Automorphisms of non-spherical buildings have unbounded displacement

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Abstract

If ϕ is a nontrivial automorphism of a thick building Δ of purely infinite type, we prove that there is no bound on the distance that ϕ moves a chamber. This has the following group-theoretic consequence: If G is a group of automorphisms of Δ with bounded quotient, then the center of G is trivial.
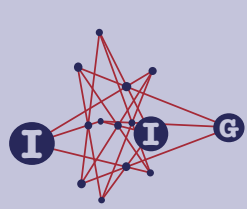
Introduction

A well-known folklore result says that a nontrivial automorphism ϕ of a thick Euclidean building X has unbounded displacement. Here we are thinking of X as a metric space, and the assertion is that there is no bound on the distance that ϕ moves a point. [For the proof, consider the action of ϕ on the boundary X_∞ at infinity. If ϕ had bounded displacement, then ϕ would act as the identity on X_∞, and one would easily conclude that $\phi = \mathrm{id}$.] In this note we generalize this result to buildings that are not necessarily Euclidean. We work with buildings Δ as combinatorial objects, whose set $\mathcal{C}$ of chambers has a discrete metric ("gallery distance"). We say that Δ is of *purely infinite type* if every irreducible factor of its Weyl group is infinite.

Theorem. *Let ϕ be a nontrivial automorphism of a thick building Δ of purely infinite type. Then ϕ, viewed as an isometry of the set $\mathcal{C}$ of chambers, has unbounded displacement.*

It is possible to prove the theorem by using the Davis realization of Δ as a CAT(0) metric space [3] and arguing as in the Euclidean case. (But more

work is required in the non-Euclidean case.) We instead give an elementary combinatorial proof based on a result about Coxeter groups (Lemma 2.2) that may be of independent interest. We prove the lemma in Section 2, after a review of the Tits cone in Section 1. We then prove the theorem in Section 3, and we obtain the following (almost immediate) corollary: If G is a subgroup of $\mathrm{Aut}(\Delta)$ such that there is a bounded set of representatives for the G-orbits in $\mathcal{C}$, then the center of G is trivial.

We conclude the paper by giving a brief discussion in Section 4 of displacement in the spherical case. We are grateful to Hendrik Van Maldeghem for providing us with some counterexamples in this connection (see Example 4.1 and Remark 4.5).

1. Preliminaries on the Tits cone

In this section we review some facts about the Tits cone associated to a Coxeter group [1, 2, 5, 11, 15]. We will use [1] as our basic reference, but much of what we say can also be found in one or more of the other cited references.

Let (W, S) be a Coxeter system with S finite. Then W admits a canonical representation, which turns out to be faithful (see Lemma 1.2 below), as a linear reflection group acting on a real vector space V with a basis $\{e_s \mid s \in S\}$. There is an induced action of W on the dual space V^*. We denote by C_0 the simplicial cone in V^* defined by

$$C_0 := \{x \in V^* \mid \langle x, e_s \rangle > 0 \text{ for all } s \in S\} \,;$$

here $\langle -, - \rangle$ denotes the canonical evaluation pairing between V^* and V. We call C_0 the *fundamental chamber*. For each subset $J \subseteq S$, we set

$$A_J := \{x \in V^* \mid \langle x, e_s \rangle = 0 \text{ for } s \in J \text{ and } \langle x, e_s \rangle > 0 \text{ for } s \in S \smallsetminus J\} \,.$$

The sets A_J are the (relatively open) *faces* of C in the standard terminology of polyhedral geometry. They form a partition of the closure $\overline{C}_0$ of C_0 in V^*.

For each $s \in S$, we denote by H_s the hyperplane in V^* defined by the linear equation $\langle -, e_s \rangle = 0$. If follows from the explicit definition of the canonical representation of W (which we have not given) that H_s is the fixed hyperplane of s acting on V^*. The complement of H_s in V^* is the union of two open halfspaces $U_{\pm}(s)$ that are interchanged by s. Here
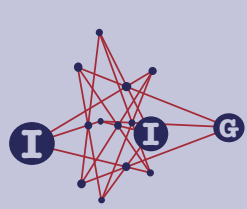
$$U_+(s) := \{x \in V^* \mid \langle x, e_s \rangle > 0\} \,,$$

and

$$U_-(s) := \{x \in V^* \mid \langle x, e_s \rangle < 0\} \,.$$

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The hyperplanes H_s are called the *walls* of C_0. We denote by $\mathcal{H}_0$ the set of walls of C_0.

The *support* of the face $A = A_J$, denoted $\operatorname{supp} A$, is defined to be the intersection of the walls of C_0 containing A, i.e., $\operatorname{supp} A = \bigcap_{s \in J} H_s$. Note that A is open in $\operatorname{supp} A$ and that $\operatorname{supp} A$ is the linear span of A.

Although our definitions above made use of the basis $\{e_s \mid s \in S\}$ of V, there are also intrinsic geometric characterizations of walls and faces. Namely, the walls of C_0 are the hyperplanes H in V^* such that H does not meet C_0 and $H \cap \overline{C}_0$ has nonempty interior in H. And the faces of C_0 correspond to subsets $\mathcal{H}_1 \subseteq \mathcal{H}_0$. Given such a subset, let $L := \bigcap_{H \in \mathcal{H}_1} H$; the corresponding face A is then the relative interior (in L) of the intersection $L \cap \overline{C}_0$.

We now make everything W-equivariant. We call a subset C of V^* a *chamber* if it is of the form $C = wC_0$ for some $w \in W$, and we call a subset A of V^* a *cell* if it is of the form $A = wA_J$ for some $w \in W$ and $J \subseteq S$. Each chamber C is a simplicial cone and hence has well-defined walls and faces, which can be characterized intrinsically as above. If $C = wC_0$ with $w \in W$, the walls of C are the transforms wH_s ($s \in S$), and the faces of C are the cells wA_J ($J \subseteq S$). Finally, we call a hyperplane H in V^* a *wall* if it is a wall of some chamber, and we denote by $\mathcal{H}$ the set of all walls; thus

$$\mathcal{H} = \{wH_s \mid w \in W,\, s \in S\}.$$

The set of all faces of all chambers is equal to the set of all cells. The union of these cells is called the *Tits cone* and will be denoted by X in the following. Equivalently,

$$X = \bigcup_{w \in W} w\overline{C}_0.$$

We now record, for ease of reference, some standard facts about the Tits cone. The first fact is Lemma 2.58 in [1, Section 2.5]. See also the proof of Theorem 1 in [2, Section V.4.4].

Lemma 1.1. *For any $w \in W$ and $s \in S$, we have*

$$wC_0 \subseteq U_+(s) \iff l(sw) > l(w)$$

and

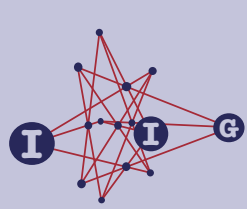$$wC_0 \subseteq U_-(s) \iff l(sw) < l(w).$$

Here $l(-)$ is the length function on W with respect to S. □

This immediately implies:

Lemma 1.2. *W acts simply transitively on the set of chambers.* □

The next result allows one to talk about separation of cells by walls. It is part of Theorem 2.80 in [1, Section 2.6], and it can also be deduced from Proposition 5 in [2, Section V.4.6].

Lemma 1.3. *If H is a wall and A is a cell, then either A is contained in H or A is contained in one of the two open halfspaces determined by H.* □

We turn now to reflections. The following lemma is an easy consequence of the stabilizer calculation in [1, Theorem 2.80] or [2, Section V.4.6].

Lemma 1.4. *For each wall $H \in \mathcal{H}$, there is a unique nontrivial element $s_H \in W$ that fixes H pointwise.* □

We call s_H the *reflection* with respect to H. In view of a fact stated above, we have $s_{H_s} = s$ for all $s \in S$. Thus S is the set of reflections with respect to the walls in $\mathcal{H}_0$. It follows immediately from Lemma 1.4 that

$$s_{wH} = w s_H w^{-1} \tag{1}$$

for all $H \in \mathcal{H}$ and $w \in W$. Hence wSw^{-1} is the set of reflections with respect to the walls of wC_0.

Corollary 1.5. *For $s \in S$ and $w \in W$, H_s is a wall of wC_0 if and only if $w^{-1}sw$ is in S.*

Proof. H_s is a wall of wC_0 if and only if s is the reflection with respect to a wall of wC_0. In view of the observations above, this is equivalent to saying $s \in wSw^{-1}$, i.e., $w^{-1}sw \in S$. □

Finally, we record some special features of the infinite case.

**Lemma 1.6.** *Assume that (W, S) is irreducible and W is infinite.*

(1) *If two chambers C, D have the same walls, then $C = D$.*

(2) *The Tits cone X does not contain any pair $\pm x$ of opposite nonzero vectors.*

Proof. (1) We may assume that $C = C_0$ and $D = wC_0$ for some $w \in W$. Then Corollary 1.5 implies that C and D have the same walls if and only if w normalizes S. So the content of (1) is that the normalizer of S in W is trivial. This is a well known fact. See [2, Section V.4, Exercise 3], [4, Proposition 4.1], or [1, Section 2.5.6]. Alternatively, there is a direct geometric proof of (1); see [1, Exercises 3.121 and 3.122].

(2) This is a result of Vinberg [15, p. 1112, Lemma 15]. See also [1, Section 2.6.3] and [6, Theorem 2.1.6] for alternate proofs. □
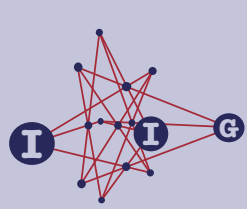
2. A lemma about Coxeter groups

We begin with a geometric version of our lemma, and then we translate it into algebraic language.

Lemma 2.1. *Let (W, S) be an infinite irreducible Coxeter system with S finite. If C and D are distinct chambers in the Tits cone, then C has a wall H with the following two properties:*

(a) *H is not a wall of D.*

(b) *H does not separate C from D.*

Proof. For convenience (and without loss of generality), we assume that C is the fundamental chamber C_0. Define $J \subseteq S$ by

$$J := \{ s \in S \mid H_s \text{ is a wall of } D \},$$

and set $L := \bigcap_{s \in J} H_s$. Thus L is the support of the face $A = A_J$ of C. By Lemma 1.6(1), $J \neq S$, hence $L \neq \{0\}$. Since L is an intersection of walls of D, it is also the support of a face B of D. Note that A and B are contained in precisely the same walls, since they have the same span L. In particular, B is not contained in any of the walls H_s with $s \in S \smallsetminus J$, so, by Lemma 1.3, B is contained in either $U_+(s)$ or $U_-(s)$ for each such s.

Suppose that $B \subseteq U_-(s)$ for each $s \in S \smallsetminus J$. Then, in view of the definition of $A = A_J$ by linear equalities and inequalities, $B \subseteq -A$. But B contains a nonzero vector x (since B spans L), so we have contradicted Lemma 1.6(2). Thus there must exist $s \in S \smallsetminus J$ with $B \subseteq U_+(s)$. This implies that $D \subseteq U_+(s)$, and the wall $H = H_s$ then has the desired properties (a) and (b). $\square$

We now prove the algebraic version of the lemma, for which we relax the hypotheses slightly. We do not even have to assume that S is finite. Recall that (W, S) is said to be *purely infinite* if each of its irreducible factors is infinite.

Lemma 2.2. *Let (W, S) be a purely infinite Coxeter system. If $w \neq 1$ in W, then there exists $s \in S$ such that:*

(a) *$w^{-1} s w \notin S$.*

(b) *$l(sw) > l(w)$.*

Proof. Let (W_i, S_i) be the irreducible factors of (W, S), which are all infinite. Suppose the lemma is true for each factor (W_i, S_i), and consider any $w \neq 1$ in W. Then w has components $w_i \in W_i$, at least one of which (say w_1) is
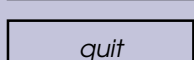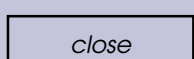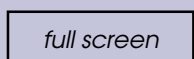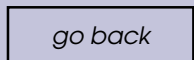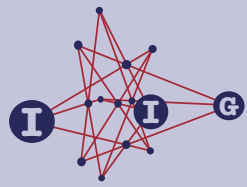
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nontrivial. So we can find $s \in S_1$ with $w_1^{-1} s w_1 \notin S_1$ and $l(sw_1) > l(w_1)$. One easily deduces (a) and (b). We are now reduced to the case where (W, S) is irreducible.

If S is finite, we apply Lemma 2.1 with C equal to the fundamental chamber C_0 and $D = wC_0$. Then $H = H_s$ for some $s \in S$. Property (a) of that lemma translates to (a) of the present lemma by Corollary 1.5, and property (b) of that lemma translates to (b) of the present lemma by Lemma 1.1.

If S is infinite, we use a completely different method. The result in this case follows from Lemma 2.3 below. □

Recall that for any Coxeter system (W, S) and any $w \in W$, there is a (finite) subset $S(w) \subseteq S$ such that every reduced decomposition of w involves precisely the generators in $S(w)$. This follows, for example, from Tits's solution to the word problem [12]. (See also [1, Section 2.3.3]).

Lemma 2.3. *Let (W, S) be an irreducible Coxeter system, and let $w \in W$ be nontrivial. If $S(w) \neq S$, then there exists $s \in S$ satisfying conditions (a) and (b) of Lemma 2.2.*

Proof. By irreducibility, there exists an element $s \in S \setminus S(w)$ that does not commute with all elements of $S(w)$. Condition (b) then follows from the fact that $s \notin S(w)$ and standard properties of Coxeter groups; see [1, Lemma 2.15]. To prove (a), suppose $sw = wt$ with $t \in S$. We have $s \notin S(w)$ but $s \in S(sw)$ (since $l(sw) > l(w)$), so necessarily $t = s$. Using induction on $l(w)$, one now deduces from Tits's solution to the word problem that s commutes with every element of $S(w)$ (see [1, Section 2.3.3]), contradicting the choice of s. □

Finally, we consider what happens if W is finite. Here the conclusion of Lemma 2.2 is false in general. For example, if w is the longest element $w_0 \in W$, then one cannot even achieve condition (b) of the lemma. Nevertheless, there is still something useful that one can say in cases where the lemma fails. We need some notation: For any subset $J \subseteq S$, we denote by W_J the subgroup generated by J, and we denote by $w_0(J)$ its longest element. We continue to write $w_0 = w_0(S)$ for the longest element of W.

Lemma 2.4. *Let (W, S) be a Coxeter system with W finite. Fix $w \in W$, let $J := \{s \in S \mid l(sw) > l(w)\}$, and suppose $w^{-1}sw \in S$ for all $s \in J$. Set $K := w^{-1}Jw \subseteq S$. Then $w_0 = ww_0(K)$. Consequently, $w_0^{-1}Jw_0 = K$, and the coset wW_K contains w_0.*

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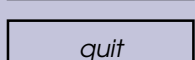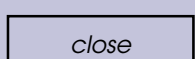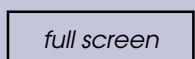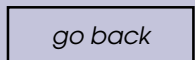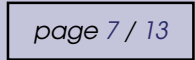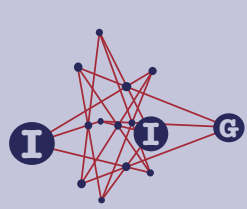
Proof. The second assertion follows at once from the first. To prove the first, set $\sigma(s) := w^{-1}sw \in K$ for $s \in J$, so that

$$sw = w\sigma(s)$$

for all $s \in J$. This equation and the definition of J imply that w is right K-reduced and hence that

$$l(wv) = l(w) + l(v)$$

for all $v \in W_K$. In particular, the (unique) longest element of the coset wW_K is $u := ww_0(K)$. To show that $u = w_0$, we need to show that $l(su) < l(u)$ for all $s \in S$. If $s \in J$, we have $su = w\sigma(s)w_0(K) \in wW_K$, so $l(su) < l(u)$ because u is the longest element of wW_K. And if $s \notin J$, then

$$l(su) = l(sww_0(K)) < l(w) + l(w_0(K)) = l(u)$$

because $l(sw) < l(w)$. $\qquad\square$

3.   Proof of the theorem

In this section we assume familiarity with basic concepts from the theory of buildings [1, 8, 9, 13, 16].

Let Δ be a building with Weyl group (W, S), let $\mathcal{C}$ be the set of chambers of Δ, and let $\delta \colon \mathcal{C} \times \mathcal{C} \to W$ be the Weyl distance function. (See [1, Section 4.8 or 5.1] for the definition and standard properties of δ.) Recall that $\mathcal{C}$ has a natural *gallery metric* $d(-, -)$ and that

$$d(C, D) = l\big(\delta(C, D)\big)$$

for $C, D \in \mathcal{C}$. Let $\phi \colon \Delta \to \Delta$ be an automorphism of Δ that is not necessarily type-preserving. Recall that ϕ induces an automorphism σ of (W, S). From the simplicial point of view, we can think of σ (restricted to S) as describing the effect of ϕ on types of vertices. From the point of view of Weyl distance, σ is characterized by the equation

$$\delta(\phi(C), \phi(D)) = \sigma(\delta(C, D))$$
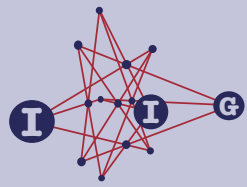
for $C, D \in \mathcal{C}$.

Our main theorem will be obtained from the following technical lemma:

Lemma 3.1. *Assume that Δ is thick. Fix a chamber $C \in \mathcal{C}$, and set $w := \delta(C, \phi(C))$. Suppose there exists $s \in S$ such that $l(sw) > l(w)$ and $w^{-1}sw \neq \sigma(s)$. Then there is a chamber D s-adjacent to C such that $d(D, \phi(D)) > d(C, \phi(C))$.*

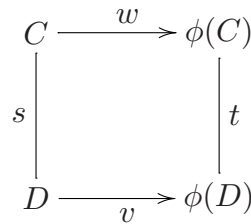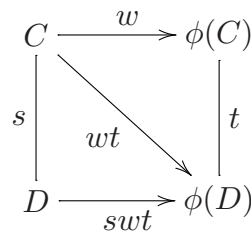*Proof.* Given a chamber D s-adjacent to C, set $v := \delta(D, \phi(D))$. We then have the situation illustrated in the following schematic diagram, where $t := \sigma(s)$:
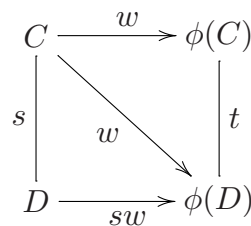
$$
\begin{array}{ccc}
C & \xrightarrow{\ w\ } & \phi(C) \\
\big| s & & \big| t \\
D & \xrightarrow{\ v\ } & \phi(D)
\end{array}
$$

Our task is to choose D so that $l(v) > l(w)$.

Case 1. $l(wt) > l(w)$. Then $l(swt) > l(wt)$ because the conditions $l(swt) < l(wt)$, $l(wt) > l(w)$, and $l(sw) > l(w)$ would imply (e.g., by the deletion condition for Coxeter groups) $swt = w$, and the latter is excluded by assumption. In this case we choose D s-adjacent to C arbitrarily. We then have $\delta(C, \phi(D)) = wt$ and $\delta(D, \phi(D)) = swt$:

$$
\begin{array}{ccc}
C & \xrightarrow{\ w\ } & \phi(C) \\
\big| s & \searrow^{wt} & \big| t \\
D & \xrightarrow{\ swt\ } & \phi(D)
\end{array}
$$

Thus $v = swt$ and $l(v) = l(w) + 2$.

Case 2. $l(wt) < l(w)$. Then there is a unique chamber E_0 t-adjacent to $\phi(C)$ such that $\delta(C, E_0) = wt$. For all other E that are t-adjacent to $\phi(C)$, we have $\delta(C, E) = w$. Using thickness, we may choose D so that $\phi(D) \neq E_0$, and then $\delta(C, \phi(D)) = w$ and $\delta(D, \phi(D)) = sw$:

$$
\begin{array}{ccc}
C & \xrightarrow{\ w\ } & \phi(C) \\
\big| s & \searrow^{w} & \big| t \\
D & \xrightarrow{\ sw\ } & \phi(D)
\end{array}
$$

Thus $v = sw$ and $l(v) = l(w) + 1$. $\qquad\square$

Suppose now that (W, S) is purely infinite and ϕ is nontrivial. Then we can start with any chamber C such that $\phi(C) \neq C$, and Lemma 2.2 shows that the hypothesis of Lemma 3.1 is satisfied. We therefore obtain a chamber D such that $d(D, \phi(D)) > d(C, \phi(C))$. Our main theorem as stated in the introduction follows at once. We restate it here for ease of reference:
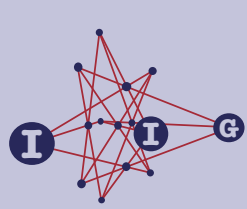
Theorem 3.2. *Let ϕ be a nontrivial automorphism of a thick building Δ of purely infinite type. Then ϕ, viewed as an isometry of the set $\mathcal{C}$ of chambers, has unbounded displacement, i.e., the set $\{d(C, \phi(C)) \mid C \in \mathcal{C}\}$ is unbounded.* $\square$

Remark 3.3. Note that, in view of the generality under which we proved Lemma 2.2, the building Δ is allowed to have infinite rank.

Remark 3.4. In view of the existence of translations in Euclidean Coxeter complexes, the thickness assumption in the theorem cannot be dropped.

Corollary 3.5. *Let Δ and $\mathcal{C}$ be as in the theorem, and let G be a group of automorphisms of Δ. If there is a bounded set of representatives for the G-orbits in $\mathcal{C}$, then G has trivial center.*

Proof. Let $\mathcal{M}$ be a bounded set of representatives for the G-orbits in $\mathcal{C}$, and let $z \in G$ be central. Then there is an upper bound M on the distances $d(C, zC)$ for $C \in \mathcal{M}$; we can take M to be the diameter of the bounded set $\mathcal{M} \cup z\mathcal{M}$, for instance. Now every chamber $D \in \mathcal{C}$ has the form $D = gC$ for some $g \in G$ and $C \in \mathcal{M}$, hence

$$d(D, zD) = d(gC, zgC) = d(gC, gzC) = d(C, zC) \le M.$$

Thus z has bounded displacement and therefore $z = 1$ by the theorem. $\square$

Remark 3.6. Although Corollary 3.5 is stated for faithful group actions, we can also apply it to actions that are not necessarily faithful and conclude (under the hypothesis of the corollary) that the center of G acts trivially.

Remark 3.7. Note that the hypothesis of the corollary is satisfied if the action of G is chamber transitive. In particular, it is satisfied if the action is strongly transitive and hence corresponds to a BN-pair in G. In this case, however, the result is trivial (and does not require the building to be of purely infinite type). Indeed, the stabilizer of every chamber is a parabolic subgroup and hence is self-normalizing, so it automatically contains the center of G. To obtain other examples, consider a cocompact action of a group on a locally finite thick Euclidean building (e.g., a thick tree). The corollary then implies that the center of the group must act trivially.
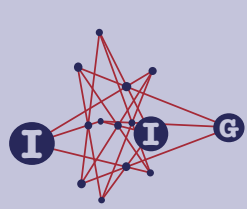
4. Spherical buildings

The conclusion of Theorem 3.2 is obviously false for spherical buildings, since the metric space $\mathcal{C}$ is bounded in this case. But one can ask instead whether or not

$$\operatorname{disp} \phi = \operatorname{diam} \Delta, \tag{2}$$

where $\operatorname{diam} \Delta$ denotes the diameter of the metric space $\mathcal{C}$, and $\operatorname{disp} \phi$ is the *displacement* of ϕ; the latter is defined by

$$\operatorname{disp} \phi := \sup\{d(C, \phi(C)) \mid C \in \mathcal{C}\}.$$

Note that, in the spherical case, equation (2) holds if and only if there is a chamber C such that $\phi(C)$ and C are opposite. This turns out to be false in general. The following counterexample was pointed out to us by Hendrik Van Maldeghem.

Example 4.1. Let k be a field and n an integer ≥ 2. Let Δ be the building associated to the vector space $V = k^{2n}$. Thus the vertices of Δ are the subspaces U of V such that $0 < U < V$, and the simplices are the chains of such subspaces. A chamber is a chain

$$U_1 < U_2 < \cdots < U_{2n-1}$$

with $\dim U_i = i$ for all i, and two such chambers (U_i) and (U_i') are opposite if and only if $U_i + U_{2n-i}' = V$ for all i. Now choose a non-degenerate alternating bilinear form B on V, and let ϕ be the (type-reversing) involution of Δ that sends each vertex U to its orthogonal subspace $U^\perp$ with respect to B. For any chamber (U_i) as above, its image under ϕ is the chamber (U_i') with $U_{2n-i}' = U_i^\perp$ for all i. Since $U_1 \leq U_1^\perp = U_{2n-1}'$, these two chambers are not opposite.
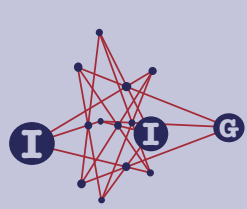
Even though (2) is false in general, one can still use Lemma 3.1 to obtain lower bounds on $\operatorname{disp} \phi$. Recall first the notion of *opposite residue* in a spherical building [1, Section 5.7.1]. Let (W, S) be a Coxeter system with W finite. The longest element w_0 of W has order 2 and normalizes S. We therefore have an involution σ_0 of S, given by $s \mapsto w_0 s w_0$ for $s \in S$. We call two subsets J and K of S *opposite* if $K = \sigma_0(J)$. And we say that a J-residue $\mathcal{R}$ and a K-residue $\mathcal{S}$ of a spherical building Δ are *opposite* if their types J and K are opposite and there are chambers $C \in \mathcal{R}$ and $D \in \mathcal{S}$ such that C and D are opposite. This is equivalent to saying that the simplices corresponding to $\mathcal{R}$ and $\mathcal{S}$ are opposite in some (or every) apartment containing them.

Proposition 4.2. *Let ϕ be a nontrivial automorphism of a thick spherical building Δ. Then Δ contains a proper residue $\mathcal{R}$ such that $\phi(\mathcal{R})$ and $\mathcal{R}$ are opposite. Equivalently, Δ contains a nonempty simplex A such that $\phi(A)$ and A are opposite.*

Proof. As before let σ be the automorphism of (W, S) induced by ϕ. We again start with an arbitrary chamber that is moved by ϕ, and we repeatedly apply Lemma 3.1 as long as possible. When the process stops, we have a chamber C such that $w := \delta(C, \phi(C))$ is nontrivial and satisfies $w^{-1}sw = \sigma(s)$ for all $s \in J := \{s \in S \mid l(sw) > l(w)\}$. In particular, w satisfies the hypotheses of Lemma 2.4. Letting $\mathcal{R}$ be the J-residue containing C, its image $\phi(\mathcal{R})$ is a

K-residue with $K = \sigma(J) = w^{-1}Jw$. Lemma 2.4 therefore implies that $\phi(\mathcal{R})$ and $\mathcal{R}$ are opposite residues. Moreover, they are proper residues (and therefore correspond to nonempty simplices) because $w \neq 1$ and hence $J \neq S$. $\qquad \square$

Remark 4.3. Proposition 4.2 was originally proved by Leeb [7, Sublemma 5.22], who stated the conclusion in the following equivalent form: The geometric realization X of Δ contains a point x such that $\phi(x)$ and x are opposite. His proof used spherical geometry in the apartments of X.

As an illustration of the proposition, consider the rank 2 case. Then Δ is a generalized m-gon for some m, and its diameter is m. Proposition 4.2 in this case yields the following result.

Corollary 4.4. *Let ϕ be a nontrivial automorphism of a thick generalized m-gon. Then the following hold:*

(a) $\operatorname{disp} \phi \geq m - 1$.

(b) *If ϕ is type preserving and m is odd, or if ϕ is type reversing and m is even, then $\operatorname{disp} \phi = m$.*

Proof. (a) Choose A as in the proposition. It is either a vertex or an edge. If it is an edge, then $\operatorname{disp} \phi = m$. Otherwise, it is a vertex, and then any edge C having A as one of its vertices is mapped to an edge $\phi(C)$ with $d(C, \phi(C)) \geq m - 1$.

(b) Recall that opposite vertices have the same type if m is even and different types if m is odd. So the hypotheses of (b) imply that no vertex of Δ can be mapped to an opposite vertex. The simplex A in the proof of (a) must therefore be an edge, implying $\operatorname{disp} \phi = m$. $\qquad \square$

(See also Tent [10] for a direct proof of the corollary.)

For spherical buildings of higher rank, Leeb's result (Proposition 4.2) yields the following less satisfying lower bound on displacement:
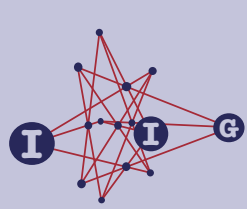
$$\operatorname{disp} \phi \geq \operatorname{diam} \Delta - r \, ,$$

where r is the maximal diameter of a proper residue of Δ. Note that r depends only on the type of Δ and is 1 in the rank 2 case.

We conclude by mentioning another family of examples, again pointed out to us by Van Maldeghem.

Remark 4.5. For even $m = 2n$, type-preserving automorphisms ϕ of generalized m-gons with $\operatorname{disp}\phi = m - 1$ arise as follows. Assume that there exists a vertex x in the generalized m-gon Δ such that the ball $B(x, n)$ is fixed pointwise by ϕ. Here $B(x, n)$ is the set of vertices with $d(x, y) \leq n$, where $d(-, -)$ now denotes the usual graph metric, obtained by minimizing lengths of paths. Recall that there are two types of vertices in Δ and that opposite vertices always have the same type since m is even. Let y be any vertex that does not have the same type as x. Then y is at distance at most $n - 1$ from some vertex in $B(x, n)$. Since ϕ fixes $B(x, n)$ pointwise, $d(y, \phi(y)) \leq 2n - 2$. So C and $\phi(C)$ are not opposite for any chamber C having y as a vertex. Since this is true for any vertex y that does not have the same type as x, $\operatorname{disp}\phi \neq m$ and hence, by Corollary 4.4(a), $\operatorname{disp}\phi = m - 1$ if $\phi \neq \operatorname{id}$. Now it is a well-known fact (see for instance [14, Corollary 5.4.7]) that every *Moufang* m-gon possesses nontrivial type-preserving automorphisms ϕ fixing some ball $B(x, n)$ pointwise. (In the language of incidence geometry, these automorphisms are called central or axial collineations, depending on whether x is a point or a line in the corresponding rank 2 geometry.) So for $m = 4$, 6, or 8, all Moufang m-gons admit type-preserving automorphisms ϕ with $\operatorname{disp}\phi = m - 1$.

References

[1] **P. Abramenko** and **K. S. Brown**, *Buildings: Theory and Applications*, Grad. Texts in Math. **248**, Springer, New York, 2008.

[2] **N. Bourbaki**, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines*, Actualités Scientifiques et Industrielles, No. **1337**, Hermann, Paris, 1968.

[3] **M. W. Davis**, Buildings are $\mathrm{CAT}(0)$, in *Geometry and cohomology in group theory (Durham, 1994)*, London Math. Soc. Lecture Note Ser., vol. **252**, Cambridge Univ. Press, Cambridge, 1998, pp. 108–123.

[4] **V. V. Deodhar**, On the root system of a Coxeter group, *Comm. Algebra* **10** (1982), no. 6, 611–630.

[5] **J. E. Humphreys**, *Reflection Groups and Coxeter Groups*, Cambridge Stud. Adv. Math., vol. **29**, Cambridge University Press, Cambridge, 1990.

[6] **D. Krammer**, The Conjugacy Problem for Coxeter Groups, *Groups Geom. Dyn.* **3** (2009), 71–171.
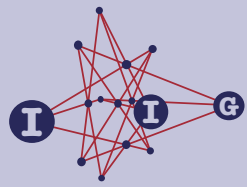
[7] **B. Leeb**, *A Characterization of Irreducible Symmetric Spaces and Euclidean Buildings of Higher Rank by their Asymptotic Geometry*, Bonner Mathematische Schriften [Bonn Mathematical Publications] **326**, Universität Bonn, Mathematisches Institut, Bonn, 2000.

[8] **M. Ronan**, *Lectures on Buildings*, Perspectives Math., vol. **7**, Academic Press Inc., Boston, MA, 1989.

[9] **R. Scharlau**, Buildings, in *Handbook of Incidence Geometry*, North-Holland, Amsterdam, 1995, pp. 477–645.

[10] **K. Tent**, Free polygons, twin trees, and $\mathrm{CAT}(1)$-spaces, to appear in *Pure Appl. Math. Q.*

[11] **J. Tits**, Groupes et géométries de Coxeter, *unpublished manuscript*, 1961.

[12] _____, Le problème des mots dans les groupes de Coxeter, *Symposia Mathematica (INDAM, Rome)* **1** (1967/68), Academic Press, London, 1969, pp. 175–185.

[13] _____, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math., vol. **386**, Springer-Verlag, Berlin, 1974.

[14] **H. Van Maldeghem**, *Generalized Polygons*, Monogr. Math., vol. **93**, Birkhäuser Verlag, Basel, 1998.

[15] **È. B. Vinberg**, Discrete linear groups that are generated by reflections, *Izv. Akad. Nauk SSSR Ser. Mat.* **35** (1971), 1072–1112.

[16] **R. M. Weiss**, *The Structure of Spherical Buildings*, Princeton University Press, Princeton, NJ, 2003.

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