Innovations in Incidence Geometry
Volume 10 (2009), Pages 33–80
ISSN 1781-6475

From symmetric spaces to buildings, curve complexes and outer spaces

Lizhen Ji*

Abstract

In this article, we explain how spherical Tits buildings arise naturally and play a basic role in studying many questions about symmetric spaces and arithmetic groups, why Bruhat-Tits Euclidean buildings are needed for studying S-arithmetic groups, and how analogous simplicial complexes arise in other contexts and serve purposes similar to those of buildings.

We emphasize the close relationships between the following: (1) the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ of a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, (2) a parametrization by the simplices of $\Delta_{\mathbb{Q}}(\mathbf{G})$ of the boundary components of the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$ of the symmetric space X associated with $\mathbf{G}$, which gives the Borel-Serre compactification of the quotient of X by every arithmetic subgroup Γ of $\mathbf{G}(\mathbb{Q})$, (3) and a realization of $\overline{X}^{\mathrm{BS}}$ by a truncated submanifold X_T of X. We then explain similar results for the curve complex $\mathcal{C}(S)$ of a surface S, Teichmüller spaces T_g, truncated submanifolds $T_g(\varepsilon)$, and mapping class groups Mod_g of surfaces. Finally, we recall the outer automorphism groups $\mathrm{Out}(F_n)$ of free groups F_n and the outer spaces X_n, construct truncated outer spaces $X_n(\varepsilon)$, and introduce an infinite simplicial complex, called the core graph complex and denoted by $\mathcal{CG}(F_n)$, and we then parametrize boundary components of the truncated outer space $X_n(\varepsilon)$ by the simplices of the core graph complex $\mathcal{CG}(F_n)$. This latter result suggests that the core graph complex is a proper analogue of the spherical Tits building.

The ubiquity of such relationships between simplicial complexes and structures at infinity of natural spaces sheds a different kind of light on the importance of Tits buildings.

Keywords: Tits building, symmetric space, curve complex, outer space, arithmetic group, mapping class group, outer automorphism group of free group
MSC 2000: 20E42, 32G15, 20E06,14L30

*Partially supported by NSF grant DMS-0604878

Contents

1 Introduction

Symmetric spaces and Lie groups, in particular symmetric spaces of noncompact type and their quotients by arithmetic groups, real and p-adic semisimple Lie groups, are fundamental objects in many different subjects of mathematics such as differential geometry, topology, analysis, Lie group theory, algebraic geometry, and number theory. Both Tits buildings and Bruhat-Tits buildings arise naturally and play a crucial role in better understanding structures at infinity of these spaces, for example, the topology of the boundaries of compactifications of such spaces.

In this article, we try to illustrate this through some applications. For example, we show how the classification of geodesics in a symmetric space X of noncompact type naturally leads to the spherical Tits building $\Delta(G)$ of an associated semisimple Lie group G. We explain that the Borel-Serre compactification $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ of quotients $\Gamma\backslash X$ of X by arithmetic subgroups Γ, or rather the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$ of X, is related to the Tits building

$\Delta_{\mathbb{Q}}(\mathbf{G})$ of an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ [11]. We also explain that the Solomon-Tits theorem on the homotopy type of $\Delta_{\mathbb{Q}}(\mathbf{G})$ can be used to show that arithmetic subgroups Γ are virtual duality groups, but not virtual Poincaré duality groups if $\Gamma\backslash X$ are not compact [11]. We discuss a realization of the partial compactification $\overline{X}^{\mathrm{BS}}$ by a truncated submanifold X_T of X [40, 35], which is convenient for many applications and generalizations. Furthermore, we use the Borel-Serre compactification $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ and its relationship to the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ to determine completely the ends of locally symmetric spaces $\Gamma\backslash X$ (Proposition 4.7).

S-arithmetic subgroups such as $\mathsf{GL}(n, \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_k}])$, where $p_1, \dots, p_k$ are prime numbers, are natural generalizations of arithmetic subgroups such as $\mathsf{GL}(n, \mathbb{Z})$. We explain how Bruhat-Tits buildings are naturally needed in order to study S-arithmetic subgroups. A Bruhat-Tits building can be compactified by adding a corresponding spherical Tits building, and this compactification was used to prove that S-arithmetic subgroups of semisimple linear algebraic groups are duality groups [12].

More importantly, we also try to explain and emphasize in this article the following basic point, in a variety of settings, that *natural simplicial complexes can be used to understand geometry and structures at infinity of noncompact spaces*, as in the above case of buildings for symmetric spaces and locally symmetric spaces.

One setting is that of the Teichmüller space T_g associated with a closed, orientable surface $S = S_g$ of genus $g \geq 2$. Its structure at infinity, in particular the boundary of a Borel-Serre type partial compactificattion $\overline{T_g}^{\mathrm{BS}}$ can be described in terms of an infinite simplicial complex, called the *curve complex* and denoted by $\mathcal{C}(S)$ [21]. Together with an analogue of the Solomon-Tits Theorem for the curve complex $\mathcal{C}(S)$, it can be shown the mapping class group Mod_g is a virtual duality group [20] but not a virtual Poincaré duality group [24]. The partial compactification $\overline{T_g}^{\mathrm{BS}}$ can also be realized by a truncated submanifold $T_g(\varepsilon)$, whose boundary components can be easily seen to be parametrized by the simplices of the curve complex $\mathcal{C}(S)$.

Another setting is that of the outer automorphism group $\mathrm{Out}(F_n)$ of a free group F_n on n-generators. It acts properly on the associated outer space X_n of marked metric graphs with fundamental group equal to F_n [17]. In order to prove that $\mathrm{Out}(F_n)$ is a virtual duality group, a Borel-Serre type partial compactification $\overline{X_n}^{\mathrm{BS}}$ of X_n was constructed [8]. An infinite simplicial complex, called the *free factor complex* and which is similar to a Tits building was constructed for F_n, and an analogue of the Solomon-Tits theorem was also proved in [22]. On the other hand, the expected relationship between the free factor complex and the boundary of $\overline{X_n}^{\mathrm{BS}}$ was not clear. In the last section of this paper, we (1) construct another simplicial complex, called the *core graph complex* and denoted by $\mathcal{CG}(F_n)$, which is similar to the curve complex in its role in describing the geometry at infinity of X_n, and which can be identified with the free factor complex, (2) give a realization of $\overline{X_n}^{\mathrm{BS}}$ in terms of a truncated subspace $X_n(\varepsilon)$, (3) decompose the boundary of $X_n(\varepsilon)$ and hence also of $\overline{X_n}^{\mathrm{BS}}$ into components, which are parametrized by the simplices of the core graph complex $\mathcal{CG}(F_n)$. Since the quotient of X_n by $\mathrm{Out}(F_n)$ is non-compact, we also mention the expected result that $\mathrm{Out}(F_n)$ is not a virtual Poincaré duality group (Theorem 9.2), motivated by the results for non-uniform arithmetic subgroups and mapping class groups.

The basic arrangement of topics in this paper can be seen from the table of contents at the beginning. For each section, there is also a summary at the beginning of results discussed in that section. For more applications of buildings in geometry and topology, see the survey [25]. Though there is some overlap with the survey paper [25], the present paper is complementary. In fact, it can be considered as a supplement (or a sequel) to [25]. In this paper, arithmetic subgroups of algebraic groups have played an important role and motivated results for natural generalizations such as mapping class groups and outer automorphism groups of free groups. For more results and references about arithmetic groups and other generalizations, see the book [27].

Acknowledgments. This paper is based on a lecture by the author at the conference *Buildings and Groups* held in Ghent, Belgium, in May 2007. I would like to thank the organizers for the invitation to this exciting conference and their hospitality, and the referee for carefully reading an earlier version of this paper and for making many constructive suggestions. I would also like to thank R. Spatzier for pointing out references on the rank rigidity problem of Hadamard manifolds. Finally I would like to thank Diane M. Vavrichek for reading the whole paper carefully.

2 Symmetric spaces and the geodesic compactifications

In this section, we first recall the definition of symmetric spaces and their relationships with Lie groups. Then we define the geodesic compactification $X \cup X(\infty)$ of a symmetric space X of noncompact type, and explain relationships between the boundary points in $X(\infty)$ and the parabolic subgroups of G. We will then be able, in the next section, to introduce the geometric realization of the spherical Tits building $\Delta(G)$.

By definition, a complete Riemannian manifold M is called a *symmetric space* if for every point $x \in M$, the geodesic symmetry s_x, which reverses every geodesic passing through x, is defined locally and gives a local diffeomorphism of M at x, is a *global isometry* of M.

Let G be the identity component of the isometry group $\mathrm{Isom}(M)$ of M. Then G acts transitively on X. For any base point $x \in X$, denote the stabilizer of x in G by K. Then X can be identified with the homogeneous space G/K, and the Riemannian metric on X is equal to a G-invariant metric induced from an inner product on the tangent space T_xX.

The Euclidean spaces $\mathbb{R}^n$ with the standard metric and their quotients by discrete isometry groups are symmetric spaces. Besides these flat symmetric spaces, irreducible symmetric spaces are classified into two types: compact and non-compact types.

In terms of curvature, a symmetric space of compact type has nonnegative sectional curvature and strictly positive Ricci curvature, and is compact. Furthermore, its fundamental group is finite.

On the other hand, a symmetric space of non-compact type has nonpositive sectional curvature and strictly negative Ricci curvature. It is also simply connected. Hence it is a so-called *Hadamard manifold*, and in particular it is diffeomorphic to its tangent space T_xX (more precisely, the exponential map $\exp_x : T_xX \to X$ is a diffeomorphism.)

In this paper, we are mainly concerned with symmetric spaces of noncompact type. In terms of the identification $X = G/K$ above, the Lie group $G = \mathrm{Isom}(X)^0$ is a connected semisimple noncompact Lie group with trivial center, and the stabilizer K is a maximal compact subgroup of G.

Conversely, for any semisimple Lie group G with finitely many connected components and finite center, let K be a maximal compact subgroup of G, which is unique up to conjugation by elements of G. Then the homogeneous space $X = G/K$ with a G-invariant metric is a Riemannian symmetric space of noncompact type.

The main example of symmetric spaces for us arises as follows. Let $\mathbf{G} \subset \mathsf{GL}(n, \mathbb{C})$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. Then the real locus $G = \mathbf{G}(\mathbb{R})$ is a real semisimple Lie group with finitely many connected components and finite center. Note that even if $\mathbf{G}$ is connected, G is not necessarily connected.

Besides symmetric spaces X of noncompact type, we will also study their quotients $\Gamma\backslash X$ by discrete isometry groups Γ. We will concentrate on the case that $\Gamma\backslash X$ have finite volume with respect to the measure induced from the invariant metric. In terms of the identification $X = G/K$, Γ is a discrete subgroup of G such that the quotient $\Gamma\backslash G$ has finite volume with respect to any Haar measure of G, i.e., Γ is a *lattice subgroup* of G. If $G = \mathbf{G}(\mathbb{R})$ is the real locus of $\mathbb{Q}$-algebraic group as above, then an important class of lattice subgroups consists of arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$.

Besides the above connections with Lie groups and algebraic groups, symmetric spaces X and locally symmetric spaces $\Gamma\backslash X$ are important in other fields such as harmonic analysis, differential geometry, algebraic geometry, number theory, and rigidity theory. For example, the Poincaré upper half plane $\mathbb{H}^2$, i.e., the simply connected surface with constant negative sectional curvature -1, is the symmetric space associated with $G = \mathsf{SL}(2, \mathbb{R})$ and $K = \mathsf{SO}(2)$, and the quotient $\mathsf{SL}(2, \mathbb{Z})\backslash\mathbb{H}^2$ is the moduli space of all elliptic curves.

Understanding the geometry of X has led to many important notions, and relationships between the geometry of X and group theoretic structures of G have played an important role in many problems. One important notion is the notion of maximal flat subspaces, often called *maximal flats* or simply *flats* in the following. By definition, a *maximal flat subspace* of X is a maximal totally geodesic flat subspace. It is known that all maximal flats of X are isometric to each other, and their common dimension is called the *rank* of X. Since every geodesic in X is a totally geodesic flat subspace of dimension 1, the rank of X is at least 1.

If $X = G/K$ and $G = \mathbf{G}(\mathbb{R})$ is the real locus of a semisimple linear algebraic group defined over $\mathbb{Q}$ and hence over $\mathbb{R}$, then the rank of X is equal to the $\mathbb{R}$-rank of $\mathbf{G}$ as an algebraic group.

The geodesic compactification

Since X is non-compact, a natural problem is to understand its geometry at infinity. One way to understand its geometry at infinity is to study its compactifications and their relationship to its geometry. One such compactification is the geodesic compactification $X \cup X(\infty)$ which is obtained by adding as the boundary the set of equivalence classes of directed geodesics.

More specifically, two unit speed directed geodesics $\gamma_1(t), \gamma_2(t)$ of X are called *equivalent* if

$$\limsup_{t \to +\infty} d(\gamma_1(t), \gamma_2(t)) < +\infty.$$

Denote the equivalence class containing a geodesic γ by $[\gamma]$, and the set of all equivalence classes of unit speed geodesics in X by $X(\infty)$.

Since X is simply connected and nonpositively curved, it can be shown that for any base point $x \in X$, every equivalence class of geodesics contains exactly one geodesic passing through x. Therefore, $X(\infty)$ can be identified with the unit sphere of the tangent space T_xX and hence $X(\infty)$ is called the *sphere at infinity*.

The sphere at infinity $X(\infty)$ can be attached to the space X to form the space $X \cup X(\infty)$ with the following topology:

(1) it restricts to the original topology of X,

(2) it restricts to the topology of $X(\infty)$ when $X(\infty)$ is identified with the unit sphere in the tangent space T_xX,

(3) a sequence $x_j \in X$ converges to a boundary point represented by an equivalence class $[\gamma]$ if and only if, for a base point $x \in X$, the distance $d(x_j, x_0) \to +\infty$, and the unit speed geodesic passing through x and x_j and pointing towards x_j converges to a geodesic in the equivalence class $[\gamma]$.

It can be shown easily that this topology on $X \cup X(\infty)$ is independent of the choice of the basepoint $x \in X$ in (3) and the space $X \cup X(\infty)$ is homeomorphic to the closed unit ball in T_xX. This compactification is called the *geodesic compactification*.

Since the isometric action of G on X preserves the equivalence relation on geodesics, it extends to a continuous action of G on the compactification $X \cup X(\infty)$.

A natural problem is to understand relationship between the boundary $X(\infty)$ and structures of G and X.

It is known that for any point in X, its stabilizer in G is a maximal compact subgroup. Conversely, by a well-known theorem of Cartan, every compact subgroup of G has at least one fixed point in X. It is also known that every maximal compact subgroup of G has *exactly one* fixed point in X. Therefore, X can be identified with the set of all maximal compact subgroups of G, and every two maximal compact subgroups of G are conjugate.

The stabilizers of the boundary points in $X(\infty)$ are described as follows.

Proposition 2.1. *For every point $z \in X(\infty)$, its stabilizer in G is a proper parabolic subgroup of G. Conversely, every proper parabolic subgroup P of G fixes at least one point in $X(\infty)$.*

This gives a classification of geodesics of X, or rather their equivalence classes, in terms of parabolic subgroups of G, whose structures are well developed and understood. On the other hand, we can also directly classify geodesics in terms of geometry. Under the above correspondence, a generic geodesic, i.e., a geodesic contained in a unique flat, corresponds to a minimal parabolic subgroup of G. See [18] for details.

For every proper parabolic subgroup P of G, denote by σ_P the set of points of $X(\infty)$ fixed by P. In general, σ_P consists of more than one point. Then, for every pair of parabolic subgroups P_1 and P_2,

$$P_1 \subseteq P_2 \text{ if and only if } \sigma_{P_1} \supseteq \sigma_{P_2}.$$

It is also known that, for every two parabolic subgroups P_1 and P_2, if $\sigma_{P_1} \cap \sigma_{P_2} \neq \emptyset$, then either $\sigma_{P_1} \subseteq \sigma_{P_2}$ or $\sigma_{P_2} \subseteq \sigma_{P_1}$. In particular, $X(\infty)$ admits a decomposition into these subsets σ_P, where P ranges over all proper parabolic subgroups of G.

Proposition 2.2. *For every proper parabolic subgroup P, σ_P is a simplex, and the decomposition*

$$X(\infty) = \bigcup_P \sigma_P, \text{ where } P \text{ ranges over all proper parabolic subgroups of } G,$$

gives $X(\infty)$ the structure of an infinite simplicial complex.

For each P, the interior of the simplex σ_P, i.e., the open simplex whose closure is equal to σ_P, is the set of points in $X(\infty)$ whose stabilizers are *exactly* equal to P. See [4, 18, 10] for proofs of these statements.

Another important property of the action of G near the infinity $X(\infty)$ of the compactification $X \cup X(\infty)$ is the following.

Proposition 2.3. *If $\Gamma \subset G$ is a uniform lattice subgroup, i.e., Γ is a discrete subgroup with a compact quotient $\Gamma\backslash G$, then the action of Γ on $X \cup X(\infty)$ is small near infinity in the following sense. For every boundary point $z \in X(\infty)$ and every compact subset $C \subset X$, for any neighborhood V of z in $X \cup X(\infty)$, there exists another neighborhood U of z in $X \cup X(\infty)$ such that for any $\gamma \in \Gamma$, if $\gamma C \cap U \neq \emptyset$, then $\gamma C \subset V$.*

The smallness of the action at infinity says roughly that translates γC are shrunk to points near $X(\infty)$. In particular, if a sequence x_j of X converges to a point $z \in X(\infty)$, then any other sequence x'_j within bounded distance of the sequence x_j, i.e., $\limsup_j d(x_j, x'_j) < +\infty$, also converges to the same point z.

The geodesic compactification $X \cup X(\infty)$ has applications to various problems. For example, the above proposition can be used to prove the integral Novikov conjecture of Γ (see §6 below).

The Dirichlet problem on $X \cup X(\infty)$

The construction of the geodesic compactification $X \cup X(\infty)$ works for any Hadamard manifold. An important problem for Hadamard manifolds is the solvability of the Dirichlet problem on $X \cup X(\infty)$: given a continuous function f on $X(\infty)$, find a harmonic function u on X with boundary value equal to f, i.e., u can be extended continuously to the boundary $X(\infty)$ and $u|_{X(\infty)} = f$.

When the Hadamard manifold is given by a simply connected Riemannian manifold with sectional curvature pinched by two negative numbers, this Dirichlet problem is solvable (see [1] and the references therein). For more general rank 1 Hadamard manifolds which admit compact quotients, the Dirichlet problem is also solvable [3] (see also [5] for the identification of the Poisson boundary of X with $X(\infty)$). (Note that in this case, the rank of a Hadamard manifold M is defined to the minimum of the dimension of vector spaces of parallel Jacobi fields along geodesics in M.)

On the other hand, if X is a symmetric space of noncompact type and of rank at least 2, then the above Dirichlet problem is not solvable. The basic reason is that the Poisson boundary of X can be identified with a proper subset of $X(\infty)$. For example, consider $X = \mathbb{H}^2 \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the Poincaré upper half plane. Then the rank of X is equal to 2. Suppose that the Dirichlet problem is solvable on $X \cup X(\infty)$. Then, for every continuous function f on $X(\infty)$, we obtain a bounded harmonic function u_f on X with the boundary value f. Now every bounded harmonic function $u_f(x, y)$ on the product $\mathbb{H}^2 \times \mathbb{H}^2$ splits as a product $u_1(x)u_2(y)$, where u_1, u_2 are bounded harmonic functions on $\mathbb{H}^2$. (Note that this splitting follows from the result that the Poisson boundary of X is given by the distinguished boundary $\mathbb{H}^2(\infty) \times \mathbb{H}^2(\infty)$. See [18] for references on the Poisson boundary of symmetric spaces of noncompact type.) If a sequence x_j goes to a boundary point in $\mathbb{H}^2(\infty)$ but y is fixed, then (x_j, y) will converge to a boundary point in $X(\infty)$ independent of y. Then the assumption that the boundary value of $u_f(x, y) = u_1(x)u_2(y)$ is equal to f implies that $u_2(y)$ must be independent of y if $\lim_{j\to+\infty} u_1(x_j) \neq 0$. Similarly, we can conclude that if a sequence y_j goes to a boundary point and $\lim_{j\to+\infty} u_2(y_j) \neq 0$, then $u_1(x)$ is constant. This implies that $u(x, y)$ is a constant function. This clearly contradicts the condition that the boundary value of u_f is equal to f if f is not constant. Therefore, in this case, the Dirichlet problem is not solvable for any non-constant continuous function f on $X(\infty)$.

It is perhaps also worthwhile to point out that it is still an open problem if a general Hadamard manifold of higher rank that is not a product is a symmetric space, though the answer is positive if it admits a finite volume quotient. See [41, 2] for summaries and references of this rank rigidity result and related generalizations and open problems.

3 Tits buildings and their geometric realizations

In this section, we recall briefly the notion of spherical buildings and the construction of spherical Tits buildings associated with semisimple real Lie groups. Then we explain a geometric realization of the Tits buildings in terms of the simplicial structures on the boundaries of the geodesic compactifications and their applications to the Mostow strong rigidity of locally symmetric spaces. We also explain how to use Tits buildings to parametrize boundary components of other compactifications of symmetric spaces. Then we recall the Tits building of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and its relationship with the Tits building of the Lie group $G = \mathbf{G}(\mathbb{R})$.

In the study of a semisimple real Lie group G, an important role is played by a finite group, the *Weyl group* W. Briefly, let $G = NAK$ be the Iwasawa decomposition of G, let $\mathcal{N}_K(A)$ be the normalizer of A in K and let $\mathcal{Z}_K(A)$ be the centralizer of A in K. Then the quotient $\mathcal{N}_K(A)/\mathcal{Z}_K(A)$ is the Weyl group W of G. It is known that W is generated by elements of order 2 , which correspond to reflections in flats in X with respect to the so-called root hyperplanes.

Let $\mathfrak{a}$ be the Lie algebra of A. Then $\mathcal{N}_K(A)$ and $\mathcal{Z}_K(A)$ can be canonically identified with the normalizer and centralizer of $\mathfrak{a}$ in K respectively.

The Killing form of $\mathfrak{g}$ restricts to an inner product on $\mathfrak{a}$, and W acts isometrically on $\mathfrak{a}$. The roots of the Lie algebra $\mathfrak{g}$ of G with respect to $\mathfrak{a}$ are linear functionals on $\mathfrak{a}$. Each root defines a root hyperplane, and the collection of these hyperplanes is invariant under W.

It is known that the reflections with respect to these root hyperplanes generate the Weyl group W. The Weyl group is an important example of a finite Coxeter group.

There is a finite complex naturally associated to the action of W on $\mathfrak{a}$. In fact, the root hyperplanes and their intersections give a decomposition of $\mathfrak{a}$ into simplicial cones, i.e., the Weyl chambers and Weyl chamber faces. This decomposition is called the *Weyl chamber decomposition* of $\mathfrak{a}$. The intersection of these simplicial cones with the unit sphere of $\mathfrak{a}$ gives a finite simplicial complex, which is called the *Coxeter complex associated with the Weyl group* and its underlying topological space is the unit sphere.

In general, for every finite group Coxeter group, there is associated a finite simplicial complex, called the *Coxeter complex* of the Coxeter group.

With these preparations, we are ready to define spherical Tits buildings.

Definition 3.1. A simplicial complex Δ is called *a spherical Tits building* if it contains a family of sub-complexes called apartments, which satisfies the following conditions:

(1) Every apartment is a finite Coxeter complex.

(2) Any two simplices are contained in some apartment.

(3) Given two apartments Σ and Σ' and simplices $\sigma, \sigma' \in \Sigma \cap \Sigma'$, there exists an isomorphism of Σ onto Σ' which keeps σ, σ' pointwise fixed.

The condition (3) implies that there is a common finite Coxeter group whose complex is isomorphic to every apartment. Since the Coxeter complex and hence the apartments are triangulations of spheres, the building is called spherical.

To construct a Tits building Δ, we can proceed as follows:

(1) Construct a simplicial complex.

(2) Pick out a collection of subcomplexes and show that they satisfy the conditions for apartments.

Tits building of a semisimple Lie group

Let G be a semisimple Lie group. Define a simplicial complex $\Delta(G)$ as follows:

(1) The simplices of $\Delta(G)$ are parametrized by the proper parabolic subgroups of G. For each parabolic subgroup P, denote the corresponding simplex by σ_P.

(2) The vertices (or rather simplices of dimension 0) of $\Delta(G)$ are parametrized by the maximal proper parabolic subgroups of G.

(3) Let $P_1, \ldots, P_k$ be distinct maximal proper parabolic subgroups of G. Then their corresponding simplices form the vertices of a $(k-1)$-simplex σ_P if and only if the intersection $P_1 \cap \cdots \cap P_k$ is the parabolic subgroup P.

Since G has uncountably infinitely many parabolic subgroups, $\Delta(G)$ is an infinite simplicial complex. If the rank of G is equal to 1, then every proper parabolic subgroup of G is maximal, and hence $\Delta(G)$ is a disjoint union of uncountably many points. For example, when $G = \mathsf{SL}(2, \mathbb{R})$, the building $\Delta(G)$ can be identified with the circle S^1 with the discrete topology, in which every apartment consists of a pair of distinct points.

On the other hand, if the rank of G is at least 2, then once $\Delta(G)$ is shown to be a spherical building, it is connected. In fact, every apartment is connected and every two simplices of $\Delta(G)$ are contained in an apartment, so in particular, every two vertices are connected by an apartment.

A natural collection of apartments of $\Delta(G)$ can be constructed as follows. For every maximal compact subgroup K of G, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. For every maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, let $A = \exp \mathfrak{a}$ be the corresponding subgroup.

It is known that there are only finitely many parabolic subgroups $P_1, \dots, P_l$ containing A, and they correspond to the Weyl chambers in the decomposition of $\mathfrak{a}$ described above. Therefore, the subcomplex of $\Delta(G)$ consisting of the simplices $\sigma_{P_1}, \dots, \sigma_{P_l}$ is isomorphic to the Coxeter complex of the Weyl group W of G. This subcomplex is defined to be an apartment of $\Delta(G)$.

By changing the choices of K and A, we obtain the desired collection of apartments. Since all such subgroups K and A are conjugate in G, all these apartments are isomorphic.

By using the Bruhat decomposition, it can be shown that, for every two parabolic subgroups P_1, P_2, there exists a maximal compact subgroup K and a corresponding subgroup $A = \exp \mathfrak{a}$ as above such that $A \subset P_1, P_2$. It is important to note that the maximal compact subgroup K will change and depend on the pair of parabolic subgroups P_1, P_2.

Combining the above discusssions, it can be shown that these apartments satisfy the conditions in Definition 3.1 above and that $\Delta(G)$ is a spherical Tits building. See [14, 18, 42] for more details.

A geometric realization of Tits buildings and its applications

After defining the Tits building $\Delta(G)$ of G, we now relate it to the geometry of the symmetric space X.

For $K \subset G$, let $x \in X$ be the corresponding point. First, it is known that, for every maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ as above, the orbit of $A = \exp \mathfrak{a}$ through the point x, $A \cdot x$, is a flat of X. Conversely, every flat of X containing the basepoint x is of this form $A \cdot x$. To get other flats of X, we need to vary the maximal compact subgroup K.

The closure $\overline{A \cdot x}$ of the flat $A \cdot x$ in the geodesic compactification $X \cup X(\infty)$ is homeomorphic to the closed unit ball in $\mathfrak{a}$, and the Weyl chamber decomposition of $\mathfrak{a}$ induces a simplicial complex decomposition of the boundary $\partial \overline{A \cdot x}$ which can be naturally identified with the Coxeter complex of the Weyl group W of G.

Using this we can obtain the following geometric realization of $\Delta(G)$ (see [18, Proposition 3.20]).

Proposition 3.2. *The simplicial complex coming from the decomposition of $X(\infty)$ into cells σ_P in Proposition* 2.2 *is isomorphic to the spherical Tits building $\Delta(G)$*

of G. In particular, the underlying topological space of $\Delta(G)$ can be identified with the sphere at infinity $X(\infty)$.

In the above proposition, the simplex σ_P is exactly the set of points $X(\infty)$ fixed by the parabolic subgroup P.

One major application of this identification is the celebrated Mostow strong rigidity theorem for locally symmetric spaces in [36].

Theorem 3.3. *Let $\Gamma\backslash X$ and $\Gamma'\backslash X'$ be two compact irreducible locally symmetric spaces of noncompact type. (A locally symmetric space is said to be irreducible if it does not admit any finite cover which splits as a product.) Assume that X is not equal to the Poincaré upper half plane $\mathbb{H}^2$. If $\Gamma\backslash X$ and $\Gamma'\backslash X'$ are homotopic, or equivalently, they have the same fundamental group, i.e., $\Gamma \cong \Gamma'$, then $\Gamma\backslash X$ and $\Gamma'\backslash X'$ are isometric after scaling the metrics on irreducible factors of X and X'.*

The proof is divided into two cases depending on whether the rank of G is equal to 1 or not. The proof of the higher rank case makes use of Tits buildings, and the basic idea is as follows.

A homotopy equivalence $\varphi\colon \Gamma\backslash X \to \Gamma'\backslash X'$ induces an equivariant quasi-isometry between X and X'. Since the actions of Γ and Γ' on the geodesic compactifications $X \cup X(\infty)$ and $X' \cup X'(\infty)$ respectively are small at infinity, it is reasonable to expect that the map φ induces a well-defined map $\varphi_\infty\colon X(\infty) \to X'(\infty)$. (Note that though φ is not uniquely defined, the smallness of the action allows us to remove the ambiguity.) In fact, this is true but is the most difficult part of the whole proof, and the proof makes essential use of the realization in Proposition 3.2 of the spherical Tits building $\Delta(G)$ in terms of the simplicial structure on the sphere at infinity $X(\infty)$. As a consequence, it follows that the map φ_∞ induces an isomorphism of the spherical Tits buildings $\Delta(G)$ and $\Delta(G')$. Using the rigidity of spherical Tits buildings in [42], it follows that $\Gamma\backslash X$ and $\Gamma'\backslash X'$ are isometric to each other after scaling. See [25] for a more detailed outline.

Remark 3.4. As emphasized in [25], the identification in Proposition 3.2 naturally gives an enhanced Tits building, the so-called topological Tits building. In the proof of the Mostow strong rigidity, this notion of topological Tits buildings is used implicitly and plays an important role.

Remark 3.5. In [36], the Mostow strong rigidity theorem was only proved for compact locally symmetric spaces as stated above. It also holds for noncompact but finite volume locally symmetric spaces and was proved later by Margulis when the rank of the symmetric spaces is at least 2, and by Prasad when the rank of the locally symmetric spaces is equal to 1. For a summary of the history of rigidity properties of locally symmetric spaces and other related results, see [28].

Remark 3.6. A closely related Mostow strong rigidity result for lattices in non-archimedean (i.e., p-adic) linear semisimple algebraic groups was proved by Prasad in [39]. The proof also uses similar ideas by replacing the symmetric spaces of real semisimple Lie groups with the Bruhat-Tits buildings of linear semisimple algebraic groups over non-archimedean fields. It might be also worthwhile to point out that rigidity of lattices of p-adic semisimple Lie groups was first considered in [39]. A significant generalization of the Mostow strong rigidity was given by Kleiner and Leeb in [32] on quasi-isometry rigidity of symmetric spaces and Euclidean buildings. In both these papers, the Tits buildings appearing at the infinity have also played an important role.

For other generalizations of the Mostow strong rigidity and a characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic (or Tits) geometry, see the paper of Leeb [34].

Relationships similar to that in Proposition 3.2 between Tits buildings (or equivalently parabolic subgroups) and boundaries of compactifications can also be established for other compactifications of symmetric spaces. The importance of such relationships is that parabolic subgroups can be used to understand the structure at infinity of X. For a systematical discussion, see the books [10, 18].

For a symmetric space X of non-compact type, there are finitely many non-isomorphic Satake compactifications. They are partially ordered, and there is a unique maximal one, called the *maximal Satake compactification* and denoted by $\overline{X}^S_{\max}$.

The maximal Satake compactification $\overline{X}^S_{\max}$ is important for many applications. For example, it contains the maximal Furstenberg boundary $G/P_{\min}$, where $P_{\min}$ is a minimal parabolic subgroup of G, in its boundary $\partial\overline{X}^S_{\max}$, which parametrizes the set of chambers, i.e., simplices of top dimension, of $\Delta(G)$. In the proof of the Mostow strong rigidity theorem outlined above, to show that the map φ_∞ induces an isomorphism between $\Delta(G)$ and $\Delta(G')$, the starting pointing is to prove that φ_∞ maps the maximal Furstenberg boundary of X to the maximal Furstenberg boundary of X'.

The compactification $\overline{X}^S_{\max}$ is also important in determining the Martin compactification of X, which deals with structures of the cone of positive eigenfunctions on X, for example, how to represent a general positive eigenfunction as a superposition of extremal ones of the cone (see [18]).

The boundary of $\overline{X}^S_{\max}$ can be described as follows. With respect to a fixed maximal compact subgroup K, every proper parabolic subgroup P of G admits a Langlands decomposition:

$$P = N_P A_P M_P \cong N_P \times A_P \times M_P\,. \tag{1}$$

Then $K_P = M_P \cap K$ is a maximal compact subgroup of M_P, and

$$X_P = M_P / K_P$$

is a symmetric space of lower dimension and is called the *boundary symmetric space at infinity* associated with P. Then the Langlands decomposition of P induces the *horospherical decomposition* of X with respect to P:

$$X = N_P \times A_P \times X_P\,,$$

which generalizes the familiar x, y coordinates of the Poincaré upper half plane $\mathbb{H}^2$.

For every proper parabolic subgroup P, its boundary symmetric space X_P is canonically contained in the boundary $\partial \overline{X}^S_{\max}$, and the following disjoint decomposition holds:

$$\partial \overline{X}^S_{\max} = \coprod_P X_P\,,$$

where P runs over all proper parabolic subgroups of G.

In the compactification $\overline{X}^S_{\max}$, for every two parabolic subgroups P_1, P_2, X_{P_1} is contained in the closure of X_{P_2} if and only if P_1 is contained in P_2. Since each X_P is diffeomorphic to some Euclidean space, the boundary $\partial \overline{X}^S_{\max}$ is a cell complex dual to the Tits building $\Delta(G)$. See [18] for more discussion about this duality.

For both the geodesic compactification $X \cup X(\infty)$ and the maximal Satake compactification $\overline{X}^S_{\max}$, the boundary can be decomposed into boundary components parametrized by parabolic subgroups. It is naturally expected that similar relations hold for other compactifications of X. See [10] for more details.

The Tits building of a semisimple linear algebraic group defined over $\mathbb{Q}$

As mentioned before, given a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, its real locus $G = \mathbf{G}(\mathbb{R})$ is a semisimple Lie group with finitely many connected components and finite center. Therefore, there is naturally a spherical Tits building $\Delta(G)$ associated with G.

This building is important for understanding the geometry at infinity of the symmetric space X associated with G. However, to study the geometry at infinity of quotients of X by arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$, we need another spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{G}$ which also encodes the fact that $\mathbf{G}$ is defined over $\mathbb{Q}$.

Assume that $\mathbf{G}$ contains at least one proper parabolic subgroup defined over $\mathbb{Q}$, which is equivalent to assuming that the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, which is in turn equivalent to assuming that quotients of X by arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$ are noncompact (see Proposition 4.4 below). Otherwise, the building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is empty. Under this assumption, the definition of $\Delta_{\mathbb{Q}}(\mathbf{G})$ is similar to that of $\Delta(G)$ by considering only parabolic subgroups of $\mathbf{G}$ defined over $\mathbb{Q}$, which are often called $\mathbb{Q}$-parabolic subgroups.

Specifically, $\Delta_{\mathbb{Q}}(\mathbf{G})$ is an infinite simplicial complex satisfying the following properties:

(1) The simplices of $\Delta_{\mathbb{Q}}(\mathbf{G})$ are parametrized by the set of all proper $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$. For each $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, denote the corresponding simplex by $\sigma_{\mathbf{P}}$.

(2) The vertices (i.e. simplices of dimension 0) of $\Delta_{\mathbb{Q}}(\mathbf{G})$ are parametrized by the maximal proper $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$.

(3) Let $\mathbf{P}_1, \dots, \mathbf{P}_k$ be distinct maximal proper parabolic subgroups of $\mathbf{G}$. Then their corresponding zero dimensional simplices form the vertices of a $(k-1)$-simplex $\sigma_{\mathbf{P}}$ if and only if the intersection $\mathbf{P}_1 \cap \cdots \cap \mathbf{P}_k$ is equal to the parabolic subgroup $\mathbf{P}$.

In addition, a natural collection of apartments in $\Delta_{\mathbb{Q}}(\mathbf{G})$ can be described in a manner similar to the description of the apartments of $\Delta(G)$. Briefly, for every maximal $\mathbb{Q}$-split torus $\mathbf{T}$ of $\mathbf{G}$, there are only finitely many $\mathbb{Q}$-parabolic subgroups $\mathbf{P}_1, \dots, \mathbf{P}_m$ containing $\mathbf{T}$, and their simplices $\sigma_{\mathbf{P}_1}, \dots, \sigma_{\mathbf{P}_m}$ form an apartment of $\Delta_{\mathbb{Q}}(\mathbf{G})$.

Remark 3.7. It might be helpful to point out some differences between the two buildings $\Delta_{\mathbb{Q}}(\mathbf{G})$ and $\Delta(G)$ associated with $\mathbf{G}$. It is true that for every proper $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, its real locus $P = \mathbf{P}(\mathbb{R})$ is a proper parabolic subgroup of G, and hence there a simplex σ_P in $\Delta(G)$ corresponding to $\mathbf{P}$. On the other hand, by the definition of $\Delta_{\mathbb{Q}}(\mathbf{G})$, there is also a simplex $\sigma_{\mathbf{P}}$ of $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{P}$. In general, these two simplices σ_P and $\sigma_{\mathbf{P}}$ are not equal to each other. In particular, $\dim \sigma_P \geq \dim \sigma_{\mathbf{P}}$, and the equality holds if and only if the $\mathbb{Q}$-rank of $\mathbf{P}$ is equal to its $\mathbb{R}$-rank, which holds when $\mathbf{P}$ is $\mathbb{Q}$-split. Therefore, if $\mathbf{G}$ is $\mathbb{Q}$-split, then $\Delta_{\mathbb{Q}}(\mathbf{G})$ can be naturally contained in $\Delta(G)$ as a sub-building. Otherwise, there is in general no inclusion between them.

Though these two buildings $\Delta(G)$ and $\Delta_{\mathbb{Q}}(\mathbf{G})$ are different, they have the following property, the so-called Solomon-Tits Theorem (see [14]).

Proposition 3.8. *Let r be the rank of a semisimple Lie group G. Then the Tits building $\Delta(G)$ is homotopic to a bouquet of uncountably many spheres of dimension $r-1$. Let $r_{\mathbb{Q}}$ be the $\mathbb{Q}$-rank of a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. Then the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is homotopic to a bouquet of infinitely but countably many spheres of dimension $r_{\mathbb{Q}}-1$.*

4 Arithmetic groups, Borel-Serre compactifications, and Tits buildings

In this section we first recall the definitions of arithmetic subgroups Γ of linear algebraic groups $\mathbf{G}$ and of classifying (or universal) spaces for Γ. Then we recall the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$ of a symmetric space X and an application of this compactification to understanding the ends of locally symmetric spaces $\Gamma\backslash X$. Finally we summarize duality properties of arithmetic subgroups and explain that $\overline{X}^{\mathrm{BS}}$ is a Γ-cofinite universal space for proper actions of Γ.

Let $\mathbf{G} \subset \mathsf{GL}(n,\mathbb{C})$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. Then a subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called an *arithmetic subgroup* if it is commensurable with $\mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{Q}) \cap \mathsf{GL}(n,\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathbf{G}(\mathbb{Z})$ is of finite index in both Γ and $\mathbf{G}(\mathbb{Z})$.

As above, let $G = \mathbf{G}(\mathbb{R})$, and let $K \subset G$ be a maximal compact subgroup. Then $X = G/K$ with an invariant metric is a Riemannian symmetric space of non-compact type.

It is easy to see that an arithmetic subgroup Γ is a discrete subgroup of G and hence acts properly on X. If Γ is torsion-free, then the action of Γ on X is proper and fixed point free, and $\Gamma\backslash X$ is a manifold.

Proposition 4.1. *If Γ is a torsion-free arithmetic subgroup, then the locally symmetric space $\Gamma\backslash X$ is a $K(\Gamma,1)$-space.*

Proof. By definition, a $K(\Gamma,1)$-space is a topological space B with $\pi_1(B) = \Gamma$ and $\pi_i(B) = \{1\}$ for $i \geq 2$. Since X is contractible and the action of Γ is proper and fixed point free, $\pi_1(\Gamma\backslash X) = \Gamma$ and for $i \geq 2$, $\pi_i(\Gamma\backslash X) = \pi_i(X) = \{1\}$. □

The existence of a good model of $K(\Gamma,1)$ has important consequences on cohomological properties of Γ. An immediately corollary is the following result.

Corollary 4.2. *If Γ is a torsion-free arithmetic subgroup, then the cohomological dimension of Γ is less than or equal to $\dim X$, i.e., for every Γ-module M, $H^i(\Gamma, M) = 0$ for $i > \dim X$.*

To get other finiteness properties such as FP_∞ or FL (see [14]), we need some finiteness conditions on models of $K(\Gamma, 1)$. A strong finiteness condition is that there exists a finite $K(\Gamma, 1)$-space in the following sense.

Definition 4.3. A $K(\Gamma, 1)$-space is called finite if it is given by a finite CW-complex.

If Γ is torsion-free and the quotient $\Gamma\backslash X$ is compact, then $\Gamma\backslash X$ is a compact manifold and hence admits a finite triangulation. This implies that Γ admits a finite $K(\Gamma, 1)$-space.

Therefore, a natural and important problem is to determine when the quotient $\Gamma\backslash X$ is compact, i.e., when Γ is a uniform lattice subgroup of G.

Proposition 4.4. *Let $\mathbf{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. Then for every arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$, $\Gamma\backslash X$ is compact if and only if the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to zero, or equivalently there is no proper $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$.*

This was conjectured by Godement and proved by Borel and Harish-Chandra [9], and by Mostow and Tamagawa [37].

If $\Gamma\backslash X$ is non-compact, then $\mathbf{G}$ admits proper $\mathbb{Q}$-parabolic subgroups, or equivalently the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is nontrivial, which turns out to be crucial in describing the boundary at infinity of $\Gamma\backslash X$.

When $\Gamma\backslash X$ is non-compact, it certainly is not a finite CW-complex. One natural way to overcome this problem is to find a compactification $\overline{\Gamma\backslash X}$ such that

(1) the compactification $\overline{\Gamma\backslash X}$ has the structure of a finite CW-complex,

(2) and the inclusion $\Gamma\backslash X \to \overline{\Gamma\backslash X}$ is a homotopy equivalence.

The first condition is satisfied if $\overline{\Gamma\backslash X}$ is a compact manifold with boundary or with corners, and with interior equal to $\Gamma\backslash X$.

When X is the Poincaré upper halfplane $\mathbb{H}^2$, a non-compact quotient $\Gamma\backslash X$ is a Riemann surface with finitely many cusp neighborhoods. Each cusp neighborhood is diffeomorphic to a cylinder $[0, +\infty) \times S^1$ and hence can be compactified by adding a circle S^1 at infinity. The resulting compactification $\overline{\Gamma\backslash X}$ of $\Gamma\backslash X$ is a compact surface with boundary, and the inclusion $\Gamma\backslash X \to \overline{\Gamma\backslash X}$ is clearly a homotopy equivalence.

For a general non-compact locally symmetric space $\Gamma\backslash X$ associated to an arithmetic subgroup Γ, Borel and Serre [11] defined a compactification $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ of $\Gamma\backslash X$ which has similar properties as follows.

Theorem 4.5. *Let* $\mathbf{G}$ *be a semisimple linear algebraic group defined over* $\mathbb{Q}$ *and let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be an arithmetic subgroup such that* $\Gamma\backslash X$ *is non-compact, i.e.,* Γ *is a non-uniform lattice of* G*. Then there exists a partial compactification* $\overline{X}^{\mathrm{BS}}$ *of* X *such that the following conditions are satisfied:*

(1) $\overline{X}^{\mathrm{BS}}$ *is a real analytic manifold with corners and can be deformation retracted into the interior* X*, and hence is contractible, since* X *is diffeomorphic to* $\mathbb{R}^n$*,* $n = \dim X$*, and contractible.*

(2) *The boundary* $\partial\overline{X}^{\mathrm{BS}}$ *is decomposed into boundary components* $e_{\mathbf{P}}$ *parametrized by proper* $\mathbb{Q}$*-parabolic subgroups* $\mathbf{P}$ *of* $\mathbf{G}$ *such that each boundary component is contractible, and for every two proper* $\mathbb{Q}$*-parabolic subgroups* $\mathbf{P}_1$ *and* $\mathbf{P}_2$*,* $e_{\mathbf{P}_1}$ *is contained in the closure of* $e_{\mathbf{P}_2}$ *if and only if* $\mathbf{P}_1 \subseteq \mathbf{P}_2$*. Consequently,* $\partial\overline{X}^{\mathrm{BS}}$ *has the same homotopy type as the Tits building* $\Delta_{\mathbb{Q}}(\mathbf{G})$*.*

(3) *The* Γ*-action on* X *extends to a proper, real analytic action on* $\overline{X}^{\mathrm{BS}}$ *with a compact quotient* $\Gamma\backslash\overline{X}^{\mathrm{BS}}$*. In particular,* $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ *defines a compactification of* $\Gamma\backslash X$*, which is called the Borel-Serre compactification and is also denoted by* $\overline{\Gamma\backslash X}^{\mathrm{BS}}$*.*

(4) *If* Γ *is torsion free, then* Γ *acts fixed-point freely on* $\overline{X}^{\mathrm{BS}}$*, the quotient* $\Gamma\backslash\overline{X}^{\mathrm{BS}}$ *is a compact real analytic manifold with corners, and the inclusion* $\Gamma\backslash X \to \overline{\Gamma\backslash X}^{\mathrm{BS}}$ *is a homotopy equivalence.*

A corollary to the above result is the following result.

Corollary 4.6. *If* Γ *is a torsion-free arithmetic subgroup as above, then* Γ *admits a finite* $K(\Gamma, 1)$*-space, and hence* Γ *is of type* FL*.*

Note that the torsion-free condition on Γ is necessary for the existence of a finite dimensional $B\Gamma$. Proposition 4.7 also follows from Theorem 4.5. It concerns the ends of $\Gamma\backslash X$ and is well-known but probably has not been written down explicitly before.

Proposition 4.7. *Let* $\mathbf{G}$ *be a semisimple linear algebraic group defined over* $\mathbb{Q}$*, and let* Γ *be an arithmetic subgroup of* $\mathbf{G}(\mathbb{Q})$ *as above. Denote the* $\mathbb{Q}$*-rank of* $\mathbf{G}$ *by* $r_{\mathbb{Q}}$*. Assume that* $r_{\mathbb{Q}} \geq 1$*, which is equivalent to* $\Gamma\backslash X$ *being noncompact. The structure of the ends of* $\Gamma\backslash X$ *is as follows:*

(1) *If* $r_{\mathbb{Q}} = 1$*, then the ends of* $\Gamma\backslash X$ *are parametrized by* Γ*-conjugacy classes of proper* $\mathbb{Q}$*-parabolic subgroups of* $\mathbf{G}$*.*

(2) *If* $r_{\mathbb{Q}} > 1$*, then* $\Gamma\backslash X$ *has only one end, i.e., the infinity of* $\Gamma\backslash X$ *is connected.*

Proof. By definition, the ends of a complete connected noncompact Riemannian manifold M are basically the unbounded connected components of the complement of compact subsets. Specifically, for any compact subset $C \subset M$, denote by $n(C)$ the number of unbounded connected components of the complement $M - C$. Clearly $n(C) < +\infty$. For any two compact subsets C_1, C_2 of M with $C_1 \subset C_2$, it is also clear that $n(C_1) \leq n(C_2)$. The number $e(M)$ of ends of M is defined by

$$e(M) = \sup_C n(C) \,,$$

where C ranges over compact subsets of M.

Assume that $e(M) < +\infty$. Then there exists a compact subset C_0 such that $n(C_0) = e(M)$, and the unbounded connected components of $M - C_0$ are called the *ends* of M. Clearly we can replace C_0 by any larger compact subset and hence ends of M are well-defined up to compact subsets.

First we assume that $r_{\mathbb{Q}} \geq 2$ and that Γ is torsion-free. In this case, the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is connected. It suffices to show that for every compact subset C of $\Gamma\backslash X$, the complement $\Gamma\backslash X - C$ has only one unbounded connected component. Since $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ is a manifold with corners with the interior equal to $\Gamma\backslash X$, for every compact subset C of $\Gamma\backslash X$, there is a neighborhood U of the boundary $\partial\overline{\Gamma\backslash X}^{\mathrm{BS}}$ such that the complement $\overline{\Gamma\backslash X}^{\mathrm{BS}} - U$ contains C and U is homotopic to the boundary $\partial\overline{\Gamma\backslash X}^{\mathrm{BS}}$ by deformation retraction. Since $\partial\overline{X}^{\mathrm{BS}}$ has the same homotopy type as the building $\Delta_{\mathbb{Q}}(\mathbf{G})$, and $\Delta_{\mathbb{Q}}(\mathbf{G})$ is connected, it follows that $\partial\overline{\Gamma\backslash X}^{\mathrm{BS}} = \Gamma\backslash\partial\overline{X}^{\mathrm{BS}}$ and hence U is connected. Since $U \cap \Gamma\backslash X$ and U are also homotopic to each other, $U \cap \Gamma\backslash X$ is also connected. Note that the complement $\Gamma\backslash X - C$ of C in $\Gamma\backslash X$ is contained in $U \cap \Gamma\backslash X$, and the complement of $U \cap \Gamma\backslash X$ in $\Gamma\backslash X$ is compact. It follows that the complement $\Gamma\backslash X - C$ has only one unbounded connected component. By the arbitrary choice of C, this implies that $\Gamma\backslash X$ has only one end.

If Γ contains some nontrivial torsion elements, then there is always a torsion-free normal subgroup Γ' of finite index. By the previous paragraph, the locally symmetric space $\Gamma'\backslash X$ has only one end. Since $\Gamma'\backslash X$ is a finite cover of $\Gamma\backslash X$, any compact subset of $\Gamma\backslash X$ is lifted to a compact subset of $\Gamma'\backslash X$. This implies that $\Gamma\backslash X$ has also only one end.

Before treating the case when the rank $r_{\mathbb{Q}} = 1$, we recall the rational Langlands decomposition and Siegel sets. For every $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, there is also a rational Laglands decomposition of the real locus $P = \mathbf{P}(\mathbb{R})$ with respect to the maximal compact subgroup $K \subset G$ obtained by taking the

$\mathbb{Q}$-structure of $\mathbf{P}$ into account:

$$P = N_P A_{\mathbf{P}} M_{\mathbf{P}} \cong N_P \times A_{\mathbf{P}} \times M_{\mathbf{P}} \,, \tag{2}$$

where $A_{\mathbf{P}}$ is isomorphic to the identity component of the real locus of a maximal $\mathbb{Q}$-split torus of $\mathbf{P}$. Note that unless the $\mathbb{Q}$-rank of $\mathbf{P}$ agrees with the $\mathbb{R}$-rank of $\mathbf{P}$, this decomposition is different from the real Langlands decomposition in equation (1).

Assume now that $r = 1$. Then every proper $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$ is both a minimal and maximal $\mathbb{Q}$-parabolic subgroup, and in particular, the split component $A_{\mathbf{P}}$ has dimension 1. Identify $A_{\mathbf{P}}$ with $\mathbb{R}^1$ so that the positive chamber corresponds to $(0, +\infty)$. For any $t \in \mathbb{R}^1$, denote the subset of $A_{\mathbf{P}}$ corresponding to $[t, +\infty)$ by $A_{\mathbf{P},t}$. For any compact subset ω of $N_P M_{\mathbf{P}}$, the subset $\omega A_{\mathbf{P},t} x \subset X$ is called a *Siegel set* of X associated with $\mathbf{P}$, where $x \in X$ is the fixed point of K.

The reduction theory for arithmetic groups implies the following results (see [10]):

(1) There are only finitely many conjugate classes of proper $\mathbb{Q}$-parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$.

(2) Let $\mathbf{P}_1, \dots, \mathbf{P}_m$ be representatives of Γ-conjugacy classes of the proper $\mathbb{Q}$-parabolic subgroups, and let $\omega_1 A_{\mathbf{P}_1,t_1} x, \dots, \omega_m A_{\mathbf{P}_m,t_m} x$ be the Siegel sets associated with them. When $\omega_1, \dots, \omega_m$ are sufficiently large, the complement of the images of these Siegel sets in $\Gamma\backslash X$ is a bounded subset. Furthermore, when $t_i \gg 0$, the image of these Siegel sets are disjoint in $\Gamma\backslash X$.

Choose the compact sets ω_i to be sufficiently large and also connected. Pick any compact subset C of $\Gamma\backslash X$. It follows easily from the horospherical decomposition of X induced from the Langlands decomposition of $\mathbf{P}_i$ that for $t_i \gg 0$, the image of $\omega_i A_{\mathbf{P}_i,t_i} x$ in $\Gamma\backslash X$ for $i = 1, \dots, m$ is contained in $\Gamma\backslash X - C$. Since these images are disjoint, this implies that the image of each $\omega_i A_{\mathbf{P}_i,t_i} x$ in $\Gamma\backslash X$ is an end of $\Gamma\backslash X$, and the ends of $\Gamma\backslash X$ are parametrized by $\mathbf{P}_1, \dots, \mathbf{P}_m$, i.e., the Γ-conjugacy classes of the proper $\mathbb{Q}$-parabolic subgroups. (Note that the complement of the union of these images in $\Gamma\backslash X$ is bounded under the above largeness assumption on ω_i.) $\square$

Realization of $\overline{X}^{\mathbf{BS}}$ by a truncated subspace X_T

As pointed out above, one of the motivations for constructing the Borel-Serre compactification $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ in [11] is to get a finite $K(\Gamma, 1)$-space using the fact that the inclusion $\Gamma\backslash X \to \overline{\Gamma\backslash X}^{\mathrm{BS}}$ is a homotopy equivalence and $\overline{\Gamma\backslash X}^{\mathrm{BS}}$ is a manifold with corners.

Another natural method is to construct a compact submanifold with corners, denoted by $(\Gamma\backslash X)_T$, where T is the truncation parameter, such that the inclusion

$$(\Gamma\backslash X)_T \to \Gamma\backslash X$$

is a homotopy equivalence. Then the submanifold $(\Gamma\backslash X)_T$ can be used for this same purpose.

When $\Gamma\backslash X$ is a Riemann surface, i.e., when X is the Poincaré upper half-plane $\mathbb{H}^2$, then $(\Gamma\backslash X)_T$ is a compact surface with boundary obtained by truncating sufficiently small cusp neighborhoods, where T measures the depth of the neighborhoods. The inverse image of $(\Gamma\backslash X)_T$ in $X = \mathbb{H}^2$, denoted by X_T, is the complement of Γ-equivariant horodiscs at the rational boundary points, i.e., X_T is obtained from X by removing horodiscs from rational boundary points in an equivariant way. See [40] for a picture.

For a general $\Gamma\backslash X$, such a truncated submanifold $(\Gamma\backslash X)_T$ has also been constructed. See the papers [35, 40] for the history and references. Briefly, X contains a Γ-equivariant submanifold with corners, denoted by X_T, such that the quotient $\Gamma\backslash X_T$ is compact and both the partial compactification $\overline{X}^{\mathrm{BS}}$ and X can be Γ-equivariantly deformation retracted onto X_T. In fact, $\overline{X}^{\mathrm{BS}}$ is also equivariantly diffeomorphic to X_T.

The arithmetic subgroup Γ leaves X_T invariant, and the quotient $\Gamma\backslash X_T$ gives the desired submanifold $(\Gamma\backslash X)_T$ of $\Gamma\backslash X$ which is diffeomorphic to $\Gamma\backslash X$ and hence gives a finite $K(\Gamma, 1)$-space if Γ is torsion-free.

Duality properties of arithmetic subgroups

Besides giving a finite $K(\Gamma, 1)$-space, the partial Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$ can also be used to show that non-uniform arithmetic subgroups Γ are not virtual Poincaré duality groups.

Recall from [15] that a group Γ is called a *duality group of dimension d* if there is an integer $d \geq 0$ and a Γ-module D such that, for every Γ-module M and $i \geq 0$, there is the following isomorphism:

$$H^i(\Gamma, M) \cong H_{d-i}(\Gamma, D \otimes M)\,. \tag{3}$$

The module D is called the *dualizing module* of Γ. If D can be taken to $\mathbb{Z}$, then Γ is called a *Poincaré duality group*, i.e., the following isomorphism holds:

$$H^i(\Gamma, M) \cong H_{d-i}(\Gamma, M)\,. \tag{4}$$

If Γ is the fundamental group of a closed nonpositively curved Riemannian manifold M, then it follows from the Poincaré duality of M that Γ is a Poincaré duality group (see [15]). As pointed out before, a symmetric space of noncompact type is nonpositively curved. In particular, if Γ is a torsion-free arithmetic subgroup and $\Gamma\backslash X$ is compact, then $\Gamma\backslash X$ is a closed nonpositively curved Riemannian manifold and hence Γ is a Poincaré duality group.

Another important result of [11] is the following:

Theorem 4.8. *Assume that the $\mathbb{Q}$-rank r of $\mathbf{G}$ is positive. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Then Γ is a duality group of dimension $\dim X - r$, and the dualizing module D is equal to $H_{r-1}(\Delta_{\mathbb{Q}}(\mathbf{G}))$. In particular, D is of infinite rank and Γ is not a Poincaré duality group.*

The Solomon-Tits theorem (Proposition 3.8) implies that $H_i(\Delta_{\mathbb{Q}}(\mathbf{G}))$ is not equal to zero if and only if $i = r-1$, and when $i = r-1$, $H_i(\Delta_{\mathbb{Q}}(\mathbf{G}))$ is infinitely generated. This result and the fact that the boundary of the partial Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$ is homotopic to $\Delta_{\mathbb{Q}}(\mathbf{G})$ are crucial in the proof of the above theorem.

Universal spaces for proper actions of $\boldsymbol{\Gamma}$ and $\overline{\boldsymbol{X}}^{\mathbf{BS}}$

Another important property of $\overline{X}^{\mathrm{BS}}$ is that it is a universal space for proper actions of arithmetic subgroups.

Recall that when we discussed $\Gamma\backslash X$ as a $K(\Gamma,1)$-space, we assumed that Γ was torsion-free. This is important since if Γ contains nontrivial torsion elements, then it does not admit finite dimensional $K(\Gamma,1)$-spaces. On the other hand, many natural arithmetic subgroups contain nontrivial torsion elements, for example, $\mathsf{SL}(n,\mathbb{Z})$ and $\mathsf{Sp}(n,\mathbb{Z})$.

A $K(\Gamma,1)$-space is the classifying space for Γ and is also denoted by $B\Gamma$. Its universal covering space $E\Gamma = \widetilde{B\Gamma}$ is the universal space for proper and fixed point free actions of Γ and is characterized up to homotopy equivalence as follows:

(1) $E\Gamma$ is contractible.

(2) The Γ action on $E\Gamma$ is proper and fixed point free.

Clearly, given $E\Gamma$, we can take the quotient $\Gamma\backslash E\Gamma$ as $B\Gamma$.

For groups Γ containing torsion elements, a natural replacement for $E\Gamma$ is the universal space for proper actions, usually denoted by $\underline{E}\Gamma$, which is characterized up to homotopy equivalence as follows:

(1) For every finite subgroup F of Γ, the set of fixed points $(\underline{E}\Gamma)^F$ is nonempty and contractible. In particular, $\underline{E}\Gamma$ is contractible.

(2) Γ acts properly on $\underline{E}\Gamma$.

If $\underline{E}\Gamma$ is a Γ-CW complex and the quotient $\Gamma\backslash\underline{E}\Gamma$ is a finite CW-complex, then $\underline{E}\Gamma$ is called Γ-*cofinite*. As in the earlier case where we had to find a finite $K(\Gamma, 1)$-space, it is also an important problem for various applications to find Γ-cofinite $\underline{E}\Gamma$-spaces.

Assume that Γ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ as above, and that $X = G/K$ is the associated symmetric space.

Since X is a symmetric space of non-compact type, it follows from the famous Cartan fixed point theorem that X is an $\underline{E}\Gamma$-space. In fact, for any finite subgroup F of Γ, the fixed point set X^F is a totally geodesic submanifold and hence contractible.

If $\Gamma\backslash X$ is compact, then the existence of a Γ-equivariant triangulation shows that X is a $\underline{E}\Gamma$-space.

If $\Gamma\backslash X$ is non-compact, then we have the following result [26, Theorem 3.2].

Theorem 4.9. *Assume that Γ is a non-uniform lattice of G as above. Then the partial Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$ is a Γ-cofinite $\underline{E}\Gamma$-space.*

Instead of using the partial Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$, we can also show more directly that the truncated submanifold X_T of X is a Γ-cofinite $\underline{E}\Gamma$-space.

5 S-arithmetic groups and Bruhat-Tits buildings

In this section, we first explain two natural generalizations of arithmetic subgroups: finitely generated linear groups and S-arithmetic subgroups. Then we show that p-adic Lie groups and Bruhat-Tits buildings are needed to understand them. After this, we introduce the Bruhat-Tits building by observing that we can realize a symmetric space $X = G/K$ as the space of maximal compact subgroups of G. For various applications, for example, to construct cofinite universal spaces of proper actions for S-arithmetic subgroups, the fact that Bruhat-Tits buildings are CAT(0)-spaces plays an important role. Then we discuss duality properties of S-arithmetic subgroups. It also turns out that as a CAT(0)-space, the set of equivalence classes of rays in a Bruhat-Tits building can naturally be identified with a spherical Tits building, which appears naturally as the boundary of a compactification of the former building.

Arithmetic subgroups Γ of a linear algebraic group $\mathbf{G} \subset \mathsf{GL}(n, \mathbb{C})$ are special finitely generated subgroups of the Lie group $G = \mathbf{G}(\mathbb{R})$.

There are two methods to produce more general classes of subgroups. The first method is to take finitely many elements $\gamma_1, \dots, \gamma_m$ of $\mathbf{G}(\mathbb{Q})$ and consider the subgroup $\Gamma = \langle \gamma_1, \dots, \gamma_m \rangle$ generated by them. In general, Γ is not a discrete subgroup of $\mathbf{G}$ and is not related to $\mathbf{G}(\mathbb{Q}) \cap \mathsf{GL}(n, \mathbb{Z})$.

For example, take $\mathbf{G} = \mathsf{GL}(n, \mathbb{C})$, and $\gamma_1, \dots, \gamma_m \in \mathsf{GL}(n, \mathbb{Q})$. If all the matrix entries of $\gamma_1, \dots, \gamma_m$ are integral, then Γ is contained in $\mathsf{GL}(n, \mathbb{Z})$, and in particular, Γ is a discrete subgroup of $\mathsf{GL}(n, \mathbb{R})$. Otherwise, it is not a discrete subgroup of $\mathsf{GL}(n, \mathbb{R})$ in general.

On the other hand, let $p_1, \dots, p_k$ be all the prime numbers which appear in the denominators of the entries of $\gamma_1, \dots, \gamma_m$, and let $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_k}]$ be the ring of rational numbers of the form p/q, where $p, q \in \mathbb{Z}$ and q is a product of primes only from $p_1, \dots, p_k$. Then Γ is contained in $\mathsf{GL}(n, \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_k}])$. Since $Z[\frac{1}{p_1}, \dots, \frac{1}{p_k}]$ is not a discrete subgroup of $\mathbb{R}$, $\mathsf{GL}(n, \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_k}])$ is not a discrete subgroup of $\mathsf{GL}(n, \mathbb{R})$.

This motivates the second method. For any finite set S of primes, $S = \{p_1, \dots, p_k\}$, a subgroup Γ of $\mathbf{G}(\mathbb{Q})$ is called an *S-arithmetic subgroup* if it is commensurable with $\mathbf{G}(\mathbb{Q}) \cap \mathsf{GL}(n, \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_k}])$.

If $\mathbf{G}$ is semisimple, then S-arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$ are finitely generated. Therefore, the first method produces a larger class of groups.

Let $\mathbb{Q}_p$ be the field of p-adic numbers, and $\mathbf{G}(\mathbb{Q}_p)$ the associated p-adic Lie group. Then an important fact is that under the diagonal embedding

$$\Gamma \subset \mathbf{G}(\mathbb{Q}) \to \mathbf{G}(\mathbb{R}) \times \prod_{p \in S} \mathbf{G}(\mathbb{Q}_p)\,, \tag{5}$$

Γ is a discrete subgroup of $\mathbf{G}(\mathbb{R}) \times \prod_{p \in S} \mathbf{G}(\mathbb{Q}_p)$.

Since $\mathbb{R}$ and $\mathbb{Q}_p$ are both completions of $\mathbb{Q}$ with respect to different norms, this shows the importance of treating all of them, both archimedean norms and non-archimedean norms, simultaneously.

As the previous sections showed, symmetric spaces and spherical Tits buildings are useful for many questions about Lie groups G and their arithmetic subgroups. It is a natural and important problem to find analogues of symmetric spaces for p-adic Lie groups $\mathbf{G}(\mathbb{Q}_p)$.

First we note that the spherical Tits buildings can be defined in a manner similar to what we did in the case of semisimple Lie groups and algebraic groups. In fact, the algebraic group $\mathbf{G}$ can be considered as an algebraic group defined over $\mathbb{Q}_p$, and the notion of parabolic subgroups of $\mathbf{G}$ defined over $\mathbb{Q}_p$ can be introduced. The Tits building of $\mathbf{G}(\mathbb{Q}_p)$, denoted by $\Delta(\mathbf{G}(\mathbb{Q}_p))$, is an infinite simplicial complex whose simplices are parametrized by parabolic subgroups of $\mathbf{G}$ defined over $\mathbb{Q}_p$ and satisfy properties similar to those for the Tits building $\Delta(G)$ of a semisimple real Lie group G in §3. Since $\mathbf{G}(\mathbb{Q}_p)$ acts on the set of $\mathbb{Q}_p$-parabolic subgroups by conjugation, it also acts on the building $\Delta(\mathbf{G}(\mathbb{Q}_p))$.

The analogue of the symmetric space $X = G/K$ for $\mathbf{G}(\mathbb{Q}_p)$ is the Bruhat-Tits building, denoted by $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ in this paper, which is a Euclidean building (also called an affine building).

The definition and construction of $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ is much more complicated than that of the spherical Tits building $\Delta(\mathbf{G}(\mathbb{Q}_p)$. To motivate the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$, we note that for the semisimple Lie group $G = \mathbf{G}(\mathbb{R})$, its symmetric space $X = G/K$ can be identified with the space of maximal compact subgroups of G. An important fact here is that all maximal compact subgroups of G are conjugate. But this statement is not true for $\mathbf{G}(\mathbb{Q}_p)$. In general, there is more than one conjugacy class of maximal compact subgroups of $\mathbf{G}(\mathbb{Q}_p)$.

For simplicity, we assume that $\mathbf{G}$ is a simply connected semisimple linear algebraic group defined over $\mathbb{Q}$. Let r be the $\mathbb{Q}_p$-rank of $\mathbf{G}$. In this case, there are exactly $(r+1)$-conjugacy classes of maximal compact open subgroups of $\mathbf{G}(\mathbb{Q}_p)$. Let $P_1, \ldots, P_{r+1}$ be representatives of such conjugacy classes. Then it is natural to consider the homogeneous spaces $\mathbf{G}(\mathbb{Q}_p)/P_1, \ldots, \mathbf{G}(\mathbb{Q}_p)/P_{r+1}$, each of which is a disjoint union of points. For various purposes, it is desirable to enhance them into simplicial complexes so that they remain as vertices and then to combine these finitely many simplicial complexes suitably into *one single simplicial complex*. Basically, the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ can be visualized this way.

One original way to construct the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ is to use the method of BN-pairs and, instead of using parabolic subgroups of $\mathbf{G}(\mathbb{R})$ as was done for the spherical Tits building $\Delta(G)$, to use the so-called *parahoric groups*. This depends crucially on the fact that $\mathbb{Q}_p$ has a nontrivial discrete valuation.

Under the assumption that $\mathbf{G}$ is simply connected, maximal compact open subgroups of $\mathbf{G}(\mathbb{Q}_p)$ are maximal parahoric subgroups, and maximal parahoric subgroups are also maximal compact open subgroups of $\mathbf{G}(\mathbb{Q}_p)$. More importantly, the structure of parahoric subgroups $\mathbf{G}(\mathbb{Q}_p)$ is similar to the structure of parabolic subgroups of $\mathbf{G}(\mathbb{R})$. For example, the following result reminds one of the structure of parabolic subgroups of G and the notion of standard parabolic subgroups (see [25] and references there for more details).

Proposition 5.1. *Under the above assumptions on $\mathbf{G}$, all minimal parahoric subgroups of $\mathbf{G}(\mathbb{Q}_p)$ are conjugate. Fix any minimal parahoric subgroup B. Then there are exactly $r+1$ maximal parahoric subgroups $P_1, \ldots, P_{r+1}$ which contain B, and $\{P_{i_1} \cap \cdots \cap P_{i_j}\}$ are exactly the parahoric subgroups which contain B when*

$\{i_1, \ldots, i_j\}$ runs through non-empty subsets of $\{1, \ldots, r+1\}$. They are called the standard parahoric subgroups. Furthermore, any parahoric subgroup of $\mathbf{G}(\mathbb{Q}_p)$ is conjugate to such a standard parahoric subgroup.

Bruhat-Tits buildings are important examples of Euclidean buildings. We recall some of their basic properties. See [15] for more details.

Recall from §3 that an important notion in the definition of a spherical Tits buildings is that of apartments, which are finite Coxeter complexes. Recall that their underlying topological spaces are spheres, and that buildings are obtained by gluing these apartments together suitably. For Euclidean buildings, finite Coxeter complexes are replaced by infinite Euclidean Coxeter complexes, whose underlying spaces are Euclidean spaces.

Briefly, let V be a Euclidean space. An *affine reflection group* W on V is a group of affine isometries generated by reflections with respect to affine hyperplanes such that the set $\mathcal{H}$ of affine hyperplanes fixed by reflections in W is locally finite.

The linear parts of the affine transformations in W define a finite (linear) reflection group $\overline{W}$, or a finite Coxeter group. An infinite affine reflection group is called a *Euclidean reflection group*.

The hyperplanes in $\mathcal{H}$ divide V into chambers, and W acts simply transitively on the set of chambers. If the reflection group W is irreducible, then the chambers and their faces form a simplicial complex. Otherwise, they form a polysimplicial complex which is a product of simplicial complexes. Such a polysimplicial complex is called a *Euclidean Coxeter complex*. For simplicity, in the following we assume that the affine reflection groups W and the Euclidean Bruhat-Tits buildings introduced below are irreducible. Otherwise, we get polysimplicial complexes instead of simplicial complexes.

Definition 5.2. A polysimplicial complex Δ is called a *Euclidean building* if it contains a family of subsets called apartments, which satisfies the following conditions:

(1) Every apartment is an infinite Euclidean Coxeter complex.

(2) Any two simplices are contained in some apartment.

(3) Given two apartments Σ and Σ' and simplices $\sigma, \sigma' \in \Sigma \cap \Sigma'$, there exists an isomorphism of Σ onto Σ' which keeps σ, σ' pointwise fixed.

The Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ is a Euclidean building, and its simplices are parametrized by parahoric subgroups of $\mathbf{G}(\mathbb{Q}_p)$ such that maximal parahoric subgroups correspond to vertices, i.e., simplices of zero dimension, and minimal parahoric subgroups correspond to chambers, i.e., top dimensional simplices. Since $\mathbf{G}(\mathbb{Q}_p)$ acts on the set of parahoric subgroups by conjugation, it acts on $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$.

Since the underlying space of each Euclidean Coxeter complex is a Euclidean space, it has a metric, or a distance function. Fix a Euclidean metric on every apartment such that all apartments are isometric and their induced metrics agree on their intersections. Then these metrics can be glued to give a metric on the building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ which becomes a geodesic space (recall that a geodesic space is a metric space such that the distance between any two points is realized by a geodesic connecting them) [14, Chap. VI, §3].

The group $\mathbf{G}(\mathbb{Q}_p)$ acts on the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ by isometries. Furthermore, the action is proper. (Note that the stabilizers of the simplices coincide with their corresponding parahoric subgroups up to finite index, and are hence compact.)

Proposition 5.3. *Any Euclidean building Δ as a metric space is a CAT(0)-space, and hence has nonpositive curvature and is contractible. In particular, it is simply connected.*

Recall that a CAT(0)-space M is a geodesic length space such that every triangle in M is thinner than a corresponding triangle in $\mathbb{R}^2$ of the same side lengths [13]. This proposition implies that if a compact group acts isometrically on Δ, then it has at least one fixed point, which has important applications to understanding structures of compact open subgroups of $\mathbf{G}(\mathbb{Q}_p)$ as mentioned in Proposition 5.1.

The Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ is non-compact. Since it is a proper CAT(0)-space, it admits a compactification by adding the set of equivalence classes of geodesics as was discussed for Hadamard manifolds in §2. It turns out that this set can be canonically identified with the spherical Tits building $\Delta(\mathbf{G}(\mathbb{Q}_p))$. It should be emphasized that the topology on $\Delta(\mathbf{G}(\mathbb{Q}_p))$ induced from the compactification is not the simplicial one. See [12, 14].

Remark 5.4. As pointed out before, maximal totally geodesic flat subspaces of a symmetric space X play a fundamental role in understanding the geometry of X, and X is the union of such flats. Apartments in a Euclidean building, in particular, the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$, are also maximal totally geodesic subspaces, and the building is also the union of such flats. Furthermore, both X and $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ are CAT(0)-spaces and the Cartan fixed point theorem holds for them. In this sense, the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ is a good replacement for the symmetric space X. The fact that they can be compactified by adding the spherical buildings of $\mathbf{G}(\mathbb{R})$ and $\mathbf{G}(\mathbb{Q}_p)$ respectively is another indication of their similarity.

Finiteness properties and duality properties of S-arithmetic groups

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an S-arithmetic subgroup, i.e., a subgroup commensurable with $\mathbf{G}(\mathbb{Q}) \cap \mathsf{GL}(n, \mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_k}])$ as in the previous section.

By equation (5), Γ is a discrete subgroup of $\mathbf{G}(\mathbb{R}) \times \prod_{p\in S} \mathbf{G}(\mathbb{Q}_p)$. Since each $\mathbf{G}(\mathbb{Q}_p)$ acts isometrically and properly on the building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$, Γ acts isometrically and properly on the product

$$X_S = X \times \prod_{p\in S} \Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p)) .$$

Since each of the factors is a proper CAT(0)-space, the product X_S is also a proper CAT(0)-space. An immediately corollary of this action is the following.

Proposition 5.5. *The space X_S is an $\underline{E}\Gamma$-space; in particular, if Γ is torsion-free, then the quotient $\Gamma\backslash X_S$ is a $K(\Gamma, 1)$-space and hence the cohomological dimension of Γ is less than or equal to $\dim X_S$. If the $\mathbb{Q}$-rank of $\mathbf{G}$ is 0, then $\Gamma\backslash X_S$ is compact, and X_S is a Γ-cofinite $\underline{E}\Gamma$-space.*

On the other hand, if the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, then $\Gamma\backslash X_S$ is non-compact. As recalled earlier, X admits the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$. Define a partial compactification of X_S by

$$\overline{X_S}^{\mathrm{BS}} = \overline{X}^{\mathrm{BS}} \times \prod_{p\in S} \Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p)) .$$

Then the Γ action on X_S extends to $\overline{X_S}^{\mathrm{BS}}$ and the quotient $\Gamma\backslash\overline{X_S}^{\mathrm{BS}}$ is compact [12]. The following result of Borel and Serre [12] is an important application of $\overline{X_S}^{\mathrm{BS}}$ together with the homotopy equivalence between the boundary $\partial\overline{X}^{\mathrm{BS}}$ and the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ (§4) and the compactification of the Bruhat-Tits building $\Delta^{\mathrm{BT}}(\mathbf{G}(\mathbb{Q}_p))$ by attaching a spherical building $\Delta(\mathbf{G}(\mathbb{Q}_p))$ (§5).

Theorem 5.6. *If Γ is a torsion-free S-arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ as above, then Γ is a duality group. If the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, then Γ is not a Poincaré duality group.*

If Γ is torsion-free and the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, then $\Gamma\backslash\overline{X_S}^{\mathrm{BS}}$ is a finite $K(\Gamma, 1)$-space. As in the case of arithmetic subgroups, many natural S-arithmetic subgroups contain torsion elements, for example, $\mathsf{SL}(n, \mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_k}])$. A natural problem is to find good models of Γ-cofinite $\underline{E}\Gamma$-spaces. By methods similar to the proof of [26, Theorem 3.2] (see Theorem 4.9), we can prove the following.

Proposition 5.7. *For a non-uniform S-arithmetic subgroup Γ as above, $\overline{X_S}^{\mathrm{BS}}$ is a Γ-cofinite $\underline{E}\Gamma$-space.*

6 Integral Novikov conjectures and the Borel conjecture

Besides applications to cohomological properties mentioned in the previous section, the action of S-arithmetic subgroups on X_S is also important for other results such as the integral Novikov conjecture for S-arithmetic subgroups and the stable Borel conjecture.

In this section, we first recall the Borel conjecture in geometric topology and a weaker version, the stable Borel conjecture. Then we explain both the original version of the Novikov conjecture on homotopy invariance of higher signatures and the integral Novikov conjecture in terms of the modern formulation by the assembly map. Then we explain the integral Novikov conjecture for S-arithmetic subgroups and finitely generated linear groups by making use of their actions on Bruhat-Tits buildings.

An important conjecture in geometric topology is the *Borel conjecture*, which states that *if M and N are two aspherical closed manifolds with the same fundamental group, i.e., they are homotopic, then M and N are homeomorphic.*

In the above statement, a manifold M is called *aspherical* if $\pi_i(M) = \{1\}$ for $i \geq 2$. If a $K(\Gamma, 1)$-space can be realized by a closed manifold M, then M is an aspherical manifold.

It is an important problem to find conditions under which a $K(\Gamma, 1)$-space can be realized by a closed manifold. The Borel conjecture is basically about the uniqueness up to homeomorphism for such realizations.

Clearly the Borel conjecture only depends on the fundamental group $\pi_1(M)$. It is still open and has motivated a lot of work in geometric topology.

A weaker version of the Borel conjecture is the *stable Borel conjecture* which states that *if M and N are two aspherical closed manifolds with the same fundamental group, then $M \times \mathbb{R}^3$ and $N \times \mathbb{R}^3$ are homeomorphic.*

Another closely related conjecture is the integral Novikov conjecture. We briefly recall the motivations and different versions of Novikov conjecture.

The original Novikov conjecture concerns homotopy invariance of higher signatures. Briefly, let M be an oriented closed manifold, and $\Gamma = \pi_1(M)$ its fundamental group. Let $B\Gamma$ be the classifying space of Γ, i.e., a $K(\Gamma, 1)$-space. Let $f\colon M \to B\Gamma$ be the classifying map corresponding to the universal covering space $\tilde{M} \to M$. For any $\alpha \in H^*(M, \mathbb{Q})$, define a *higher signature*

$$\mathrm{Sgn}_\alpha(M) = \langle \mathcal{L}(M) \cup f^*(\alpha), [M] \rangle$$

associated with α, where $\mathcal{L}(M)$ is the Hirzebruch class of M. When $\alpha = 1$, $\mathrm{Sgn}_\alpha(M)$ is equal to the usual signature, by the Hirzebruch index theorem.

The *original Novikov conjecture* says that for any α, $\mathrm{Sgn}_\alpha(M)$ is an oriented homotopy invariant of M.

This Novikov conjecture is equivalent to the rational injectivity of the assembly map in L-theory (or surgery theory)

$$A\colon H_*(B\Gamma, \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}\Gamma)\,, \tag{6}$$

i.e., that the map $A \otimes \mathbb{Q}\colon H_*(B\Gamma, \mathbb{L}(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$ is injective, where $H_*(B\Gamma, \mathbb{L}(\mathbb{Z}))$ is a generalized homology theory with coefficients in the spectra $\mathbb{L}(\mathbb{Z})$, and $L_*(\mathbb{Z}\Gamma)$ are surgery groups. The stronger statement that A is injective is called the *integral Novikov conjecture*. It is known that the integral Novikov conjecture for Γ implies the stable Borel conjecture [29]. The Borel conjecture is related to the conjecture that the assembly map A is an isomorphism.

Besides the assembly map in surgery theory as recalled here, there are also assembly maps in other theories such as algebraic K-theory. See [29] and the references there for more precise statements.

If a group Γ contains nontrivial torsion elements, then the integral Novikov conjecture does not hold in general. In this case, we need to use the universal space $\underline{E}\Gamma$ for proper actions of Γ, and to replace the above assembly map by

$$A\colon H_*^{\Gamma}(\underline{E}\Gamma, \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}\Gamma)\,, \tag{7}$$

which is reduced to the previous case in equation (6) when Γ is torsion free, in which case $\underline{E}\Gamma$ is equal to $E\Gamma$. If A is injective in equation (7), we say that the *generalized Novikov conjecture* holds for Γ.

To understand the assembly map A, a good model of $\underline{E}\Gamma$ is important. As explained above, the Borel-Serre partial compactification $\overline{X_S}^{\mathrm{BS}}$ can be used to construct a Γ-cofinite $\underline{E}\Gamma$-space.

Using actions of S-arithmetic subgroups Γ on X_S, in particular the fact that X_S is a CAT(0)-space, together with finiteness of the asymptotic dimension of Γ, a large scale geometric invariant of Γ, the following result was proved in [29].

Theorem 6.1. *If Γ is an S-arithmetic subgroup of a linear algebraic $\mathbf{G}$ defined over $\mathbb{Q}$, then the generalized integral Novikov conjecture holds for Γ. Consequently, if Γ is further assumed to be torsion-free, then the stable Borel conjecture holds also for Γ.*

It is worthwhile to emphasize that X_S is the product of Riemannian symmetric spaces and Bruhat-Tits buildings. Therefore, buildings play a crucial role in the above theorem.

As mentioned at the beginning of this section, a class of groups larger than the class of S-arithmetic subgroups is the class of finitely generated subgroups of $\mathsf{GL}(n,\mathbb{Q})$. Assume that Γ is such a group. Then there exists a finite set of primes, $S=\{p_1,\dots,p_k\}$, such that $\Gamma\subset\mathsf{GL}(n,\mathbb{Z}[\frac{1}{p_1},\dots,\frac{1}{p_k}])$.

By using the action of Γ on a similar space X_S associated with $\mathsf{GL}(n)$ and the finite set of prime numbers S, the following more general result was also proved in [29].

Theorem 6.2. *For every finitely generated subgroup of* $\mathsf{GL}(n,\mathbb{Q})$*, the generalized integral Novikov conjecture holds.*

7 Mapping class groups and Teichmüller spaces

In this section we recall the definition of mapping class groups and Teichmüller spaces of surfaces. Then we explain that the Teichmüller spaces are universal spaces for proper actions of mapping class groups.

In the previous sections, we have studied arithmetic subgroups and some close analogues. Another class of groups closely related to arithmetic subgroups is that of mapping class groups of surfaces.

Let M be an oriented manifold. Denote by $\mathrm{Diff}(M)$ the group of all diffeomorphisms of M, by $\mathrm{Diff}(M)^+$ the subgroup of all orientation preserving diffeomorphisms of M, and by $\mathrm{Diff}^0(M)$ the identity component of $\mathrm{Diff}(M)$. Then $\mathrm{Diff}^0(M)$ is a normal subgroup of both $\mathrm{Diff}(M)$ and $\mathrm{Diff}^+(M)$. The quotient

$$\mathrm{Mod}(M)=\mathrm{Diff}(M)/\,\mathrm{Diff}^0(M)$$

is called the *extended mapping class group* of M, and

$$\mathrm{Mod}^+(M)=\mathrm{Diff}^+(M)/\,\mathrm{Diff}^0(M)$$

is the *mapping class group* of M.

We are mainly interested in the case when M is a surface. Assume that M is a closed oriented surface S of genus g. Then we denote $\mathrm{Mod}(S)$ by Mod_g, and $\mathrm{Mod}^+(S)$ by Mod_g^+. When $g=1$, i.e., $M=\mathbb{R}^2/\mathbb{Z}^2$, then

$$\mathrm{Mod}(\mathbb{R}^2/\mathbb{Z}^2)\cong\mathsf{GL}(2,\mathbb{Z}),\ \text{and}\ \mathrm{Mod}^+(\mathbb{R}^2/\mathbb{Z}^2)\cong\mathsf{SL}(2,\mathbb{Z})\,.$$

Therefore, Mod_g and Mod_g^+ are natural generalizations of the basic arithmetic subgroups $\mathsf{GL}(2,\mathbb{Z})$ and $\mathsf{SL}(2,\mathbb{Z})$. It is known that for $g\geq 2$, Mod_g is not isomorphic to arithmetic subgroups of semisimple Lie groups [23].

For the mapping class groups Mod_g, the analogue of the symmetric spaces is the Teichmüller space T_g of closed surfaces.

Briefly, let $S = S_g$ be a closed orientable surface of genus g. A *marked hyperbolic metric* on S is a hyperbolic surface (Σ, ds) together with a homotopy class of diffeomorphisms $\varphi\colon S \to \Sigma$. A diffeomorphism φ induces an isomorphism between $\pi_1(S)$ and $\pi_1(\Sigma)$ and only depends on the homotopy class of φ. In fact, the converse is also true. The reason for this is that S and Σ are aspherical two dimensional manifolds, and every isomorphism between $\pi_1(S)$ and $\pi_1(\Sigma)$ can be realized by a diffeomorphism from S to Σ. Therefore, a marking on a hyperbolic surface is to fix an isomorphism from $\pi_1(S)$ to $\pi_1(\Sigma)$.

By definition, two marked hyperbolic metrics $(\Sigma_1, ds_1; \varphi_1)$ and $(\Sigma_2, ds_2; \varphi_2)$ on S are equivalent if there exists an isometry $\Phi\colon (\Sigma_1, ds_1) \to (\Sigma_2, ds_2)$ such that $\Phi \circ \varphi_1$ and φ_2 are homotopic to each other.

Then the set of equivalence class of all marked hyperbolic metrics on S is called the *Teichmüller space* of S and is denoted by T_g.

Another way to define T_g is as follows. Let $\mathcal{H}$ be the set of all hyperbolic metrics on S. Clearly $\mathrm{Diff}(S)$ and $\mathrm{Diff}^0(S)$ act on $\mathcal{H}$. Then T_g can be identified with the quotient $\mathcal{H}/\mathrm{Diff}^0(S)$.

Clearly Mod_g acts on T_g by changing the marking via composition. In terms of the realization $T_g = \mathcal{H}/\mathrm{Diff}^0(S)$, this action comes from the action of $\mathrm{Diff}(S)$ on $\mathcal{H}$. Furthermore, this action is proper. See [23] and the references there for more about Teichmüller spaces and mapping class groups.

An important property of T_g is that it has a natural complex structure with respect to which it can be realized as bounded contractible domain in $\mathbb{C}^{3g-3}$, in particular it is diffeomorphic to $\mathbb{R}^{6g-6}$ and contractible. (In fact, the Fenchel-Nielson coordinates give explicit identifications of T_g with $\mathbb{R}^{6g-6}$.) The group Mod_g^+ acts on T_g by holomorphic automorphisms.

The Teichmüller space T_g admits several natural metrics, for example, the Weil-Peterson metric. With respect to the Weil-Peterson metric, T_g is a CAT(0)-space. See [46] for a summary and references. The positive solution to the Nielson realization problem gives that every finite subgroup of Mod_g has at least one fixed point in T_g. Then one can prove the following result.

Proposition 7.1. *The Teichmüller space T_g is an $\underline{E}\,\mathrm{Mod}_g$-space, i.e., a universal space for proper actions of Mod_g.*

The quotient $\mathrm{Mod}_g^+ \backslash T_g$ is equal to the moduli space $\mathcal{M}_g$ of all closed hyperbolic surfaces of genus g, i.e., the moduli space of projective curves of genus g. Note that taking the quotient by Mod_g^+ divides out the marking in the space T_g of marked hyperbolic metrics and only isometry classes of hyperbolic surfaces remain.

8 Truncated Teichmüller spaces, Borel-Serre compactifications, and curve complexes

In this section, we define the curve complex $\mathcal{C}(S)$ of a surface S and explain how it is related to the boundary of the Borel-Serre partial compactification $\overline{T_g}^{\mathrm{BS}}$ of the Teichmüller space T_g. We give a realization of the Borel-Serre partial compactification $\overline{T_g}^{\mathrm{BS}}$ by a truncated subspace $T_g(\varepsilon)$, and this gives a model of a cofinite universal space for proper actions of Mod_g. Then we explain an analogue of the Solomon-Tits theorem for curve complexes and use it together with the truncated subspace $T_g(\varepsilon)$ to obtain duality properties of mapping class groups.

It is known that the moduli space $\mathcal{M}_g$ is non-compact, and hence the action of Mod_g^+ on T_g is similar to the action of a *non-umiform* arithmetic subgroup Γ on the associated symmetric space X. This implies that T_g is not a Mod_g^+-cofinite $\underline{E}\,\mathrm{Mod}_g^+$-space.

In fact, starting with any closed hyperbolic metric, we can pinch along a simple closed geodesic and produce a family of hyperbolic surfaces in $\mathcal{M}_g$ which does not have any accumulation point in $\mathcal{M}_g$. More specifically, fix any simple closed curve c in S. For each marked hyperbolic surface $(\Sigma, ds; \varphi)$, $\varphi(c)$ gives a homotopy class of simple closed curves in Σ. Since (Σ, ds) has strictly negative sectional curvature, there is a unique closed geodesic in this homotopy class. Denote the length of this geodesic by $\ell_\Sigma(c)$.

Then for any ε sufficiently small, the marked hyperbolic $(\Sigma, ds; \varphi)$ can be deformed to a marked hyperbolic metric $(\Sigma_\varepsilon, ds_\varepsilon; \varphi)$ such that $\ell_{\Sigma_\varepsilon}(c) = \varepsilon$. Certainly, the image in $\mathcal{M}_g$ of this family of hyperbolic surfaces $(\Sigma_\varepsilon, ds_\varepsilon; \varphi)$ has no accumulation point in M_g.

Similarly, we can also pinch along several disjoint simple closed geodesics. It turns out that pinching along geodesics is the only reason for the noncompactness of the quotient $\mathrm{Mod}_g^+ \backslash T_g$. Specifically, for any small fixed positive constant ε, define a subspace $T_g(\varepsilon)$ by

$$T_g(\varepsilon) = \{(\Sigma, ds; \varphi) \mid \ell_\Sigma(c) \geq \varepsilon, \text{ for every simple closed curve } c\}. \tag{8}$$

Clearly $T_g(\varepsilon)$ is invariant under Mod_g. It is known that the quotient $\mathrm{Mod}_g \backslash T_g(\varepsilon)$ is compact [38].

Motivated by symmetric and locally symmetric spaces, a natural question is that of how to compactify T_g and its quotients for various applications, for example, to get an analogue of the Borel-Serre compactification, which will give a cofinite $\underline{E}\,\mathrm{Mod}_g$-space.

In fact, an analogue of the Borel-Serre compactification was constructed by Harvey [21]. He outlined a construction of a partial compactification of T_g which is a real analytic manifold with corners, denoted by $\overline{T_g}^{\mathrm{BS}}$, such that the quotient $\mathrm{Mod}_g \backslash \overline{T_g}^{\mathrm{BS}}$ is compact.

The boundary $\partial \overline{T_g}^{\mathrm{BS}}$ consists of contractible pieces that are parametrized by an infinite simplicial complex, called the *curve complex* of S, and denoted by $\mathcal{C}(S)$.

Briefly, for each simple closed curve c in S, denote by $[c]$ the homotopy class of c. Then the vertices of $\mathcal{C}(S)$ correspond to homotopy classes $[c]$ of simple closed curves. Any collection of distinct homtopy classes $[c_1], \dots, [c_k]$, forms the vertices of a $(k-1)$-simplex if and only if they contain disjoint representatives.

Since we can pinch along disjoint simple closed geodesics to go to the boundary at infinity of $\mathrm{Mod}_g \backslash T_g$ and this is basically the only way to go to infinity, it is reasonable that the curve complex $\mathcal{C}(S)$ describes the structure at infinity of T_g and its quotients such as M_g. For example, the curve complex was used in the original application of [21] to parametrize the boundary components of $\overline{T_g}^{\mathrm{BS}}$. In this sense, it is an analogue of the spherical Tits buildings for symmetric spaces.

As in the case of the Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$ for symmetric spaces, $\overline{T_g}^{\mathrm{BS}}$ can also be realized by the truncated subspace $T_g(\varepsilon)$. See [23] for details and references. In fact, it is easy to see that $T_g(\varepsilon)$ is a manifold with corners, and its boundary faces are parametrized by simplices of $\mathcal{C}(S)$.

The following result can also be proved [31].

Theorem 8.1. *The truncated space $T_g(\varepsilon)$ is a Mod_g-cofinite $\underline{E}\,\mathrm{Mod}_g$-space.*

As mentioned earlier, in some applications, the Solomon-Tits theorem on the homotopy type of spherical Tits buildings is important. An analogue of this is also true.

Theorem 8.2. *The curve complex $\mathcal{C}(S)$ has the homotopy type of bouquet spheres of dimension $2g-2$. Furthermore, there are infinitely many spheres in the bouquet.*

The first statement was proved by Harvey (see [20]), and the statement that the bouquet contains at least one sphere and the statement that its contains infinitely many spheres were proved recently in [24].

By combining the above theorem and the relation between $\overline{T_g}^{\mathrm{BS}}$ (or rather $T_g(\varepsilon)$) and $\mathcal{C}(S)$, the following result can be proved (see [20, 23, 24]).

Theorem 8.3. *Any torsion free, finite index subgroup of the mapping class group Mod_g is a duality group, but is not a Poincaré duality group.*

Remark 8.4. The above discussions show that the curve complex $\mathcal{C}(S)$ is an analogue of the spherical Tits buildings in many ways. The curve complexes and the spherical Tits buildings also enjoy the common property that they are not locally finite in general. In fact, a Tits building $\Delta(\mathbf{G})$ is not locally finite unless it is a finite simplicial complex. The reason for this is that if the field k is infinite and the group $\mathbf{G}$ is of rank strictly greater than 1, then every minimal k-parabolic subgroup of $\mathbf{G}$ is contained in infinitely many k-parabolic subgroups. For the curve complexes, the reason is that, given a simple closed curve c, there are infinitely many different simple closed curves that are disjoint from it. Therefore, there are infinitely many edges in $\mathcal{C}(S)$ coming out of the vertex $[c]$.

On the other hand, an important difference is that the buildings $\Delta(G)$ we discussed earlier contain apartments, which form a distinguished class of finite simplicial subcomplexes and are fundamental in buildings, but the curve complexes $\mathcal{C}(S)$ do not contain similar finite simplicial complexes in general.

Remark 8.5. As pointed out in §3 (Theorem 3.3 and Remark 3.6), spherical Tits buildings have played an important role in the Mostow strong rigidity of lattices in semisimple Lie groups and quasi-rigidity properties of symmetric spaces and Euclidean buildings. The curve complexes have also been crucial in proving similar quasi-rigidity properties of mapping class groups. See [6, 19].

9 Outer autormorphism groups and outer spaces

In this section, we first introduce another class of groups: the outer automorphism groups of free groups $\mathrm{Out}(F_n)$, which are related to arithmetic subgroups and mapping class groups. Then we introduce the outer spaces X_n, on which the outer automorphism groups act.

Briefly, let F_n be the free group on n generators, let $\mathrm{Aut}(F_n)$ be the group of all automorphisms of F_n, and let $\mathrm{Inn}(F_n)$ be the subgroup of inner automorphisms. Clearly, $\mathrm{Inn}(F_n)$ is a normal subgroup of $\mathrm{Out}(F_n)$. Define the *group of outer automorphisms* F_n, or the *outer automorphism group* of F_n, by

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\,\mathrm{Inn}(F_n)\,.$$

It is known that when $n = 2$,

$$\mathrm{Out}(F_2) = \mathsf{GL}(2, \mathbb{Z})\,.$$

For $n \geq 3$, there is a surjective map $\mathrm{Out}(F_n) \to \mathsf{GL}(n, \mathbb{Z})$ with a large kernel. Therefore, $\mathrm{Out}(F_n)$ is yet another natural generalization of the basic arithmetic subgroup $\mathsf{GL}(2, \mathbb{Z})$.

As pointed out earlier, the mapping class group Mod_1 for the closed surface of genus 1 is also equal to $\mathsf{GL}(2,\mathbb{Z})$. Therefore, these three classes of groups are closely related.

A natural and important problem is to prove results for $\mathrm{Out}(F_n)$ that are analogous to those for arithmetic groups and mapping class groups.

The analogue of the symmetric spaces and Teichmüller spaces for $\mathrm{Out}(F_n)$ is the outer space or the reduced outer space, which were first introduced by Culler and Vogtmann [17]. We briefly recall their definitions. See [7, 43, 44] for surveys on various results about $\mathrm{Out}(F_n)$ and outer spaces, and references.

A *metric graph* (G,ℓ) is a graph G which assigns a nonnegative length $\ell(e)$ to every edge e in G such that it is not degenerate in the sense that for every loop in the graph, its total length, i.e., the sum of lengths of edges in it, is positive. (Note that some edges could have length 0).

For each positive integer n, we are only interested in metric graphs (G,ℓ) satisfying the following conditions:

(1) G is of rank n, i.e., $\pi_1(G) \cong F_n$,

(2) G is normalized, i.e., the total sum of all edge lengths is equal to 1, and

(3) G is reduced. In other words, G does not contain any separating edge (called a bridge), or any node of valence $0, 1$ or 2.

A basic reduced graph of rank n is given by the rose R_n with n petals, i.e., a wedge of n circles S^1. In this case, the valence at the unique vertex is equal to $2n$ and hence large.

A *marked metric graph* G is a normalized, reduced metric graph (G,ℓ) together with a homotopy class of maps $\varphi\colon R_n \to G$ such that φ induces an isomorphism $\pi_1(R_n) \to \pi_1(G)$, i.e., φ is a homotopy equivalence. Two marked metric graphs are called equivalent if there exists an isometry between them which commutes with the markings.

Then *the reduced outer space* X_n is the space of equivalence classes of all such marked, reduced metric graphs of rank n with a suitable topology. (Note the similarity with the definition of the Teichmüller space of marked hyperbolic metrics.)

It turns out that X_n is an infinite simplicial complex. In fact, for every marking of a graph G, $\varphi\colon R_n \to G$, the set of all possible choices of normalized metrics on G is parametrized by a simplex, which is denoted by Σ_φ below.

Clearly $\mathrm{Out}(F_n)$ acts on X_n by changing the markings of the marked graphs. It was shown [17] that $\mathrm{Out}(F_n)$ acts properly on X_n, and X_n is contractible.

Furthermore, X_n is an $\underline{E}\,\mathrm{Out}(F_n)$-space, i.e., a universal space for proper actions of $\mathrm{Out}(F_n)$ [16, 45].

It can be seen easily that the quotient $\mathrm{Out}(F_n)\backslash X_n$ is non-compact. In fact, by making the total length of one loop go to zero, we get a sequence of points in the quotient which has no accumulation point.

It was also proved in [17] that the spine K_n of X_n is a simplicial complex of dimension $2n-3$, and is an $\mathrm{Out}(F_n)$-equivariant deformation retract of X_n with compact quotient $\mathrm{Out}(F_n)\backslash K_n$. It can also be shown that K_n is an $\mathrm{Out}(F_n)$-cofinite $\underline{E}\,\mathrm{Out}(F_n)$-space [17, 33, 45]. As a corollary, the virtual cohomological dimension of $\mathrm{Out}(F_n)$ is equal to $2n-3$.

For some other purposes, K_n is too small a model of an $\mathrm{Out}(F_n)$-cofinite $\underline{E}\,\mathrm{Out}(F_n)$-space. In [8], an analogue of the Borel-Serre type partial compactification $\overline{X}^{\mathrm{BS}}$ is constructed, and used to prove the following result.

Theorem 9.1. *Every torsion-free subgroup of* $\mathrm{Out}(F_n)$ *of finite index is a duality group of dimension* $2n-3$.

Since the quotient $\mathrm{Out}(F_n)\backslash X_n$ is non-compact, $\mathrm{Out}(F_n)$ is an analogue of a non-uniform arithmetic subgroup, and the following naturally expected result is proved in [30].

Theorem 9.2. *Every torsion-free subgroup of* $\mathrm{Out}(F_n)$ *of finite index is not a Poincaré duality group.*

10 Truncated outer spaces, Borel-Serre compactifications, and core graph complexes

Given the previous results on relationships between the Borel-Serre partial compactification of symmetric spaces and Tits buildings, and Teichmüller spaces and curve complexes, it is natural to look for an analogue to the spherical Tits buildings for the boundary of a Borel-Serre type partial compactification $\overline{X_n}^{\mathrm{BS}}$ of the outer space X_n.

As mentioned above, an analogue of the Borel-Serre partial compactification $\overline{X_n}^{\mathrm{BS}}$ is defined in [8]. A candidate for an analogue of Tits buildings was proposed in [22] and called the *complex of free factors* or *free factor complex*, but its relationship to the Borel-Serre partial compactification $\overline{X_n}^{\mathrm{BS}}$ of X_n in [8] is not clear. Questions about establishing this kind of relationship have been raised in [22, p. 459–460] and [43, p. 25, Problem 21].

In the following, we outline the construction of a truncated subspace $X_n(\varepsilon)$ of X, which is similar to the truncated symmetric space X_T and the truncated Teichmüller space $T_g(\varepsilon)$.

It was pointed out before that these truncated spaces are realizations of the Borel-Serre partial compactifications of X and T_g respectively. It can be shown that the space $X_n(\varepsilon)$ is also $\mathrm{Out}(F_n)$-equivariantly homeomorphic by cellular maps to the Borel-Serre partial compactification $\overline{X_n}^{\mathrm{BS}}$ constructed in [8].

We also outline the construction of a simplicial complex, called the *core graph complex*, $\mathcal{CG}(F_n)$, which is isomorphic to the complex of free factors of [22], and show how this core graph complex can be used to parametrize boundary components of $X_n(\varepsilon)$. Therefore, this provides an analogue to the relationship between the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ and the boundary of the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$.

In the definition of $T_g(\varepsilon)$, simple closed geodesics play an important role. The key point in defining a truncated outer space X_n is to find a replacement for collections of disjoint simple closed geodesics in surfaces for metric graphs.

A natural analogue of a simple closed geodesic in a graph G is a subgraph of G that contains no nodes of valence 0 or 1, which is roughly speaking a loop without spikes coming out.

But there is an important difference between closed geodesics in hyperbolic surfaces and the above loops in graphs: If two simple closed geodesics in a surface agree on any small segment, then they agree everywhere; but this is not true for the loops in graphs.

In the boundary of $T_g(\varepsilon)$, boundary faces of codimension 1 are defined by requiring that exactly one marked simple closed geodesic has length equal to ε, and boundary faces of higher codimension $k \geq 2$ are defined by requiring that exactly k marked disjoint simple closed geodesics have length equal to ε.

To obtain an analogue to the conditions for the boundary face of $T_g(\varepsilon)$ induced from the lengths of more than one more simple closed geodesics to get boundary faces of higher codimension, we need to use the notion of core subgraphs. Following [8], a *core subgraph* C of a graph G is a subgraph which contains no nodes of valence 0 or 1, or separating edges. It is not necessarily connected. Roughly, C is the union of subgraphs without isolated nodes or spikes, or more intuitively a union of loops which could have overlaps on some edges. An important invariant of a core subgraph C is the *rank*, and is denoted by $r(C)$: if the connected components of C are joined by minimal number of bridges to get a new connected subgraph $\tilde{C}$, then $r(C)$ is defined to be the rank of the fundamental group $\pi_1(\tilde{C})$. Roughly speaking, the rank $r(C)$ is equal to the number of loops in it.

Note that if C_1, C_2 are two disjoint core subgraphs of rank r_1 and r_2 respectively, then $C = C_1 \cup C_2$ is a core subgraph, and $r(C) = r_1 + r_2$.

Let (G, ℓ) be a metric graph. For any core subgraph C of G, let $\ell(C)$ denote the sum of lengths of all edges of C.

For a sufficiently small positive number ε, define a *truncated subspace* $X_n(\varepsilon)$ by

$$X_n(\varepsilon) = \{(G, \ell) \in X_n \mid \ell(C) \geq 3^{r(C)-1}\varepsilon\,,\ \text{for every core subgraph } C \text{ of } G\}. \tag{9}$$

If C consists of one loop, then the condition is

$$\ell(C) \geq \varepsilon\,.$$

If C is of rank 2 and is the disjoint union of two rank 1 core subgraphs C_1, C_2, then

$$\ell(C) = \ell(C_1) + \ell(C_2)\,,$$

and the condition

$$\ell(C) \geq 3\varepsilon$$

is not implied by the conditions:

$$\ell(C_1) \geq \varepsilon, \quad \ell(C_2) \geq \varepsilon\,.$$

If C_1 and C_2 contain some common edges, for metrics in which the common edges have very short lengths, the same reasoning shows that the lower bound on $\ell(C)$ gives a new condition.

The above discussion explains one reason for the choice of the inequality $\ell(C) \geq 3^{r(C)-1}\varepsilon$. In this way, each core subgraph gives a new restriction, as in the case of the truncated Teichmüller space $T_g(\varepsilon)$, where simultaneous restrictions on the lengths of disjoint simple closed geodesics give new restrictions. Of course, the base number 3 in the above inequality (equation (9)) can be replaced by any number strictly greater than 2.

When $n = 2$, for each marking $\varphi\colon R_2 \to G$ of graphs which contain 3 edges and 2 loops, all possible normalized reduced metrics on G form a 2-simplex Σ_φ. It can be seen easily that the intersection

$$\Sigma_\varphi(\varepsilon) = \Sigma_\varphi \cap X_n(\varepsilon)$$

is a hexagon obtained by cutting off a small triangle near every vertex.

In general, for $n \geq 2$ and each fixed marking $\varphi\colon R_n \to G$ of graphs, all possible normalized reduced metrics on G form a (k-1)-simplex Σ_φ, where k is the number of edges in G. Similarly, the intersection $\Sigma_\varphi(\varepsilon) = \Sigma_\varphi \cap X_n(\varepsilon)$ is a convex polytope obtained by cutting off suitable neighborhoods of some proper faces of Σ_φ. In fact, the boundary faces of $\Sigma_\varphi(\varepsilon)$ are parametrized by chains of proper core subgraphs of G. Specifically, for every such chain

$$C_1 \subset C_2 \subset \cdots \subset C_k\,, \tag{10}$$

the corresponding face is determined by the equalities:

$$\ell(C_i) = 3^{r(C_i)-1}\varepsilon, \quad i = 1, \dots, k\,, \tag{11}$$

and for any other core subgraph C,

$$\ell(C) > 3^{r(C)-1}\varepsilon\,.$$

Note that no condition in equation (11) is a redundant, and that this is the key observation in defining $X_n(\varepsilon)$ as pointed out above.

Using the above description of the case $n = 2$, and the description of boundary faces of $\Sigma_\varphi(\varepsilon)$ in terms of chains of core subgraphs, and induction on n and the length of chains, it can be shown that the closure $\overline{\Sigma_\varphi}$ of each simplex Σ_φ in the Borel-Serre partial compactification $\overline{X_n}^{\mathrm{BS}}$ defined in [8] has the same decomposition into polyhedra as the decomposition of $\Sigma_\varphi(\varepsilon)$ into the boundary faces which were described above. This implies that the closure $\overline{\Sigma_\varphi}$ in $\overline{X_n}^{\mathrm{BS}}$ is homeomorphic to $\Sigma_\varphi(\varepsilon)$. From this it follows that $X_n(\varepsilon)$ has the same structure as $\overline{X_n}^{\mathrm{BS}}$ and is equivariantly homeomorphic to $\overline{X_n}^{\mathrm{BS}}$ by cellular maps.

For both $X_n(\varepsilon)$ and $\overline{X_n}^{\mathrm{BS}}$, taking each face of $\Sigma_\varphi(\varepsilon)$ or $\overline{\Sigma_\varphi}$ as a boundary component will result in so many boundary components that they might not correspond to simplices of an analogue of a spherical Tits building. The basic point here is that we need to *glue* suitable collections of such faces together into boundary components in order that they correspond to *simplices* of an analogue of a spherical Tits building.

To explain and motivate this, we examine the case $n = 2$. It is known that X_2 can be canonically identified with the Poincaré upper half plane $\mathbb{H}^2$ (see [43]) and the simplicial complex X_2 gives a triangulation of $\mathbb{H}^2$ by ideal triangles with vertices at rational boundary points, i.e., points in $\mathbb{Q} \cup \{i\infty\} \subset \partial\mathbb{H}^2$.

Under this identification, the truncated space $X_2(\varepsilon)$ is a truncated subspace of $\mathbb{H}^2$ obtained by removing suitable horodiscs at these rational boundary points, whose sizes are determined by ε. In other words, $X_2(\varepsilon)$ is exactly the realization of the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$ given by the truncated subspace X_T as discussed in §4 when $X = \mathbb{H}^2$.

The spherical Tits building for $\mathbf{G} = \mathsf{SL}(2, \mathbb{C})$ defined over $\mathbb{Q}$ is equal to $\mathbb{Q} \cup \{i\infty\}$, and the boundary components of $\mathbb{H}^2_T$ are horocycles. In terms of $X_2(\varepsilon)$, each horocycle of a rational boundary point z is the union of a face of the truncated simplices $\Sigma_\varphi(\varepsilon)$, where z is an ideal vertex of Σ_φ and the face of $\Sigma_\varphi(\varepsilon)$ corresponds to this vertex.

We next outline the construction of the core graph complex $\mathcal{CG}(F_n)$ mentioned above and indicate how it is isomorphic to the complex of free factors in [22] and how to use it to decompose the boundary $\partial X_n(\varepsilon)$ into boundary components which are parametrized by simplices of $\mathcal{CG}(F_n)$.

By definition, the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ is the union of finite subcomplexes, which are finite Coxeter complexes and called called apartments. We can construct the building $\Delta_{\mathbb{Q}}(\mathbf{G})$ by gluing these apartments together. We will construct the core graph complex $\mathcal{CG}(F_n)$ by gluing a collection of some finite complexes together.

Consider graphs G with only nodes of valence 3. Then G has $3n-3$ edges, where we assume as before that $\pi_1(G) \cong F_n$. In this case, the graph G has the maximal numbers of edges under the assumption that G does not contain vertices of valences $0, 1$ or 2. As pointed out earlier, every marked graph $\varphi\colon R_n \to G$ corresponds to a simplex Σ_φ in the outer space X_n which parametrizes all possible normalized metrics on the graph G. This implies that every marking on such a graph G with only nodes of valence 3 corresponds to a simplex Σ_φ of maximal dimension in X_n.

For every marking $\varphi\colon R_n \to G$, define a finite simplicial complex A_φ as follows:

(1) The vertices of A_φ correspond to core subgraphs contained in G.

(2) Let C_1, C_2 be two core subgraphs. Then their corresponding vertices form the vertices of a 1-simplex if and only if they form a chain, i.e., either $C_1 \subset C_2$ or $C_2 \subset C_1$. More generally, the vertices corresponding to a collection of core subgraphs $C_1, \ldots, C_k$ are the vertices of a $(k-1)$-simplex if and only if the core subgraphs $\{C_i\}$ form an increasing chain after reordering if necessary.

Define the *core graph complex* $\mathcal{CG}(F_n)$ of F_n by

$$\mathcal{CG}(F_n) = \coprod_{\varphi} A_\varphi / \sim\,, \tag{12}$$

where φ ranges over all nonequivalent markings $\varphi\colon R_n \to G$, and the identification $\sim$ is defined as follows. Note that every loop of G induces an element of $F_n \cong \pi_1(R_n) \cong \pi_1(G)$ which can be used as one element of a set of generators of F_n, under the map φ and the identification $F_n \cong \pi_1(R_n)$. Thus a core subgraph of G gives a free factor of F_n.[1] Then a simplex of A_φ corresponds to a chain of free factors of F_n. For two markings φ and φ', we identify two vertices from A_φ and $A_{\varphi'}$ under the relation $\sim$ if they correspond to the same free factor of F_n; similarly, we identify two simplices of A_φ and $A_{\varphi'}$ if their corresponding chains of free factors agree.

Remark 10.1. A more geometric way to define and understand the equivalence relation $\sim$ in defining $\mathcal{CG}(F_n)$ in equation (12) is as follows. Given a marked graph $\varphi\colon R_n \to G$, every core subgraph of G is pulled back to a subgraph of R_n. Similarly a chain of core subgraphs of G is pulled back under φ to a chain of subgraphs of R_n. Given two marked graphs $\varphi_1\colon R_n \to G_1$ and $\varphi_2\colon R_n \to G_2$, and two chains of core subgraphs in G_1 and G_2 respectively, then the corresponding simplices in A_{φ_1} and A_{φ_2} are defined to be equivalent if their pull-backs in R_n under φ_1 and φ_2 are homotopy equivalent chains of subgraphs.

This is closely related to the definition of the curve complex $\mathcal{C}(S)$, where a simplex is defined to a collection of homotopy classes of disjoint simple closed curves. It is worthwhile to point out that in the markings for the Teichmüller space T_g, $\varphi\colon S \to \Sigma$, the surfaces S and Σ are homeomorphic, and we can consider simple closed curves on a common surface. On the other hand, for marked graphs, G is usually not *homeomorphic* to R_n. This is the reason why we need to pull back chains of core subgraphs of G_1, G_2 to R_n and require them to be homotopic.

Since every free factor of F_n and every chain of free factors arise from core subgraphs this way from a core subgraph and a chain of core subgraphs, it follows that the core graph complex $\mathcal{CG}(F_n)$ is isomorphic to the factor complex in [22].

Remark 10.2. For each marked graph $\varphi\colon R_n \to G$ with only nodes of valency 3, the finite simplicial complex A_φ seems to be an analogue of an apartment of a spherical Tits building. One reason for this is that the spherical Tits building can also be obtained from the apartments by identifying along simplices of smaller dimension. As a consequence, the spherical Tits building has the same dimension as the apartments. Clearly, $\mathcal{CG}(F_n)$ also has the same dimension as A_φ.

Now we use the simplices in $\mathcal{CG}(F_n)$ to decompose the boundary $\partial X_n(\varepsilon)$ into boundary components. For a simplex $\sigma \in \mathcal{CG}(F_n)$, consider all subcomplexes A_φ that contain σ. We pointed out earlier that the faces of the polytope $\Sigma_\varphi(\varepsilon)$

[1] Geometrically, we can visualize a rose R_n obtained from G by collapsing some edges so that a core subgraph C of G is mapped to the union of $r(C)$ petals (each loop in C is mapped to one petal). Note that the petals obtained in this way are usually different from the petals in R_n under the marking $\varphi\colon R_n \to G$. Therefore, the marking φ plays a crucial role here.

correspond to chains of core subgraphs (see the discussion near equation (10)). Let $f_{\varphi,\sigma}$ be the face of $\Sigma_\varphi(\varepsilon)$ corresponding to the chain of core subgraphs which determines σ. Define the *boundary component* of $\partial_\sigma X_n(\varepsilon)$ associated with the simplex σ by

$$\partial_\sigma X_n(\varepsilon) = \bigcup_\varphi f_{\varphi,\sigma}\,, \tag{13}$$

where φ ranges over all markings with $A_\varphi \supset \sigma$. The topology on $\partial_\sigma X_n(\varepsilon)$ is induced from the topology of the ambient space X_n.

Therefore, we have decomposed the boundary of $X_n(\varepsilon)$ and hence of the Borel-Serre partial compactification $\overline{X_n}^{\mathrm{BS}}$ defined in [8] into *boundary components* which are *parametrized* by *simplices of the core graph complex $\mathcal{CG}(F_n)$*, or equivalently of the free factor complex in [22]. This gives one answer to some questions in [22, p. 459–460] and [43, p. 25, Problem 21] as mentioned above.

It can be easily seen that when $n = 2$ and X_2 is identified with the Poincaré upper half plane $\mathbb{H}^2$ as before, every boundary component $\partial_\sigma X_2(\varepsilon)$ is a horocycle of $\mathbb{H}^2$ at a rational boundary point. This is exactly a boundary component of the truncated submanifold X_T, which gives a realization of the Borel-Serre partial compactification $\overline{X}^{\mathrm{BS}}$ from §4, where $X = \mathbb{H}^2$.

Note that since a boundary component $\partial_\sigma X_2(\varepsilon)$ is a horocycle, it is clearly contractible. It seems that for general $n \geq 2$, every boundary component $\partial_\sigma X_n(\varepsilon)$ is also contractible. Intuitively this can be seen as follows. It seems that in a contractible simplicial complex, the union of all simplices containing a given *ideal* simplex, or rather a *missing* boundary simplex, is also contractible. Now, since each simplex can be deformation retracted to the link of the ideal simplex and the retractions are compatible, they can be glued to a deformation retract of the union of these simplices of the link. This would imply that the link of the missing simplex is also contractible. The boundary component $\partial_\sigma X_n(\varepsilon)$ is such a link and is hence contractible.

Assume that all boundary components $\partial_\sigma X_n(\varepsilon)$ are contractible. This implies that the boundary $\partial X_n(\varepsilon)$ of the truncated subspace $X_n(\varepsilon)$, or equivalently the Borel-Serre partial compactification $\overline{X_n}^{\mathrm{BS}}$ in [8], has the same homotopy type as the core graph complex $\mathcal{CG}(F_n)$, or equivalently, the free factor complex. By [22, Theorem 1.1], the free factor complex and hence the core graph complex $\mathcal{CG}(F_n)$ has the homotopy type of a bouquet of spheres of dimension $n-2$. This is an analogue of the Solomon-Tits Theorem for buildings and curve complexes. Then the parametrization of the boundary components of the truncated subspace $X_n(\varepsilon)$ by the simplices of the core subgraph complex $\mathcal{CG}(F_n)$ would be a close analogue to the parametrization of the boundary components of the Borel-Serre partial compactification of symmetric spaces and Teichmüller spaces by simplices of the spherical Tits buildings and curve complexes respectively. This gives a more satisfactory answer to the questions raised in [22, p. 459–460] and [43, p. 25, Problem 21] and establishes yet another similarity between the *three important classes of groups*: arithmetic subgroups Γ of semisimple algebraic groups, mapping class groups Mod_g of surfaces, and outer automorphism groups $\mathrm{Out}(F_n)$ of free groups.

References

[1] **M. Anderson** and **R. Schoen**, Positive harmonic functions on complete manifolds of negative curvature, *Ann. of Math.* **121** (1985), 429–461.

[2] **W. Ballmann**, Manifolds of nonpositive sectional curvature and manifolds without conjugate points, in *Proc. of International Congress of Mathematicians*, (Berkeley, Calif., 1986), pp. 484–490, Amer. Math. Soc., 1987.

[3] ———, On the Dirichlet problem at infinity for manifolds of nonpositive curvature, *Forum Math.* **1** (1989), 201–213.

[4] **W. Ballmann**, **M. Gromov** and **V. Schroeder**, *Manifolds of Nonpositive Curvature*, Progr. Math. vol. **61**, Birkhäuser Boston, 1985.

[5] **W. Ballmann** and **F. Ledrappier**, The Poisson boundary for rank one manifolds and their cocompact lattices, *Forum Math.* **6** (1994), 301–313.

[6] **J. Behrstock**, **B. Kleiner**, **Y. Minsky** and **L. Mosher**, Geometry and rigidity of mapping class groups, *preprint*. `arXiv:0801.2006`.

[7] **M. Bestvina**, The topology of $\mathrm{Out}(F_n)$, **in** *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002), pp. 373–384, Higher Ed. Press, 2002.

[8] **M. Bestvina** and **M. Feighn**, The topology at infinity of $\mathrm{Out}(F_n)$, *Invent. Math.* **140** (2000), 651–692.

[9] **A. Borel** and **Harish-Chandra**, Arithmetic subgroups of algebraic groups, *Ann. of Math.* **75** (1962), 485–535.

[10] **A. Borel** and **L. Ji**, *Compactifications of Symmetric and Locally Symmetric Spaces*, *Math. Theory Appl.*, Birkhäuser Boston, 2006.

[11] **A. Borel** and **J.-P. Serre**, Corners and arithmetic groups, *Comment. Math. Helv.* **48** (1973), 436–491.

[12] ______, Cohomologie d'immeubles et de groupes S-arithmétiques, *Topology* **15** (1976), 211–232.

[13] **M. R. Bridson** and **A. Haefliger**, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. **319**, Springer-Verlag, Berlin, 1999.

[14] **K. Brown**, *Buildings*, Springer Monogr. Math.. Springer-Verlag, 1998.

[15] ______, *Cohomology of Groups*, Corrected reprint of the 1982 original. Grad. Texts in Math. **87**, Springer-Verlag, 1994.

[16] **M. Culler**, Finite groups of outer automorphisms of a free group, **in** *Contributions to group theory*, pp. 197–207, Contemp. Math., vol. **33**, Amer. Math. Soc., 1984.

[17] **M. Culler** and **K. Vogtmann**, Moduli of graphs and automorphisms of free groups, *Invent. Math.* **84** (1986), 91–119.

[18] **Y. Guivarch**, **L. Ji** and **T. C. Taylor**, *Compactifications of Symmetric Spaces*, Progr. Math. **156**, Birkhäuser Boston, 1998.

[19] **U. Hamenstädt**, Geometry of the mapping class groups III: Quasi-isometric rigidity, *preprint*. `arXiv:math/0512429`.

[20] **J. Harer**, The cohomology of the moduli space of curves, **in** *Theory of moduli*, pp. 138–221, Lecture Notes in Math. vol. **1337**, Springer, 1988.

[21] **W. Harvey**, Boundary structure of the modular group, **in** *Riemann surfaces and related topics*, pp. 245–251, Ann. of Math. Stud. **97**, Princeton Univ. Press, 1981.

[22] **A. Hatcher** and **K. Vogtmann**, The complex of free factors of a free group, *Quart. J. Math. Oxford Ser.* **49** (1998), 459–468.

[23] **N. Ivanov**, *Mapping Class Groups*, in *Handbook of geometric topology*, pp. 523–633, North-Holland, 2002.

[24] **N. Ivanov** and **L. Ji**, Infinite topology of curve complex and non-Poincaré duality of Teichmüller modular groups, *Enseign. Math.* **54** (2008), 381–395.

[25] **L. Ji**, Buildings and their applications in geometry and topology, *Asian J. Math.* **10** (2006), 11–80.

[26] ______, Integral Novikov conjectures and arithmetic groups containing torsion elements, *Commun. Anal. Geom.* **15** (2007), 509-533.

[27] ———, *Arithmetic Groups and Their Generalizations: What, Why and How*, AMS/IP Stud. Adv. Math. **43**, Amer. Math. Soc., Providence, RI; Intern. Press, Cambridge, MA, 2008.

[28] ———, A summary of some work of Gregory Margulis, *Pure Appl. Math. Q.* **4** (2008), 1–69.

[29] ———, The integral Novikov conjectures for linear groups containing torsion elements, *J. Topol.* **1** (2008), 306–316.

[30] ———, Steinberg representations, and duality properties of arithmetic groups, mapping class groups and outer automorphism groups of free groups, **in** *Representation Theory*, pp. 71–98, Contemp. Math., vol. **478**, Amer. Math. Soc., Providence, RI, 2009.

[31] **L. Ji** and **S. Wolpert**, A cofinite universal space for proper actions for mapping class groups, **in** *In the Tradition of Ahlfors-Bers, V*, eds. M. Bonk, J. Gilman, H. Masur, Y. Minsky and M. Wolf, Contemp. Math., vol. **510**, pp. 151–163, Amer. Math. Soc., 2010.

[32] **B. Kleiner** and **B. Leeb**, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Publ. Math. Inst. Hautes Études Sci.* **86** (1997), 115–197.

[33] **S. Krstic** and **K. Vogtmann**, Equivariant outer space and automorphisms of free-by-finite groups, *Comment. Math. Helv.* **68** (1993), 216–262.

[34] **B. Leeb**, *A Characterization of Irreducible Symmetric Spaces and Euclidean Buildings of Higher Rank by Their Asymptotic Geometry*, Bonner Math. Schriften **326**, Universität Bonn, Mathematisches Institut, Bonn, 2000.

[35] **E. Leuzinger**, An exhaustion of locally symmetric spaces by compact submanifolds with corners, *Invent. Math.* **121** (1995), 389–410.

[36] **G. Mostow**, *Strong rigidity of locally symmetric spaces*, Annals of Math. Stud., No. **78**, Princeton University Press, 1973.

[37] **G. Mostow** and **T. Tamagawa**, On the compactness of arithmetically defined homogeneous spaces, *Ann. of Math.* **76** (1962), 446–463.

[38] **D. Mumford**, A remark on Mahler's compactness theorem, *Proc. Amer. Math. Soc.* **28** (1971), 289–294.

[39] **G. Prasad**, *Lattices in semisimple groups over local fields*, in *Studies in algebra and number theory*, pp. 285–356, Adv. in Math. Suppl. Stud., vol. **6**, Academic Press, 1979.

[40] **L. Saper**, Tilings and finite energy retractions of locally symmetric spaces, *Comment. Math. Helv.* **72** (1997), 167–202.

[41] **R. Spatzier**, An invitation to rigidity theory, in *Modern dynamical systems and applications*, pp. 211–231, Cambridge Univ. Press, 2004.

[42] **J. Tits**, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math., Vol. **386**, Springer-Verlag, 1974.

[43] **K. Vogtmann**, Automorphisms of free groups and outer space, *Geom. Dedicata* **94** (2002), 1–31.

[44] ______, The cohomology of automorphism groups of free groups, **in** *Proceedings of the International Congress of Mathematicians*, Vol. II, pp. 1101–1117, Eur. Math. Soc., 2006.

[45] **T. White**, Fixed points of finite groups of free group automorphisms, *Proc. Amer. Math. Soc.* **118** (1993), 681–688.

[46] **S. Wolpert**, Geometry of the Weil-Petersson completion of Teichmüller space, **in** *Surveys in differential geometry*, Vol. VIII, pp. 357–393, Surv. Differ. Geom., VIII, Int. Press, 2003.

Lizhen Ji

Department of Mathematics, University of Michigan, East Hall, 530 Church Street, Ann Arbor, MI 48109

e-mail: `lji@umich.edu`

website: `http://www.math.lsa.umich.edu/~lji`