathcal{D}_4(q)$. The 0-elements of $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ are the vertices of the graph $\mathsf{Alt}(m+2, q)$, which has $q^{(m+2)(m+1)/2}$ vertices. The complement of a secant hyperplane of $\mathcal{D}_4(q)$ has $q^3(q^3+1)$ points. □

The following finishes the proof of Proposition 6.2.

Corollary 6.14. *Cases (2) and (3) of Lemma 6.8 are impossible.*

Proof. The 0-vertices of $\mathcal{C}_\alpha$ are the right cosets of $\langle P_0^*, P_1, \ldots, P_m\rangle$ in the group $\langle P_0, P_0^*, P_1, \ldots, P_m\rangle$ namely, by Lemmas 6.3 and 6.5, the right cosets of G_{a,α^+} in G_S. Similarly, the 0^*-vertices are the right cosets of G_{A,α^-} in G_S. So, by

Lemma 6.13, either $|G_S : G_{a,\alpha^+}| = q^{(m+2)(m+1)/2}$ or $m = 2$ and $|G_S : G_{a,\alpha^+}| = q^3(q^3+1)$. This immediately rules out case (2), since $G_S = G$ in that case, and

$$|G : G_{a,\alpha^+}| \geq \frac{q^n \prod_{i=0}^{m+1}(q^{n+1-i}-1)}{\prod_{i=0}^{m+1}(q^{m+2-i}-1)}.$$

In case (3), $|G_{A,S} : G_{A,\alpha^-}| = q^{n-m-1}$ and G_{A,α^-} is the stabilizer in G_S of a 0^*-element of the geometry $\mathcal{G}_\alpha$ associated to $\mathcal{C}_\alpha$. Also, $G_{a,S} = G_{a,\alpha^+}$. Therefore $|\sigma(S)| = |G_S : G_{a,\alpha^+}|$, by Lemma 6.7. Suppose first that $\mathcal{G}_\alpha = \mathrm{Far}(\mathcal{D}_{m+2}(q))$. Then,

$$|\sigma(S)| = q^{(m+2)(m+1)/2}. \tag{i}$$

We recall that $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ is formed by the elements of the building $\mathcal{D} := \mathcal{D}_{m+2}(q)$ that, compatibly with their type, have maximal distance from a given element S_0 of type 0 (if m is even) or 0^* (when m is odd). The automorphism group of $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ is the stabilizer of S_0 in $\mathsf{P\Gamma O}^+_{2m+4}(q)$. It contains a flag-transitive subgroup Ω with the following properties:

(a) The group Ω is the semidirect product $V : L$ of the additive group V of $V(m+2,q) \wedge V(m+2,q)$ with the group L induced by the irreducible natural action of $\mathsf{SL}(m+2,q)$ on $V(m+2,q) \wedge V(m+2,q)$.

(b) $|\mathrm{Aut}(\mathrm{Far}(\mathcal{D}_{m+2}(q))) : \Omega|$ divides $(q-1) \cdot |\mathrm{Aut}(\mathsf{GF}(q)|$.

(c) If either $m > 2$ or $q > 2$, then Ω is contained in every flag-transitive subgroup of $\mathrm{Aut}(\mathrm{Far}(\mathcal{D}_{m+2}(q)))$.

(Property (c) follows from the theorem of Higman [8] applied to residues of 0-elements.) According to (c), and since $q > 2$ by assumption,

$$G_S \geq \Omega. \tag{ii}$$

We now turn to the polar space $\mathcal{P}$ associated to $\mathcal{D}$. The 0- and 0^*-elements of $\mathcal{D}$ are the maximal singular subspaces of $\mathcal{P}$ and the 0-elements of $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ correspond to the maximal singular subspaces of $\mathcal{P}$ that meet S_0 trivially. The 0^*-elements of $\mathrm{Far}(\mathcal{D}_{m+2}(q))$ are the maximal singular subspaces of $\mathcal{P}$ that meet S_0 in precisely one point. If X is one of them, computing the stabilizer Ω_X of X in Ω is a routine exercise, which we leave for the reader. The following can be obtained as a byproduct of that computation:

(d) $V\Omega_X$ is the unique proper subgroup of Ω that properly contains Ω_X and we have $|\Omega : V\Omega_X| = q^{m+1} + \cdots + q + 1$ and $|V\Omega_X : \Omega_X| = q^{(m+1)m/2}$.

The group G_{A,α^-} is indeed the stabilizer in $G_S \leq \mathrm{Aut}(\mathrm{Far}(\mathcal{D}_{m+2}(q)))$ of a 0^*-element of $\mathrm{Far}(\mathcal{D}_{m+2}(q))$. According to (ii), $G_{A,\alpha^-} \geq \Omega_A$. However, G_S contains

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$G_{A,S}$ and $|G_{A,S} : G_{A,\alpha^-}| = q^{n-m-1} \geq q$, as $m < n-1$. In view of (d) and (b), we get $n - m - 1 = (m+1)m/2$. So, $\sigma(X) = \sigma(S)$ for every $X \in \sigma^*(S)$ and we get a contradiction as in the proof of Corollary 5.18. (Note that $\mathrm{Tr}_{\{0,1,0^*\}}(\mathcal{G})$ is a geometry as considered in section 5, whence it satisfies the Intersection Property.)

Finally, suppose $m = 2$ with $\mathcal{G}_\alpha$ the complement of a secant hyperplane of $\mathcal{D}_4(q)$. Then $\mathcal{G}_\alpha$ has $q^3(q^3+1)$ points, each point is incident to q^3+q^2+q+1 dual points and every dual point of $\mathcal{G}_\alpha$ is incident to q^3 points. Therefore, $\mathcal{G}_\alpha$ has $(q^3+1)(q^3+q^2+q+1)$ dual points. Accordingly, $|G_A : G_{A,\alpha^-}| = (q^3+1)(q^3+q^2+q+1)$. On the other hand, $|G_{A,S} : G_{A,\alpha^-}| = q^{n-m-1} = q^{n-3}$. Hence q^{n-3} divides $(q^3+1)(q^3+q^2+q+1)$, that is $n = 3$. However, $2 = m < n-1$ by assumption. We have reached a final contradiction. □

7. End of the proof of Theorem 1.4

When $q > 2$, we obtain Theorem 1.3 by applying Proposition 5.6 first, then Proposition 6.2 as many times as we need and, finally, Proposition 4.1 (or 2.1, if $n = 3$). When $q = 2$, the statement of Theorem 1.3 is the main result of Huybrechts and Pasini [11].

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