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Codistances in buildings

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Abstract

A codistance in a building is a twinning of this building with one chamber. We study this local situation and prove that affine Bruhat-Tits buildings defined over p-adic numbers do not admit a codistance.

Keywords: twin building, p-adic building, affine building

MSC 2000: 51E24

1 Introduction

Twin buildings have been introduced by Ronan and Tits (see [13]) to give a geometric interpretation of the Kac-Moody groups. Roughly, a twin building is a pair of buildings of the same type endowed with a *codistance* between the chambers of the different buildings. The codistance takes values in the corresponding Coxeter group (the Weyl group of the buildings). The idea is that, given a chamber c in one building, there is a set $c^{\rm op}$ of chambers in the other building opposite c, and the codistance to c measures the distance to this set $c^{\rm op}$ "in a consistent way", i.e., according to some rules which are complementary to the rules of the Weyl distance. As always happens with important mathematical objects, people try to find axioms which characterize the objects, but which, at the same time, are easier to check, or ostensibly weaker. For twin buildings, the notion of a 2-*twinning*, introduced by the first author in [7], was important in this direction, but Abramenko and the second author showed in [2] that a 1-*twinning* (which is weaker than a 2-twinning) is not sufficient for a twinning (and they give a sufficient additional axiom). Now, 1- and 2-twinnings are still global conditions in that all chambers play the same role. Another, local way, to weaken the axioms is to fix one chamber and require the codistance axioms only with respect to that chamber; this approach yields precisely the axioms of a *codistance* on a given building. Up to now, it has been an open question whether such a codistance function on a building suffices for a twinning and there are indeed results in this direction [5]. On the other hand, also the other extreme seemed to be possible, i.e., that every building admits a codistance, as was recently considered by Ronan [10]. Ronan's question is the starting point of the present note, the purpose of which is to disprove that every building admits a codistance. In particular, it will follow from our Corollary 6.5 below that affine buildings arising from p-adic fields do not admit any codistance (but our result applies to a wider class of affine buildings, including non-classical ones).

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2 Preliminaries

We quickly review the notions of Coxeter groups, Coxeter systems and buildings.

A *Coxeter system* (W, S) consists of a group W presented by a set S of involutions where the (only) relations express the order of the products of the generators. The *length* of $w \in W$ is the minimal number of elements of S needed to write w as a product.

Let (W, S) be a Coxeter system and let $\ell\colon W \to \mathbb{N}$ denote the associated length function. A subset J of S is called *spherical* if $W_J := \langle J \rangle$ is a finite subgroup. If J is a spherical subset of S, then there is a unique longest element in W_J, which is an involution and which we denote by r_J.

Let (W, S) be a Coxeter system. A *building of type* (W, S) is a pair $\mathbf{B} = (C, \delta)$ where C is a set whose elements are called *chambers* and where $\delta\colon C \times C \to W$ is a distance function satisfying the following axioms for arbitrary chambers x, y, z:

(BU1) $\delta(x, y) = \delta(y, x)^{-1}$;

(BU2) if $\delta(x, y) = w \in W$ and $\delta(y, z) = s \in S$, then $\delta(x, z) \in \{w, ws\}$, and if $\ell(ws) > \ell(w)$, then $\delta(x, z) = ws$;

(BU3) If $\delta(x, y) = w \in W$ and $s \in S$, then there exists $u \in C$ such that $\delta(y, u) = s$ and $\delta(x, u) = ws$.

Let $\mathbf{B} = (C, \delta)$ be a building of type (W, S). Let c be a chamber and $J \subset S$. The *J-residue of* c is the set $R_J(c) := \{d \in C \mid \delta(c, d) \in W_J\}$. A *$J$-residue of* $\mathbf{B}$ is a J-residue of some chamber. A *residue of cotype* $j \in S$ is an $S \setminus \{j\}$-residue.

Let $s \in S$. An *s-panel* is an $\{s\}$-residue of $\mathbf{B}$; a *panel* is an s-panel for some

$s \in S$. We call $c, d \in C$ *s-adjacent* if they are both contained in an s-panel. They are called *adjacent* if they are s-adjacent for some $s \in S$.

A subset C' of C is called *thick* (*firm, thin, meager*) if for each panel P containing a chamber of C' the set $|P \cap C'|$ contains at least 3 (at least 2, precisely 2, at most 2, respectively) elements. The building $\mathbf{B} = (C, \delta)$ is called *thick* (*thin*) if C is thick (thin). Note that a building is firm by definition.

Let $c, d \in C$. We put $\ell(c, d) := \ell(\delta(c, d))$. A *gallery of length k from c to d* is a sequence of chambers $c = c_0, \ldots, c_k = d$ such that any two consecutive chambers are adjacent. It is called *minimal* if $k = \ell(\delta(c, d))$. A subset C' of C is called *convex* if it contains all minimal galleries between any two chambers of C'. An *apartment* of $\mathbf{B}$ is a thin convex subset of C.

The following lemmas are well known facts about buildings (see [9, 3]):

Lemma 2.1. *Residues are convex.*

Lemma 2.2. *Let C' be a firm convex set. Then $\mathbf{B}' := (C', \delta|_{C' \times C'})$ is a building of type (W, S). In particular, each apartment is a thin building.*

Let J be a subset of S and let R be a J-residue. Then $\mathbf{B}_R := (R, \delta|_{R \times R})$ is a building of type (W_J, J). Moreover, if Σ is an apartment of $\mathbf{B}$ intersecting R non-trivially, then $R \cap \Sigma$ is an apartment of $\mathbf{B}_R$.

Lemma 2.3. *Let R be a J-residue and let x be a chamber. Then there exists a unique chamber c in R such that $\delta(y, x) = \delta(y, c)\delta(c, x)$ and $\ell(y, x) = \ell(y, c) + \ell(c, x)$, for all chambers $y \in R$.*

The unique chamber c of the previous lemma is called the *projection of x onto R* and it is denoted by $\operatorname{proj}_R x$.

Lemma 2.4. *Let C' be a convex set of chambers and R a residue containing some chamber of C'. Then $\operatorname{proj}_R x \in C'$ for all chambers x in C'.*

A subset α of C is called a *root* if there exists a chamber $c \in \alpha$, an apartment Σ of $\mathbf{B}$ containing c, and a panel P containing c satisfying

$$\alpha := \{x \in \Sigma \mid \operatorname{proj}_P x = c\}.$$

If $c' \neq c$ is the second chamber in P, then the root $\alpha' = \{x \in \Sigma \mid \operatorname{proj}_P x = c'\}$ is called a *complementary root to α*. Note that $\alpha' = \Sigma \setminus \{\alpha\}$. The set of panels in Σ intersecting both α and α' nontrivially is called a *wall*.

Lemma 2.5. *If two apartments Σ_1, Σ_2 intersect in a root α, then $\Sigma := (\Sigma_1 \setminus \alpha) \cup (\Sigma_2 \setminus \alpha)$ is also an apartment of $\mathbf{B}$.*

Proof. For $i = 1,2$ let $\alpha_i := \Sigma_i \setminus \alpha$ and r_i the unique reflection of Σ_i interchanging α and α_i. Put $X = \Sigma_1 \cup \Sigma_2$ and consider the chamber system $\mathcal{X} := (X, (\sim_s)_{s \in S})$. It is easily verified that there is an automorphism $\bar{r}_i$ of $\mathcal{X}$ whose restriction to Σ_i is r_i and whose restriction to $X \setminus \Sigma_i$ is the idendity for $i = 1,2$. Now, the automorphism $\bar{r}_2 \circ \bar{r}_1$ maps Σ_1 onto $\alpha_1 \cup \alpha_2$ which shows that the chamber system induced on $\alpha_1 \cup \alpha_2$ is isomorphic to the chamber system induced on Σ_1. This yields the claim. $\square$

3 Systems of sectors and blow-ups

Let $\mathbf{B} = (C, \delta)$ be a building of type (W, S) and let $R \subset C$ be a J-residue where $J \subset S$. The building $\mathbf{B}$ is called a *blow-up of* R if $\Sigma \cap R \neq \emptyset$ for each apartment Σ of $\mathbf{B}$. A *system of R-sectors* is a family $\mathbf{S} = (S_c)_{c \in R}$ such that for all $c \in R$ one has $c \in S_c \subset \mathrm{proj}_R^{-1}(c)$.

Let $\mathbf{B}, R, J$ be as before and let $\mathbf{S} = (S_c)_{c \in R}$ be a system of R-sectors. An apartment $\overline{\Sigma}$ of R is called $\mathbf{S}$-*admissible* if

$$\Sigma := \bigcup_{c \in \overline{\Sigma}} S_c$$

is an apartment of $\mathbf{B}$. The system $\mathbf{S}$ of R-sectors is *admissible* if all apartments of R are $\mathbf{S}$-admissible.

Proposition 3.1. *Let $\mathbf{B}$ be a building of type (W, S), let $J \subset S$ and let R be a J-residue of $\mathbf{B}$. Let $\mathbf{S} = (S_c)_{c \in R}$ be an admissible system of R-sectors and put $C_{\mathbf{S}} := \bigcup_{c \in R} S_c$.*

Then $C_{\mathbf{S}}$ is a convex and firm subset of chambers and the building $\mathbf{B}_{\mathbf{S}} := (C_{\mathbf{S}}, \delta|_{C_{\mathbf{S}} \times C_{\mathbf{S}}})$ is a blow-up of R.

Proof. Let c be any chamber in R and let $\overline{\Sigma}$ be an apartment of R containing c. Since $\mathbf{S}$ is an admissible system of R-sectors we know that $\Sigma := \bigcup_{d \in \overline{\Sigma}} S_d$ is an apartment containing S_c from which it readily follows that S_c is a meager subset of C.

Now, let $x, y \in C_{\mathbf{S}}$ and put $c := \mathrm{proj}_R\, x$, $d := \mathrm{proj}_R\, y$ and let $\bar{\Sigma}$ be an apartment of R containing c and d. Since $\mathbf{S}$ is admissible the set $\Sigma := \bigcup_{e \in \overline{\Sigma}} S_e$ is an apartment of $\mathbf{B}$. Moreover, we have $x, y \in \Sigma \subset C_{\mathbf{S}}$. Since Σ is convex, it follows that each minimal gallery from x to y is contained in $C_{\mathbf{S}}$. Hence $C_{\mathbf{S}}$ is a convex subset of chambers. Moreover, the intersection of any panel containing x with Σ has cardinality 2 and therefore any panel containing x contains at least 2 chambers in $C_{\mathbf{S}}$. This shows that $C_{\mathbf{S}}$ is a firm subset of chambers. Hence $\mathbf{B}_{\mathbf{S}}$ is a building.

Now, let x be a chamber in $C_{\mathbf{S}}$ which is not contained in R and put $c := \operatorname{proj}_R x$. Let y be a chamber adjacent to x such that $\ell(c,y) = \ell(c,x) - 1$. Since $C_{\mathbf{S}}$ is convex it follows that $y \in C_{\mathbf{S}}$. Moreover, we have $c = \operatorname{proj}_R y$ and hence x and y are both in S_c. Let x' be any chamber in $C_{\mathbf{S}} \cap P$ where P is the unique panel containing x and y. As $x \neq y \in P$ and $c = \operatorname{proj}_R x = \operatorname{proj}_R y$ it follows that $\operatorname{proj}_R x' = c$. We deduce that x, y and x' are all contained in S_c which is a meager subset. Thus $x' = x$ or $x' = y$. We conclude that any apartment of $C_{\mathbf{S}}$ containing x also contains y. An easy induction on the $\ell(x, \operatorname{proj}_R x)$ shows now that any apartment Σ of $C_{\mathbf{S}}$ containing x also contains $\operatorname{proj}_R x$ and hence $\Sigma \cap R$ is non-empty. This completes the proof of the proposition. $\square$

4 A local criterion for admissibility

Throughout this section we assume that $\mathbf{B} = (C, \delta)$ is a building of type (W, S) and that R is a J-residue of $\mathbf{B}$.

Lemma 4.1. *Let $\mathbf{S} = (S_c)_{c \in R}$ be a system of R-sectors and let $\overline{\Sigma}_1, \overline{\Sigma}_2$ be two $\mathbf{S}$-admissible apartments of R intersecting in a root $\overline{\alpha}$. For $i = 1, 2$ let $\overline{\alpha}_i := \overline{\Sigma}_i \setminus \overline{\alpha}$. Then $\overline{\Sigma} := \overline{\alpha}_1 \cup \overline{\alpha}_2$ is also an $\mathbf{S}$-admissible apartment.*

Proof. By Lemma 2.5 the set $\overline{\Sigma}$ is an apartement of R. For $i = 1, 2$ let $\Sigma_i := \bigcup_{c \in \overline{\Sigma}_i} S_c$ and $\alpha := \bigcup_{c \in \overline{\alpha}} S_c$. It follows that $\Sigma_1 \cap \Sigma_2 = \alpha$. We have $\Sigma := \bigcup_{c \in \overline{\Sigma}} S_c = (\Sigma_1 \setminus \alpha) \cup (\Sigma_2 \setminus \alpha)$ which is an apartment by Lemma 2.5. Hence $\overline{\Sigma}$ is an $\mathbf{S}$-admissible apartment. $\square$

Let $\mathbf{S} = (S_c)_{c \in R}$ be a system of R-sectors. A chamber $c \in R$ is called $\mathbf{S}$-*admissible* if each apartment of R containing c is $\mathbf{S}$-admissible.

Proposition 4.2. *Let $\mathbf{S} = (S_c)_{c \in R}$ be a system of R-sectors and let $c \in R$ be $\mathbf{S}$-admissible. Then any chamber $d \in R$ adjacent to c is also $\mathbf{S}$-admissible.*

Proof. We may assume $c \neq d$ and we denote the unique panel containing both chambers by P. Let $\overline{\Sigma}$ be an apartment of R containing d, let $\overline{\alpha}$ be the root of $\overline{\Sigma}$ such that $P \cap \overline{\alpha} = \{d\}$ and put $-\overline{\alpha} := \overline{\Sigma} \setminus \overline{\alpha}$. Let e be the unique chamber in the intersection of $-\overline{\alpha}$ and P. We have to show that $\overline{\Sigma}$ is admissible. This is obvious if $c = e$ because c is assumed to be admissible. Hence we may assume that $c \neq e$. Let $\overline{\Sigma}_1$ be an apartment of R containing c and $\overline{\alpha}$ and put $\overline{\alpha}_1 := \overline{\Sigma}_1 \setminus \overline{\alpha}$. Now, the intersection of $\overline{\Sigma}_1$ and $\overline{\Sigma}$ is the root $\overline{\alpha}$ and therefore, by Lemma 2.5, $\overline{\Sigma}_2 := -\overline{\alpha} \cup \overline{\alpha}_1$ is an apartment of R. As $\overline{\Sigma}_i$ contains the chamber c for $i = 1, 2$ it follows that both apartments are $\mathbf{S}$-admissible. Now the claim follows from Lemma 4.1. $\square$

Corollary 4.3. *Let $\mathbf{S}$ be a system of R-sectors. If there exists an $\mathbf{S}$-admissible chamber in R, then $\mathbf{S}$ is admissible.*

Proof. This follows by an obvious induction using the previous proposition. □

5 Codistances

Let $\mathbf{B} = (C, \delta)$ be a building of type (W, S). A *codistance* on $\mathbf{B}$ is a mapping $\delta^*: C \to W$ such that the following holds for any chamber c, where $w := \delta^*(c)$:

(CD1) If $s \in S$ is such that $\ell(ws) = \ell(w) - 1$, then $\delta^*(d) = ws$ for all chambers $d \neq c$ which are s-adjacent to c.

(CD2) If $s \in S$ is such that $\ell(ws) = \ell(w) + 1$, then there exists a unique chamber d which is s-adjacent to c and such that $\delta^*(d) = ws$.

Let δ^* be a codistance on $\mathbf{B}$. The following lemma is immediate from the axioms.

Lemma 5.1. *Let $c, d \in C$ be such that $\delta(d, c) = \delta^*(c)$, then $\delta^*(d) = 1_W$.*

Lemma 5.2. *Let $c \in C$ be such that $\delta^*(c) = 1_W$, then $\Sigma := \{x \in C \mid \delta^*(x) = \delta(c, x)\}$ is an apartment of $\mathbf{B}$.*

Proof. Let $d \in \Sigma$ and $w := \delta^*(d)$. Let $s \in S$ and let P be the s-panel containing d. If $\ell(ws) = \ell(w) - 1$, then $d \neq \operatorname{proj}_P c \in \Sigma$. If $\ell(ws) = \ell(w) + 1$ then $d = \operatorname{proj}_P c$ and there exists a unique chamber x in P with $\delta^*(x) = ws = \delta(c, x)$. This shows that Σ is a firm subset of $\mathcal{C}$. Hence, it suffices to show that for each $w \in W$ there exists precisely one element $d \in \Sigma$ such that $\delta(c, d) = w$. This is seen by an obvious induction on $\ell(w)$. □

The following lemma is Proposition 3.3 in [5].

Lemma 5.3. *Let R be a spherical residue of $\mathbf{B}$, then there is a unique chamber $c \in R$ such that $\delta^*(x)\delta(x, c) = \delta^*(c)$ and $\ell(\delta^*(x)) + \ell(x, c) = \ell(\delta^*(c))$, for all $x \in R$.*

Let $\mathbf{B} = (C, \delta)$ be a building of type (W, S) and let $\delta^*: C \to W$ be a codistance. Put $\delta^{*\mathrm{op}} := \{c \in C \mid \delta^*(c) := 1_W\}$ and let R be a spherical residue of type J such that $R \cap \delta^{*\mathrm{op}} \neq \emptyset$. For each chamber c in R put

$$S_c := \{x \in C \mid \delta^*(x) = \delta^*(c)\delta(c, x) \text{ and } \operatorname{proj}_R x = c\}$$

and let $\mathbf{S} := (S_c)_{c \in R}$ be the corresponding system of R-sectors. Let $d \in R$ be the unique chamber such that $\delta^*(d) = r_J$ (cf. Lemma 5.3).

Proposition 5.4. *With the notation above, the chamber d is an $\mathbf{S}$-admissible chamber. In particular, the system of R-sectors $\mathbf{S}$ is admissible.*

Proof. Let $\overline{\Sigma}$ be an apartment of R containing d. Let d' be the unique chamber in $\overline{\Sigma}$ opposite d. As $\delta^*(d) = r_J$, it follows that $\delta^*(d') = 1_W$ and that $\delta^*(c) = \delta(d', c)$ for all $c \in \overline{\Sigma}$. Since $\delta^*(d') = 1_W$ the set $\Sigma := \{x \in C \mid \delta^*(x) = \delta(d', x)\}$ is an apartment of $\mathbf{B}$ by Lemma 5.2. Moreover, we have $\overline{\Sigma} = R \cap \Sigma$.

Let $c \in \overline{\Sigma}$ and $x \in S_c$. Then we have $\delta^*(x) = \delta^*(c)\delta(c, x)$ and $\operatorname{proj}_R x = c$. As $\delta^*(c) = \delta(d', c) \in W_J$ and $\operatorname{proj}_R x = c$ it follows $\delta^*(x) = \delta^*(c)\delta(c, x) = \delta(d', c)\delta(c, x) = \delta(d', x)$ and hence x is a chamber of Σ.

Conversely, if x is a chamber of Σ then $c := \operatorname{proj}_R x$ is contained in $\Sigma \cap R = \overline{\Sigma}$. Moreover we have $\delta(y, x) = \delta(y, c)\delta(c, x)$ for all chambers y in R. This holds in particular for $y = d'$. Since x is in Σ we have $\delta^*(x) = \delta(d', x)$ and hence $\delta^*(x) = \delta(d', c)\delta(c, x)$. As $\delta(d', c) = \delta^*(c)$ we conclude that x is contained in S_c. This shows that $\Sigma = \cup_{c \in \overline{\Sigma}} S_c$ and in particular that $\overline{\Sigma}$ is an $\mathbf{S}$-admissible apartment of R. Hence, the chamber d is an $\mathbf{S}$-admissible chamber. The second assertion follows now from Corollary 4.3. This completes the proof. □

Theorem 5.5. *Let $\mathbf{B} = (C, \delta)$ be a building of type (W, S) and let J be a spherical subset of S. Let $\delta^* \colon C \to W$ be a codistance, let c be a chamber such that $\delta^*(c) = 1_W$ and let R be the J-residue containing c. Then there exists a convex and firm subset C' of C such that the the building $\mathbf{B}' := (C', \delta|_{C' \times C'})$ is a blow-up of R.*

Proof. Let d be the unique chamber in R such that $\delta^*(d) = r_J$. For each chamber e of R put $S_e := \{x \in C \mid \operatorname{proj}_R x = e \text{ and } \delta^*(x) = \delta^*(e)\delta(e, x)\}$. It follows by the previous proposition that the system of R-sectors $(S_e)_{e \in R}$ is admissible. Setting $C' := \cup_{e \in R} S_e$ it follows by Proposition 3.1 that C' is a convex and firm subset of C and that the building $\mathbf{B}' := (C', \delta|_{C' \times C'})$ is a blow-up of R. □

6 Codistances in affine buildings

In this section, we consider buildings $\mathbf{B} = (C, \delta)$ of affine type. These are buildings where the associated Coxeter group is an affine reflection group, see for instance [3, 4, 9, 12]. The corresponding thin buildings can be realized in real Euclidean spaces, in which one can interpret several notions of the theory of buildings as certain sets of points. In particular, a chamber can be identified with the ordinary convex closure of a simplex in any apartment; a wall can be identified with a hyperplane of an apartment and a root is a half-space in an apartment bounded by a wall. We say that two walls are *parallel* if there is an apartment containing both of them and if they are parallel in that apartment as hyperplanes. We refer to [4] for details.

Let $j \in S$ and let R be a residue of cotype j of an affine building $\mathbf{B} = (C, \delta)$ of type (W, S). Let Σ be an apartment of $\mathbf{B}$ intersecting R nontrivially. Then we call R *special* if for every wall W in Σ there is a parallel wall W' containing a panel both of whose chambers belong to R. This definition is independent of Σ. The wall W' above is said to *cut R in halves*.

Let R be a special residue in $\mathbf{B}$, and let Σ be an apartment meeting R non-trivially (by which we mean that Σ and R have chambers in common). Let $c \in C$ belong to R and Σ. The intersection of all roots in Σ whose bounding walls cut R in halves and which contain c is called a *sector emerging from R*. The sectors emerging from R form the chamber set of a spherical building of the same type as R. Two chambers of this spherical building are adjacent if the corresponding sectors are contained in a common apartment and if they contain respective adjacent chambers of $\mathbf{B}$. This spherical building is called the *building at infinity* and denoted by $\mathbf{B}_\infty$. Its isomorphism type is independent of the special residue R (and the canonical isomorphism between the buildings at infinity defined using R and another special residue R' maps a sector to a sector in such a way that the intersection of these two sectors contains a sector itself). In particular, $\mathbf{B}_\infty$ can be represented with sectors emerging from any special residue. Also, the family of apartments of $\mathbf{B}$ is in bijective correspondence with the family of apartments of $\mathbf{B}_\infty$. We refer to [4] and [12].

Lemma 6.1. *Let $\mathbf{B} = (C, \delta)$ be a building of affine type. Suppose there is a special thick residue R such that $\mathbf{B}$ is a blow-up of R. Then the building $\mathbf{B}_\infty = (C_\infty, \delta_\infty)$ at infinity is isomorphic to R.*

Proof. There is a natural epimorphism $\eta\colon \mathbf{B}_\infty \to R$ mapping a chamber c_∞ of $\mathbf{B}_\infty$ emerging from R to the unique chamber in $c_\infty \cap R$. Clearly, this mapping preserves adjacency of chambers, and hence maps non-opposite chambers onto non-opposite chambers. Now let c^1_∞ and c^2_∞ be two opposite chambers of $\mathbf{B}_\infty$, and let Σ_∞ be an apartment of $\mathbf{B}$ containing c^1_∞ and c^2_∞. Then there is a unique apartment Σ of $\mathbf{B}$ containing sectors d^1_∞ and d^2_∞ at bounded distance from c^1_∞ and c^2_∞, respectively. Since $\mathbf{B}$ is a blow-up of R, $\Sigma \cap R$ is nonempty and hence c^1_∞ and c^2_∞ are contained in Σ (as all 'translates' in Σ of d^1_∞ and d^2_∞ are at bounded distance from d^1_∞ and d^2_∞, respectively, and some translate emerges from R). Since c^1_∞ and c^2_∞ are opposite, also the chambers $R \cap c^1_\infty$ and $R \cap c^2_\infty$ are opposite (as the epimorphism mentioned above is bijective in the apartment Σ).

Hence η preserves opposition and non-opposition. By Corollary 5.2 of [1], η extends uniquely to an isomorphism. $\square$

Actually, the condition of R being thick is redundant in the previous lemma, as it follows rather easily that an epimorphism between two spherical buildings of the same type, with the property that opposition, non-opposition and adjacency of chambers is preserved, extends uniquely to an isomorphism. Since we will not need this more general version, we do not insist on a detailed proof.

Lemma 6.2. *Let $\mathbf{B} = (C, \delta)$ be an affine building, let C' be a firm convex subset of C and put $\mathbf{B}' := (C', \delta|_{C' \times C'})$. Then the building at infinity of $\mathbf{B}'$ is a subbuilding of the building at infinity of $\mathbf{B}$.*

Proof. This follows directly from the fact that every sector of $\mathbf{B}'$ is a sector of $\mathbf{B}$. $\square$

Let $\mathbf{B} = (C, \delta)$ be an affine building and let $\delta^* \colon C \to W$ be a codistance. Choose a spherical residue R of $\mathbf{B}$ containing a chamber in $\delta^{*\mathrm{op}}$. By the previous sections, there is a convex sub-building $\mathbf{B}'$ of $\mathbf{B}$ of the same type which is a blow-up of R. The building at infinity of $\mathbf{B}'$ is canonically isomorphic with R and it is a subbuilding of the building at infinity of $\mathbf{B}$. So we have shown:

Theorem 6.3. *Let $\mathbf{B} = (C, \delta)$ be a thick affine building, and let δ^* be a codistance on $\mathbf{B}$. Then every special residue containing a chamber of $\delta^{*\mathrm{op}}$ is a subbuilding of the building at infinity of $\mathbf{B}$ of the same type.*

Our ultimate goal is to prove that an irreducible Bruhat-Tits building defined over the p-adic numbers, and of rank at least 3, does not admit a codistance. We will need the following fact about irreducible spherical Moufang buildings of rank at least 2. A proof for the rank 2 case is Lemma 5.2.2 of [14]. The arguments given there make implicit use of Tits' rigidity theorem [11, Theorem 4.1.1]. They generalize to the higher rank case without any problem.

Proposition 6.4. *Let $\mathbf{B} = (C, \delta)$ be an irreducible spherical Moufang building of rank at least 2 and let C' be a thick convex subset of C. Then $\mathbf{B}' := (C', \delta|_{C' \times C'})$ is a spherical Moufang building of the same type. Moreover, if $\alpha \subset C'$ is a root of $\mathbf{B}'$ and U'_α is the corresponding root group with respect to $\mathbf{B}'$, then α is also a root of $\mathbf{B}$ and U'_α injects canonically into the corresponding root group with respect to $\mathbf{B}$.*

The irreducible spherical Moufang buildings of rank at least 2 are classified and one easy consequence of this classification is the fact that one can associate a characteristic to each such building. Its characteristic is defined to be 0 (p, respectively) if the root groups are torsion free (p-groups, respectively). The characteristic of the building at infinity $\mathbf{B}_\infty$ of a Bruhat-Tits building $\mathbf{B}$ coincides with the characteristic of the global field, whereas the characteristic of a special residue coincides with the characteristic of the residue field. Combining Theorem 6.3 and Proposition 6.4 we obtain the following consequence.

Corollary 6.5. *Let $\mathbf{B} = (C, \delta)$ be a Bruhat-Tits building defined over a field $\mathbb{F}$ with discrete valuation. If $\mathbf{B}$ admits a codistance, then the characteristic of the residue field of $\mathbb{F}$ is the same as the characteristic of $\mathbb{F}$.*

In particular, Bruhat-Tits buildings defined over p-adic fields do not admit any codistance, since the characteristic of the residue field is finite, and the characteristic of the field is 0.

Remark 6.6. Replacing in the arguments leading to the proof of Theorem 6.3 the building at infinity by the "complex $\mathfrak{C}_k(R)$ defined by the chambers at distance k from a certain special residue", we can slightly do better than Theorem 6.3 and prove that, if an affine building admits a codistance, then any special residue is a substructure of the complex $\mathfrak{C}_k(R)$. Of course, this is only useful in the cases of affine buildings not of Bruhat-Tits type, and these only exist in low rank. In cases $\widetilde{\mathsf{A}}_2$ and $\widetilde{\mathsf{C}}_2$, there is a precise notion of geometry at distance k from a given special residue, which can play the role of the complex $\mathfrak{C}_k(R)$ here. Using Ronan's construction of buildings [8], possibly rephrased for type $\widetilde{\mathsf{A}}_2$ as in [6], one sees that many non-classical buildings of these types do not admit a codistance.

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