Subquadrangles of order $s$ of generalized quadrangles of order $(s, s^2)$

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Abstract

In this paper we consider the subquadrangles of order $s$ for all known classes of generalized quadrangles of order $(s, s^2)$.

Keywords: generalized quadrangle, subquadrangle, flock quadrangle, translation quadrangle

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1. Introduction

We survey results on subquadrangles of order $s$ for all known classes of generalized quadrangles of order $(s, s^2)$. An interesting application of this theory goes as follows. If $S$ is a generalized quadrangle of order $(s, s^2)$, if $S'$ is a subquadrangle of order $s$ of $S$, and if $p$ is a point of $S$ not in $S'$, then all points of $S'$ collinear with $p$ form an ovoid of $S'$, that is, a set $O$ of $s^2+1$ points of $S'$ such that each line of $S'$ contains exactly one point of $O$. If $S'$ is the classical generalized quadrangle $Q(4, s)$ arising from a nonsingular quadric in $PG(4, s)$, then, by a standard argument, $O$ defines a projective plane of order $s^2$. In such a way new projective planes were discovered; see e.g. Thas and Payne [19]. Also, several theorems of this survey will provide interesting and strong characterizations of the classical generalized quadrangle $Q(5, s)$ arising from a nonsingular elliptic quadric in $PG(5, s)$, respectively of the Kantor-Knuth generalized quadrangles. As a byproduct a nice characterization of a class of translation generalized quadrangles of order $(s, s^2)$, with $s$ odd, will be given.

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2. Definitions

A (finite) generalized quadrangle (GQ) is an incidence structure $S = (P, B, I)$ in which $P$ and $B$ are disjoint (non-empty) sets of objects called points and lines respectively, and for which $I \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $t + 1$ lines, $t \geq 1$, and two distinct points are incident with at most one line;

(ii) each line is incident with $s + 1$ points, $s \geq 1$, and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in P \times B$ for which $x I M I y I L$.

The integers $s$ and $t$ are the parameters of the GQ and $S$ is said to have order $(s, t)$. If $s = t$, then $S$ is said to have order $s$.

A subquadrangle (or also subGQ) $S' = (P', B', I')$ of $S$ is a GQ such that $P' \subseteq P$, $B' \subseteq B$ and $I'$ is the restriction of $I$ to $(P' \times B') \cup (B' \times P')$; see [13, Chapter 2].

If $S = (P, B, I)$ is a GQ of order $(s, t)$, then the GQ $S^D = (B, P, I)$ of order $(t, s)$ is called the (point-line) dual of $S$. As there is a point-line duality for GQs of order $(s, t)$, we assume without further notice that the dual of a given theorem or definition has also been given.

For background on GQs we refer to the monograph by Payne and Thas [13].

3. The classical examples

We give a brief description of three families of examples known as the classical GQs, all of which are associated with classical groups and were first recognized as GQs by Tits; see Dembowski [6].

(i) Consider a nonsingular quadric $Q$ of projective index 1, that is, of Witt index 2, of the projective space $\text{PG}(d, q)$, with $d = 3, 4$ of $5$. Then the points of $Q$ together with the lines of $Q$ (which are the subspaces of maximal dimension on $Q$) form a GQ $Q(d, q)$ with parameters

\[
\begin{align*}
    s &= q, t = 1, \quad \text{when } d = 3, \\
    s &= t = q, \quad \text{when } d = 4, \\
    s &= q, t = q^2, \quad \text{when } d = 5.
\end{align*}
\]
(ii) Let $H$ be a nonsingular Hermitian variety of the projective space $PG(d, q^2)$, $d = 3$ or $d = 4$. Then the points of $H$ together with the lines on $H$ form a $\text{GQ } H(d, q^2)$ with parameters

$$s = q^2, t = q, \text{ when } d = 3,$$

$$s = q^2, t = q^3, \text{ when } d = 4.$$ 

(iii) The points of $PG(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a $\text{GQ } W(q)$ with parameters

$$s = t = q.$$ 

The $\text{GQ } Q(5, q)$ is isomorphic to the dual of $H(3, q^2)$, the $\text{GQ } Q(4, q)$ is isomorphic to the dual of $W(q)$, and $Q(4, q)$ (and then also $W(q)$) is self-dual if and only if $q$ is even; see [13, Section 3.2].

All subquadrangles of order $q$ of $Q(5, q)$ are the $Q(4, q)$ subGQs on it.

4. Translation and flock generalized quadrangles

Let $S = (P, B, I)$ be a $\text{GQ}$ of order $(s,t)$, with $s \neq 1 \neq t$. A collineation $\theta$ of $S$ is an elation about the point $p$ if $\theta = \text{id}$ or if $\theta$ fixes all points incident with $p$ and no point of $P \setminus p^\perp$ ($p^\perp$ is the set of all points collinear with $p$). If there is a group $G$ of elations about $p$ acting regularly on $P \setminus p^\perp$, then we say that $S$ is an elation generalized quadrangle (EGQ) with elation group $G$ and elation point or base point or center $p$. Briefly, we write that $(S^{(p)}, G)$ or $S^{(p)}$ is an EGQ. If the group $G$ is abelian, then we say that the EGQ $(S^{(p)}, G)$ is a translation generalized quadrangle (TGQ) with translation group $G$ and translation point or base point or center $p$.

For any TGQ $S^{(p)}$ each line incident with $p$ is an axis of symmetry, that is, there is a (maximal) group of $s$ collineations of $S$ fixing $L^\perp$ elementwise (see [13, Chapter 8]).

In $PG(2n + m - 1, q)$ consider a set $O(n, m, q)$ of $q^m + 1$ $(n - 1)$-dimensional subspaces $\pi_0, \pi_1, \ldots, \pi_{q^m}$, every three of which generate a $\text{PG}(3n-1, q)$ and such that each element $\pi_i$ of $O(n, m, q)$ is contained in an $(n + m - 1)$-dimensional subspace $\tau_i$ having no point in common with any $\pi_j$ for $j \neq i$. It is easy to check that $\tau_i$ is uniquely determined, with $i = 0, 1, \ldots, q^m$. The space $\tau_i$ is called the tangent space of $O(n, m, q)$ at $\pi_i$. For $n = m = 1$ such a set $O(1, 1, q)$ is an oval in $\text{PG}(2, q)$ and more generally, for $n = m$, such a set $O(n, n, q)$ is called a pseudo-oval or a generalized oval or an $[n - 1]$-oval of $\text{PG}(3n - 1, q)$. For $m = 2n = 2$ such a set $O(1, 2, q)$ is an ovoid of $\text{PG}(3, q)$ and more generally, for $n \neq m$ such
a set \( O(n, m, q) \) is called a pseudo-ovoid or a generalized ovoid or an \([n-1]-ovoid\) or an egg of \( PG(2n + m - 1, q) \).

Now embed \( PG(2n + m - 1, q) \) as a hyperplane in a \( PG(2n + m, q) \), and construct a point-line geometry \( T(n, m, q) \) as follows.

- **Points** are of three types:
  
  (i) the points of \( PG(2n + m, q) \setminus PG(2n + m - 1, q) \);
  
  (ii) the \((n+m)\)-dimensional subspaces of \( PG(2n + m, q) \) which contain a tangent space \( \tau_i \) but are not contained in \( PG(2n + m - 1, q) \);
  
  (iii) a symbol \((\infty)\).

- **Lines** are of two types:
  
  (a) the \( n \)-dimensional subspaces of \( PG(2n + m, q) \) which contain an element \( \pi_i \) but are not contained in \( PG(2n + m - 1, q) \);
  
  (b) the elements of \( O(n, m, q) \).

- **Incidence** is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of \( PG(2n + m, q) \). A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of \( O(n, m, q) \) contained in it. The point \((\infty)\) is incident with all the lines of type (b) and with no lines of type (a).

Payne and Thas [13] prove that \( T(n, m, q) \) is a TGQ of order \((q^n, q^m)\) with center \((\infty)\), and that conversely every TGQ is isomorphic to a \( T(n, m, q) \).

In the case where \( n = m = 1 \), so \( O(1, 1, q) = O \) is an oval of \( PG(2, q) \), the GQ \( T(1, 1, q) \) is the Tits GQ \( T_2(O) \). When \( m = 2n = 2 \), so \( O(1, 2, q) = O \) is an ovoid of \( PG(3, q) \), the GQ \( T(1, 2, q) \) is the Tits GQ \( T_3(O) \). Note that \( T_2(O) \cong Q(4, q) \) if and only if \( O \) is a conic, while \( T_3(O) \cong Q(5, q) \) if and only if \( O \) is an elliptic quadric (see [13, Chapter 3]).

In the extension \( PG(2n+m-1, q^n) \) of \( PG(2n+m-1, q) \), with \( m \in \{n, 2n\} \), we consider \( n \) \((\frac{m}{n}+1)\)-dimensional spaces \( PG(1)(\frac{m}{n}+1, q^n) = \xi_i \), with \( i = 1, 2, \ldots, n \), which are conjugate with respect to the extension \( GF(q^n) \) of \( GF(q) \), that is, which form an orbit of the Galois group corresponding to this extension, and which span \( PG(2n+m-1, q^n) \). In \( \xi_i \) we consider an oval \( O_1 \) for \( m = n \) and an ovoid \( O_1 \) for \( m = 2n \). Let \( O_1 = \{x_0^{(1)}, x_1^{(1)}, \ldots, x_{q^n}^{(1)}\} \). Further, let \( x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)} \), with \( i = 0, 1, \ldots, q^m \), be conjugate with respect to the extension \( GF(q^n) \) of \( GF(q) \). The points \( x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)} \) generate an \((n-1)\)-dimensional space \( \pi_i \) over \( GF(q) \), with \( i = 0, 1, \ldots, q^m \). Then \( O = \{\pi_0, \pi_1, \ldots, \pi_{q^m}\} \) is a generalized oval of \( PG(3n-1, q) \) for \( m = n \), and a generalized ovoid of \( PG(4n-1, q) \) for \( m = 2n \).
Here, we speak of a regular or elementary pseudo-oval, respectively a regular or elementary pseudo-ovoid. In such a case the corresponding GQ is isomorphic to a GQ of Tits \((T_2(\mathcal{O}_1)), \text{ respectively } T_3(\mathcal{O}_1))\). If \(\mathcal{O}_1\) is a conic or an elliptic quadric, then \(T(n, m, q)\) is isomorphic to a classical GQ \((\mathcal{Q}(4, q^n), \text{ respectively } \mathcal{Q}(5, q^n))\) and \(\mathcal{O}(n, m, q)\) is called classical, see [13, Chapter 3]; for \(n = m\) the classical \(\mathcal{O}(n, n, q)\) is called a pseudo-conic and for \(m = 2n\) a pseudo-quadratic.

For \(m = n\), any known \([n - 1]\)-oval is regular, for \(m = 2n\) and \(q\) even any known \([n - 1]\)-ovoid is regular, but for \(m = 2n\) and \(q\) odd there are \([n - 1]\)-ovoids which are not regular; see Thas [16].

One can prove that \(n \leq m \leq 2n\), and that for \(q\) even \(m \in \{n, 2n\}\); see [13, Chapter 8].

Let either \(n = m\) with \(q\) odd, or let \(n \neq m\). Then by [13, Section 8.7] the \(q^m + 1\) tangent spaces of an \(\mathcal{O}(n, m, q) = \mathcal{O}\) form an \(\mathcal{O}^* = \mathcal{O}^*(n, m, q)\) in the dual space of \(\text{PG}(2n + m - 1, q)\). So in addition to \(T(n, m, q) = T(\mathcal{O})\) there arises a TGQ \(T(\mathcal{O}^*)\) with the same parameters. The TGQ \(T(\mathcal{O}^*)\) is called the translation dual of \(T(\mathcal{O})\), and \(\mathcal{O}^*\) is called the translation dual of \(\mathcal{O}\). For \(m = 2n\) and \(q\) odd there are examples for which \(T(\mathcal{O}) \not\cong T(\mathcal{O}^*)\); see Payne [11].

Let \(\mathcal{O}(n, 2n, q)\) be an egg of \(\text{PG}(4n - 1, q)\). We say that \(\mathcal{O}(n, 2n, q)\) is good at the element \(\pi_i\) of \(\mathcal{O}(n, 2n, q)\) if any \(\text{PG}(3n - 1, q)\) containing \(\pi_i\) and at least two other elements of \(\mathcal{O}(n, 2n, q)\) contains exactly \(q^n + 1\) elements of \(\mathcal{O}(n, 2n, q)\). In such a case the corresponding TGQ \(T(n, 2n, q)\) contains at least \(q^{3n} + q^{2n}\) subquadrangles of order \(q^n\); for \(q\) odd the \(q^{3n} + q^{2n}\) subquadrangles of order \(q^n\) defined by the element \(\pi_i\) at which \(\mathcal{O}(n, 2n, q)\) is good are isomorphic to the classical GQ \(\mathcal{Q}(4, q^n)\), see Thas [16]. The TGQ \(T(n, 2n, q)\) is isomorphic to a GQ of Tits of order \((q^n, q^{2n})\) if and only if the corresponding egg \(\mathcal{O}(n, 2n, q)\) is good at each of its elements; see Payne and Thas [13]. If \(q\) is even and \(\mathcal{O}(n, 2n, q)\) is good at \(\pi \in \mathcal{O}(n, 2n, q)\), then the translation dual \(\mathcal{O}^*(n, 2n, q)\) is good at the tangent space \(\tau\) of \(\mathcal{O}(n, 2n, q)\) at \(\pi\); see Thas [16]. For each known egg \(\mathcal{O}(n, 2n, q)\), either \(\mathcal{O}(n, 2n, q)\) or \(\mathcal{O}^*(n, 2n, q)\) is good at one of its elements.

Let \(\mathcal{K}\) be the quadratic cone with vertex \(x\) in \(\text{PG}(3, q)\). A partition \(\mathcal{F}\) of \(\mathcal{K}\) into \(q\) disjoint conics is called a flock of \(\mathcal{K}\). Then with \(\mathcal{F}\) there corresponds a GQ \(S(\mathcal{F})\) of order \((q^2, q)\); see Thas [15]. The GQ \(S(\mathcal{F})\) is isomorphic to the classical GQ \(\mathcal{H}(3, q^2)\) if and only if the flock is linear, that is, if and only if all planes of the conics of the flock contain a common line; see Payne and Thas [13] and Payne [12]. The GQ \(S(\mathcal{F})\) is an EGG with center \((\infty)\). By Payne and Thas [14] the center \((\infty)\) is uniquely determined if \(S(\mathcal{F})\) is not classical.

If the point-line dual of \(S(\mathcal{F})\) is a TGQ \(T(\mathcal{O})\) with center \(L\) (then \((\infty) I L\)), where \(\mathcal{O} = \mathcal{O}(n, 2n, q)\), then the translation dual \(\mathcal{O}^*\) of \(\mathcal{O}\) is good at the tangent space \(\tau_i\) of \(\mathcal{O}\) at \(\pi_i = L \in \mathcal{O}\); see Thas [16]. Conversely, by Thas [17], if the translation dual \(\mathcal{O}^*\) of an egg \(\mathcal{O} = \mathcal{O}(n, 2n, q)\), with \(q\) odd, is good at one of its
elements, then \( T(\mathcal{O}) \) is the point-line dual of a flock GQ \( S(\mathcal{F}) \). If \( q \) is even and if the TGQ \( T(\mathcal{O}) \) is the point-line dual of a flock GQ (then \( \mathcal{O} \) and \( \mathcal{O}^* \) are good), then, by Johnson [7], the GQ \( T(\mathcal{O}) \) is classical.

Finally, we remark that every known GQ of order \( (s, s^2) \) is either a TGQ \( T(\mathcal{O}) \) or the point-line dual of a flock GQ \( S(\mathcal{F}) \).

5. Subquadrangles of translation generalized quadrangles \( T(\mathcal{O}) \), with \( \mathcal{O} \) good

In this section we consider the subquadrangles of order \( q^n \) of translation generalized quadrangles \( T(\mathcal{O}) \) of order \( (q^n, q^{2n}) \), with \( \mathcal{O} = \mathcal{O}(n, 2n, q) \) and \( \mathcal{O} \) good at some element \( \pi \).

**Theorem 5.1** (Brown and Thas [4, 5]). Let \( S = T(\mathcal{O}) \), with \( \mathcal{O} = \mathcal{O}(n, 2n, q) \), be a TGQ. If \( \mathcal{O} \) is good at some element \( \pi \) and if \( S \) has a subquadrangle of order \( s = q^n \) which does not contain the point \( (\infty) \), then \( S \cong Q(5, s) \).

**Theorem 5.2** (Brown and Thas [5]). Let \( S = (P, B, \mathbb{I}) \) be a TGQ of order \( (s, s^2) \), \( s \) odd, with \( \mathcal{O} = T(\mathcal{O}) \) and \( \mathcal{O} = \mathcal{O}(n, 2n, q) \) good at some element \( \pi \). If \( S' \) is a subquadrangle of order \( s = q^n \) of \( S \) containing the point \( (\infty) \), then \( S' \cong Q(4, s) \) and either \( S \cong Q(5, s) \) or \( S' \) is one of the \( s^3 + s^2 \) subquadrangles of order \( s \) (isomorphic to \( Q(4, s) \)) containing the line \( \pi \) of \( S \).

**Theorem 5.3** (Brown and Thas [5]). Let \( \mathcal{O} = \mathcal{O}(n, 2n, q) \), with \( q \) odd, be an egg in \( PG(4n-1, q) \) which is good at \( \pi \). If there is a subspace \( PG(3n-1, q) \) of \( PG(4n-1, q) \) which contains at least four elements of \( \mathcal{O} \), but which does not contain \( \pi \), then \( \mathcal{O} \) is a Kantor-Knuth egg (see Section 7). If there is a subspace \( PG(3n-1, q) \) which contains at least five elements of \( \mathcal{O} \), but which does not contain \( \pi \), then \( \mathcal{O} \) is classical.

**Remark 5.4.** Independently and in a completely different way, Lavrauw [8] also proved the second part of Theorem 5.3.

**Theorem 5.5** (Thas [21], see also [22, 23]). Let \( S = (P, B, \mathbb{I}) \) be a TGQ of order \( (s, s^2) \), \( s > 1 \), with \( \mathcal{O} = T(\mathcal{O}) \) and \( \mathcal{O} = \mathcal{O}(n, 2n, q) \). If \( S' \) is a subquadrangle of order \( s \) of \( S \) containing the point \( (\infty) \), then \( S' \) is a TGQ \( T(\mathcal{O}') \) with \( \mathcal{O}' = \mathcal{O}'(n, n, q) \) a pseudo-oval on \( \mathcal{O} \).

**Open problem 5.6.** If in Theorem 5.5, \( s \) is even, does the existence of \( S' \) imply that \( S \) is isomorphic to a \( T_3(\mathcal{O}) \) of Tits?

We mention several recent results on this problem in the next section.
Let us end this section with the following results on good TGQs with a large automorphism group, followed by an application to group actions on subGQs.

Let $S = T(O)$ be a TGQ of order $(q^n, q^{2n})$, where $q$ is odd, and where $O$ is good at some element $PG(n - 1, q)$. Let $K$ be the kernel [13, Chapter 8] of $S$. Then by Thas [24], $(q^n + 1)(q^n - 1)q^{6n}(|K| - 1)$ is a divisor of $|Aut(S)|$. If $S$ is classical, then for an arbitrary line $L$ of $S$, $|Aut(S)_L| = (q^n + 1)(q^n - 1)q^{6n}2h$, where $q^n = p^h$ for the prime $p$.

**Theorem 5.7** (Thas [27]). Let $S = T(O)$ be a TGQ of order $(q^n, q^{2n})$, where $q$ is an odd prime power, with $O$ the generalized ovoid in $PG(4n - 1, q)$ corresponding to $S$. Suppose that $O$ is good at some element $PG(n - 1, q)$, and let $L$ be the line $PG(n - 1, q)$ of $S$. If

$$(q^n + 1)^2(q^n - 1)q^{6n} \text{ divides } |Aut(S)_L|,$$

then $S$ is isomorphic to the classical GQ $Q(5, q^n)$.

**Corollary 5.8** (Thas [27]). Let $F$ be a semifield flock of $PG(3, q^n)$, $q$ odd. If $q^n + 1$ divides the size of $Aut(F)_H$, where $H$ is any flock plane, or, equivalently, if $q^n + 1$ divides $\frac{|Aut(F)|}{q^n}$, then $F$ is linear.

From Theorem 5.7, Thas deduced the following theorem on subGQs:

**Theorem 5.9** (Thas [27]). Let $S = T(O)$ be a TGQ of order $(s, s^2)$, $s$ odd, so that the generalized ovoid $O$ is good at some element. If $Aut(S)$ acts transitively on its subGQs of order $s$, then $S \cong Q(5, s)$.

### 6. Subquadrangles of translation generalized quadrangles $T(O)$ in even characteristic

Brown and Lavrauw [2] generalized the following result of Brown on ovoids of $PG(3, q)$, $q$ even, by obtaining a similar result for generalized ovoids.

**Theorem 6.1** (Brown [1]). If an ovoid of $PG(3, q)$, $q$ even, has a conic plane section, the ovoid must be an elliptic quadric. Equivalently, if a $T_3(O)$ of order $(q, q^2)$, where $O$ is an ovoid of $PG(3, q)$, $q$ even, has a classical subGQ of order $q$ containing the point $(\infty)$, then $T_3(O) \cong Q(5, q)$.

In terms of translation generalized quadrangles, Brown and Lavrauw prove the following.

**Theorem 6.2** (Brown and Lavrauw [2]). If a $T(O)$ of order $(q^n, q^{2n})$, where $O$ is a generalized ovoid of $PG(4n - 1, q)$, $q$ even, has a classical subGQ of order $q$ containing the point $(\infty)$, then $T(O) \cong Q(5, q^n)$. 
In [26], Thas then generalized the latter result to elation generalized quadrangles.

**Theorem 6.3** (Thas [26]). Let \( S^{(x)} \) be an EGQ of order \((q, q^2)\), \( q \) even, having a classical subGQ \( S' \) of order \( q \) containing \( x \). Then \( S^{(x)} \cong Q(5, q) \).

Let \( \mathcal{O} \) be a generalized oval in \( \text{PG}(3n - 1, q) = \langle \mathcal{O} \rangle \), \( q \) even. Then the \( q^n + 1 \) tangent spaces to the elements of \( \mathcal{O} \) meet in an \((n - 1)\)-dimensional subspace \( \Omega \) of \( \text{PG}(3n - 1, q) \), called the nucleus of \( \mathcal{O} \) (cf. [13, Chapter 8]). It is easily seen that for any \( \pi \in \mathcal{O} \), \( \mathcal{O}' = (\mathcal{O} \setminus \{\pi\}) \cup \{\Omega\} \) is again a generalized oval. If \( \mathcal{O} \) is a generalized conic, that is, if \( T(\mathcal{O}) \cong Q(4, q^n) \), then \( \mathcal{O}' \) is called a pointed generalized conic or a generalized pointed conic.

The following result generalizes another recent result of Brown and Lavrauw [3] from TGQs to EGQs.

**Theorem 6.4** (Thas [26]). Let \( S^{(x)} \) be an EGQ of order \((q^n, q^{2n})\), \( q \) even, containing a subGQ \( T(\mathcal{O}) \), where \( \mathcal{O} \) is a pointed generalized conic in \( \text{PG}(3n - 1, q) \), having \( x \) as translation point. Then either \( q^n = 4 \) and \( S^{(x)} \cong Q(5, 4) \), or \( q^n = 8 \) and \( S^{(x)} \cong T_3(\mathcal{O}') \) with \( \mathcal{O}' \) the Suzuki-Tits ovoid of \( \text{PG}(3, 8) \).

Both Theorem 6.3 and Theorem 6.4 rely on the following observation, which works in any characteristic (part (ii) being a direct corollary of part (i)).

**Theorem 6.5** (Thas [26]).

(i) Let \( S^{(x)} \) be an EGQ of order \((s, s^2)\), \( s > 1 \), containing a subGQ \( S' \) of order \( s \) which has at least one axis of symmetry \( L \) incident with \( x \). Then \( S^{(x)} \) is a TGQ for the translation point \( x \).

(ii) Let \( S^{(x)} \) be an EGQ of order \((s, s^2)\), \( s > 1 \), containing a subGQ \( S' \) of order \( s \) which is a TGQ with translation point \( x \). Then \( S^{(x)} \) is a TGQ with translation point \( x \).

### 7. Subquadrangles of Flock GQs \( S(\mathcal{F}) \)

Let \( \mathcal{F} = \{C_1, C_2, \ldots, C_q\} \) be a flock of the quadratic cone \( K \) of \( \text{PG}(3, q) \), with \( q \) even. If the planes \( \xi_1, \xi_2, \ldots, \xi_q \) of the respective conics \( C_1, C_2, \ldots, C_q \) all contain a common point, then the flock is linear; see Thas [15].

Let \( \mathcal{F} = \{C_1, C_2, \ldots, C_q\} \) be a flock of the quadratic cone \( K \) of \( \text{PG}(3, q) \), with \( q \) odd. If the planes \( \xi_1, \xi_2, \ldots, \xi_q \) of the respective conics \( C_1, C_2, \ldots, C_q \) all contain a common interior point of \( K \), then the flock is linear; see Thas [15].

Let \( \mathcal{F} = \{C_1, C_2, \ldots, C_q\} \) be a flock of the quadratic cone \( K \) of \( \text{PG}(3, q) \), with \( q \) odd, and suppose that the planes \( \xi_1, \xi_2, \ldots, \xi_q \), with \( C_i \subseteq \xi_i \), all contain a common exterior point of \( K \). If \( m \) is any given non-square of \( GF(q) \), then coordinates
can be chosen in such a way that \( \mathcal{K} \) has equation \( X_0X_1 = X_2^2 \) and the planes \( \xi_i \) have equation

\[
a_iX_0 - ma_i^2X_1 + X_3 = 0,  \tag{1}
\]

with \( \{a_1, a_2, \ldots, a_q\} = GF(q) \) and \( \sigma \) an automorphism of \( GF(q) \). Conversely, given any non-square \( m \) of \( GF(q) \), the planes \( \xi_i \) with equation (1), where \( \{a_1, a_2, \ldots, a_q\} = GF(q) \) and where \( \sigma \) is any automorphism of \( GF(q) \), define a flock \( \mathcal{F} \) of the cone \( \mathcal{K} \) with equation \( X_0X_1 = X_2^2 \). Also, the planes \( \xi_i \) all contain the exterior point \((0, 0, 1, 0)\) of \( \mathcal{K} \). This flock \( \mathcal{F} \) is linear if and only if \( \sigma = 1 \). The flocks \( \mathcal{F} \) are called Kantor-Knuth flocks. For proofs we refer to Thas [15].

The point-line dual of the GQ which corresponds to a Kantor-Knuth flock is a TGQ \( T(O) \), with \( O \) isomorphic to its translation dual \( O^* \), with \( O \) good at some element \( \pi \), and with \( O^* \) good at the tangent space \( \tau \) of \( O \) at \( \pi \); see Payne [11]. Such an egg is called a Kantor-Knuth egg.

In the next theorems all subquadrangles of order \( s \) of flock GQs \( S(\mathcal{F}) \) of order \((s^2, s)\) are determined.

**Theorem 7.1** (O’Keefe and Penttila [9]). If \( S(\mathcal{F}) \) is a flock GQ of order \((s^2, s)\), \( s \) even, with center \((\infty)\), then \( S(\mathcal{F}) \) contains exactly \( s^3 + s^2 \) subquadrangles of order \( s \) containing the point \((\infty)\).

An easy way to see this goes as follows. Payne proved in [11] that for all sets \( \{L, M, N\} \) of two by two nonconcurrent lines for which there is a line \( U \) \( \perp \) \( \{L, M, N\} \) \( \subseteq U\), we have that \( |\{L, M, N\}\perp| = s + 1 \). That is, the set \( \{L, M, N\} \) is 3-regular. Now let \( \{L, M, N\} \) be such a set of lines. If \( B' \) is the set of lines of \( S(\mathcal{F}) \) incident with points of the form \( X \in \{L, M, N\}\perp \) \( Y \in \{L, M, N\}\perp \), and \( P' \) is the set of points which are incident with at least two distinct lines of \( B' \), then endowed with the induced incidence \( \mathcal{I}' \), \( S' = (P', B', I') \) is a subGQ of order \( s \) of \( S(\mathcal{F}) \), see Payne and Thas [13, §2.6.2]. Now a standard counting argument yields \( s^3 + s^2 \) of such subGQs (notice that any point of a GQ of order \((s^2, s)\) is contained in at most \( s^3 + s^2 \) subGQs of order \( s \), see e.g. [4]).

**Remark 7.2.** Payne [10] proves that each of these \( s^3 + s^2 \) subquadrangles is a \( T_2(O) \) of Tits, for some oval \( O \) of PG(2, q).

**Theorem 7.3** (Brown and Thas [5]). Let \( S(\mathcal{F}) \) be a flock GQ of order \((s^2, s)\), \( s \) odd, with center \((\infty)\). If \( S' \) is a subquadrangle of order \( s \) of \( S(\mathcal{F}) \) containing the point \((\infty)\), then \( \mathcal{F} \) is a Kantor-Knuth flock. Hence either \( S \cong H(3, s^2) \) or \( S' \) is one of the \( s^3 + s^2 \) subquadrangles of order \( s \) containing the point \((\infty)\). In all cases \( S' \cong W(s) \).

An application of Theorem 7.3 is the next theorem.
Theorem 7.4 (Thas [27]). Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$ be a good egg in $\text{PG}(4n - 1, q)$, $q$ odd. If $T(\mathcal{O})$ contains a subGQ of order $q^n$ that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong S(\mathcal{F})^D$ with $\mathcal{F}$ a Kantor-Knuth flock.

For $q$ even, one can use Theorem 6.2 and [25, Chapter 7] to show that if $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is an egg in $\text{PG}(4n - 1, q)$ and $T(\mathcal{O})$ contains a subGQ of order $q^n$ that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong Q(5, q^n)$, see [27].

Theorem 7.5 (Brown and Thas [4], Thas and Thas [20]). Let $S(\mathcal{F})$ be a flock GQ of order $(s^2, s)$, with center $(\infty)$. If $S(\mathcal{F})$ contains a subquadrangle of order $s$ which does not contain $(\infty)$, then $S \cong H(3, s^2)$ and $S' \cong W(s)$.

Remark 7.6. The even case of Theorem 7.5 is due to Brown and Thas, the odd case to Thas and Thas.

8. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with $\mathcal{O}^*$ good

As for $q$ even the egg $\mathcal{O}(n, 2n, q)$ is good if and only if $\mathcal{O}^*(n, 2n, q)$ is good, we may assume that $q$ is odd.

If $\mathcal{O}^*(n, 2n, q)$, with $q$ odd, is good, then by Thas [17] the TGQ $T(\mathcal{O})$ with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is isomorphic to the point-line dual of a flock GQ $S(\mathcal{F})$. If $\mathcal{O}^*(n, 2n, q)$ is good at the tangent space $\tau$ of $\mathcal{O}(n, 2n, q)$ at $\pi$, then $\pi$ corresponds to the point $(\infty)$ of $S(\mathcal{F})$. So by Theorems 7.3 and 7.5 we have the following result.

Theorem 8.1 (Brown and Thas [4, 5], Thas and Thas [20]). Let $S = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and $q$ odd, be a TGQ such that the translation dual $\mathcal{O}^*$ of $\mathcal{O}$ is good at the tangent space $\tau$ of $\mathcal{O}$ at $\pi$. If $S$ contains a subquadrangle $S'$ of order $q^n$, then either $S \cong Q(5, q^n)$ (and then $S' \cong Q(4, q^n)$) or $\mathcal{O}$ is a nonclassical Kantor-Knuth egg and $S'$ is one of the $q^{3m} + q^{2n}$ subquadrangles of order $q^n$ (isomorphic to $Q(4, q^n)$) containing the line $\pi$ of $T(\mathcal{O})$.

9. Byproduct: A characterization of TGQs

Part of the proof of Theorem 7.5 (odd case) was also used by Thas and Thas [20] to prove the following characterization theorem of TGQs whose point-line duals arise from flocks.
Theorem 9.1 (Thas and Thas [20]). If \( S = (P, B, I) \) is a flock GQ of order \((s^2, s)\), with \( s \) odd, then the point-line dual of \( S \) is a TGQ if and only if \( S \) has a regular point \( x \), with \((\infty) \neq x \sim (\infty)\), where \((\infty)\) is the elation point of \( S \).

Remark 9.2. Thas [18] proved the following result for \( q \) even. Let \( S \) be a flock GQ of order \((s^2, s)\), with \( s \) even. Then the point-line dual of \( S \) is isomorphic to the classical GQ \( H(3, s^2) \) if and only if \( S \) has a regular point \( x \), with \((\infty) \neq x \sim (\infty)\), where \((\infty)\) is the elation point of \( S \).

References


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