



# Buildings with isolated subspaces and relatively hyperbolic Coxeter groups

Pierre-Emmanuel Caprace\*

## Abstract

Let  $(W, S)$  be a Coxeter system. We give necessary and sufficient conditions on the Coxeter diagram of  $(W, S)$  for  $W$  to be relatively hyperbolic with respect to a collection of finitely generated subgroups. The peripheral subgroups are necessarily parabolic subgroups (in the sense of Coxeter group theory). As an application, we present a criterion for the maximal flats of the Davis complex of  $(W, S)$  to be isolated. If this is the case, then the maximal affine sub-buildings of any building of type  $(W, S)$  are isolated.

**Keywords:** Coxeter group, building, isolated flat, relative hyperbolicity

**MSC 2000:** 20F67; 20E42, 20F55, 20F69

## 1 Introduction

Let  $X$  be a complete  $\text{CAT}(0)$  space. A  $k$ -flat in  $X$  is a subset which is isometric to the  $k$ -dimensional Euclidean space. Since we will mainly be interested in isolated flats, it is convenient to define a flat as a  $k$ -flat for some  $k \geq 2$ . In particular, geodesic lines are not considered to be flats. Let  $\mathcal{F}$  be a collection of closed convex subsets of  $X$ . We say that the elements of  $\mathcal{F}$  are *isolated* in  $X$  if the following conditions hold:

- (A) There is a constant  $D < \infty$  such that each flat  $F$  of  $X$  lies in a tubular  $D$ -neighbourhood of some  $C \in \mathcal{F}$ .

---

\*F.N.R.S. research associate. This work was partially supported by IPDE.

- (B) For each positive  $r < \infty$  there is a constant  $\rho = \rho(r) < \infty$  so that for any two distinct elements  $C, C' \in \mathcal{F}$  we have

$$\text{diam}(\mathcal{N}_r(C) \cap \mathcal{N}_r(C')) < \rho,$$

where  $\mathcal{N}_r(C)$  denotes the tubular  $r$ -neighbourhood of  $C$ .

We say that  $X$  has *isolated flats* if  $\mathcal{F}$  consists of flats.

Let now  $(W, S)$  be a Coxeter system with  $S$  finite. Given a subset  $J \subset S$ , we set  $W_J = \langle J \rangle$ ; the group  $W_J$  as well as any of its  $W$ -conjugate, is called a *parabolic subgroup* of  $W$ . We also set  $J^\perp = (S \setminus J) \cap \mathcal{L}_W(W_J)$ . The set  $J$  is called *spherical* (resp. *irreducible affine*, *affine*, *Euclidean*) if  $W_J$  is finite (resp. an irreducible affine Coxeter group, a direct product of irreducible affine Coxeter groups, a direct product of finite and affine Coxeter groups). We say that  $J$  is *minimal hyperbolic* if it is non-spherical and non-affine but every proper subset is spherical or irreducible affine. Let  $X(W, S)$  be the Davis complex of  $(W, S)$ . Thus  $X(W, S)$  is a proper CAT(0) space [Dav98] and its isometry group contains  $W$  as a cocompact lattice.

According to a theorem of G. Moussong [Mou88], the group  $W$  is Gromov hyperbolic if and only if  $S$  has no irreducible affine subset of cardinality  $\geq 3$  and if for each non-spherical  $J \subset S$ , the set  $J^\perp$  is spherical. Our main result gives necessary and sufficient conditions for  $W$  to be relatively hyperbolic with respect to a collection of parabolic subgroups:

**Theorem A.** *Let  $(W, S)$  be a Coxeter system with  $S$  finite, let  $\mathcal{P}$  be a collection of parabolic subgroups of  $W$  and let  $\mathcal{T}$  be the set of types of elements of  $\mathcal{P}$ . Then the following conditions are equivalent:*

- (i)  $\mathcal{T}$  satisfies the following conditions:

**(RH1)** *For each irreducible affine subset  $J \subset S$ , there exists  $K \in \mathcal{T}$  such that  $J \subset K$ . Similarly, for each pair of irreducible non-spherical subsets  $J_1, J_2 \subset S$  with  $[J_1, J_2] = 1$ , there exists  $K \in \mathcal{T}$  such that  $J_1 \cup J_2 \subset K$ .*

**(RH2)** *For all  $K_1, K_2 \in \mathcal{T}$  with  $K_1 \neq K_2$ , the intersection  $K_1 \cap K_2$  is spherical.*

- (ii)  $W$  is relatively hyperbolic with respect to  $\mathcal{P}$ .
- (iii) In the Davis complex  $X(W, S)$ , the residues whose type belongs to  $\mathcal{T}$  are isolated.
- (iv) In any building of type  $(W, S)$ , the residues whose type belongs to  $\mathcal{T}$  are isolated.

Basic definitions and properties of relatively hyperbolic groups may be consulted in the standard references [Bow99] or [Far98].

**Remark 1.1.** Throughout this paper, the term ‘*parabolic subgroup*’ will be used only in the sense which was defined above; this agrees with the standard conventions in the theory of Coxeter groups. Given a group  $G$  which is relatively hyperbolic with respect to subgroups  $H_1, \dots, H_n$ , it is customary to call any conjugate of some  $H_i$  a *parabolic subgroup* of  $G$ . In order to avoid any confusion in the present paper, we shall instead call these the *peripheral subgroups* of  $G$ . Thus the term ‘*parabolic*’ will be exclusively used in its Coxeter group acceptance.

Notice that we do not assume the buildings to be locally compact in (iv). Conditions (RH1) and (RH2) can be checked concretely on the Coxeter diagram of  $(W, S)$ . Combining Theorem A with the following, one obtains in particular a complete characterization of those Coxeter groups which are relatively hyperbolic with respect to any family of finitely generated subgroups:

**Theorem B.** *Let  $(W, S)$  be a Coxeter system with  $S$  finite. If  $W$  is relatively hyperbolic with respect to finitely generated subgroups  $H_1, \dots, H_m$ , then each  $H_i$  is a parabolic subgroup of  $W$ .*

It should be noted that there exist non-affine Coxeter groups which are not relatively hyperbolic with respect to any family of parabolic subgroups. Consider, for example, the Coxeter group  $W$  with Coxeter generating set  $S = \{s_1, \dots, s_n\}$  defined by the following relations:  $[s_i, s_j] = 1$  for  $|i - j| \geq 2$  and  $o(s_i s_j) = 4$  for  $|i - j| = 1$ . It is easily verified, using Theorem A, that for  $n > 7$ , the group  $W$  is not relatively hyperbolic with respect to any collection of proper parabolic subgroups. For  $n = 7$ , one checks that the set

$$\mathcal{T} = \{\{1, 2, 3, 5, 6, 7\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}\}$$

satisfies (RH1) and (RH2).

It happens however quite often that a Coxeter group is relatively hyperbolic with respect to a maximal proper parabolic subgroup:

**Corollary C.** *Suppose that there exists an element  $s_0 \in S$  such that  $\{s_0\}^\perp$  is spherical. Then  $W$  is relatively hyperbolic with respect to the parabolic subgroups whose type belongs to the set*

$$\mathcal{T} = \{S \setminus \{s_0\}\} \cup \{J \subset S \mid J \text{ is affine and contains } s_0\}.$$

Of particular interest is the special case when the peripheral subgroups of a relatively hyperbolic group are virtually nilpotent (or more generally amenable).

As for any discrete group acting properly and cocompactly on a CAT(0) space, any amenable subgroup of a Coxeter group is virtually abelian. Theorem A yields the following characterization:

**Corollary D.** *The following assertions are equivalent:*

- (i) *For all non-spherical  $J_1, J_2 \subset S$  such that  $[J_1, J_2] = 1$ , the group  $\langle J_1 \cup J_2 \rangle$  is virtually abelian.*
- (ii) *For each minimal hyperbolic  $J \subset S$ , the set  $J^\perp$  is spherical.*
- (iii) *The Davis complex  $X(W, S)$  has isolated flats.*
- (iv) *The group  $W$  is relatively hyperbolic with respect to a collection of virtually abelian subgroups of rank at least 2.*
- (v) *In any building of type  $(W, S)$ , the maximal residues of (non-spherical) Euclidean type (i.e. the maximal non-spherical Euclidean sub-buildings) are isolated.*

The list of all minimal hyperbolic Coxeter systems is known (see [Bou68, Ch. V, pp.133–134] or [Hum90, II.6.9]); in fact, it is not difficult to see that a minimal hyperbolic subset of  $S$  has cardinality at most 10. Many Coxeter systems  $(W, S)$  such that  $W$  is not Gromov hyperbolic do satisfy condition (ii). In particular  $S$  may contain affine subsets.

Theorem A is deduced from the detailed study of flats in buildings and Coxeter groups which is made in [CH06]. The equivalence between (ii) and (iii) is a consequence of [HK05, Appendix]. In fact, the theorem above allows one to apply varied algebraic and geometric consequences of the isolation of subspaces established in [loc. cit.] and [DS05]. We collect a few of them in the special case of virtually abelian peripheral subgroups:

**Corollary E.** *Assume that  $(W, S)$  satisfies the equivalent conditions of Corollary D. Let  $Y$  be a building of type  $(W, S)$ ,  $\mathcal{F}$  be the collection of maximal residues of non-spherical Euclidean type and  $\Gamma < \text{Isom}(Y)$  be a subgroup acting properly discontinuously and cocompactly. Then:*

- (i)  *$\Gamma$  is relatively hyperbolic with respect to the family of stabilizers of elements of  $\mathcal{F}$ ; each of these stabilizers is a cocompact lattice of a Euclidean building.*
- (ii)  *$W$  and  $\Gamma$  are biautomatic.*
- (iii) *Every connected component of  $\partial_T X(W, S)$  (resp.  $Y$ ) is either an isolated point or a Euclidean sphere (resp. a spherical building).*
- (iv) *Every asymptotic cone of  $X(W, S)$  (resp.  $Y$ ) is tree-graded with respect to a family of closed convex subsets which are flats (resp. Euclidean buildings); furthermore any quasi-isometry of  $X(W, S)$  (resp.  $Y$ ) permutes these pieces.*

We refer to [DS05] for more information on asymptotic cones and tree-graded spaces. It is known that all Coxeter groups are automatic [BH93], but the problem of determining which Coxeter groups are biautomatic is still incompletely solved: it follows from [CM05] that  $W$  is biautomatic whenever  $S$  has no irreducible affine subset of cardinality  $\geq 3$ . Corollary E shows biautomaticity in many other cases.

Let us finally mention that the construction of cocompact lattices in  $\text{Isom}(Y)$  is a delicate problem, unless the Coxeter system  $(W, S)$  is *right-angled* (i.e.  $o(st) \in \{1, 2, \infty\}$  for all  $s, t \in S$ ). This question seems especially interesting when  $(W, S)$  is 2-spherical, namely  $o(st) < \infty$  for all  $s, t \in S$ . Besides the classical case of Euclidean buildings, some known constructions provide examples of lattices when  $W$  is 2-spherical and Gromov hyperbolic [KV06]. However, I don't know any example of a cocompact lattice in  $\text{Isom}(Y)$  in the case when  $W$  is a 2-spherical Coxeter group which is neither Euclidean nor Gromov-hyperbolic. The situation is completely different when  $(W, S)$  is right-angled. In that case indeed, graphs of groups provide a large family of examples of cocompact lattices to which Corollary E may be applied.

In order to state this properly, let  $A$  be a finite simple graph with vertex set  $I$  and for each  $i \in I$ , let  $P_i$  be a group. Let  $\Gamma = \Gamma(A, (P_i)_{i \in I})$  be the group which is the quotient of the free product of the  $(P_i)_{i \in I}$  by the normal subgroup generated by all commutators of the form  $[g_i, g_j]$  with  $g_i \in P_i$ ,  $g_j \in P_j$  and  $\{i, j\}$  spanning an edge of  $A$ . Let also  $(W, \{s_i\}_{i \in I})$  be the Coxeter system such that  $o(s_i s_j) = 2$  (resp.  $o(s_i s_j) = \infty$ ) for each edge (resp. non-edge)  $\{i, j\}$  of  $A$ . Then  $\Gamma$  acts simply transitively on the chambers of a building  $Y(A, (P_i)_{i \in I})$  of type  $(W, \{s_i\}_{i \in I})$  by [Dav98, Theorem 5.1 and Corollary 11.7]. If each  $P_i$  is finite, then this building is locally compact and, hence,  $\Gamma$  is a cocompact lattice in its automorphism group. For example, if the graph  $A$  is a  $n$ -cycle with  $n \geq 5$ , then this building is a Bourdon building and  $\Gamma$  is a Bourdon lattice (these are the Fuchsian buildings and their lattices defined and studied by M. Bourdon in [Bou97]). Moreover, if each  $P_i$  is infinite cyclic, then  $\Gamma$  is a right-angled Artin group.

For the Coxeter system  $(W, \{s_i\}_{i \in I})$ , condition (ii) of Corollary D may be expressed as follows: *for each 3-subset  $J \subset I$  which is not a triangle, the subgraph induced on  $J^\perp = \{i \in I \mid \{i, j\} \text{ is an edge for each } j \in J\}$  is a complete graph.* Let now  $S_{\text{aff}}$  be the set of all subsets of  $J$  of the form  $\{i_1, j_1, \dots, i_n, j_n\}$  where  $\{i_k, j_k\}$  is a non-edge for each  $k$  but all other pairs of elements are edges. Note that for  $J = \{i_1, j_1, \dots, i_n, j_n\}$ , the subgroup  $\Gamma_J$  of  $\Gamma$  generated by all  $P_i$ 's with  $i \in J$  has the following structure:

$$\Gamma_J \simeq (P_{i_1} * P_{j_1}) \times \cdots \times (P_{i_n} * P_{j_n}).$$

Then Corollary E implies that, under the assumption (ii), the group  $\Gamma$  is relatively hyperbolic with respect to the family of all conjugates of subgroups of the form  $\Gamma_J$  with  $J \in \mathcal{S}_{\text{aff}}$ .

## 2 On parabolic subgroups of Coxeter groups

Recall that a subgroup of  $W$  of the form  $W_J$  for some  $J \subset S$  is called a *standard parabolic subgroup*. Any of its conjugates is called a *parabolic subgroup* of  $W$ . A basic fact on Coxeter groups is that any intersection of parabolic subgroups is itself a parabolic subgroup. This allows to define the *parabolic closure*  $\text{Pc}(R)$  of a subset  $R \subset W$ : it is the smallest parabolic subgroup of  $W$  containing  $R$ .

**Lemma 2.1.** *Let  $G$  be a reflection subgroup of  $W$  (i.e. a subgroup of  $W$  generated by reflections). Then there is a set of reflections  $R \subset G$  such that  $(G, R)$  is a Coxeter system. Furthermore, if  $(G, R)$  is irreducible (resp. spherical, affine of rank  $\geq 3$ ), then so is  $\text{Pc}(G)$ .*

*Proof.* For the first assertion, see [Deo89]. Any two reflections in  $R$  which do not commute lie in the same irreducible component of  $\text{Pc}(R)$ . Therefore, if  $(G, R)$  is irreducible, then all elements of  $R$  are in the same irreducible component of  $\text{Pc}(R)$ . Since  $G = \langle R \rangle$  and  $\text{Pc}(R)$  is the minimal parabolic subgroup containing  $G$ , we deduce that  $\text{Pc}(R)$  is irreducible. If  $G$  is finite, then it is contained in a finite parabolic subgroup (see [Bou68]), hence  $\text{Pc}(G)$  is spherical. Finally, if  $(G, R)$  is affine of rank  $\geq 3$ , then so is  $\text{Pc}(G)$  by [Cap06, Proposition 16].  $\square$

**Lemma 2.2.** *Let  $P \subset W$  be an infinite irreducible parabolic subgroup. Then the normalizer of  $P$  in  $W$  splits as a direct product:  $\mathcal{N}_W(P) = P \times \mathcal{Z}_W(P)$  and  $\mathcal{Z}_W(P)$  is also a parabolic subgroup of  $W$ .*

*Proof.* See [Deo82, Proposition 5.5].  $\square$

**Lemma 2.3.** *Let  $G_1, G_2$  be finitely generated reflection subgroups of  $W$  which are irreducible, i.e.  $(G_i, R_i)$  is irreducible for  $R_i \subset G_i$  as in Lemma 2.1, and assume that  $G_1$  is infinite and that  $[G_1, G_2] = \{1\}$ . Then either*

$$\text{Pc}(\langle G_1 \cup G_2 \rangle) \simeq \text{Pc}(G_1) \times \text{Pc}(G_2)$$

or  $\text{Pc}(G_1) = \text{Pc}(G_2)$  is an irreducible affine Coxeter group of rank  $\geq 3$ .

*Proof.* By Lemma 2.1, the parabolic closure  $\text{Pc}(G_1)$  is infinite and irreducible. Given a reflection  $r \in G_2$ , then  $r$  centralizes  $G_1$  by hypothesis, hence it normalizes  $\text{Pc}(G_1)$ . Thus either  $r \in \text{Pc}(G_1)$  or  $r$  centralizes  $\text{Pc}(G_1)$  by Lemma 2.2. Therefore, either  $G_2 \subset \text{Pc}(G_1)$  or  $G_2$  centralizes  $\text{Pc}(G_1)$  and  $G_2 \cap \text{Pc}(G_1) = \{1\}$ .

If  $G_2$  centralizes  $\text{Pc}(G_1)$  and  $G_2 \cap \text{Pc}(G_1) = \{1\}$ , then  $\text{Pc}(G_2)$  centralizes  $\text{Pc}(G_1)$  and  $\text{Pc}(G_1) \cap \text{Pc}(G_2) = \{1\}$  by Lemma 2.2. Hence we are done in this case.

Assume now that  $G_2 \subset \text{Pc}(G_1)$ . Then, since  $G_1$  normalizes  $\text{Pc}(G_2)$ , we deduce from Lemma 2.2 that  $\text{Pc}(G_1) = \text{Pc}(G_2)$ . It is not difficult to see that  $G_1$  contains an element of infinite order  $w_1$  such that  $\text{Pc}(w_1) = \text{Pc}(G_1)$  (take for example  $w_1$  to be the Coxeter element in the Coxeter system  $(G_1, R_1)$  provided by Lemma 2.1). Similarly, let  $w_2 \in G_2$  be such that  $\text{Pc}(w_2) = \text{Pc}(G_2) = \text{Pc}(G_1)$ . Thus  $w_1$  and  $w_2$  are mutually centralizing. Moreover, we have  $\langle w_1 \rangle \cap \langle w_2 \rangle < G_1 \cap G_2 = \{1\}$ : indeed, any infinite irreducible Coxeter group is center-free by Lemma 2.2. Thus  $\langle w_1, w_2 \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ . By [Kra09, Corollary 6.3.10], this implies that  $\text{Pc}(G_1) = \text{Pc}(G_2)$  is affine, and clearly of rank  $\geq 3$  since it contains  $\mathbb{Z} \times \mathbb{Z}$ .  $\square$

Let  $X = X(W, S)$  denote the Davis complex.

**Lemma 2.4.** *Let  $r, r', s, t$  be reflections. Assume that the wall  $X^t$  separates  $X^r$  from  $X^{r'}$  and that  $s$  commutes with both  $r$  and  $r'$ . Then either  $s$  also commutes with  $t$  or  $s$  belongs to the parabolic closure of  $\langle r, r' \rangle$ .*

*Proof.* Let  $H < W$  be the infinite dihedral subgroup generated by  $r$  and  $r'$ . By assumption  $s$  centralizes  $H$ , whence  $s$  normalizes the parabolic  $\text{Pc}(H)$ . By Lemma 2.1, the parabolic subgroup  $\text{Pc}(H)$  is irreducible and non-spherical. Since  $s$  is a reflection, we deduce from Lemma 2.2 that either  $s$  belongs to  $\text{Pc}(H)$  or  $s$  centralizes  $\text{Pc}(H)$ . This finishes the proof because, by [Cap06, Lemma 17], the reflection  $t$  belongs to  $\text{Pc}(H)$ .  $\square$

### 3 On Euclidean flats in the CAT(0) realization of Tits buildings

Let now  $F$  be a flat in  $X = X(W, S)$ ; we remind the reader that  $\dim(F) \geq 2$  according to the convention adopted in this paper. We use the notation and terminology of [CH06]. In particular, we denote by  $\mathcal{M}(F)$  the set of all walls which separate points of  $F$ . Furthermore, for any set of walls  $M$ , we denote by  $W(M)$  the subgroup of  $W$  generated by all reflections through walls in  $M$ . For any  $m \in \mathcal{M}(F)$ , the set  $m \cap F$  is a Euclidean hyperplane of  $F$  [CH06, Lemma 4.1].

Two elements  $m, m'$  of  $\mathcal{M}(F)$  are called  $F$ -parallel if the hyperplanes  $m \cap F$  and  $m' \cap F$  are parallel in  $F$ . The following result collects some key facts on flats in  $X$  established in [CH06]:

**Proposition 3.1.** *Let  $F$  be a flat in  $X = X(W, S)$ . Then the  $F$ -parallelism in  $\mathcal{M}(F)$  induces a finite partition*

$$\mathcal{M}(F) = M_0 \cup M_1 \cup \cdots \cup M_k$$

such that

$$\mathrm{Pc}(W(\mathcal{M}(F))) \simeq \mathrm{Pc}(W(M_0)) \times \cdots \times \mathrm{Pc}(W(M_k)),$$

where each  $W(M_i)$  is a direct product of infinite irreducible Coxeter groups and  $W(M_0)$  is a direct product of irreducible affine Coxeter groups. Moreover, the set  $M_i$  is non-empty for each  $i \geq 1$  and if  $M_0 = \emptyset$ , then  $k \geq \dim F$ .

*Proof.* Let  $\mathcal{M}(F) = M_1 \cup \cdots \cup M_l$  be the partition of  $\mathcal{M}(F)$  into  $F$ -parallelism classes. Since the dimension of  $F$  is at least 2, we have  $l \geq 2$ . By Lemma 2.1 and [CH06, Lemma 4.3], the reflection subgroup  $W(M_i)$  is a direct product of infinite irreducible Coxeter groups. In particular  $W(M_i)$  is center-free by Lemma 2.2.

Let  $M_0 = \mathcal{M}_{\mathrm{Eucl}}(F) \subset \mathcal{M}(F)$  be the subset defined after Remark 4.4 in [CH06]. The group  $W(M_0)$  is a direct product of finitely many irreducible affine Coxeter groups of rank  $\geq 3$  by [CH06, Proposition 4.7]. Moreover, by [CH06, Lemma 4.6(ii)], we have either  $M_i \cap \mathcal{M}_{\mathrm{Eucl}}(F) = \emptyset$  or  $M_i \cap \mathcal{M}_{\mathrm{Eucl}}(F) = M_i$  for each  $i \in \{1, \dots, l\}$ . Thus, without loss of generality, we may and shall assume that  $M_0 = \mathcal{M}_{\mathrm{Eucl}}(F) = M_{k+1} \cup M_{k+2} \cup \cdots \cup M_l$  for some  $k \geq 0$  (with  $k = l$  if  $\mathcal{M}_{\mathrm{Eucl}}(F) = \emptyset$ ).

Let now  $i \leq k$ . By [CH06, Lemma 4.6(iii)], for each  $j \neq i$ , the subgroups  $W(M_i)$  and  $W(M_j)$  centralize each other. Since moreover, both of them are center-free, we obtain

$$W(\mathcal{M}(F)) \simeq W(M_0) \times \cdots \times W(M_k).$$

The fact that this decomposition passes to the parabolic closure follows easily from Lemma 2.3 and the fact that finitely generated infinite irreducible Coxeter groups have trivial center.

Finally, if  $M_0 = \mathcal{M}_{\mathrm{Eucl}}(F)$  is empty, then the fact that  $k \geq \dim F$  follows from the proof of [CH06, Theorem 5.2], but only the fact that  $k \geq 2$  is relevant to our later purposes.  $\square$

An immediate consequence is the following:



**Corollary 3.2.** *Assume that condition (i) of Corollary D holds. Then, for each flat  $F$  in  $X(W, S)$ , the group  $\text{Pc}(W(\mathcal{M}(F)))$  is a direct product of irreducible affine Coxeter groups.  $\square$*

For the sake of future references, we also record the following important fact:

**Proposition 3.3.** *Let  $F$  be a flat in  $X$  and let  $P$  denote the parabolic closure of  $W(\mathcal{M}(F))$ . Then:*

- (i) *Given any residue  $R$  whose stabilizer is  $P$  and any wall  $m$  which separates some point of  $F$  to its projection to  $R$ , the wall  $m$  separates  $F$  from  $R$  and the reflection  $r_m$  centralizes  $P$ .*
- (ii) *The flat  $F$  is contained in some residue whose stabilizer is  $P$ .*

*Proof.* We denote by  $\pi_R$  the nearest point projection onto  $R$  (one may use either the combinatorial projection as defined in [Tit74, §3.19], or the CAT(0) orthogonal projection; both play an equivalent role for our present purposes). Let  $x \in F$  be any point. Since walls and half-spaces in  $X$  are closed and convex and since  $R$  is  $P$ -invariant, it follows that the set  $\mathcal{M}(x, \pi_R(x))$  of all walls which separate  $x$  from  $\pi_R(x)$  intersects  $\mathcal{M}(R)$  trivially. In other words, the geodesic segment joining  $x$  to  $\pi_R(x)$  does not cross any wall of  $\mathcal{M}(R)$ . Therefore, since  $\mathcal{M}(F) \subset \mathcal{M}(R)$ , any wall in  $\mathcal{M}(x, \pi_R(x))$  separates  $F$  from  $R$  and, hence, meets every element of  $\mathcal{M}(F)$ .

Pick an element  $m \in \mathcal{M}(x, \pi_R(x))$  and let  $\mu \in \mathcal{M}(F)$  be any wall. If the reflections  $r_m$  and  $r_\mu$  do not commute, then the wall  $r_m(\mu)$  is distinct from  $m$ . Furthermore  $r_m(\mu)$  also separates a point of  $F$  to its projection to  $R$  since otherwise, we would have  $r_m(\mu) \in \mathcal{M}(R)$  and hence  $\text{Pc}(r_\mu, r_m r_\mu r_m) \subset P$ , which in turn yields the absurd conclusion that  $r_m$  belongs to  $P$ . By the above, this wall therefore separates  $F$  from  $R$  and hence, it belongs to  $\mathcal{M}(x, \pi_R(x))$ . Since the latter set of walls is finite, this shows that the subset of  $\mathcal{M}(F)$  consisting of all those walls  $\mu$  such that  $r_m$  does not commute with  $r_\mu$  is finite.

By [CH06, Lemma 4.3], given any wall  $\mu_0 \in \mathcal{M}(F)$ , there exist infinitely many pairs of (pairwise distinct) walls  $\{\mu, \mu'\} \subset \mathcal{M}(F)$  such that  $\mu_0$  separates  $\mu$  from  $\mu'$ . The preceding paragraph therefore shows that, given  $\mu_0$ , we may choose  $\mu$  and  $\mu'$  in such a way that  $r_m$  commutes with both  $r_\mu$  and  $r_{\mu'}$ . Notice that  $r_m$  does not belong to the parabolic subgroup  $P$  which, by definition, contains the parabolic closure of  $\langle r_\mu, r_{\mu'} \rangle$ . Therefore Lemma 2.4 implies that  $r_m$  commutes with  $r_{\mu_0}$ . This implies that  $r_m$  centralizes  $W(\mathcal{M}(F))$  and, hence, normalizes  $P$ . Once again, since  $r_m$  does not belong to  $P$ , it follows from Lemma 2.2 that  $r_m$  centralizes  $P$ , thereby establishing (i).

For assertion (ii), choose  $R$  amongst the residues whose stabilizer is  $P$  in such a way that it minimizes the distance to  $F$ . If some point of  $F$  does not belong to  $R$ , there exists a wall which separates that point from its projection to  $R$ . By (i) this wall separates  $F$  from  $R$  and is perpendicular to every wall of  $R$ . Transforming  $R$  by the reflection through that wall, we obtain another residue whose stabilizer is  $P$ , but closer to  $F$ . This contradicts the minimality assumption made on  $R$ .  $\square$

## 4 Relative hyperbolicity

### 4.1 Peripheral subgroups are parabolic

The purpose of this section is to prove Theorem B. We will need a subsidiary result on Coxeter groups. In order to state it properly, we make use of some additional terminology which we now introduce.

Given an element  $w \in W$  and a half-space  $\mathcal{H}$  of the Cayley graph  $\text{Cay}(W, S)$  (or of the Davis complex  $X(W, S)$ ), we say that  $\mathcal{H}$  is  $w$ -essential if  $w\mathcal{H} \subsetneq \mathcal{H}$  or  $w^{-1}\mathcal{H} \subsetneq \mathcal{H}$ . Notice that an element  $w \in W$  admits a  $w$ -essential half-space if and only if it has infinite order.

The reflection of  $W$  associated to  $\mathcal{H}$  is denoted by  $r_{\mathcal{H}}$ .

**Lemma 4.1.** *Let  $H < W$  be a finitely generated subgroup. Suppose that for any  $w \in H$  and any  $w$ -essential half-space  $\mathcal{H}$ , the reflection  $r_{\mathcal{H}}$  belongs to  $H$ . Then  $H$  contains a parabolic subgroup of  $W$  as a normal subgroup of finite index.*

*Proof.* Let  $P < H$  be the subgroup of  $H$  generated by all reflections  $r_{\mathcal{H}}$  associated to a  $w$ -essential half-space  $\mathcal{H}$  for some element  $w \in H$ . Thus  $P$  is a reflection subgroup of  $W$  contained in  $H$ . In particular  $P$  is itself a Coxeter group, see Lemma 2.1.

A crucial point, which follows from [Kra09, Th. 5.8.1] and [CH06, Lem. 5.3], is that  $W$  admits a finite index torsion-free subgroup  $W'$  such that for all  $w \in W'$ , we have

$$\text{Pc}(w) = \langle r_{\mathcal{H}} \mid \mathcal{H} \text{ is a } w\text{-essential half-space} \rangle.$$

In particular  $W' \cap H$  is contained in  $P$  and hence  $P$  has finite index in  $H$ .

We now choose  $w \in P$  in such a way that in the Coxeter group  $P$ , the parabolic closure  $\text{Pc}_P(w)$  of  $w$  relative to  $P$  is the whole  $P$ . Such an element  $w$  always exists, see [CF, Cor. 3.3]. Let also  $P'$  denote the parabolic subgroup of  $W$  generated by all those reflections  $r_{\mathcal{H}}$  such that  $\mathcal{H}$  is a  $w$ -essential half-space. By the definition of  $P$ , we have  $P' \subset P$ . By the property recalled in the preceding

paragraph, the group  $P'$  has finite index in  $\text{Pc}_P(w) = P$ . It follows that  $P'$  is a parabolic subgroup of  $W$  which is contained as a finite index subgroup in  $P$ . In particular  $P'$  is a parabolic subgroup of  $W$  which is contained as a finite index subgroup in  $H$ . Since any intersection of parabolic subgroups is parabolic, and since  $P'$  has finitely many conjugates in  $H$ , the desired result follows.  $\square$

*Proof of Theorem B.* Consider the graph  $K$  with vertex set  $V = W \cup (\bigcup_{i=1}^m W/H_i)$  and edge set defined as follows. Two elements of  $W$  are joined by an edge if their quotient is an element of  $S$ ; an element  $w \in W$  is joined to a coset  $v \in W/H_i$  if and only if  $w \in v$ . Then  $K$  is a connected graph on which  $W$  acts by automorphisms, and containing the Cayley graph of  $(W, S)$  as an induced subgraph. By relative hyperbolicity, this graph is hyperbolic. Furthermore for any  $n$  and any two vertices  $x, y$ , the collection of arcs of length  $n$  joining  $x$  to  $y$  is finite. In other words, the graph  $K$  is *fine* in the terminology of [Bow99]. Upon adding an edge between any two vertices at distance 2 in the Cayley graph of  $(W, S)$ , we may assume that the graph  $K$  has no cut-vertex.

We now apply Bowditch's construction of a hyperbolic 2-complex  $X(K)$  starting from  $K$ , see Theorem 3.8 (and also Lemmas 2.5 and 3.3) in [Bow99]. As explained in *loc. cit.* the action of  $W$  on the boundary of the space  $X(K)$  is a geometrically finite convergence action, and the peripheral subgroups (namely the  $W$ -conjugates of the  $H_i$ 's) are the stabilizers of the parabolic points in  $\partial X(K)$ . In particular any infinite order element  $h \in H_i$  acts as a parabolic element on  $X(K)$ ; it has a unique fixed point  $\xi_i \in \partial X(K)$  and the limit set of  $\langle h \rangle$  is precisely  $\{\xi_i\}$ .

Let now  $r \in W$  be a reflection. Then  $r$  acts on the Cayley graph  $\text{Cay}(W, S)$  as a reflection. Clearly  $r$  also acts as a reflection on  $K$ , in the sense that it interchanges two non-empty convex subgraphs whose union is the whole  $K$ . It follows from the construction of  $X(K)$  that  $r$  acts on  $X(K)$  as a *quasi-reflection*: the two half-spaces  $\mathcal{H}, \mathcal{H}'$  of  $K$  which are interchanged by  $r$  correspond in  $X(K)$  to two subcomplexes  $X(\mathcal{H}), X(\mathcal{H}')$  interchanged by  $r$ . It follows immediately that these two subcomplexes are quasi-convex; the fixed point set of  $r$  at infinity thus coincides with  $\partial X(\mathcal{H}) \cap \partial X(\mathcal{H}')$ .

Let now  $w \in H_i$  be an infinite order element and let  $\xi_i \in \partial X(K)$  be the parabolic point fixed by  $H_i$ . Let also  $\mathcal{H}$  be a half-space of  $K$  such that  $w(\mathcal{H}) \subsetneq \mathcal{H}$  and denote by  $\mathcal{H}'$  the complementary half-space. Then, for  $n > 0$  large enough we have  $w(X(\mathcal{H})) \subset X(\mathcal{H})$ . Since  $w^n \cdot x$  tends to  $\xi_i$  with  $w \rightarrow \infty$  for each  $x \in X(K)$ , we deduce that  $\xi_i$  belongs to  $\partial X(\mathcal{H})$ . Applying the same argument to  $h^{-1}$ , we deduce on the other hand that  $\xi_i$  belongs to  $\partial X(\mathcal{H}')$ . Thus  $\xi_i \in \partial X(\mathcal{H}) \cap \partial X(\mathcal{H}')$ . It follows that the reflection  $r_{\mathcal{H}}$  fixes  $\xi_i$  and, hence, that  $r_{\mathcal{H}}$  belongs to  $H_i$ .

By Lemma 4.1, each peripheral subgroup  $H_i$  contains a parabolic subgroup  $P_i$  as a finite index normal subgroup. Since  $P_i$  has finite index in  $H_i$  it follows that  $\xi_i$  is the unique fixed point of  $P_i$  in  $\partial X(K)$ . In particular the normalizer of  $P_i$  in  $W$  fixes  $\xi_i$ . It follows that  $H_i = \mathcal{N}_W(P_i)$ . Upon replacing  $P_i$  by a finite index subgroup, we may assume that each irreducible component of  $P_i$  is infinite. It then follows from Lemma 2.2 that  $\mathcal{N}_W(P_i) = H_i$  is itself parabolic.  $\square$

## 4.2 Relatively hyperbolic Coxeter groups

The aim of this section is the proof of Theorem A. We treat the different implications successively.

(ii)  $\Rightarrow$  (iii).

Follows from [HK05, Theorems A.0.1 and A.0.3].

(ii)  $\Rightarrow$  (i).

If (RH1) fails, then  $W$  contains a free abelian subgroup which is not contained in any element of  $\mathcal{P}$ . If (RH2) fails, then some infinite order element of  $W$  is contained in two distinct elements of  $\mathcal{P}$ . Therefore, relative hyperbolicity of  $W$  with respect to  $\mathcal{P}$  implies that (RH1) and (RH2) both hold.

(i)  $\Rightarrow$  (iii).

We start with a trivial observation. By condition (RH2), for each irreducible affine subset  $J \subset S$ , there is a unique  $J_0 \in \mathcal{T}$  containing  $J$ . Similarly for each irreducible non-spherical subset  $J \subset S$  such that  $J^\perp$  is non-spherical, there is a unique  $J_0 \in \mathcal{T}$  containing  $J$ .

For each  $J \in \mathcal{T}$ , we choose a residue of type  $J$  in  $X$ , which we denote by  $R_J$ . We define  $\mathcal{F}$  to be the set of all residues of the form  $w.R_J$  with  $J \in \mathcal{T}$  and  $w$  runs over a set of coset representatives of  $\mathcal{N}_W(W_J)$  in  $W$ . Note that  $\mathcal{F}$ , and hence  $\mathcal{T}$ , is non-empty unless  $W$  is Gromov hyperbolic; of course, we may and shall assume without loss of generality that  $W$  is not Gromov hyperbolic.

Let now  $F$  be a flat in  $X$ . Up to replacing it by a conjugate, we may assume without loss of generality that  $\text{Pc}(W(\mathcal{M}(F)))$  is standard. Let  $I \subset S$  be such that  $\text{Pc}(W(\mathcal{M}(F))) = W_I$  and  $I_0$  be the unique element of  $\mathcal{T}$  containing  $I$ . Let  $F_0 \in \mathcal{F}$  be the  $W_{I_0}$ -invariant residue belonging to  $\mathcal{F}$ . By Proposition 3.3, any wall  $m$  separating a point of  $F$  from its projection to  $F_0$  actually separates  $F$  from  $F_0$ ; furthermore the reflection  $r_m$  through  $m$  centralizes  $W_{I_0}$ . Let  $M$  denote the set consisting of all these walls.

Recall that  $W(\mathcal{M}(F)) = W_I$  is a parabolic subgroup which is a direct product of irreducible non-spherical subgroups. By (i) and Lemma 2.2, and since  $I_0$  is

the unique element of  $\mathcal{F}$  containing  $I$ , it follows that the centralizer of  $W_I$  is contained in the centralizer of  $W_{I_0}$ . Furthermore, by the definition of  $\mathcal{F}$ , the latter centralizer is a finite extension of  $W_{I_0} = W(\mathcal{M}(F_0))$ . Since the walls in  $M$  may not belong to  $\mathcal{M}(F_0)$ , it finally follows that  $W(M)$  centralizes  $W(\mathcal{M}(F_0))$ . Hence  $W(M)$  is finite, and so is  $\text{Pc}(W(M))$  by Lemma 2.1. In particular, the cardinality of  $M$  is bounded above by the maximal number of reflections in a finite standard parabolic subgroup. This shows that the combinatorial distance from any point  $x \in F$  to  $F_0$  is uniformly bounded. Therefore, condition (A) is satisfied.

Now we prove (B). Let  $F, F' \in \mathcal{F}$  be residues such that  $\mathcal{N}_r(F) \cap \mathcal{N}_r(F')$  is unbounded for some  $r > 0$ . Then the visual boundaries  $\partial_\infty(F)$  and  $\partial_\infty(F')$  have a common point. In other words, there exists a geodesic ray  $\rho \subset F$  and  $\rho' \subset F'$  such that  $\rho$  and  $\rho'$  are at bounded Hausdorff distance. Let  $\mathcal{M}(\rho)$  be the set of walls which separate two points of  $\rho$ . Since  $\mathcal{M}(\rho)$  is infinite whereas for any  $x \in \rho$ , the set of walls which separate  $x$  from  $\rho'$  is uniformly bounded, it follows that  $\mathcal{M}(\rho) \cap \mathcal{M}(\rho')$  is infinite. Therefore  $\mathcal{M}(\rho) \cap \mathcal{M}(\rho')$  contains two walls  $m, m'$  which do not meet [Cap06, Lemma 13].

Denote by  $P, P'$  the respective stabilizers of  $F, F'$  in  $W$ . Notice that  $P$  and  $P'$  are parabolic subgroups whose reflections consist of the sets  $\mathcal{M}(F)$  and  $\mathcal{M}(F')$  respectively. The preceding paragraph shows that  $P$  and  $P'$  share a common infinite dihedral subgroup. By (RH2), this implies that  $P$  and  $P'$  coincide. In view of the definition of  $\mathcal{F}$ , we deduce that  $F$  and  $F'$  must coincide.

This shows that for any two distinct  $F, F' \in \mathcal{F}$  and each  $r > 0$ , the set  $\mathcal{N}_r(F) \cap \mathcal{N}_r(F')$  is bounded. The fact that its diameter depends only on  $r$ , but not on the specific choice of  $F$  and  $F'$ , follows from the cocompactness of the  $W$ -action on  $X$ . Hence (B) holds.

(iv)  $\Rightarrow$  (iii).

Clear since the Davis complex  $X(W, S)$  is a (thin) building of type  $(W, S)$ .

(iii)  $\Rightarrow$  (iv).

Let  $Y$  be a building of type  $(W, S)$  and  $\mathcal{F}$  be the set of all  $J$ -residues of  $Y$  with  $J \in \mathcal{T}$ . Furthermore, given an apartment  $A$  in  $Y$ , we set

$$\mathcal{F}_A = \{A \cap F \mid F \in \mathcal{F}, A \cap F \neq \emptyset\}.$$

Since (iii) holds and since an apartment in  $Y$  is nothing but an isometrically embedded copy of the Davis complex  $X(W, S)$ , it follows that the elements of  $\mathcal{F}_A$  are isolated in  $A$ . Moreover, the constant  $D$  which appears in condition (A) depends only on  $(W, S)$ .

Let  $F$  be a flat in  $Y$ . Then  $F$  is contained in an apartment  $A$  by [CH06,

Theorem 6.3]. Therefore condition (A) holds for  $Y$  since the elements of  $\mathcal{F}_A$  are isolated in  $A$ .

Let now  $J, J' \in \mathcal{T}$  and  $F, F' \in \mathcal{F}$  be residues of type  $J$  and  $J'$  respectively. Assume that  $\mathcal{N}_r(F) \cap \mathcal{N}_r(F')$  is unbounded for some  $r > 0$ . Let  $A$  be an apartment contained a chamber  $c$  of  $F$  and let  $c'$  be the combinatorial projection of  $c$  onto  $F'$  (see [Tit74, §3.19]). We denote by  $\rho_{c,A}$  the combinatorial retraction of  $Y$  onto  $A$  centered at  $c$ . Recall that this maps any chamber  $x$  of  $Y$  to the unique chamber  $x'$  of  $A$  such that  $\delta_Y(c, x') = \delta_Y(c, x)$ , where  $\delta_Y: \text{Ch}(Y) \times \text{Ch}(Y) \rightarrow W$  denotes the Weyl distance.

By assumption, there exists an unbounded sequence  $c' = c'_0, c'_1, \dots$  of chambers of  $F'$  such that  $c'_n$  lies at uniformly bounded distance from  $F$ . Since combinatorial retractions do not increase distances and since  $\rho_{c,A}$  maps any chamber in  $F$  to a chamber in  $A \cap F$ , it follows that the sequence  $(x'_n)$  defined by  $x'_n = \rho_{c,A}(c'_n)$  lies at uniformly bounded distance from  $A \cap F$ . Furthermore, by a standard property of the combinatorial projection, namely the *gate property* (see [Tit74, Ch. 3]), for each chamber  $x' \in F'$  there exists a minimal gallery joining  $c$  to  $x'$  via  $c'$ . Therefore, it follows that for each  $n$ , the chamber  $x'_n$  lies in the  $J'$ -residue containing  $x'_0$ , say  $F''$ . This shows in particular that  $\mathcal{N}_r(A \cap F) \cap \mathcal{N}_r(A \cap F'')$  is unbounded. Since  $F''$  is a residue of type  $J'$  and since the residues whose type belong to  $\mathcal{T}$  are isolated in  $A$  by assumption, it follows that  $A \cap F = A \cap F''$  and hence  $F = F''$  and  $J = J'$ . In particular we obtain  $x'_0 \in F$  because  $c \in F$  and  $x'_0 \in F''$ . Since  $\delta_Y(c, x'_0) = \delta_Y(c, c')$ , we deduce that  $c' \in F$ , whence  $F = F'$  since  $J = J'$ .

This shows that for any two distinct  $F, F' \in \mathcal{F}$  and each  $r > 0$ , the set  $\mathcal{N}_r(F) \cap \mathcal{N}_r(F')$  is bounded. The fact that its diameter depends only on  $r$ , but not on the specific choice of  $F$  and  $F'$ , follows from the corresponding fact for apartments in  $Y$ . Hence (B) holds.  $\square$

*Proof of Corollary C.* It is immediate to check that the set  $\mathcal{T}$  satisfies (RH1) and (RH2).  $\square$

### 4.3 Isolated flats

**Lemma 4.2.** *The following conditions are equivalent:*

- (i) *The collection  $\mathcal{T}$  of maximal Euclidean subsets of  $S$  satisfies (RH1) and (RH2).*
- (ii) *For all non-spherical  $J_1, J_2 \subset S$  such that  $[J_1, J_2] = 1$ , the group  $\langle J_1 \cup J_2 \rangle$  is virtually abelian.*

(iii) For each minimal hyperbolic  $J \subset S$ , the set  $J^\perp$  is spherical.

*Proof.* The main point is that, given a Coxeter system  $(W, S)$ , it is well known the group  $W$  is virtually abelian if and only if it is a direct product of finite and affine Coxeter groups, i.e. if  $S$  is Euclidean (see e.g. [MV00]).

(i)  $\Rightarrow$  (iii).

Let  $\mathcal{T}$  be a collection of subsets of  $S$  satisfying (RH1) and (RH2). If (iii) fails, then there exist a minimal hyperbolic subset  $J$  and a non-spherical irreducible subset  $I \subset J^\perp$ . By (RH1) there exists  $K \in \mathcal{T}$  such that  $I \cup J \subset K$ . Then  $\langle K \rangle$  is not virtually abelian since it contains  $\langle J \rangle$ , hence (i) fails as well.

(iii)  $\Leftrightarrow$  (ii).

If (ii) fails then there exists a non-spherical and non-affine subset  $J \subset S$  such that  $J^\perp$  is non-spherical. Now any minimal non-spherical and non-affine subset  $I$  of  $J$  is minimal hyperbolic, and since  $I \subset J$  we have  $I^\perp \supset J^\perp$ . Thus (iii) fails as well.

(ii)  $\Rightarrow$  (i).

The condition (ii) clearly implies that for each irreducible non-spherical subset  $J$ , either  $J$  is affine and  $J \cup J^\perp$  is Euclidean or  $J$  is non-affine and  $J^\perp$  is spherical. In particular, every irreducible affine subset is contained in a *unique* maximal Euclidean subset. In other words the collection  $\mathcal{T}$  of maximal Euclidean subsets of  $S$  satisfies (RH2). Moreover (RH1) clearly holds as well.  $\square$

*Proof of Corollary D.* In view of Theorem A, Lemma 4.2 and [HK05, Th. 1.2.1], it is enough to prove that  $W$  is relatively hyperbolic with respect to its maximal virtually abelian subgroups if and only if it is relatively hyperbolic with respect to its maximal parabolic subgroups of Euclidean type. Since any parabolic subgroup of Euclidean type is virtually abelian, the ‘if’ part is clear. Conversely, assume that  $W$  is relatively hyperbolic with respect to its maximal virtually abelian subgroups. Then conditions (i) and (ii) hold. In view of [CM05, Prop. 3.2], this implies that the parabolic closure of any virtually abelian subgroup of rank  $\geq 2$  is of Euclidean type. In particular, if  $A < W$  is a maximal virtually abelian subgroup, then  $A = \text{Pc}(A)$ .  $\square$

*Proof of Corollary E.* Assertions (i) and (iii) follow from [HK05, Theorem A.0.1]. For (iv), one applies [DS05, Proposition 5.4]; one needs the fact that any asymptotic cone of a Euclidean building is itself a Euclidean building: this is established in [KL97, Theorem 1.2.1]. The fact that  $W$  is biautomatic follows from

[HK05, Theorem 1.2.2(5)]. The biautomaticity of  $\Gamma$  can then be deduced either directly from [Swi06] or from (i) together with [Reb01] and the fact that cocompact lattices of Euclidean buildings are biautomatic by [Swi06].  $\square$

## References

- [BH93] **B. Brink** and **R. B. Howlett**, A finiteness property and an automatic structure for Coxeter groups, *Math. Ann.* **296** (1993), 179–190.
- [Bou68] **N. Bourbaki**, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie*. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines. Actualités Scientifiques et Industrielles, No. **1337**, Hermann, Paris, 1968.
- [Bou97] **M. Bourdon**, Immeubles hyperboliques, dimension conforme et rigidité de Mostow, *Geom. Funct. Anal.* **7** (1997), 245–268.
- [Bow99] **B. H. Bowditch**, Relatively hyperbolic groups, *preprint*, available at <http://eprints.soton.ac.uk/29769/01/bhb-relhyp.pdf>, 1999.
- [Cap06] **P.-E. Caprace**, Conjugacy of 2-spherical subgroups of Coxeter groups and parallel walls, *Algebr. Geom. Topol.* **6** (2006), 1987–2029.
- [CF] **P.-E. Caprace** and **K. Fujiwara**, Rank-one isometries of buildings and quasi-morphisms of Kac-Moody groups, *Geom. Funct. Anal.* **19** (2010), 1296–1319.
- [CH06] **P.-E. Caprace** and **F. Haglund**, On geometric flats in the  $\text{CAT}(0)$  realization of Coxeter groups and Tits buildings, *Canad. J. Math.* **61** (2009), 740–761.
- [CM05] **P.-E. Caprace** and **B. Mühlherr**, Reflection triangles in Coxeter groups and biautomaticity, *J. Group Theory* **8** (2005), 467–489.
- [Dav98] **M. Davis**, Buildings are  $\text{CAT}(0)$ , in *Geometry and cohomology in group theory* (Durham, 1994) (Cambridge) (H. Kropholler et al., ed.), London Math. Soc. Lecture Note Ser. **252**, Cambridge Univ. Press, 1998, pp. 108–123.
- [Deo82] **V. V. Deodhar**, On the root system of a Coxeter group, *Comm. Algebra* **10** (1982), 611–630.



- [Deo89] ———, A note on subgroups generated by reflections in Coxeter groups, *Arch. Math. (Basel)* **53** (1989), 543–546.
- [DS05] **C. Drutu** and **M. Sapir**, Tree-graded spaces and asymptotic cones of groups, *Topology* **44** (2005), 959–1058.
- [Far98] **B. Farb**, Relatively hyperbolic groups, *Geom. Funct. Anal.* **8** (1998), 810–840.
- [HK05] **G. C. Hruska** and **B. Kleiner**, Hadamard spaces with isolated flats, *Geom. Topol.* **9** (2005), 1501–1538.
- [Hum90] **J. E. Humphreys**, *Reflection Groups and Coxeter Groups*, Cambridge Stud. Adv. Math., vol. **29**, Cambridge University Press, Cambridge, 1990.
- [KV06] **R. Kangaslampi** and **A. Vdovina**, Triangular hyperbolic buildings, *C. R. Math. Acad. Sci. Paris* **342** (2006), 125–128.
- [KL97] **B. Kleiner** and **B. Leeb**, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Publ. Math. Inst. Hautes Études Sci.* **86** (1997), 115–197.
- [Kra09] **D. Krammer**, The conjugacy problem for Coxeter groups, *Groups Geom. Dyn.* **3** (2009), 71–171.
- [MV00] **G. A. Margulis** and **È. B. Vinberg**, Some linear groups virtually having a free quotient, *J. Lie Theory* **10** (2000), 171–180.
- [Mou88] **G. Moussong**, *Hyperbolic Coxeter Groups*, Ph.D. thesis, Ohio State University, 1988.
- [Reb01] **D. Y. Rebbechi**, *Algorithmic Properties of Relatively Hyperbolic Groups*, Ph.D. thesis, Rutgers University, 2001, arXiv:math.GR/0302245.
- [Swi06] **J. Swiatkowski**, Regular path systems and (bi)automatic groups, *Geom. Dedicata* **118** (2006), 23–48.
- [Tit74] **J. Tits**, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math., Vol. **386**, Springer-Verlag, Berlin, 1974.

Pierre-Emmanuel Caprace

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, CHEMIN DU CYCLOTRON 2, 1348 LOUVAIN-LA-NEUVE, BELGIUM.

e-mail: pe.caprace@uclouvain.be