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Collinear triples in permutations

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Abstract

Let $\alpha : \mathbb{F}_q \rightarrow \mathbb{F}_q$ be a permutation and $\Psi(\alpha)$ be the number of collinear triples in the graph of α , where \mathbb{F}_q denotes a finite field of q elements. When q is odd, Cooper and Solymosi once proved $\Psi(\alpha) \geq (q-1)/4$ and conjectured the sharp bound should be $\Psi(\alpha) \geq (q-1)/2$. In this note we confirm this conjecture.

Keywords: collinear triple, permutation, Kakeya set

MSC 2000: 11T99

1. Introduction

Denote by \mathbb{F}_q the finite field of q elements with q odd. Let $\alpha : \mathbb{F}_q \rightarrow \mathbb{F}_q$ be a permutation and $\Psi(\alpha)$ be the number of collinear triples in

$$G_\alpha = \{(i, \alpha(i)) : i \in \mathbb{F}_q\},$$

the graph of α . Cooper and Solymosi [5] once obtained the lower bound

$$\Psi(\alpha) \geq \frac{q-1}{4}, \tag{1}$$

and conjectured the best one should be

$$\Psi(\alpha) \geq \frac{q-1}{2}. \tag{2}$$

Later Cooper [4] showed that the problem of counting collinear triples in a permutation and the finite plane Kakeya problem are intimately connected, and improved (1) slightly to

$$\Psi(\alpha) \geq \frac{5q-1}{14}.$$

The main purpose of this note is to indicate that the Cooper-Solymosi conjecture (2) is true. We also mention that Ball [1] has proved (2) in a very nice way.

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2. Proof

A subset in \mathbb{F}_q^2 containing a line in each direction is called a Kakeya set. Given the permutation α , one can construct a corresponding Kakeya set

$$K_\alpha \doteq L(\infty, (0, 0)) \cup \bigcup_{i \in \mathbb{F}_q} L(i, (0, \alpha(i))),$$

where $L(s, x)$ denotes the line in \mathbb{F}_q^2 through x with slope s . Writing μ_x for the number of these lines passing through x , it follows from the incidence formula of Faber [6] that

$$\#K_\alpha = \frac{q(q+1)}{2} + \sum_{x \in K_\alpha} \binom{\mu_x - 1}{2}. \quad (3)$$

By duality (cf. [4]), a point x in K_α with $\mu_x \geq 3$ corresponds to a collinear μ_x -tuple in G_α , and vice versa. Thus denoting by Γ_α the hypergraph on the vertex set G_α whose edges are the maximal collinear subsets of G_α , (3) turns out to be

$$\#K_\alpha = \frac{q(q+1)}{2} + \sum_{e \in E(\Gamma_\alpha)} \binom{|e| - 1}{2}.$$

By considering

$$\Psi(\alpha) = \sum_{e \in E(\Gamma_\alpha)} \binom{|e|}{3} \geq \sum_{e \in E(\Gamma_\alpha)} \binom{|e| - 1}{2},$$

to confirm (2) it suffices to prove

$$\#K_\alpha \geq \frac{q(q+1)}{2} + \frac{q-1}{2}. \quad (4)$$

Coincidentally, for any Kakeya set $K \subset \mathbb{F}_q^2$, Faber [6] has proved the bound

$$\#K \geq \frac{q(q+1)}{2} + \frac{q}{3},$$

and conjectured the sharp one should be

$$\#K \geq \frac{q(q+1)}{2} + \frac{q-1}{2}. \quad (5)$$

Recently, by exploiting the Jamison-Brouwer-Schrijver bound [3, 7] on the size of blocking sets in Desarguesian affine planes, (5) was established independently by Blokhuis and Mazzocca [2] and Ball [1]. As an immediate corollary, (4) is true, and so is (2).



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