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Line-transitive, point-imprimitive linear spaces: the grid case

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Abstract

For a fixed integer k, all but a finite number of line-transitive linear spaces with lines of size k are point-primitive. In this paper, we study the finite class of examples where a line-transitive group is point-imprimitive, that is, preserves a non-trivial partition of the point set. We restrict to the case where (i) the number of unordered point-pairs, on a given line, contained in the same class of the partition is at most eight, and (ii) some non-identity group element fixes setwise each class of the partition, and also fixes a point. This family of linear spaces was studied by Ngo Dac Tuan and the third author in 2003, leaving several problems unresolved. We prove that all examples in this family are known, namely Desarguesian projective planes of appropriate orders, and an additional example on 91 points. The result is obtained by a combination of theoretical analysis, and exhaustive computer search.

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1 Introduction

A finite *linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ consists of a finite set $\mathcal{P}$ of points, and a set $\mathcal{L} = \{\lambda_1, \ldots, \lambda_b\}$ of subsets of $\mathcal{P}$, called lines, such that each pair of points lies in a unique line, and each line contains at least two points. The *automorphism*

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group $\mathrm{Aut}(\mathcal{S})$ of $\mathcal{S}$ consists of all permutations of $\mathcal{P}$ that leave $\mathcal{L}$ invariant, and $\mathcal{S}$ is called *line-transitive* if $\mathrm{Aut}(\mathcal{S})$ is transitive on $\mathcal{L}$. More generally, we say that a linear space $\mathcal{S}$ has a group theoretic property if some subgroup of $\mathrm{Aut}(\mathcal{S})$ has the property. In particular, for a line-transitive linear space $\mathcal{S}$, all lines have the same size k say, and hence $\mathcal{S}$ is a 2-$(v, k, 1)$ design with $v = |\mathcal{P}|$ the number of points of the linear space.

The major result that inspired this investigation is due to Delandtsheer and Doyen [8] (see Theorem 1.1) and shows that, in a sense, line-transitive linear spaces are almost always point-primitive. That is, for a given line-size, there is only a finite number of linear spaces which are line-transitive but not point-primitive. Put in a different way, linear spaces with these properties are counterexamples to the (wrong) assertion that line-transitivity implies point-primitivity. The result of Delandtsheer and Doyen shows that these examples are exceptional and hence deserve attention. In this paper, we study these exceptional examples of line-transitive point-imprimitive linear spaces.

Let us describe the history of the study of these counterexamples briefly. The theorem of Delandtsheer and Doyen shows that if a line-transitive group of automorphisms of a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ with line-size k leaves invariant a non-trivial partition of $\mathcal{P}$, then $|\mathcal{P}| \leq \left(\binom{k}{2} - 1\right)^2$. It was shown in [3, 12] that this upper bound could not be sharp if $k \neq 8$, and in [11] that there are exactly 467 pairwise non-isomorphic linear spaces that attain the bound when $k = 8$.

Line-transitive, point-imprimitive linear spaces were investigated further in [13] in the special case where (i) the number of inner pairs of a given line, that is, the number of unordered point-pairs of the line that lie together in the same class of an invariant partition, is at most eight, and (ii) some non-identity automorphism fixes setwise each class of the partition, and also fixes a point. In this situation, Praeger and Tuan [13, Theorem 1.6] showed that there is only a small number of feasible parameter sets.

Our aim in this paper is to complete this classification by dealing with a rather difficult case, namely case (c) of [13, Theorem 1.6], about which not much could be said until now. In Theorem 1.2 below, we show that this case does indeed lead to examples, but that all these examples are known.

We remark that parallel to this study, a massive investigation of the general case for line-transitive, point-imprimitive linear spaces has been undertaken. It is general in the sense that the number of inner pairs on a line is no longer restricted to be small, and there are no restrictions on the classwise stabiliser. That study is reported in [2], and is independent of this work. It relies only on [13, Theorem 1.6], not on Theorem 1.2 below. Similarly, the classification achieved in this paper does not follow from the classification in [2] as our classification assumes no bound on $k/\gcd(k, v)$.

1.1 Line-transitive, point-imprimitive linear spaces

A partition $\mathcal{C}$ of a finite set $\mathcal{P}$ is a set of pairwise disjoint subsets whose union equals the set $\mathcal{P}$. The subsets $C \in \mathcal{C}$ are called *parts* or *classes* of $\mathcal{C}$. The partition $\mathcal{C}$ is called *trivial* if either $\mathcal{C}$ consists of only one class or $\mathcal{C}$ contains only one-element classes, otherwise $\mathcal{C}$ is said to be *non-trivial*. A *$\mathcal{C}$-inner pair* (or simply an inner pair, if the partition $\mathcal{C}$ is clear from the context) is an unordered pair of points $\alpha, \beta \in \mathcal{P}$ which belong to the same class of $\mathcal{C}$. Otherwise, if the points belong to two different classes of $\mathcal{C}$, we say that they form a *$\mathcal{C}$-outer pair*.

Let G be a group that acts on a set $\mathcal{P}$. Then G is said to leave a partition $\mathcal{C}$ invariant if, for all $g \in G$ and all $C \in \mathcal{C}$, the image C^g is also a class of $\mathcal{C}$. In particular, $\mathcal{C}$-inner pairs are mapped to $\mathcal{C}$-inner pairs under elements of G. Let G be transitive on $\mathcal{P}$. If there exists a non-trivial G-invariant partition of $\mathcal{P}$ then G is said to be *imprimitive* on $\mathcal{P}$. Otherwise, G is *primitive* on $\mathcal{P}$.

We remark that in studying a group G of automorphisms of a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, the group may not coincide with $\mathrm{Aut}(\mathcal{S})$, the full automorphism group of $\mathcal{S}$. Also, a linear space is said to be *trivial* if it has only one line, or if all its lines have only two points; otherwise it is called non-trivial. Thus $\mathcal{S}$ is non-trivial if $2 < k < |\mathcal{P}|$.

The result from [8] mentioned above, and stated as Theorem 1.1 below, shows among other things that, for a linear space admitting a line-transitive, point-imprimitive automorphism group, the number of points is bounded above by a function of the line size. This means that for a given line size k there is only a finite number of line-transitive, point-imprimitive linear spaces.

Theorem 1.1 (Delandtsheer-Doyen parameters [8]). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a non-trivial linear space admitting a line-transitive automorphism group G that leaves invariant a non-trivial partition $\mathcal{C}$ of $\mathcal{P}$ with d classes of size c. Let x be the number of $\mathcal{C}$-inner pairs of a line, and let k be the line size. Then there exists another positive integer y such that*

$$c = \frac{\binom{k}{2} - x}{y} \quad \text{and} \quad d = \frac{\binom{k}{2} - y}{x}. \tag{1}$$

We call the pair (x, y) the *Delandtsheer-Doyen parameters* corresponding to $\mathcal{C}$. By [13, Theorem 1.1(a)],

$$x = \frac{\binom{k}{2}(c-1)}{cd-1} \quad \text{and} \quad y = \frac{\binom{k}{2}(d-1)}{cd-1}, \tag{2}$$

and hence the integer triples (c, d, k) and (x, y, k) mutually determine each other. Moreover, if (c, d, k) corresponds to (x, y, k) then (d, c, k) has (y, x, k) as its mate, though in general there will not exist a G-invariant partition of $\mathcal{P}$ with c classes of size d. Recent attempts, for example in [4], have aimed at classifying line-transitive, point-imprimitive linear spaces for small values of k. This article is a step towards a classification in the case where the number x of inner pairs on a line is small. We remark further that while the first of the Delandtsheer-Doyen parameters x has a combinatorial interpretation as the number of inner pairs on a line, no similar meaning is known for the second parameter y in general. However, there is an interpretation for y when G preserves a grid structure on $\mathcal{P}$. We will explore this in section 3.

1.2 The main result

For a transitive group G on a set $\mathcal{P}$ and a G-invariant partition $\mathcal{C} = \{C_1, \ldots, C_d\}$ of $\mathcal{P}$, the kernel of G on $\mathcal{C}$ is the subgroup $G_{(\mathcal{C})}$ of elements $g \in G$ with $C_i^g = C_i$ for $i = 1, \ldots, d$. We say that $\mathcal{C}$ is *G-normal* if $G_{(\mathcal{C})}$ is transitive on each of the $\mathcal{C}$-classes.

In [13], Praeger and Tuan examined line-transitive point-imprimitive linear spaces where the first Delandtsheer-Doyen parameter is small and the partition is normal. The current work aims at resolving the open cases in part (c) of their Theorem 1.6, thereby proving the result promised in the remarks following the statement of [13, Theorem 1.6]. Part (a) of [13, Theorem 1.6] addresses the situation where the only group element fixing a point, and fixing each class of the partition setwise, is the identity element. Parts (b) and (c) (which contain the unresolved cases) address the opposite situation, namely where there is a non-identity element that fixes a point and fixes each class setwise. Note that we do not include as an assumption that the partition is normal. Instead we prove this as a consequence of the existence of this element. Our result is as follows. Its proof will be presented in section 4.

Theorem 1.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a non-trivial linear space admitting a line-transitive group G that leaves invariant a non-trivial point-partition $\mathcal{C}$. Assume further that the number of $\mathcal{C}$-inner pairs on a line is at most 8, and that some non-trivial element of G fixes each $\mathcal{C}$-class setwise and also fixes at least one point. Then either*

(i) *$\mathcal{S}$ is a Desarguesian projective plane of order 4, 9 or 16, or*

(ii) *$\mathcal{S}$ is the Colbourn-McCalla design constructed in [6] with 91 points and line size 6.*

Moreover, each of these linear spaces has a line-transitive, point-imprimitive subgroup of automorphisms satisfying the hypotheses above.

2 Some parameters for line-transitive, point-imprimitive linear spaces

2.1 Base lines

Let G act on a set of points $\mathcal{P}$. For a nonnegative integer s, we denote the set of s-subsets of $\mathcal{P}$ by $\mathcal{P}^{\{s\}}$. For $S \subseteq \mathcal{P}$, let S^G denote the G-orbit of S, that is, the set of images S^g for all elements g in G, and let G_S denote the setwise stabiliser of S in G. Under certain conditions, the G-orbit of a k-subset $\lambda \subseteq \mathcal{P}$ can be taken as the set of lines of a linear space admitting G as a line-transitive automorphism group. In this case, we say that λ is a *base line* for the G-line-transitive linear space $(\mathcal{P}, \lambda^G)$. We will characterize base lines shortly. First, we examine some properties of G-orbits on pairs. In a linear space the unique line containing two points α and β is denoted by $\lambda(\alpha, \beta)$. Recall that the number of lines of a linear space is generally denoted by b and equals $\binom{v}{2}/\binom{k}{2}$, where $v = |\mathcal{P}|$ and k is the line size.

Lemma 2.1. *Let G be a line-transitive automorphism group of a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ with line size k and $|\mathcal{L}| = b$. Then the length of every G-orbit $\mathcal{O}$ on $\mathcal{P}^{\{2\}}$ is divisible by b. In particular, the number of orbits of G on $\mathcal{P}^{\{2\}}$ is at most $\binom{k}{2}$.*

Proof. For every pair of points $\{\alpha, \beta\}$, $G_{\{\alpha,\beta\}}$ fixes $\lambda(\alpha, \beta)$, and hence $G_{\{\alpha,\beta\}} \leq G_{\lambda(\alpha,\beta)}$. Thus $b = |\lambda(\alpha, \beta)^G| = |G : G_{\lambda(\alpha,\beta)}|$ divides $|G : G_{\{\alpha,\beta\}}| = \left|\{\alpha, \beta\}^G\right|$. In particular $|\mathcal{O}|/b$ is a positive integer, where $\mathcal{O} := \{\alpha, \beta\}^G$. Counting all pairs of points yields the upper bound on the number of those orbits: $\binom{v}{2} = b \cdot \sum_{\mathcal{O}} |\mathcal{O}|/b \geq \left(\binom{v}{2}/\binom{k}{2}\right) \cdot \sum_{\mathcal{O}} 1$, and hence $\binom{k}{2} \geq \sum_{\mathcal{O}} 1$. □

We call a transitive permutation group G on $\mathcal{P}$ *feasible* (for a particular linear space parameter set) if the length of every orbit on $\mathcal{P}^{\{2\}}$ is divisible by b. Lemma 2.2 gives a criterion for testing whether a k-subset λ is a base line for a G-line-transitive linear space. The criterion involves the integers $\mu(\mathcal{O}, S)$ defined in (3), for G-orbits $\mathcal{O}$ on $\mathcal{P}^{\{2\}}$ and k-subsets S of $\mathcal{P}$.

$$\mu(\mathcal{O}, S) = \left|\mathcal{O} \cap S^{\{2\}}\right| = \left|\left\{\{\gamma, \delta\} \in \mathcal{O} : \{\gamma, \delta\} \subseteq S\right\}\right|. \qquad (3)$$

Note that $\mu(\mathcal{O}, S)$ does not depend on the choice of S within its G-orbit $S^G = \{S^g \mid g \in G\}$.

Lemma 2.2 (Orbit Lemma)**.** *Let G be a group acting on a set $\mathcal{P}$ of v points. Let k be a positive integer greater than 2 such that $b = \binom{v}{2}/\binom{k}{2}$ is an integer. Then $\lambda \in \mathcal{P}^{\{k\}}$ is a base line of a linear space with point set $\mathcal{P}$ if and only if both*

(i) $\left|\lambda^G\right| = b$ *and*

(ii) $\mu(\mathcal{O}, \lambda) = |\mathcal{O}|/b$ *for each G-orbit* $\mathcal{O}$ *on* $\mathcal{P}^{\{2\}}$.

A proof can be found, for example, in [3, Proposition 1.3]: one shows that $\mu(\mathcal{O}, \lambda)/|\mathcal{O}|$ is a constant, say a, independent of the G-orbit $\mathcal{O}$ in $\mathcal{P}^{\{2\}}$, if and only if each pair of points lies in a constant number of the k-subsets in λ^G. An easy counting argument then shows that this number is $a|\lambda^G|$. Thus λ is a base line for a linear space if and only if the constant a is $1/b$ where $b = |\lambda^G|$, that is, if and only if (i) and (ii) both hold.

We mention some elementary facts about automorphisms of linear spaces.

Lemma 2.3. *Let G be a group of automorphisms of a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, and let $\lambda \in \mathcal{L}$, $A \subset \mathcal{P}$, and $g \in G$.*

(a) *If $A \subseteq \lambda$ and $|A| \geq 2$, then $G_A \leq G_\lambda$.*

(b) *The image $\lambda^g = \lambda$ if and only if λ is a union of $\langle g \rangle$-orbits.*

(c) *If the order $|g|$ of g is 2 then g fixes some line setwise.*

Proof. For part (a), let α, β be distinct points of A. Since λ is the unique line containing $\{\alpha, \beta\}$ it follows that λ is the unique line containing A. Therefore $G_A \leq G_\lambda$. Part (b) is obvious. For part (c), let $|g| = 2$ and let α, β be distinct points interchanged by g. Then g fixes $\{\alpha, \beta\}$ setwise and hence by part (a), g fixes setwise the line $\lambda(\alpha, \beta)$. □

2.2 Intersection numbers

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a line-transitive point-imprimitive linear space with respect to a group G of automorphisms and a non-trivial partition $\mathcal{C}$ with d classes of size c, and let $\lambda \in \mathcal{L}$. Then the *intersection numbers* defined as

$$d_i = \left|\left\{\, C \in \mathcal{C} :\ |C \cap \lambda| = i \right\}\right|, \tag{4}$$

for $0 \leq i \leq k$ are independent of the choice of the line λ and satisfy

$$\sum_{i=1}^{k} d_i \binom{i}{2} = x\,. \tag{5}$$

The *intersection type* is the vector $(1^{d_1}, \ldots, k^{d_k})$, where we usually omit components i^{d_i} if $d_i = 0$. The *spectrum* is the set of non-zero intersection numbers

$$\operatorname{spec} \mathcal{S} := \{ i > 0 \mid d_i \neq 0 \}\,.$$

We sometimes write $\operatorname{spec}_{\mathcal{C}} \mathcal{S}$ if we need to specify the partition $\mathcal{C}$.

3 Grid structures on points

Throughout this section, $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ denotes a linear space that admits a line-transitive group G leaving invariant a non-trivial partition $\mathcal{C} = \{C_1, \dots, C_d\}$ of the point set $\mathcal{P}$ with d classes of size c. Sometimes G leaves invariant a second partition $\mathcal{D}$ such that $|C \cap D| = 1$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. In this case $\mathcal{D} = \{D_1, \dots, D_c\}$ with each $|D_j| = d$ and every point α of $\mathcal{P}$ may be labelled by the unique ordered pair of integers (i, j) such that $\{\alpha\} = C_i \cap D_j$. We write $\alpha = \alpha_{i,j}$. Thus we can parametrize the points by the Cartesian product

$$\{1, \dots, d\} \times \{1, \dots, c\}$$

such that

$$\mathcal{P} = \{\alpha_{i,j} \mid 1 \le i \le d,\ \ 1 \le j \le c\}\,.$$

The classes of the partitions $\mathcal{C}$ and $\mathcal{D}$ can then be thought of as the rows and columns respectively of the matrix

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,c} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,c} \\ \vdots & & & \vdots \\ \alpha_{d,1} & \alpha_{d,2} & \cdots & \alpha_{d,c} \end{pmatrix}, \tag{6}$$

that is,

$$C_i = \{\alpha_{i,j} \mid 1 \le j \le c\}\,, \quad 1 \le i \le d$$

and

$$D_j = \{\alpha_{i,j} \mid 1 \le i \le d\}\,, \quad 1 \le j \le c\,.$$

We call such a set $\mathcal{P}$ together with a distinguished ordered pair of partitions $(\mathcal{C}, \mathcal{D})$ as above a *grid of type* $d \times c$ and write $\mathcal{P} = \mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$. We call $\mathcal{C}$ the *row-partition* and $\mathcal{D}$ the *column-partition* of the grid. Furthermore, the intersection types of a line with respect to the partitions $\mathcal{C}$ and $\mathcal{D}$ are called the *row intersection type* and *column intersection type*, respectively.

Desarguesian projective planes for which the number v of points is not a prime power provide line-transitive examples: write $v = d \cdot c$ with d and c coprime, and take G to be a Singer group, that is, a cyclic group $\langle g \rangle$ of automorphisms that is regular on points (and hence also on lines). Then the orbit sets of $\langle g^d \rangle$ and $\langle g^c \rangle$ on points are G-invariant point-partitions $\mathcal{C}$ and $\mathcal{D}$ that form a grid $\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$.

A grid is said to be *trivial* if at least one of the numbers c or d is 1, that is, if $\mathcal{C}$ and $\mathcal{D}$ are trivial partitions. We also say that G *preserves a grid* if G preserves a pair of non-trivial partitions that forms a grid. In fact if G preserves

$\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$, then G may be identified with a subgroup of $\mathrm{Sym}_d \times \mathrm{Sym}_c$ in such a way that, for $g = (g_1, g_2) \in G$, the image of a point α_{ij} is $\alpha_{i'j'}$, where $C_{i'} = C_i^{g_1}$ and $D_{j'} = D_j^{g_2}$. The permutations g_1, g_2 are the permutations induced by g on $\mathcal{C}$ and $\mathcal{D}$, and they act as row and column permutations respectively on the array (6). Thus we write $g_{\mathrm{rows}} := g_1$ and $g_{\mathrm{cols}} := g_2$. (This is the natural *product action* of $\mathrm{Sym}_d \times \mathrm{Sym}_c$, where Sym_n denotes the symmetric group of degree n.) We formalise this, and the obvious converse, below.

Lemma 3.1. *A permutation group $G \leq \mathrm{Sym}_{dc}$ preserves a grid structure $\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$ if and only if G is isomorphic to a subgroup of $\mathrm{Sym}_d \times \mathrm{Sym}_c$ in its product action. In this case every element g of G can be written as an ordered pair $g = (g_{\mathrm{rows}}, g_{\mathrm{cols}}) \in \mathrm{Sym}_d \times \mathrm{Sym}_c$ where $g_{\mathrm{rows}} \in \mathrm{Sym}_d$ and $g_{\mathrm{cols}} \in \mathrm{Sym}_c$ act as row and as column permutations, respectively.*

If an automorphism group G of a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ leaves invariant a non-trivial grid structure $\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$ on points, then $\mathcal{P}$ can be identified with the points of $\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$, and we simply write $\mathcal{S} = \left(\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}, \mathcal{L}\right)$ and identify G with a subgroup of $\mathrm{Sym}_d \times \mathrm{Sym}_c$.

Bearing in mind the arrangement of the point set in (6), we say that a pair of distinct points

$$\{\alpha_{i,j}, \alpha_{u,v}\}$$

is *horizontal* if $i = u$, *vertical* if $j = v$, and *skew* if $i \neq u, j \neq v$. If G preserves the grid (6), then G also preserves the properties on point-pairs of being horizontal, vertical, or skew. This means that a G-orbit on point-pairs consists of pairs of one kind only. We say that an orbit on point-pairs is horizontal (or vertical or skew, respectively) if the pairs in that orbit all have that property.

Apart from the product action of the group $\mathrm{Sym}_d \times \mathrm{Sym}_c$ referred to in Lemma 3.1, there is a second action of this group relevant to our study of linear spaces and grids. An action of $\mathrm{Sym}_d \times \mathrm{Sym}_c$ on the set $\mathcal{M}_{d\times c}$ of $\{0,1\}$-matrices of size $d \times c$ is defined as follows: if $A = (a_{i,j}) \in \mathcal{M}_{d\times c}$ and $g = (g_{\mathrm{rows}}, g_{\mathrm{cols}}) \in \mathrm{Sym}_d \times \mathrm{Sym}_c$, then

$$A^g = \left(a_{i^{g_{\mathrm{rows}}^{-1}},\, j^{g_{\mathrm{cols}}^{-1}}}\right).$$

For two $(d \times c)$-matrices A and B we write $A \equiv B$ if there is an element $g \in \mathrm{Sym}_d \times \mathrm{Sym}_c$ with $A^g = B$. The equivalence classes of $\{0,1\}$-matrices under this relation are known as isomorphism types of incidence structures. The relation to linear spaces is as follows:

Assume that a line-transitive group G on a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ preserves a grid structure $\mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$ on the point set $\mathcal{P}$. Then each subset $S \subseteq \mathcal{P}$ corresponds to a matrix in $\mathcal{M}_{d\times c}$, the *characteristic matrix* of S,

$$\chi(S) = (a_{i,j}),$$

where $a_{i,j} = 1$ if $\alpha_{i,j} \in S$ and $a_{i,j} = 0$ otherwise. The map χ is a one-to-one correspondence between the powerset of $\mathcal{P}$ (the set of subsets of $\mathcal{P}$) and the set $\mathcal{M}_{d\times c}$. Moreover, the actions of $\mathrm{Sym}_d \times \mathrm{Sym}_c$ on the powerset of $\mathcal{P}$ and on $\mathcal{M}_{d\times c}$ are equivalent, that is,

$$\chi(S^g) = \chi(S)^g, \text{ for all } S \subseteq \mathcal{P}, g \in G.$$

For a line $\lambda \in \mathcal{L}$, the equivalence class of $\chi(\lambda)$ is called *the mask* of $\mathcal{S}$ relative to $(\mathcal{C}, \mathcal{D})$. The mask determines both the row and the column intersection types. However, in a search for line-transitive linear spaces preserving a grid structure, we may find none, one, or several different masks corresponding to a given pair of row and column intersection types, see [7, Chapter 5] for an example. Thus, since the mask is an invariant of the linear space under automorphisms which preserve the grid structure, the mask provides an additional tool to assist in the search. Such a search might proceed as follows.

Suppose we are searching for all linear spaces $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ (if any exist) with line size k admitting a given line-transitive group G that preserves a grid structure $\mathcal{P} = \mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$. The values c, d, k are known and determine the Delandtsheer-Doyen parameters (x, y) for the G-invariant partition $\mathcal{C}$ by (2), and (y, x) are the Delandtsheer-Doyen parameters for the G-invariant partition $\mathcal{D}$. From the values of x and y we first determine a list IntTypes of feasible pairs of row and column intersection types. We then determine, for each pair in IntTypes, a list of all masks that correspond to the pair. Then we search for linear spaces corresponding to each of the masks in turn (or prove that none exist).

Note that it is possible to have several non-isomorphic linear spaces corresponding to the same group and mask, as demonstrated by the line-transitive linear spaces on 729 points with line size 8 classified in [11]. In the next subsection we give an example of this type of analysis.

3.1 An example: masks corresponding to given intersection types

Grids arise in the second case of [13, Theorem 1.4], and we restate part of this result below in the language of grids. A permutation group H on a set $\mathcal{P}$ is said to be *semiregular* if only the identity element of H fixes a point of $\mathcal{P}$, and H is *regular* if it is semiregular and transitive. The point-partition $\mathcal{C}$ is *minimal* if $|C_i| > 1$ and the only G-invariant refinement of $\mathcal{C}$ is the trivial partition with all parts of size 1. Our notation with regard to groups is as follows: for two groups A and B, $A : B$ denotes a semidirect product of A by B. Also D_{2n} will denote the dihedral group of order $2n$.

Lemma 3.2. *Let $\mathcal{S}$, G and $\mathcal{C}$ be as above with Delandtsheer-Doyen parameters (x, y), $k \geq 2x$ and $\operatorname{spec}_{\mathcal{C}} \mathcal{S} = \{1, 2\}$. Assume further that $\mathcal{C}$ is minimal and G-normal and that $G_{(\mathcal{C})}$ is not semiregular on $\mathcal{P}$. Then*

(i) *there is a G-invariant partition $\mathcal{D}$ of $\mathcal{P}$ such that $\mathcal{P} = \mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$ is a grid of type $d \times c$;*

(ii) *$c = p^a$ for some odd prime p, and $G_{(\mathcal{C})} = Z_{p^a} : Z_2$; moreover, $G_{(\mathcal{C})}$ contains c involutions and the c classes of $\mathcal{D}$ are the sets of fixed points of these involutions;*

(iii) *the row intersection type is $(1^{k-2x}, 2^x)$ with $k - 2x \geq 2$, and the column intersection type is $(1^{2x}, (k-2x)^1)$;*

(iv) *$y = \binom{k-2x}{2}$ and $\mathcal{D}$ has Delandtsheer-Doyen parameters (y, x).*

We make a few comments about our re-statement of the result. A crucial observation that underlies our translation of [13, Theorem 1.4] into the language of grids is that, for a normal subgroup N of a transitive permutation group G, the set F of fixed points of a point stabiliser N_α generates a G-invariant point-partition $\{F^g \mid g \in G\}$. (The proof is straightforward and elementary.) We applied this with $N = G_{(\mathcal{C})}$, $N_\alpha \cong Z_2$. The assertion about the Delandtsheer-Doyen parameters with respect to $\mathcal{D}$ follows from (2). The result [13, Theorem 1.4] obtained $y = \binom{k-2x}{2}$, and the fact that each line meets some column in exactly $k - 2x$ points. Since y is the number of $\mathcal{D}$-inner pairs, this information implies that the column intersection type is $\left(1^{2x}, (k-2x)^1\right)$.

We shall analyse possibilities for the characteristic matrix of a line in the situation of Lemma 3.2, showing that they all belong to the same mask. For a matrix $M \in \mathcal{M}_{d\times c}$, the *weight* of a row or column is the number of 1's it contains; two rows (or columns) are *disjoint* if their sets of non-zero entries correspond to disjoint sets of columns (or rows, respectively), or equivalently, if their dot product is zero. Recall that each matrix in the mask is the characteristic matrix $\chi(\lambda)$ of a line λ. By Lemma 3.2(ii), each involution $u \in G_{(\mathcal{C})}$ fixes a class of $\mathcal{D}$ pointwise, and hence fixes some line setwise. Since G is line-transitive, each line is fixed setwise by an involution of G.

Lemma 3.3. *Let $\mathcal{S}$, G, $\mathcal{C}$, $\mathcal{D}$, x and y be as in Lemma 3.2, let $\lambda \in \mathcal{L}$, and let u be an involution in G that fixes λ setwise. Then $\chi(\lambda)$ has the following shape:*

(i) *there are x pairwise disjoint rows of weight 2 corresponding to $2x$ columns of weight 1;*

(ii) *the non-zero entries in the rows of weight 2 lie in columns corresponding to pairs of classes of $\mathcal{D}$ which are interchanged by u;*

(iii) *there are $k-2x$ rows of weight 1 whose unique non-zero entries all lie in the same column, namely the column corresponding to the unique class of $\mathcal{D}$ fixed by u.*

In particular there is a unique mask corresponding to $\mathcal{S}, \mathcal{C}, \mathcal{D}$.

Proof. By Lemma 3.2, $\mathcal{P} = \mathcal{G}_{d\times c}^{(\mathcal{C},\mathcal{D})}$, and for an involution $u \in G_{(\mathcal{C})}$ there is a class $D_0 \in \mathcal{D}$ such that D_0 is the set of fixed points of u. Moreover, since u fixes each $\mathcal{C}$-class setwise, D_0 must be the unique class of $\mathcal{D}$ fixed setwise by u. Since u fixes λ setwise, by Lemma 2.3, λ is a union of $\langle u\rangle$-orbits. We shall describe $\chi(\lambda)$. By Lemma 3.2, the row intersection type is $(1^{k-2x}, 2^x)$ and the column intersection type is $(1^{2x}, k-2x)$.

Since u fixes setwise the unique $\mathcal{D}$-class containing $k-2x$ points of λ, this class must be D_0. Then, since λ meets each $\mathcal{C}$-class in at most 2 points, it follows that, for each $\mathcal{C}$-class C such that $C\cap D_0 \subset \lambda$, we must have $C\cap\lambda = C\cap D_0$. This proves (iii), and we note that each of the $y = \binom{k-2x}{2}$ $\mathcal{D}$-inner pairs of λ is therefore a pair of points of $D_0\cap\lambda$.

The rows of weight 2 can only be formed by choosing the non-zero entries in columns corresponding to pairs of $\mathcal{D}$-classes interchanged by u. Moreover, since these non-zero entries can contribute no $\mathcal{D}$-inner pairs, these rows are pairwise disjoint, giving us $2x$ columns of weight 1.

Finally, all matrices in $\mathcal{M}_{d\times c}$ with these properties are equivalent under the action of $\mathrm{Sym}_d \times \mathrm{Sym}_c$, and hence the mask corresponding to $\mathcal{S}, \mathcal{C}, \mathcal{D}$ is uniquely determined. □

4 Proof of Theorem 1.2

Let $\mathcal{S}, G, \mathcal{C}$ be as in Theorem 1.2, and let $G_{(\mathcal{C})}$ denote the kernel of the action of G on $\mathcal{C}$. By assumption, the number x of $\mathcal{C}$-inner pairs is at most 8, and $G_{(\mathcal{C})}$ contains a non-identity element, g say, such that g fixes some point α. In particular, $G_{(\mathcal{C})} \neq 1$ and the $G_{(\mathcal{C})}$-orbits in $\mathcal{P}$ form a G-invariant partition $\mathcal{C}'$ that refines $\mathcal{C}$. Moreover the partition $\mathcal{C}'$ is G-normal with $G_{(\mathcal{C})}$ contained in $G_{(\mathcal{C}')}$. Also, since each $\mathcal{C}'$-inner pair is also a $\mathcal{C}$-inner pair, the number x' of $\mathcal{C}'$-inner pairs is at most x, which by assumption is at most 8.

Thus the hypotheses of [13, Theorem 1.6] hold relative to the partition $\mathcal{C}'$, and in addition we have that $G_{(\mathcal{C}')}$ is not semiregular on $\mathcal{P}$. We review and extend the information given in [13, Theorem 1.6]. Let $|\mathcal{C}'| = d'$, let the classes of $\mathcal{C}'$ have size c', let the Delandtsheer-Doyen parameters relative to $\mathcal{C}'$ be (x', y'), and the $\mathcal{C}'$-intersection type be $(1^{d_1'}, \ldots, k^{d_k'})$. Also set $b^{(r)} := b/\gcd(b,v)$, where $b = |\mathcal{L}|$. Notice that $\gcd(v, b^{(r)}) = 1$ since $v-1 = r(k-1)$.

Lemma 4.1. *If $\mathcal{S}$ is $\mathsf{PG}_2(4)$ or the Colbourn-McCalla design then $\mathrm{Aut}(\mathcal{S})$ contains a subgroup with the required properties. If $\mathcal{S}$ is not one of these two linear spaces, then all of the following hold.*

(a) *$k, c', d', x', y', d'_1, d'_2, b^{(r)}$ are as in one of lines 1–7 of Table 1;*

(b) *$\mathrm{spec}_{\mathcal{C}'}\,\mathcal{S} = \{1,2\}$, and the $\mathcal{C}'$-intersection type is $(1^{k-2x'}, 2^{x'})$;*

(c) *$G_{(\mathcal{C}')} = N : Z_2 \simeq D_{2c'}$ with $N \simeq Z_{c'}$, and $G = X \times G_{(\mathcal{C}')}$;*

(d) *moreover in lines 1–6, $\mathcal{C}' = \mathcal{C}$, $X = M : Z_h$ where $M \simeq Z_{d'}$ and $b^{(r)} \mid h \mid (d'-1)$; and in line 7, $G^{\mathcal{C}'} = X^{\mathcal{C}'}$ is imprimitive.*

We remark that in line 7, the partition $\mathcal{C}$ may equal $\mathcal{C}'$ or it may be properly refined by $\mathcal{C}'$.

Proof. As discussed above, the assumptions of [13, Theorem 1.6] hold relative to $\mathcal{C}'$. The first paragraph of the proof [13, pp. 57–58] shows that $\mathcal{C}'$ is minimal and c' is odd. Next we consider two examples. If $\mathcal{S} = \mathsf{PG}_2(4)$ then $|\mathcal{P}| = 21$, a product of two primes, and so $\mathcal{C} = \mathcal{C}'$. In the automorphism group $\mathsf{P\Gamma L}(3,4)$ of $\mathsf{PG}_2(4)$ the normaliser of a Singer Cycle has the form $Z_{21} : Z_6$ and contains subgroups $Z_7 \times D_6$ and $F_{21} \times D_6$ that satisfy the conditions with $c = 3, d = 7$, and the subgroup $F_{21} \times D_6$ also satisfies the conditions with $c = 7, d = 3$. For the Colbourn-McCalla linear space, again $|\mathcal{P}| = 91$ is a product of two primes and so $\mathcal{C}' = \mathcal{C}$. It was shown in [9] that there is a subgroup of automorphisms of the form $Z_7 \times D_{26}$ satisfying the conditions with $c = 13, d = 7$. These examples were given in [13, Theorem 1.6 (b)].

Thus we may assume that $\mathcal{S}$ is not one of these two linear spaces. Then by [13, Theorem 1.6], setting $K := G_{(\mathcal{C}')}$, all of the following hold.

(i) $\mathrm{spec}_{\mathcal{C}'}\,\mathcal{S} = \{1,2\}$, and hence the $\mathcal{C}'$-intersection type is $(1^{k-2x'}, 2^{x'})$;

(ii) $K = N : Z_2 \simeq D_{2c'}$ with $N \simeq Z_{c'}$, G has a normal subgroup $X \times K$ where $X := C_G(K)$, and either

   (a) $X = M : Z_h$ where $M \cong Z_{d'}$, for some h dividing $d' - 1$, and one of lines 1–6 of Table 1 holds; or

   (b) $G^{\mathcal{C}'}$ is imprimitive, and line 7 of Table 1 holds.

In lines 1–6, the integer d' is prime, and since d divides d' it follows that $d' = d$ and hence $\mathcal{C}' = \mathcal{C}$ for these lines. Also for lines 1 and 7, $b^{(r)} = 1$ and hence in these lines $\mathcal{S}$ is a projective plane.

We take this analysis further. Lines 1–6 of Table 1 correspond to cases 6, 8, 9, 11, 17, 18, respectively analysed on page 59 of [13], and line 7 of Table 1

line	k	c'	d'	x'	y'	d'_1	d'_2	$b^{(r)}$	*comment*
1	10	7	13	3	6	4	3	1	projective plane
2	10	41	11	4	1	2	4	5	
3	11	17	13	4	3	3	4	2	
4	12	61	13	5	1	2	5	6	
5	16	113	17	7	1	2	7	8	
6	17	43	19	7	3	3	7	3	
7	17	13	21	6	10	5	6	1	projective plane

Table 1: Parameters for Lemma 4.1

corresponds to case 14 analysed on page 60 of [13]. It is proved there that in all of these cases, $K \simeq D_{2c'}$, and X is defined as $C_G(K)$, the kernel of the natural conjugation action of G on K. It is observed that $X \cap K = 1$ and $G/X \leq \mathrm{Aut}(K)$. However it is wrongly asserted that $\mathrm{Aut}(K) \simeq Z_{c'} \cdot Z_{c'-1}$. In fact $\mathrm{Aut}(K) \simeq K \simeq D_{2c'}$ since c' is odd. Now $X \cap K = 1$ implies that $K \simeq XK/X \leq G/X \leq \mathrm{Aut}(K) \simeq K$, and hence $G = XK \simeq X \times K$ in all these cases. Moreover, since an involution in K fixes a line of $\mathcal{S}$, it follows that b divides $|G|/2$. In lines 2–6, $|G|/2 = c'd'h = vh$, and hence $b^{(r)}$ divides h. This divisibility condition also holds for line 1 since in that case $b^{(r)} = 1$. This completes the proof. □

From now on we will assume that $\mathcal{S}$ is neither $\mathsf{PG}_2(4)$ nor the Colbourn-McCalla design. Note that Lemmas 3.2 and 3.3 apply to all lines of Table 1, proving in particular that G preserves a grid structure on the point set.

Corollary 4.2. *In each of lines* 1–7 *of Table* 1*, G preserves a grid structure $\mathcal{P} = \mathcal{G}^{(\mathcal{C}',\mathcal{D})}_{d'\times c'}$ on points, where $\mathcal{D}$ is the set of X-orbits in $\mathcal{P}$ with X as in Lemma* 4.1*. Moreover, the X-action on $\mathcal{C}'$ is equivalent to its action on each of these orbits.*

Proof. Suppose that one of the lines of Table 1 holds. Then all the conditions of Lemma 3.2 hold relative to the partition $\mathcal{C}'$, and hence G preserves a second point-partition $\mathcal{D}$ such that $\mathcal{P} = \mathcal{G}^{(\mathcal{C}',\mathcal{D})}_{d'\times c'}$ is a grid of type $d' \times c'$. Moreover, each part D of $\mathcal{D}$ is the fixed point set of an involution u in $K = G_{(\mathcal{C}')}$. Now $\langle u\rangle$ is a point stabiliser in $G_{(\mathcal{C}')}$, and its normaliser in G is $X \times \langle u\rangle$. Therefore $X \times \langle u\rangle$ acts transitively on the fixed point set D of $\langle u\rangle$, and thus D is an X-orbit. Hence $\mathcal{D}$ is the set of X-orbits in $\mathcal{P}$. Clearly the X-action on $\mathcal{C}'$ is equivalent to its action on each of these orbits. □

Our next step is to prove that lines 2–6 do not lead to any examples.

Lemma 4.3. *There are no examples for lines* 2–6 *of Table* 1*. Moreover, in the case of line* 1*, $G = X \times G_{(\mathcal{C})}$, where $X \simeq Z_{13}$ and $G_{(\mathcal{C})} = D_{14}$.*

Proof. By Corollary 4.2, G preserves a grid structure $\mathcal{P} = \mathcal{G}_{d' \times c'}^{(\mathcal{C}', \mathcal{D})}$, where $\mathcal{D}$ is the set of X-orbits in $\mathcal{P}$. This implies that the partition $\mathcal{D}$ is also G-normal, and we have $G_{(\mathcal{D})} = X$. Moreover, the number of $\mathcal{D}$-inner pairs on a line is y', and for each of the lines 1–6, $y' < 8$. Thus [13, Theorem 1.6] applies for these lines with $\mathcal{C}'$, K replaced by $\mathcal{D}$, X, and we deduce that X is semiregular on $\mathcal{P}$. Thus the integer h in Lemma 4.1 is 1. Since, by Lemma 4.1, h is divisible by $b^{(r)}$, it follows that lines 2–6 lead to no examples, and in the case of line 1, we have $X \simeq Z_{13}$. □

Now we show that, in the case of line 7 of Table 1, the group G must contain at least one of only two types of line-regular subgroups. Here $F_{21} = Z_7 : Z_3$ denotes the Frobenius group of order 21.

Lemma 4.4. *If line* 7 *of Table* 1 *holds, then* G *has a line-transitive subgroup* $Y \times G_{(\mathcal{C}')}$*, where* $Y \leq X$ *with* X *as in Lemma* 4.1*,* $Y \simeq F_{21}$ *or* Z_{21}*, and* $G_{(\mathcal{C}')} \simeq D_{26}$.

Proof. By Lemma 4.1, $G^{\mathcal{C}'} = X^{\mathcal{C}'} \simeq X$ is imprimitive, and by Corollary 4.2, the set $\mathcal{D}$ of X-orbits in $\mathcal{P}$ forms a G-invariant partition, and the X-action on $\mathcal{C}'$ is equivalent to its action on each of these orbits. Thus it is sufficient to find a subgroup Y of X such that $Y \simeq F_{21}$ or Z_{21}, with Y regular on an X-orbit D. By Lemma 3.2, D is the fixed point set of an involution u in $G_{(\mathcal{C}')}$, and hence the group G^D induced by $G_D = X \times \langle u \rangle$ on D is equal to $X^D \simeq X$.

Suppose first that $X^{\mathcal{C}'}$ is quasiprimitive, that is, all non-trivial normal subgroups of X are transitive on $\mathcal{C}'$. Since $X^{\mathcal{C}'}$ is imprimitive, this means that there is a non-trivial X-invariant partition $\mathcal{C}''$ of $\mathcal{C}'$ on which X acts faithfully. This implies that $|\mathcal{C}''| = 7$ and $X \leq S_7$. Since a minimal normal subgroup S of X is transitive on $\mathcal{C}'$ of degree 21, it follows that $S \neq Z_7$ or A_7, and hence $S = X = \mathsf{PSL}(2,7)$. In this case a Frobenius subgroup Y of X of order 21 must act regularly on $\mathcal{C}'$ and D, since $X_\alpha \simeq D_8$.

Thus we may assume that X has a non-trivial normal subgroup M that is intransitive on D. Then X permutes the M-orbits in D transitively, so M has equal length orbits in D of length 3 or 7, and we may assume that M is equal to the kernel of this action. Since $G_{(\mathcal{C}')}$ centralises M, the subgroup M is normal in G and the set $\mathcal{D}'$ of M-orbits in $\mathcal{P}$ is a G-invariant partition that refines $\mathcal{D}$. By [5, Theorem 1], M is faithful on each of its orbits in $\mathcal{P}$.

Suppose that M has orbits of length 3 in $\mathcal{P}$. Then $M = Z_3$ or S_3, and so $X/C_X(M) \leq 2$. Hence $C_X(M)$ is transitive on the seven M-orbits in D, and $C_X(M) \cap M \leq Z_3$. Thus if $a \in M$ has order 3, and $z \in C_X(M)$ has order 7, then $\langle az \rangle \simeq Z_{21}$ and is regular on D.

We may therefore assume that each non-trivial, intransitive normal subgroup of X has 3 orbits of length 7 in D. In this case M acts faithfully on each orbit of degree 7. Suppose that $M \neq Z_7$. Then, for all possibilities, M has trivial centre and $\mathrm{Aut}(M)/M \leq Z_2$. Thus $|X/(MC_X(M))|$ is not divisible by 3, and so $C_X(M)$ permutes transitively the three M-orbits in D and $C_X(M) \cap M = 1$. This time, if $a \in M$ has order 7, and $z \in C_X(M)$ has order 3, then $\langle az \rangle \simeq Z_{21}$ and is regular on D. Finally suppose that $M = Z_7$. If $C_X(M)$ is transitive on D then we obtain a subgroup Z_{21} of $C_X(M)$ acting regularly as required, while if $C_X(M)$ is intransitive on D then $C_X(M) = M$ and $X \leq \mathsf{AGL}(1,7)$. In this last case we have a subgroup F_{21} of X acting regularly on D. □

We complete consideration of lines 1 and 7 in the following section.

4.1 The projective plane cases

Lemma 4.5. *The only projective plane of order* 9 *admitting a line-transitive action of the cyclic group* Z_{91} *is the Desarguesian plane* $\mathsf{PG}_2(9)$*. Moreover* $\mathsf{PG}_2(9)$ *admits a line-transitive group* $G \cong Z_{13} \times D_{14}$ *corresponding to line* 1 *of Table* 1*.*

Proof. Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a projective plane of order 9 admitting a line-transitive cyclic subgroup $G = \langle a \rangle \cong Z_{91}$ of automorphisms. Note that the point set $\mathcal{P}$ can be identified with the set G in such a way that G acts by multiplication. The G-action is line-transitive and point imprimitive, preserving the point partition $\mathcal{C} = \{C_i \mid 0 \leq i < 13\}$, where $C_i = \{a^{13j+i} \mid 0 \leq j < 7\}$. Thus the algorithm in [1] may be used to find all such projective planes up to isomorphism. A computer search using this algorithm led to 12 base lines of (possibly isomorphic) projective planes. The search for the designs took 20 seconds on a Macintosh PowerBook G4, during which around 3 million backtrack nodes were examined by the program. Using the software package `nauty` [10] (version 2.2), the 12 designs from the search were all identified as the Desarguesian projective plane $\mathsf{PG}_2(9)$. This proves the first assertion.

Finally we consider $\mathcal{S} = \mathsf{PG}_2(9)$. The normaliser of a Singer cycle in $\mathsf{P\Gamma L}(3,9)$ is $Z_{91} : Z_6$ and contains a unique subgroup of the form $Z_{91} : Z_2$. This subgroup may be identified with a subgroup Z_{91} of the multiplicative group of a field of order 3^6, extended by a field automorphism of order 2 that fixes pointwise a subfield of order 3^3. Thus this subgroup is isomorphic to $Z_{13} \times D_{14}$ (as in Lemma 4.3) and so $\mathsf{PG}_2(9)$ is the unique linear space corresponding to line 1 of Table 1. □

Lemma 4.6. *Up to isomorphism, there is a unique linear space corresponding to line* 7 *of Table* 1*, namely the Desarguesian projective plane* $\mathsf{PG}_2(16)$.

Proof. By Lemma 4.3, we may identify the point set with $\mathcal{P} = \mathcal{G}_{d' \times c'}^{(\mathcal{C}', \mathcal{D})}$; we use the notation of (6) for points. By Lemma 4.4, we may assume that the line transitive group is $G = G_{(\mathcal{D})} \times G_{(\mathcal{C}')}$ with $G_{(\mathcal{D})} = Z_{21}$ or F_{21}, and $G_{(\mathcal{C}')} = D_{26}$. We may take $G_{(\mathcal{C}')} = Z_{13} : Z_2$, with $Z_{13} = \langle t_{0,1} \rangle$ the additive group of integers modulo 13, and $Z_2 = \langle s \rangle$, where $t_{0,i}^s = t_{0,-i}$ for each integer i modulo 13. Also either $G_{(\mathcal{D})} = Z_{21} = \langle t_{1,0} \rangle$, the additive group of integers modulo 21, or $G_{(\mathcal{D})} = F_{21} = \langle t_{1,0}, s' \rangle$, where $\langle t_{1,0} \rangle$, the additive group of integers modulo 7, and $t_{i,0}^{s'} = t_{2i,0}$ for each integer i modulo 7.

First we prove that the Desarguesian projective plane $\mathsf{PG}_2(16)$ provides an example for the first type of group, but not the second type. For $\mathcal{S} = \mathsf{PG}_2(16)$, the group G is a subgroup of $\mathsf{P\Gamma L}(3, 16)$, and G is contained in the normaliser of a cyclic subgroup of order 13. Thus G must be contained in the normaliser Y of a Singer cycle in $\mathsf{P\Gamma L}(3, 16)$. Now $Y := Z_{273} : Z_{12}$, where the cyclic normal subgroup $\langle y \rangle \simeq Z_{273}$ may be identified with the multiplicative group of a field of order 2^{12} modulo the multiplicative group of its subfield of order 2^4, and the cyclic subgroup Z_{12} is generated by the Frobenius automorphism $\sigma : y \to y^2$. This means that the subgroup of G of order 13 is $\langle y^{21} \rangle$. There is an element of G of order 3 that centralises this subgroup, and the only such elements in Y are y^{91} and its inverse. Also the only subgroup of Y of order 7 is $\langle y^{39} \rangle$. Hence G contains $\langle y^{21}, y^{91}, y^{39} \rangle = \langle y \rangle \simeq Z_{273}$. Since $|G| = 273 \times 2$, it follows that G contains no subgroup $F_{21} \times D_{26}$. On the other hand the subgroup

$$\langle y, \sigma^6 \rangle = \langle y^{13} \rangle \times \langle y^{21}, \sigma^6 \rangle \simeq Z_{21} \times D_{26}$$

acts on $\mathcal{S}$ as required, giving an example.

Now we describe our analysis that proves that $\mathcal{S}$ must be $\mathsf{PG}_2(16)$.

It is not difficult to prove that each of the two groups G above has the following orbit types on horizontal, vertical and skew pairs of points: 6 orbits of length 273 on horizontal pairs, 10 orbits of length 273 on vertical pairs, and 60 orbits of length 546 on skew pairs. In particular, by Lemma 2.2, $\mu(\mathcal{O}, \lambda) = 1$ for each horizontal orbit $\mathcal{O}$.

Let λ be a line containing an orbit of $\langle s \rangle$ of length 2. Then λ is fixed by s, and $\langle s \rangle = G_\lambda$. By Lemma 3.3, the characteristic matrix of λ has one column of weight 5, namely the unique column fixed by s, and 6 pairwise disjoint rows of weight 2 corresponding to 12 columns of weight 1; each pair of columns corresponding to the two non-zero entries in one of these rows is interchanged by s. The $\mathcal{C}$-inner pairs which arise from the 6 rows of weight 2 represent the 6 different horizontal orbits (see Lemma 2.2).

A computer search using a partial implementation of the algorithm described in [1] was conducted for each group G. Since each pair of points lies in a unique line, we decided to search for the line λ containing the points $\alpha_1 = \alpha_{0,0}$ and

$\alpha_2 = \alpha_{0,1}$. Thus we began with the set $A = \{\alpha_1, \alpha_2\}$ (a horizontal pair), and we used the fact that G_λ contains the involution s that fixes this pair. During the search we made sure that we chose whole orbits under $\langle s \rangle$ for addition to the current set. We did not re-compute the setwise stabiliser G_A during the procedure when A was enlarged. This was because we did not have a fast algorithm for computing set stabilisers. The extra information that might have been gained from a possibly larger group, was offset by the time that would have been used for this computation.

For the first group with $G_{(\mathcal{D})} = Z_{21}$, the search on a Macintosh PowerBook G4 quickly yielded 12 base lines. The computing time was less than 2 minutes. A total number of around 8.3 million backtrack nodes were examined by the algorithm. It turned out that all linear spaces generated by the 12 base lines were isomorphic to the Desarguesian projective plane $\mathsf{PG}_2(16)$. This was verified by computing for each design the canonical form of its incidence matrix using the software package `nauty` [10]. Here, a canonical form of the incidence matrix of a design is simply a unique representative of all incidence matrices of designs isomorphic to the given design. Once the canonical form has been computed, isomorph checking is equivalent to equality testing of the corresponding canonical incidence matrices. In order to prove that the designs were isomorphic to $\mathsf{PG}(2, 16)$, the latter design was constructed separately and its canonical incidence matrix was computed as well. It turned out that the canonical incidence matrices of the 12 designs and the canonical incidence matrix of $\mathsf{PG}(2, 16)$ were all identical. This proves that the designs are all isomorphic to $\mathsf{PG}(2, 16)$. In order to treat incidence matrices of designs with `nauty`, a graph had to be created which encodes the design. This graph is a bipartite graph, with vertices corresponding to points and blocks of the design. The edges in the graph correspond to incident point/line pairs. The isomorphism types of projective planes correspond to the isomorphism classes of this type of graph, where the isomorphisms are required to preserve the bipartition. In order to have `nauty` compute the canonical form with respect to this class of isomorphisms, the bipartition was handed to `nauty` in the form of an initial partition of the vertices of the graph. The details of how to specify such a partition are described in the `nauty` User Guide [10].

For the second group with $G_{(\mathcal{D})} = F_{21}$, the computer found no solutions at all. The search took less than one minute time. A total number of around 3.6 million backtrack nodes were examined by the algorithm. The best that could be found were 4620 partial baselines of size 13. This completes the proof. $\square$

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