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# Subquadrangles of order $s$ of generalized quadrangles of order $(s, s^2)$

Joseph A. Thas Koen Thas*

**Abstract**

In this paper we consider the subquadrangles of order $s$ for all known classes of generalized quadrangles of order $(s, s^2)$.

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## 1 Introduction

We survey results on subquadrangles of order $s$ for all known classes of generalized quadrangles of order $(s, s^2)$. An interesting application of this theory goes as follows. If $\mathcal{S}$ is a generalized quadrangle of order $(s, s^2)$, if $\mathcal{S}'$ is a subquadrangle of order $s$ of $\mathcal{S}$, and if $p$ is a point of $\mathcal{S}$ not in $\mathcal{S}'$, then all points of $\mathcal{S}'$ collinear with $p$ form an *ovoid* of $\mathcal{S}'$, that is, a set $\mathbf{O}$ of $s^2+1$ points of $\mathcal{S}'$ such that each line of $\mathcal{S}'$ contains exactly one point of $\mathbf{O}$. If $\mathcal{S}'$ is the classical generalized quadrangle $\mathcal{Q}(4, s)$ arising from a nonsingular quadric in $\mathsf{PG}(4, s)$, then, by a standard argument, $\mathbf{O}$ defines a projective plane of order $s^2$. In such a way new projective planes were discovered; see e.g. Thas and Payne [19]. Also, several theorems of this survey will provide interesting and strong characterizations of the classical generalized quadrangle $\mathcal{Q}(5, s)$ arising from a nonsingular elliptic quadric in $\mathsf{PG}(5, s)$, respectively of the Kantor-Knuth generalized quadrangles. As a byproduct a nice characterization of a class of translation generalized quadrangles of order $(s, s^2)$, with $s$ odd, will be given.

*The second author is a Postdoctoral Fellow of the Research Foundation – Flanders (FWO), Belgium.

## 2 Definitions

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathtt{I})$ in which $P$ and $B$ are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which $\mathtt{I} \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $t + 1$ lines, $t \geq 1$, and two distinct points are incident with at most one line;

(ii) each line is incident with $s + 1$ points, $s \geq 1$, and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in P \times B$ for which $x \,\mathtt{I}\, M \,\mathtt{I}\, y \,\mathtt{I}\, L$.

The integers $s$ and $t$ are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* $(s, t)$. If $s = t$, then $\mathcal{S}$ is said to have *order* $s$.

A *subquadrangle* (or also *subGQ*) $\mathcal{S}' = (P', B', \mathtt{I}')$ of $\mathcal{S}$ is a GQ such that $P' \subseteq P$, $B' \subseteq B$ and $\mathtt{I}'$ is the restriction of $\mathtt{I}$ to $(P' \times B') \cup (B' \times P')$; see [13, Chapter 2].

If $\mathcal{S} = (P, B, \mathtt{I})$ is a GQ of order $(s, t)$, then the GQ $\mathcal{S}^D = (B, P, \mathtt{I})$ of order $(t, s)$ is called the *(point-line) dual* of $\mathcal{S}$. As there is a point-line duality for GQs of order $(s, t)$, we assume without further notice that the dual of a given theorem or definition has also been given.

For background on GQs we refer to the monograph by Payne and Thas [13].

## 3 The classical examples

We give a brief description of three families of examples known as the *classical GQs*, all of which are associated with classical groups and were first recognized as GQs by Tits; see Dembowski [6].

(i) Consider a nonsingular quadric $\mathcal{Q}$ of projective index $1$, that is, of Witt index 2, of the projective space $\mathsf{PG}(d, q)$, with $d = 3, 4$ of $5$. Then the points of $\mathcal{Q}$ together with the lines of $\mathcal{Q}$ (which are the subspaces of maximal dimension on $\mathcal{Q}$) form a GQ $\mathcal{Q}(d, q)$ with parameters

$$\begin{aligned} &s = q, t = 1, && \text{when } d = 3,\\ &s = t = q, && \text{when } d = 4,\\ &s = q, t = q^2, && \text{when } d = 5. \end{aligned}$$

(ii) Let $H$ be a nonsingular Hermitian variety of the projective space $\mathsf{PG}(d, q^2)$, $d = 3$ or $d = 4$. Then the points of $H$ together with the lines on $H$ form a GQ $H(d, q^2)$ with parameters

$$s = q^2, t = q, \quad \text{when } d = 3,$$
$$s = q^2, t = q^3, \ \text{when } d = 4.$$

(iii) The points of $\mathsf{PG}(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ with parameters

$$s = t = q.$$

The GQ $\mathcal{Q}(5, q)$ is isomorphic to the dual of $H(3, q^2)$, the GQ $\mathcal{Q}(4, q)$ is isomorphic to the dual of $W(q)$, and $\mathcal{Q}(4, q)$ (and then also $W(q)$) is self-dual if and only if $q$ is even; see [13, Section 3.2].

All subquadrangles of order $q$ of $\mathcal{Q}(5, q)$ are the $\mathcal{Q}(4, q)$ subGQs on it.

# 4 Translation and flock generalized quadrangles

Let $\mathcal{S} = (P, B, \mathtt{I})$ be a GQ of order $(s, t)$, with $s \neq 1 \neq t$. A collineation $\theta$ of $\mathcal{S}$ is an *elation* about the point $p$ if $\theta = \mathrm{id}$ or if $\theta$ fixes all lines incident with $p$ and no point of $P \setminus p^\perp$ ($p^\perp$ is the set of all points collinear with $p$). If there is a group $G$ of elations about $p$ acting regularly on $P \setminus p^\perp$, then we say that $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* $G$ and *elation point* or *base point* or *center* $p$. Briefly, we write that $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ is an EGQ. If the group $G$ is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, G)$ is a *translation generalized quadrangle* (TGQ) with *translation group* $G$ and *translation point* or *base point* or *center* $p$.

For any TGQ $\mathcal{S}^{(p)}$ each line incident with $p$ is an *axis of symmetry*, that is, there is a (maximal) group of $s$ collineations of $\mathcal{S}$ fixing $L^\perp$ elementwise (see [13, Chapter 8]).

In $\mathsf{PG}(2n + m - 1, q)$ consider a set $\mathcal{O}(n, m, q)$ of $q^m + 1$ $(n - 1)$-dimensional subspaces $\pi_0, \pi_1, \ldots, \pi_{q^m}$, every three of which generate a $\mathsf{PG}(3n-1, q)$ and such that each element $\pi_i$ of $\mathcal{O}(n, m, q)$ is contained in an $(n + m - 1)$-dimensional subspace $\tau_i$ having no point in common with any $\pi_j$ for $j \neq i$. It is easy to check that $\tau_i$ is uniquely determined, with $i = 0, 1, \ldots, q^m$. The space $\tau_i$ is called the *tangent space* of $\mathcal{O}(n, m, q)$ at $\pi_i$. For $n = m = 1$ such a set $\mathcal{O}(1, 1, q)$ is an oval in $\mathsf{PG}(2, q)$ and more generally, for $n = m$, such a set $\mathcal{O}(n, n, q)$ is called a *pseudo-oval* or a *generalized oval* or an $[n - 1]$-*oval* of $\mathsf{PG}(3n - 1, q)$. For $m = 2n = 2$ such a set $\mathcal{O}(1, 2, q)$ is an ovoid of $\mathsf{PG}(3, q)$ and more generally, for $n \neq m$ such a set $\mathcal{O}(n,m,q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathsf{PG}(2n+m-1,q)$.

Now embed $\mathsf{PG}(2n+m-1,q)$ as a hyperplane in a $\mathsf{PG}(2n+m,q)$, and construct a point-line geometry $T(n,m,q)$ as follows.

- POINTS are of three types:
  - (i) the points of $\mathsf{PG}(2n+m,q) \setminus \mathsf{PG}(2n+m-1,q)$;
  - (ii) the $(n+m)$-dimensional subspaces of $\mathsf{PG}(2n+m,q)$ which contain a tangent space $\tau_i$ but are not contained in $\mathsf{PG}(2n+m-1,q)$;
  - (iii) a symbol $(\infty)$.
- LINES are of two types:
  - (a) the $n$-dimensional subspaces of $\mathsf{PG}(2n+m,q)$ which contain an element $\pi_i$ but are not contained in $\mathsf{PG}(2n+m-1,q)$;
  - (b) the elements of $\mathcal{O}(n,m,q)$.
- INCIDENCE is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathsf{PG}(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{O}(n,m,q)$ contained in it. The point $(\infty)$ is incident with all the lines of type (b) and with no lines of type (a).

Payne and Thas [13] prove that $T(n,m,q)$ is a TGQ of order $(q^n,q^m)$ with center $(\infty)$, and that conversely every TGQ is isomorphic to a $T(n,m,q)$.

In the case where $n=m=1$, so $\mathcal{O}(1,1,q)=\mathcal{O}$ is an oval of $\mathsf{PG}(2,q)$, the GQ $T(1,1,q)$ is the Tits GQ $T_2(\mathcal{O})$. When $m=2n=2$, so $\mathcal{O}(1,2,q)=\mathcal{O}$ is an ovoid of $\mathsf{PG}(3,q)$, the GQ $T(1,2,q)$ is the Tits GQ $T_3(\mathcal{O})$. Note that $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$ if and only if $\mathcal{O}$ is a conic, while $T_3(\mathcal{O}) \cong \mathcal{Q}(5,q)$ if and only if $\mathcal{O}$ is an elliptic quadric (see [13, Chapter 3]).

In the extension $\mathsf{PG}(2n+m-1,q^n)$ of $\mathsf{PG}(2n+m-1,q)$, with $m \in \{n,2n\}$, we consider $n$ $(\frac{m}{n}+1)$-dimensional spaces $\mathsf{PG}^{(i)}(\frac{m}{n}+1,q^n)=\xi_i$, with $i=1,2,\ldots,n$, which are conjugate with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$, that is, which form an orbit of the Galois group corresponding to this extension, and which span $\mathsf{PG}(2n+m-1,q^n)$. In $\xi_1$ we consider an oval $\mathcal{O}_1$ for $m=n$ and an ovoid $\mathcal{O}_1$ for $m=2n$. Let $\mathcal{O}_1=\{x_0^{(1)},x_1^{(1)},\ldots,x_{q^m}^{(1)}\}$. Further, let $x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)}$, with $i=0,1,\ldots,q^m$, be conjugate with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. The points $x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)}$ generate an $(n-1)$-dimensional space $\pi_i$ over $\mathsf{GF}(q)$, with $i=0,1,\ldots,q^m$. Then $\mathcal{O}=\{\pi_0,\pi_1,\ldots,\pi_{q^m}\}$ is a generalized oval of $\mathsf{PG}(3n-1,q)$ for $m=n$, and a generalized ovoid of $\mathsf{PG}(4n-1,q)$ for $m=2n$.

Here, we speak of a *regular* or *elementary pseudo-oval*, respectively a *regular* or *elementary pseudo-ovoid*. In such a case the corresponding GQ is isomorphic to a GQ of Tits ($T_2(\mathcal{O}_1)$, respectively $T_3(\mathcal{O}_1)$). If $\mathcal{O}_1$ is a conic or an elliptic quadric, then $T(n, m, q)$ is isomorphic to a classical GQ ($\mathcal{Q}(4, q^n)$, respectively $\mathcal{Q}(5, q^n)$) and $\mathcal{O}(n, m, q)$ is called *classical*, see [13, Chapter 3]; for $n = m$ the classical $\mathcal{O}(n, n, q)$ is called a *pseudo-conic* and for $m = 2n$ a *pseudo-quadric*.

For $m = n$, any known $[n-1]$-oval is regular, for $m = 2n$ and $q$ even any known $[n-1]$-ovoid is regular, but for $m = 2n$ and $q$ odd there are $[n-1]$-ovoids which are not regular; see Thas [16].

One can prove that $n \le m \le 2n$, and that for $q$ even $m \in \{n, 2n\}$; see [13, Chapter 8].

Let either $n = m$ with $q$ odd, or let $n \neq m$. Then by [13, Section 8.7] the $q^m + 1$ tangent spaces of an $\mathcal{O}(n, m, q) = \mathcal{O}$ form an $\mathcal{O}^* = \mathcal{O}^*(n, m, q)$ in the dual space of $\mathsf{PG}(2n + m - 1, q)$. So in addition to $T(n, m, q) = T(\mathcal{O})$ there arises a TGQ $T(\mathcal{O}^*)$ with the same parameters. The TGQ $T(\mathcal{O}^*)$ is called the *translation dual* of $T(\mathcal{O})$, and $\mathcal{O}^*$ is called the *translation dual* of $\mathcal{O}$. For $m = 2n$ and $q$ odd there are examples for which $T(\mathcal{O}) \not\cong T(\mathcal{O}^*)$; see Payne [11].

Let $\mathcal{O}(n, 2n, q)$ be an egg of $\mathsf{PG}(4n - 1, q)$. We say that $\mathcal{O}(n, 2n, q)$ is *good* at the element $\pi_i$ of $\mathcal{O}(n, 2n, q)$ if any $\mathsf{PG}(3n - 1, q)$ containing $\pi_i$ and at least two other elements of $\mathcal{O}(n, 2n, q)$ contains exactly $q^n + 1$ elements of $\mathcal{O}(n, 2n, q)$. In such a case the corresponding TGQ $T(n, 2n, q)$ contains at least $q^{3n} + q^{2n}$ subquadrangles of order $q^n$; for $q$ odd the $q^{3n} + q^{2n}$ subquadrangles of order $q^n$ defined by the element $\pi_i$ at which $\mathcal{O}(n, 2n, q)$ is good are isomorphic to the classical GQ $\mathcal{Q}(4, q^n)$, see Thas [16]. The TGQ $T(n, 2n, q)$ is isomorphic to a GQ of Tits of order $(q^n, q^{2n})$ if and only if the corresponding egg $\mathcal{O}(n, 2n, q)$ is good at each of its elements; see Payne and Thas [13]. If $q$ is even and $\mathcal{O}(n, 2n, q)$ is good at $\pi \in \mathcal{O}(n, 2n, q)$, then the translation dual $\mathcal{O}^*(n, 2n, q)$ is good at the tangent space $\tau$ of $\mathcal{O}(n, 2n, q)$ at $\pi$; see Thas [16]. For each known egg $\mathcal{O}(n, 2n, q)$, either $\mathcal{O}(n, 2n, q)$ or $\mathcal{O}^*(n, 2n, q)$ is good at one of its elements.

Let $\mathcal{K}$ be the quadratic cone with vertex $x$ in $\mathsf{PG}(3, q)$. A partition $\mathcal{F}$ of $\mathcal{K} \setminus \{x\}$ into $q$ disjoint conics is called a *flock* of $\mathcal{K}$. Then with $\mathcal{F}$ there corresponds a GQ $\mathcal{S}(\mathcal{F})$ of order $(q^2, q)$; see Thas [15]. The GQ $\mathcal{S}(\mathcal{F})$ is isomorphic to the classical GQ $H(3, q^2)$ if and only if the flock is *linear*, that is, if and only if all planes of the conics of the flock contain a common line; see Payne and Thas [13] and Payne [12]. The GQ $\mathcal{S}(\mathcal{F})$ is an EGQ with center $(\infty)$. By Payne and Thas [14] the center $(\infty)$ is uniquely determined if $\mathcal{S}(\mathcal{F})$ is not classical.

If the point-line dual of $\mathcal{S}(\mathcal{F})$ is a TGQ $T(\mathcal{O})$ with center $L$ (then $(\infty)\,\mathrm{I}\,L$), where $\mathcal{O} = \mathcal{O}(n, 2n, q)$, then the translation dual $\mathcal{O}^*$ of $\mathcal{O}$ is good at the tangent space $\tau_i$ of $\mathcal{O}$ at $\pi_i = L \in \mathcal{O}$; see Thas [16]. Conversely, by Thas [17], if the translation dual $\mathcal{O}^*$ of an egg $\mathcal{O} = \mathcal{O}(n, 2n, q)$, with $q$ odd, is good at one of its elements, then $T(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. If $q$ is even and if the TGQ $T(\mathcal{O})$ is the point-line dual of a flock GQ (then $\mathcal{O}$ and $\mathcal{O}^*$ are good), then, by Johnson [7], the GQ $T(\mathcal{O})$ is classical.

Finally, we remark that every known GQ of order $(s, s^2)$ is either a TGQ $T(\mathcal{O})$ or the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$.

## 5 Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with $\mathcal{O}$ good

In this section we consider the subquadrangles of order $q^n$ of translation generalized quadrangles $T(\mathcal{O})$ of order $(q^n, q^{2n})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and $\mathcal{O}$ good at some element $\pi$.

**Theorem 5.1** (Brown and Thas [4, 5]). *Let $\mathcal{S} = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$, be a TGQ. If $\mathcal{O}$ is good at some element $\pi$ and if $\mathcal{S}$ has a subquadrangle of order $s = q^n$ which does not contain the point $(\infty)$, then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

**Theorem 5.2** (Brown and Thas [5]). *Let $\mathcal{S} = (P, B, \mathtt{I})$ be a TGQ of order $(s, s^2)$, $s$ odd, with $\mathcal{S} = T(\mathcal{O})$ and $\mathcal{O} = \mathcal{O}(n, 2n, q)$ good at some element $\pi$. If $\mathcal{S}'$ is a subquadrangle of order $s = q^n$ of $\mathcal{S}$ containing the point $(\infty)$, then $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and either $\mathcal{S} \cong \mathcal{Q}(5, s)$ or $\mathcal{S}'$ is one of the $s^3 + s^2$ subquadrangles of order $s$ (isomorphic to $\mathcal{Q}(4, s)$) containing the line $\pi$ of $\mathcal{S}$.*

**Theorem 5.3** (Brown and Thas [5]). *Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$, with $q$ odd, be an egg in $\mathsf{PG}(4n-1, q)$ which is good at $\pi$. If there is a subspace $\mathsf{PG}(3n-1, q)$ of $\mathsf{PG}(4n-1, q)$ which contains at least four elements of $\mathcal{O}$, but which does not contain $\pi$, then $\mathcal{O}$ is a Kantor-Knuth egg (see Section 7). If there is a subspace $\mathsf{PG}(3n-1, q)$ which contains at least five elements of $\mathcal{O}$, but which does not contain $\pi$, then $\mathcal{O}$ is classical.*

**Remark 5.4.** Independently and in a completely different way, Lavrauw [8] also proved the second part of Theorem 5.3.

**Theorem 5.5** (Thas [21], see also [22, 23]). *Let $\mathcal{S} = (P, B, \mathtt{I})$ be a TGQ of order $(s, s^2)$, $s > 1$, with $\mathcal{S} = T(\mathcal{O})$ and $\mathcal{O} = \mathcal{O}(n, 2n, q)$. If $\mathcal{S}'$ is a subquadrangle of order $s$ of $\mathcal{S}$ containing the point $(\infty)$, then $\mathcal{S}'$ is a TGQ $T(\mathcal{O}')$ with $\mathcal{O}' = \mathcal{O}'(n, n, q)$ a pseudo-oval on $\mathcal{O}$.*

**Open problem 5.6.** *If in Theorem 5.5, $s$ is even, does the existence of $\mathcal{S}'$ imply that $\mathcal{S}$ is isomorphic to a $T_3(\overline{\mathcal{O}})$ of Tits?*

We mention several recent results on this problem in the next section.

Let us end this section with the following results on good TGQs with a large automorphism group, followed by an application to group actions on subGQs.

Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order $(q^n, q^{2n})$, where $q$ is odd, and where $\mathcal{O}$ is good at some element $\mathsf{PG}(n-1, q)$. Let $\mathbb{K}$ be the kernel [13, Chapter 8] of $\mathcal{S}$. Then by Thas [24], $(q^n+1)(q^n-1)q^{6n}(|\mathbb{K}|-1)$ is a divisor of $|\mathrm{Aut}(\mathcal{S})|$. If $\mathcal{S}$ is classical, then for an arbitrary line $L$ of $\mathcal{S}$, $|\mathrm{Aut}(\mathcal{S})_L| = (q^n+1)^2(q^n-1)^2q^{6n}2h$, where $q^n = p^h$ for the prime $p$.

**Theorem 5.7** (Thas [27]). *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order $(q^n, q^{2n})$, where $q$ is an odd prime power, with $\mathcal{O}$ the generalized ovoid in $\mathsf{PG}(4n-1, q)$ corresponding to $\mathcal{S}$. Suppose that $\mathcal{O}$ is good at some element $\mathsf{PG}(n-1, q)$, and let $L$ be the line $\mathsf{PG}(n-1, q)$ of $\mathcal{S}$. If*

$$(q^n+1)^2(q^n-1)q^{6n} \quad \textit{divides} \quad |\mathrm{Aut}(\mathcal{S})_L|\,,$$

*then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(5, q^n)$.*

**Corollary 5.8** (Thas [27]). *Let $\mathcal{F}$ be a semifield flock of $\mathsf{PG}(3, q^n)$, $q$ odd. If $q^n+1$ divides the size of $\mathrm{Aut}(\mathcal{F})_\Pi$, where $\Pi$ is any flock plane, or, equivalently, if $q^n+1$ divides $\frac{|\mathrm{Aut}(\mathcal{F})|}{q^n}$, then $\mathcal{F}$ is linear.*

From Theorem 5.7, Thas deduced the following theorem on subGQs:

**Theorem 5.9** (Thas [27]). *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order $(s, s^2)$, $s$ odd, so that the generalized ovoid $\mathcal{O}$ is good at some element. If $\mathrm{Aut}(\mathcal{S})$ acts transitively on its subGQs of order $s$, then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

# 6 Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$ in even characteristic

Brown and Lavrauw [2] generalized the following result of Brown on ovoids of $\mathsf{PG}(3, q)$, $q$ even, by obtaining a similar result for generalized ovoids.

**Theorem 6.1** (Brown [1]). *If an ovoid of $\mathsf{PG}(3, q)$, $q$ even, has a conic plane section, the ovoid must be an elliptic quadric. Equivalently, if a $T_3(\mathcal{O})$ of order $(q, q^2)$, where $\mathcal{O}$ is an ovoid of $\mathsf{PG}(3, q)$, $q$ even, has a classical subGQ of order $q$ containing the point $(\infty)$, then $T_3(\mathcal{O}) \cong \mathcal{Q}(5, q)$.*

In terms of translation generalized quadrangles, Brown and Lavrauw prove the following.

**Theorem 6.2** (Brown and Lavrauw [2]). *If a $T(\mathcal{O})$ of order $(q^n, q^{2n})$, where $\mathcal{O}$ is a generalized ovoid of $\mathsf{PG}(4n-1, q)$, $q$ even, has a classical subGQ of order $q$ containing the point $(\infty)$, then $T(\mathcal{O}) \cong \mathcal{Q}(5, q^n)$.*

In [26], Thas then generalized the latter result to elation generalized quadrangles.

**Theorem 6.3** (Thas [26]). *Let $\mathcal{S}^{(x)}$ be an EGQ of order $(q, q^2)$, $q$ even, having a classical subGQ $\mathcal{S}'$ of order $q$ containing $x$. Then $\mathcal{S}^{(x)} \cong \mathcal{Q}(5, q)$.*

Let $\mathcal{O}$ be a generalized oval in $\mathsf{PG}(3n-1, q) = \langle \mathcal{O} \rangle$, $q$ even. Then the $q^n + 1$ tangent spaces to the elements of $\mathcal{O}$ meet in an $(n-1)$-dimensional subspace $\mathfrak{N}$ of $\mathsf{PG}(3n-1, q)$, called the *nucleus* of $\mathcal{O}$ (cf. [13, Chapter 8]). It is easily seen that for any $\pi \in \mathcal{O}$, $\mathcal{O}' = (\mathcal{O} \setminus \{\pi\}) \cup \{\mathfrak{N}\}$ is again a generalized oval. If $\mathcal{O}$ is a generalized conic, that is, if $T(\mathcal{O}) \cong \mathcal{Q}(4, q^n)$, then $\mathcal{O}'$ is called a *pointed generalized conic* or a *generalized pointed conic*.

The following result generalizes another recent result of Brown and Lavrauw [3] from TGQs to EGQs.

**Theorem 6.4** (Thas [26]). *Let $\mathcal{S}^{(x)}$ be an EGQ of order $(q^n, q^{2n})$, $q$ even, containing a subGQ $T(\mathcal{O})$, where $\mathcal{O}$ is a pointed generalized conic in $\mathsf{PG}(3n-1, q)$, having $x$ as translation point. Then either $q^n = 4$ and $\mathcal{S}^{(x)} \cong \mathcal{Q}(5, 4)$, or $q^n = 8$ and $\mathcal{S}^{(x)} \cong T_3(\mathcal{O}')$ with $\mathcal{O}'$ the Suzuki-Tits ovoid of $\mathsf{PG}(3, 8)$.*

Both Theorem 6.3 and Theorem 6.4 rely on the following observation, which works in any characteristic (part (ii) being a direct corollary of part (i)).

**Theorem 6.5** (Thas [26]). (i) *Let $\mathcal{S}^{(x)}$ be an EGQ of order $(s, s^2)$, $s > 1$, containing a subGQ $\mathcal{S}'$ of order $s$ which has at least one axis of symmetry $L$ incident with $x$. Then $\mathcal{S}^{(x)}$ is a TGQ for the translation point $x$*

(ii) *Let $\mathcal{S}^{(x)}$ be an EGQ of order $(s, s^2)$, $s > 1$, containing a subGQ $\mathcal{S}'$ of order $s$ which is a TGQ with translation point $x$. Then $\mathcal{S}^{(x)}$ is a TGQ with translation point $x$.*

# 7 Subquadrangles of flock GQs $\mathcal{S}(\mathcal{F})$

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ of $\mathsf{PG}(3, q)$, with $q$ even. If the planes $\xi_1, \xi_2, \ldots, \xi_q$ of the respective conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ all contain a common point, then the flock is linear; see Thas [15].

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ of $\mathsf{PG}(3, q)$, with $q$ odd. If the planes $\xi_1, \xi_2, \ldots, \xi_q$ of the respective conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ all contain a common interior point of $\mathcal{K}$, then the flock is linear; see Thas [15].

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ of $\mathsf{PG}(3, q)$, with $q$ odd, and suppose that the planes $\xi_1, \xi_2, \ldots, \xi_q$, with $\mathcal{C}_i \subseteq \xi_i$, all contain a common exterior point of $\mathcal{K}$. If $m$ is any given non-square of $\mathsf{GF}(q)$, then coordinates can be chosen in such a way that $\mathcal{K}$ has equation $X_0X_1 = X_2^2$ and the planes $\xi_i$ have equation

$$a_iX_0 - ma_i^\sigma X_1 + X_3 = 0\,, \qquad (1)$$

with $\{a_1, a_2, \ldots, a_q\} = \mathsf{GF}(q)$ and $\sigma$ an automorphism of $\mathsf{GF}(q)$. Conversely, given any non-square $m$ of $\mathsf{GF}(q)$, the planes $\xi_i$ with equation (1), where $\{a_1, a_2, \ldots, a_q\} = \mathsf{GF}(q)$ and where $\sigma$ is any automorphism of $\mathsf{GF}(q)$, define a flock $\mathcal{F}$ of the cone $\mathcal{K}$ with equation $X_0X_1 = X_2^2$. Also, the planes $\xi_i$ all contain the exterior point $(0, 0, 1, 0)$ of $\mathcal{K}$. This flock $\mathcal{F}$ is linear if and only if $\sigma = \mathbf{1}$. The flocks $\mathcal{F}$ are called *Kantor-Knuth flocks*. For proofs we refer to Thas [15]. The point-line dual of the GQ which corresponds to a Kantor-Knuth flock is a TGQ $T(\mathcal{O})$, with $\mathcal{O}$ isomorphic to its translation dual $\mathcal{O}^*$, with $\mathcal{O}$ good at some element $\pi$, and with $\mathcal{O}^*$ good at the tangent space $\tau$ of $\mathcal{O}$ at $\pi$; see Payne [11]. Such an egg is called a *Kantor-Knuth egg*.

In the next theorems all subquadrangles of order $s$ of flock GQs $\mathcal{S}(\mathcal{F})$ of order $(s^2, s)$ are determined.

**Theorem 7.1** (O'Keefe and Penttila [9]). *If $\mathcal{S}(\mathcal{F})$ is a flock GQ of order $(s^2, s)$, $s$ even, with center $(\infty)$, then $\mathcal{S}(\mathcal{F})$ contains exactly $s^3 + s^2$ subquadrangles of order $s$ containing the point $(\infty)$.*

An easy way to see this goes as follows. Payne proved in [11] that for all sets $\{L, M, N\}$ of two by two nonconcurrent lines for which there is a line $U\,\mathtt{I}(\infty)\,\mathtt{I}\,L$ so that $\{L, M, N\} \subseteq U^\perp$, we have that $|\{L, M, N\}^{\perp\perp}| = s + 1$. That is, the set $\{L, M, N\}$ is 3*-regular*. Now let $\{L, M, N\}$ be such a set of lines. If $B'$ is the set of lines of $\mathcal{S}(\mathcal{F})$ incident with points of the form $X \cap Y$, with $X \in \{L, M, N\}^\perp$ and $Y \in \{L, M, N\}^{\perp\perp}$, and $P'$ is the set of points which are incident with at least two distinct lines of $B'$, then endowed with the induced incidence $\mathtt{I}'$, $\mathcal{S}' = (P', B', \mathtt{I}')$ is a subGQ of order $s$ of $\mathcal{S}(\mathcal{F})$, see Payne and Thas [13, §2.6.2]. Now a standard counting argument yields $s^3 + s^2$ of such subGQs (notice that any point of a GQ of order $(s^2, s)$ is contained in at most $s^3 + s^2$ subGQs of order $s$, see e.g. [4]).

**Remark 7.2.** Payne [10] proves that each of these $s^3 + s^2$ subquadrangles is a $T_2(\mathcal{O})$ of Tits, for some oval $\mathcal{O}$ of $\mathsf{PG}(2, q)$.

**Theorem 7.3** (Brown and Thas [5]). *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order $(s^2, s)$, $s$ odd, with center $(\infty)$. If $\mathcal{S}'$ is a subquadrangle of order $s$ of $\mathcal{S}(\mathcal{F})$ containing the point $(\infty)$, then $\mathcal{F}$ is a Kantor-Knuth flock. Hence either $\mathcal{S} \cong H(3, s^2)$ or $\mathcal{S}'$ is one of the $s^3 + s^2$ subquadrangles of order $s$ containing the point $(\infty)$. In all cases $\mathcal{S}' \cong W(s)$.*

An application of Theorem 7.3 is the next theorem.

**Theorem 7.4** (Thas [27])**.** *Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$ be a good egg in* $\mathsf{PG}(4n-1, q)$*, $q$ odd. If $T(\mathcal{O})$ contains a subGQ of order $q^n$ that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong \mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a Kantor-Knuth flock.*

For $q$ even, one can use Theorem 6.2 and [25, Chapter 7] to show that if $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is an egg in $\mathsf{PG}(4n-1, q)$ and $T(\mathcal{O})$ contains a subGQ of order $q^n$ that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong \mathcal{Q}(5, q^n)$, see [27].

**Theorem 7.5** (Brown and Thas [4], Thas and Thas [20])**.** *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order $(s^2, s)$, with center $(\infty)$. If $\mathcal{S}(\mathcal{F})$ contains a subquadrangle of order $s$ which does not contain $(\infty)$, then $\mathcal{S} \cong H(3, s^2)$ and $\mathcal{S}' \cong W(s)$.*

**Remark 7.6.** The even case of Theorem 7.5 is due to Brown and Thas, the odd case to Thas and Thas.

# 8 Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with $\mathcal{O}^*$ good

As for $q$ even the egg $\mathcal{O}(n, 2n, q)$ is good if and only if $\mathcal{O}^*(n, 2n, q)$ is good, we may assume that $q$ is odd.

If $\mathcal{O}^*(n, 2n, q)$, with $q$ odd, is good, then by Thas [17] the TGQ $T(\mathcal{O})$ with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is isomorphic to the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. If $\mathcal{O}^*(n, 2n, q)$ is good at the tangent space $\tau$ of $\mathcal{O}(n, 2n, q)$ at $\pi$, then $\pi$ corresponds to the point $(\infty)$ of $\mathcal{S}(\mathcal{F})$. So by Theorems 7.3 and 7.5 we have the following result.

**Theorem 8.1** (Brown and Thas [4, 5], Thas and Thas [20])**.** *Let $\mathcal{S} = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and $q$ odd, be a TGQ such that the translation dual $\mathcal{O}^*$ of $\mathcal{O}$ is good at the tangent space $\tau$ of $\mathcal{O}$ at $\pi$. If $\mathcal{S}$ contains a subquadrangle $\mathcal{S}'$ of order ${q'}^n$, then either $\mathcal{S} \cong \mathcal{Q}(5, q^n)$ (and then $\mathcal{S}' \cong \mathcal{Q}(4, q^n)$) or $\mathcal{O}$ is a nonclassical Kantor-Knuth egg and $\mathcal{S}'$ is one of the $q^{3n} + q^{2n}$ subquadrangles of order $q^n$ (isomorphic to $\mathcal{Q}(4, q^n)$) containing the line $\pi$ of $T(\mathcal{O})$.*

# 9 Byproduct: A characterization of TGQs

Part of the proof of Theorem 7.5 (odd case) was also used by Thas and Thas [20] to prove the following characterization theorem of TGQs whose point-line duals arise from flocks.

**Theorem 9.1** (Thas and Thas [20]). *If $\mathcal{S} = (P, B, \mathrm{I})$ is a flock GQ of order $(s^2, s)$, with $s$ odd, then the point-line dual of $\mathcal{S}$ is a TGQ if and only if $\mathcal{S}$ has a regular point $x$, with $(\infty) \neq x \sim (\infty)$, where $(\infty)$ is the elation point of $\mathcal{S}$.*

**Remark 9.2.** Thas [18] proved the following result for $q$ even. Let $\mathcal{S}$ be a flock GQ of order $(s^2, s)$, with $s$ even. Then the point-line dual of $\mathcal{S}$ is isomorphic to the classical GQ $H(3, s^2)$ if and only if $\mathcal{S}$ has a regular point $x$, with $(\infty) \neq x \sim (\infty)$, where $(\infty)$ is the elation point of $\mathcal{S}$.

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Joseph A. Thas

GHENT UNIVERSITY, DEPARTMENT OF PURE MATHEMATICS AND COMPUTER ALGEBRA, KRIJGSLAAN 281, S22, B-9000 GHENT, BELGIUM

*e-mail*: `jat@cage.UGent.be`

*website*: `http://cage.ugent.be/~jat`

Koen Thas

GHENT UNIVERSITY, DEPARTMENT OF PURE MATHEMATICS AND COMPUTER ALGEBRA, KRIJGSLAAN 281, S22, B-9000 GHENT, BELGIUM

*e-mail*: `kthas@cage.UGent.be`

*website*: `http://cage.ugent.be/~kthas`