Innovations in Incidence Geometry
Volume 8 (2008), Pages 171–173
ISSN 1781-6475

Collinear triples in permutations

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Abstract

Let $\alpha : \mathbb{F}_q \to \mathbb{F}_q$ be a permutation and $\Psi(\alpha)$ be the number of collinear triples in the graph of α, where $\mathbb{F}_q$ denotes a finite field of q elements. When q is odd, Cooper and Solymosi once proved $\Psi(\alpha) \geq (q-1)/4$ and conjectured the sharp bound should be $\Psi(\alpha) \geq (q-1)/2$. In this note we confirm this conjecture.

Keywords: collinear triple, permutation, Kakeya set

MSC 2000: 11T99

1 Introduction

Denote by $\mathbb{F}_q$ the finite field of q elements with q odd. Let $\alpha : \mathbb{F}_q \to \mathbb{F}_q$ be a permutation and $\Psi(\alpha)$ be the number of collinear triples in

$$G_\alpha = \{(i, \alpha(i)) : i \in \mathbb{F}_q\},$$

the graph of α. Cooper and Solymosi [5] once obtained the lower bound

$$\Psi(\alpha) \geq \frac{q-1}{4}\,, \tag{1}$$

and conjectured the best one should be

$$\Psi(\alpha) \geq \frac{q-1}{2}\,. \tag{2}$$

Later Cooper [4] showed that the problem of counting collinear triples in a permutation and the finite plane Kakeya problem are intimately connected, and improved (1) slightly to

$$\Psi(\alpha) \geq \frac{5q-1}{14}\,.$$

The main purpose of this note is to indicate that the Cooper-Solymosi conjecture (2) is true. We also mention that Ball [1] has proved (2) in a very nice way.

2 Proof

A subset in $\mathbb{F}_q^2$ containing a line in each direction is called a Kakeya set. Given the permutation α, one can construct a corresponding Kakeya set

$$K_\alpha \doteq L\big(\infty, (0,0)\big) \cup \bigcup_{i \in \mathbb{F}_q} L\big(i, (0, \alpha(i))\big)\,,$$

where $L(s,x)$ denotes the line in $\mathbb{F}_q^2$ through x with slope s. Writing μ_x for the number of these lines passing through x, it follows from the incidence formula of Faber [6] that

$$\sharp K_\alpha = \frac{q(q+1)}{2} + \sum_{x \in K_\alpha} \binom{\mu_x - 1}{2}. \tag{3}$$

By duality (cf. [4]), a point x in K_α with $\mu_x \geq 3$ corresponds to a collinear μ_x-tuple in G_α, and vice versa. Thus denoting by Γ_α the hypergraph on the vertex set G_α whose edges are the maximal collinear subsets of G_α, (3) turns out to be

$$\sharp K_\alpha = \frac{q(q+1)}{2} + \sum_{e \in E(\Gamma_\alpha)} \binom{|e| - 1}{2}.$$

By considering

$$\Psi(\alpha) = \sum_{e \in E(\Gamma_\alpha)} \binom{|e|}{3} \geq \sum_{e \in E(\Gamma_\alpha)} \binom{|e| - 1}{2},$$

to confirm (2) it suffices to prove

$$\sharp K_\alpha \geq \frac{q(q+1)}{2} + \frac{q-1}{2}\,. \tag{4}$$

Coincidentally, for any Kakeya set $K \subset \mathbb{F}_q^2$, Faber [6] has proved the bound

$$\sharp K \geq \frac{q(q+1)}{2} + \frac{q}{3}\,,$$

and conjectured the sharp one should be

$$\sharp K \geq \frac{q(q+1)}{2} + \frac{q-1}{2}\,. \tag{5}$$

Recently, by exploiting the Jamison-Brouwer-Schrijver bound [3, 7] on the size of blocking sets in Desarguesian affine planes, (5) was established independently by Blokhuis and Mazzocca [2] and Ball [1]. As an immediate corollary, (4) is true, and so is (2).

Acknowledgments. The author would like to thank Simeon Ball, Aart Blokhuis, Xander Faber, Qing Xiang and Yaokun Wu for kindly pointing out the recent progresses on the finite field Kakeya problem to him. This work was supported by the Mathematical Tianyuan Foundation of China (Grant No. 10826088).

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