

page 1 / 13

go back

full screen

close

quit

Subquadrangles of order s of generalized quadrangles of order (s, s^2)

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Abstract

In this paper we consider the subquadrangles of order s for all known classes of generalized quadrangles of order (s, s^2) .

Keywords: generalized quadrangle, subquadrangle, flock quadrangle, translation quadrangle

MSC 2000: 51E12, 05B25

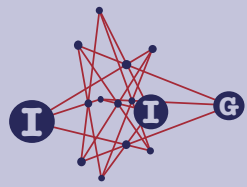
1. Introduction

We survey results on subquadrangles of order s for all known classes of generalized quadrangles of order (s, s^2) . An interesting application of this theory goes as follows. If \mathcal{S} is a generalized quadrangle of order (s, s^2) , if \mathcal{S}' is a subquadrangle of order s of \mathcal{S} , and if p is a point of \mathcal{S} not in \mathcal{S}' , then all points of \mathcal{S}' collinear with p form an *ovoid* of \mathcal{S}' , that is, a set \mathcal{O} of s^2+1 points of \mathcal{S}' such that each line of \mathcal{S}' contains exactly one point of \mathcal{O} . If \mathcal{S}' is the classical generalized quadrangle $\mathcal{Q}(4, s)$ arising from a nonsingular quadric in $\text{PG}(4, s)$, then, by a standard argument, \mathcal{O} defines a projective plane of order s^2 . In such a way new projective planes were discovered; see e.g. Thas and Payne [19]. Also, several theorems of this survey will provide interesting and strong characterizations of the classical generalized quadrangle $\mathcal{Q}(5, s)$ arising from a nonsingular elliptic quadric in $\text{PG}(5, s)$, respectively of the Kantor-Knuth generalized quadrangles. As a byproduct a nice characterization of a class of translation generalized quadrangles of order (s, s^2) , with s odd, will be given.

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2. Definitions

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathbb{I})$ in which P and B are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which $\mathbb{I} \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line incidence relation satisfying the following axioms:

- (i) each point is incident with $t + 1$ lines, $t \geq 1$, and two distinct points are incident with at most one line;
- (ii) each line is incident with $s + 1$ points, $s \geq 1$, and two distinct lines are incident with at most one point;
- (iii) if x is a point and L is a line not incident with x , then there is a unique pair $(y, M) \in P \times B$ for which $x \mathbb{I} M \mathbb{I} y \mathbb{I} L$.

The integers s and t are the *parameters* of the GQ and \mathcal{S} is said to have *order* (s, t) . If $s = t$, then \mathcal{S} is said to have *order* s .

A *subquadrangle* (or also *subGQ*) $\mathcal{S}' = (P', B', \mathbb{I}')$ of \mathcal{S} is a GQ such that $P' \subseteq P$, $B' \subseteq B$ and \mathbb{I}' is the restriction of \mathbb{I} to $(P' \times B') \cup (B' \times P')$; see [13, Chapter 2].

If $\mathcal{S} = (P, B, \mathbb{I})$ is a GQ of order (s, t) , then the GQ $\mathcal{S}^D = (B, P, \mathbb{I})$ of order (t, s) is called the *(point-line) dual* of \mathcal{S} . As there is a point-line duality for GQs of order (s, t) , we assume without further notice that the dual of a given theorem or definition has also been given.

For background on GQs we refer to the monograph by Payne and Thas [13].

3. The classical examples

We give a brief description of three families of examples known as the *classical* GQs, all of which are associated with classical groups and were first recognized as GQs by Tits; see Dembowski [6].

- (i) Consider a nonsingular quadric \mathcal{Q} of projective index 1, that is, of Witt index 2, of the projective space $\text{PG}(d, q)$, with $d = 3, 4$ or 5 . Then the points of \mathcal{Q} together with the lines of \mathcal{Q} (which are the subspaces of maximal dimension on \mathcal{Q}) form a GQ $\mathcal{Q}(d, q)$ with parameters

$$s = q, t = 1, \quad \text{when } d = 3,$$

$$s = t = q, \quad \text{when } d = 4,$$

$$s = q, t = q^2, \quad \text{when } d = 5.$$



page 2 / 13

go back

full screen

close

quit

ACADEMIA
PRESS





page 3 / 13

go back

full screen

close

quit

- (ii) Let H be a nonsingular Hermitian variety of the projective space $\text{PG}(d, q^2)$, $d = 3$ or $d = 4$. Then the points of H together with the lines on H form a GQ $H(d, q^2)$ with parameters

$$\begin{aligned} s &= q^2, t = q, & \text{when } d = 3, \\ s &= q^2, t = q^3, & \text{when } d = 4. \end{aligned}$$

- (iii) The points of $\text{PG}(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ with parameters

$$s = t = q.$$

The GQ $\mathcal{Q}(5, q)$ is isomorphic to the dual of $H(3, q^2)$, the GQ $\mathcal{Q}(4, q)$ is isomorphic to the dual of $W(q)$, and $\mathcal{Q}(4, q)$ (and then also $W(q)$) is self-dual if and only if q is even; see [13, Section 3.2].

All subquadrangles of order q of $\mathcal{Q}(5, q)$ are the $\mathcal{Q}(4, q)$ subGQs on it.

4. Translation and flock generalized quadrangles

Let $\mathcal{S} = (P, B, \mathcal{I})$ be a GQ of order (s, t) , with $s \neq 1 \neq t$. A collineation θ of \mathcal{S} is an *elation* about the point p if $\theta = \text{id}$ or if θ fixes all lines incident with p and no point of $P \setminus p^\perp$ (p^\perp is the set of all points collinear with p). If there is a group G of elations about p acting regularly on $P \setminus p^\perp$, then we say that \mathcal{S} is an *elation generalized quadrangle* (EGQ) with *elation group* G and *elation point* or *base point* or *center* p . Briefly, we write that $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ is an EGQ. If the group G is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, G)$ is a *translation generalized quadrangle* (TGQ) with *translation group* G and *translation point* or *base point* or *center* p .

For any TGQ $\mathcal{S}^{(p)}$ each line incident with p is an *axis of symmetry*, that is, there is a (maximal) group of s collineations of \mathcal{S} fixing L^\perp elementwise (see [13, Chapter 8]).

In $\text{PG}(2n + m - 1, q)$ consider a set $\mathcal{O}(n, m, q)$ of $q^m + 1$ $(n - 1)$ -dimensional subspaces $\pi_0, \pi_1, \dots, \pi_{q^m}$, every three of which generate a $\text{PG}(3n - 1, q)$ and such that each element π_i of $\mathcal{O}(n, m, q)$ is contained in an $(n + m - 1)$ -dimensional subspace τ_i having no point in common with any π_j for $j \neq i$. It is easy to check that τ_i is uniquely determined, with $i = 0, 1, \dots, q^m$. The space τ_i is called the *tangent space* of $\mathcal{O}(n, m, q)$ at π_i . For $n = m = 1$ such a set $\mathcal{O}(1, 1, q)$ is an oval in $\text{PG}(2, q)$ and more generally, for $n = m$, such a set $\mathcal{O}(n, n, q)$ is called a *pseudo-oval* or a *generalized oval* or an $[n - 1]$ -*oval* of $\text{PG}(3n - 1, q)$. For $m = 2n = 2$ such a set $\mathcal{O}(1, 2, q)$ is an ovoid of $\text{PG}(3, q)$ and more generally, for $n \neq m$ such

ACADEMIA
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page 4 / 13

go back

full screen

close

quit

a set $\mathcal{O}(n, m, q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n - 1]$ -*ovoid* or an *egg* of $\text{PG}(2n + m - 1, q)$.

Now embed $\text{PG}(2n + m - 1, q)$ as a hyperplane in a $\text{PG}(2n + m, q)$, and construct a point-line geometry $T(n, m, q)$ as follows.

- POINTS are of three types:

- (i) the points of $\text{PG}(2n + m, q) \setminus \text{PG}(2n + m - 1, q)$;
- (ii) the $(n + m)$ -dimensional subspaces of $\text{PG}(2n + m, q)$ which contain a tangent space τ_i but are not contained in $\text{PG}(2n + m - 1, q)$;
- (iii) a symbol (∞) .

- LINES are of two types:

- (a) the n -dimensional subspaces of $\text{PG}(2n + m, q)$ which contain an element π_i but are not contained in $\text{PG}(2n + m - 1, q)$;
- (b) the elements of $\mathcal{O}(n, m, q)$.

- INCIDENCE is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\text{PG}(2n + m, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{O}(n, m, q)$ contained in it. The point (∞) is incident with all the lines of type (b) and with no lines of type (a).

Payne and Thas [13] prove that $T(n, m, q)$ is a TGQ of order (q^n, q^m) with center (∞) , and that conversely every TGQ is isomorphic to a $T(n, m, q)$.

In the case where $n = m = 1$, so $\mathcal{O}(1, 1, q) = \mathcal{O}$ is an oval of $\text{PG}(2, q)$, the GQ $T(1, 1, q)$ is the Tits GQ $T_2(\mathcal{O})$. When $m = 2n = 2$, so $\mathcal{O}(1, 2, q) = \mathcal{O}$ is an ovoid of $\text{PG}(3, q)$, the GQ $T(1, 2, q)$ is the Tits GQ $T_3(\mathcal{O})$. Note that $T_2(\mathcal{O}) \cong \mathcal{Q}(4, q)$ if and only if \mathcal{O} is a conic, while $T_3(\mathcal{O}) \cong \mathcal{Q}(5, q)$ if and only if \mathcal{O} is an elliptic quadric (see [13, Chapter 3]).

In the extension $\text{PG}(2n + m - 1, q^n)$ of $\text{PG}(2n + m - 1, q)$, with $m \in \{n, 2n\}$, we consider $n \left(\frac{m}{n} + 1\right)$ -dimensional spaces $\text{PG}^{(i)}\left(\frac{m}{n} + 1, q^n\right) = \xi_i$, with $i = 1, 2, \dots, n$, which are conjugate with respect to the extension $\text{GF}(q^n)$ of $\text{GF}(q)$, that is, which form an orbit of the Galois group corresponding to this extension, and which span $\text{PG}(2n + m - 1, q^n)$. In ξ_1 we consider an oval \mathcal{O}_1 for $m = n$ and an ovoid \mathcal{O}_1 for $m = 2n$. Let $\mathcal{O}_1 = \{x_0^{(1)}, x_1^{(1)}, \dots, x_{q^m}^{(1)}\}$. Further, let $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$, with $i = 0, 1, \dots, q^m$, be conjugate with respect to the extension $\text{GF}(q^n)$ of $\text{GF}(q)$. The points $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$ generate an $(n - 1)$ -dimensional space π_i over $\text{GF}(q)$, with $i = 0, 1, \dots, q^m$. Then $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^m}\}$ is a generalized oval of $\text{PG}(3n - 1, q)$ for $m = n$, and a generalized ovoid of $\text{PG}(4n - 1, q)$ for $m = 2n$.

ACADEMIA
PRESS





page 5 / 13

go back

full screen

close

quit

Here, we speak of a *regular* or *elementary pseudo-oval*, respectively a *regular* or *elementary pseudo-ovoid*. In such a case the corresponding GQ is isomorphic to a GQ of Tits ($T_2(\mathcal{O}_1)$, respectively $T_3(\mathcal{O}_1)$). If \mathcal{O}_1 is a conic or an elliptic quadric, then $T(n, m, q)$ is isomorphic to a classical GQ ($\mathcal{Q}(4, q^n)$, respectively $\mathcal{Q}(5, q^n)$) and $\mathcal{O}(n, m, q)$ is called *classical*, see [13, Chapter 3]; for $n = m$ the classical $\mathcal{O}(n, n, q)$ is called a *pseudo-conic* and for $m = 2n$ a *pseudo-quadric*.

For $m = n$, any known $[n - 1]$ -oval is regular, for $m = 2n$ and q even any known $[n - 1]$ -ovoid is regular, but for $m = 2n$ and q odd there are $[n - 1]$ -ovoids which are not regular; see Thas [16].

One can prove that $n \leq m \leq 2n$, and that for q even $m \in \{n, 2n\}$; see [13, Chapter 8].

Let either $n = m$ with q odd, or let $n \neq m$. Then by [13, Section 8.7] the $q^m + 1$ tangent spaces of an $\mathcal{O}(n, m, q) = \mathcal{O}$ form an $\mathcal{O}^* = \mathcal{O}^*(n, m, q)$ in the dual space of $\text{PG}(2n + m - 1, q)$. So in addition to $T(n, m, q) = T(\mathcal{O})$ there arises a TGQ $T(\mathcal{O}^*)$ with the same parameters. The TGQ $T(\mathcal{O}^*)$ is called the *translation dual* of $T(\mathcal{O})$, and \mathcal{O}^* is called the *translation dual* of \mathcal{O} . For $m = 2n$ and q odd there are examples for which $T(\mathcal{O}) \not\cong T(\mathcal{O}^*)$; see Payne [11].

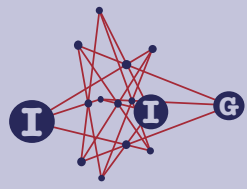
Let $\mathcal{O}(n, 2n, q)$ be an egg of $\text{PG}(4n - 1, q)$. We say that $\mathcal{O}(n, 2n, q)$ is *good* at the element π_i of $\mathcal{O}(n, 2n, q)$ if any $\text{PG}(3n - 1, q)$ containing π_i and at least two other elements of $\mathcal{O}(n, 2n, q)$ contains exactly $q^n + 1$ elements of $\mathcal{O}(n, 2n, q)$. In such a case the corresponding TGQ $T(n, 2n, q)$ contains at least $q^{3n} + q^{2n}$ subquadrangles of order q^n ; for q odd the $q^{3n} + q^{2n}$ subquadrangles of order q^n defined by the element π_i at which $\mathcal{O}(n, 2n, q)$ is good are isomorphic to the classical GQ $\mathcal{Q}(4, q^n)$, see Thas [16]. The TGQ $T(n, 2n, q)$ is isomorphic to a GQ of Tits of order (q^n, q^{2n}) if and only if the corresponding egg $\mathcal{O}(n, 2n, q)$ is good at each of its elements; see Payne and Thas [13]. If q is even and $\mathcal{O}(n, 2n, q)$ is good at $\pi \in \mathcal{O}(n, 2n, q)$, then the translation dual $\mathcal{O}^*(n, 2n, q)$ is good at the tangent space τ of $\mathcal{O}(n, 2n, q)$ at π ; see Thas [16]. For each known egg $\mathcal{O}(n, 2n, q)$, either $\mathcal{O}(n, 2n, q)$ or $\mathcal{O}^*(n, 2n, q)$ is good at one of its elements.

Let \mathcal{K} be the quadratic cone with vertex x in $\text{PG}(3, q)$. A partition \mathcal{F} of $\mathcal{K} \setminus \{x\}$ into q disjoint conics is called a *flock* of \mathcal{K} . Then with \mathcal{F} there corresponds a GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q) ; see Thas [15]. The GQ $\mathcal{S}(\mathcal{F})$ is isomorphic to the classical GQ $H(3, q^2)$ if and only if the flock is *linear*, that is, if and only if all planes of the conics of the flock contain a common line; see Payne and Thas [13] and Payne [12]. The GQ $\mathcal{S}(\mathcal{F})$ is an EGQ with center (∞) . By Payne and Thas [14] the center (∞) is uniquely determined if $\mathcal{S}(\mathcal{F})$ is not classical.

If the point-line dual of $\mathcal{S}(\mathcal{F})$ is a TGQ $T(\mathcal{O})$ with center L (then $(\infty) \text{ I } L$), where $\mathcal{O} = \mathcal{O}(n, 2n, q)$, then the translation dual \mathcal{O}^* of \mathcal{O} is good at the tangent space τ_i of \mathcal{O} at $\pi_i = L \in \mathcal{O}$; see Thas [16]. Conversely, by Thas [17], if the translation dual \mathcal{O}^* of an egg $\mathcal{O} = \mathcal{O}(n, 2n, q)$, with q odd, is good at one of its

ACADEMIA
PRESS





page 6 / 13

go back

full screen

close

quit

elements, then $T(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. If q is even and if the TGQ $T(\mathcal{O})$ is the point-line dual of a flock GQ (then \mathcal{O} and \mathcal{O}^* are good), then, by Johnson [7], the GQ $T(\mathcal{O})$ is classical.

Finally, we remark that every known GQ of order (s, s^2) is either a TGQ $T(\mathcal{O})$ or the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$.

5. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with \mathcal{O} good

In this section we consider the subquadrangles of order q^n of translation generalized quadrangles $T(\mathcal{O})$ of order (q^n, q^{2n}) , with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and \mathcal{O} good at some element π .

Theorem 5.1 (Brown and Thas [4, 5]). *Let $\mathcal{S} = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$, be a TGQ. If \mathcal{O} is good at some element π and if \mathcal{S} has a subquadrangle of order $s = q^n$ which does not contain the point (∞) , then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

Theorem 5.2 (Brown and Thas [5]). *Let $\mathcal{S} = (P, B, \mathbb{I})$ be a TGQ of order (s, s^2) , s odd, with $\mathcal{S} = T(\mathcal{O})$ and $\mathcal{O} = \mathcal{O}(n, 2n, q)$ good at some element π . If \mathcal{S}' is a subquadrangle of order $s = q^n$ of \mathcal{S} containing the point (∞) , then $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and either $\mathcal{S} \cong \mathcal{Q}(5, s)$ or \mathcal{S}' is one of the $s^3 + s^2$ subquadrangles of order s (isomorphic to $\mathcal{Q}(4, s)$) containing the line π of \mathcal{S} .*

Theorem 5.3 (Brown and Thas [5]). *Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$, with q odd, be an egg in $\text{PG}(4n - 1, q)$ which is good at π . If there is a subspace $\text{PG}(3n - 1, q)$ of $\text{PG}(4n - 1, q)$ which contains at least four elements of \mathcal{O} , but which does not contain π , then \mathcal{O} is a Kantor-Knuth egg (see Section 7). If there is a subspace $\text{PG}(3n - 1, q)$ which contains at least five elements of \mathcal{O} , but which does not contain π , then \mathcal{O} is classical.*

Remark 5.4. Independently and in a completely different way, Lavrauw [8] also proved the second part of Theorem 5.3.

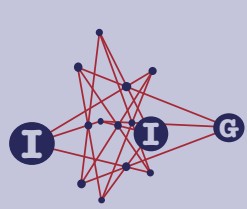
Theorem 5.5 (Thas [21], see also [22, 23]). *Let $\mathcal{S} = (P, B, \mathbb{I})$ be a TGQ of order (s, s^2) , $s > 1$, with $\mathcal{S} = T(\mathcal{O})$ and $\mathcal{O} = \mathcal{O}(n, 2n, q)$. If \mathcal{S}' is a subquadrangle of order s of \mathcal{S} containing the point (∞) , then \mathcal{S}' is a TGQ $T(\mathcal{O}')$ with $\mathcal{O}' = \mathcal{O}'(n, n, q)$ a pseudo-oval on \mathcal{O} .*

Open problem 5.6. *If in Theorem 5.5, s is even, does the existence of \mathcal{S}' imply that \mathcal{S} is isomorphic to a $T_3(\overline{\mathcal{O}})$ of Tits?*

We mention several recent results on this problem in the next section.

ACADEMIA
PRESS





page 7 / 13

go back

full screen

close

quit

Let us end this section with the following results on good TGQs with a large automorphism group, followed by an application to group actions on subGQs.

Let $S = T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}) , where q is odd, and where \mathcal{O} is good at some element $\text{PG}(n - 1, q)$. Let \mathbb{K} be the kernel [13, Chapter 8] of S . Then by Thas [24], $(q^n + 1)(q^n - 1)q^{6n}(|\mathbb{K}| - 1)$ is a divisor of $|\text{Aut}(S)|$. If S is classical, then for an arbitrary line L of S , $|\text{Aut}(S)_L| = (q^n + 1)^2(q^n - 1)^2q^{6n}2h$, where $q^n = p^h$ for the prime p .

Theorem 5.7 (Thas [27]). *Let $S = T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}) , where q is an odd prime power, with \mathcal{O} the generalized ovoid in $\text{PG}(4n - 1, q)$ corresponding to S . Suppose that \mathcal{O} is good at some element $\text{PG}(n - 1, q)$, and let L be the line $\text{PG}(n - 1, q)$ of S . If*

$$(q^n + 1)^2(q^n - 1)q^{6n} \text{ divides } |\text{Aut}(S)_L|,$$

then S is isomorphic to the classical GQ $\mathcal{Q}(5, q^n)$.

Corollary 5.8 (Thas [27]). *Let \mathcal{F} be a semifield flock of $\text{PG}(3, q^n)$, q odd. If $q^n + 1$ divides the size of $\text{Aut}(\mathcal{F})_{\Pi}$, where Π is any flock plane, or, equivalently, if $q^n + 1$ divides $\frac{|\text{Aut}(\mathcal{F})|}{q^n}$, then \mathcal{F} is linear.*

From Theorem 5.7, Thas deduced the following theorem on subGQs:

Theorem 5.9 (Thas [27]). *Let $S = T(\mathcal{O})$ be a TGQ of order (s, s^2) , s odd, so that the generalized ovoid \mathcal{O} is good at some element. If $\text{Aut}(S)$ acts transitively on its subGQs of order s , then $S \cong \mathcal{Q}(5, s)$.*

6. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$ in even characteristic

Brown and Lavrauw [2] generalized the following result of Brown on ovoids of $\text{PG}(3, q)$, q even, by obtaining a similar result for generalized ovoids.

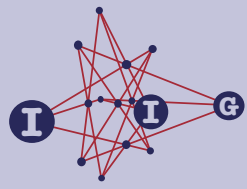
Theorem 6.1 (Brown [1]). *If an ovoid of $\text{PG}(3, q)$, q even, has a conic plane section, the ovoid must be an elliptic quadric. Equivalently, if a $T_3(\mathcal{O})$ of order (q, q^2) , where \mathcal{O} is an ovoid of $\text{PG}(3, q)$, q even, has a classical subGQ of order q containing the point (∞) , then $T_3(\mathcal{O}) \cong \mathcal{Q}(5, q)$.*

In terms of translation generalized quadrangles, Brown and Lavrauw prove the following.

Theorem 6.2 (Brown and Lavrauw [2]). *If a $T(\mathcal{O})$ of order (q^n, q^{2n}) , where \mathcal{O} is a generalized ovoid of $\text{PG}(4n - 1, q)$, q even, has a classical subGQ of order q containing the point (∞) , then $T(\mathcal{O}) \cong \mathcal{Q}(5, q^n)$.*

ACADEMIA
PRESS





page 8 / 13

go back

full screen

close

quit

In [26], Thas then generalized the latter result to elation generalized quadrangles.

Theorem 6.3 (Thas [26]). *Let $S^{(x)}$ be an EGQ of order (q, q^2) , q even, having a classical subGQ S' of order q containing x . Then $S^{(x)} \cong Q(5, q)$.*

Let \mathcal{O} be a generalized oval in $PG(3n - 1, q) = \langle \mathcal{O} \rangle$, q even. Then the $q^n + 1$ tangent spaces to the elements of \mathcal{O} meet in an $(n - 1)$ -dimensional subspace \mathfrak{N} of $PG(3n - 1, q)$, called the *nucleus* of \mathcal{O} (cf. [13, Chapter 8]). It is easily seen that for any $\pi \in \mathcal{O}$, $\mathcal{O}' = (\mathcal{O} \setminus \{\pi\}) \cup \{\mathfrak{N}\}$ is again a generalized oval. If \mathcal{O} is a generalized conic, that is, if $T(\mathcal{O}) \cong Q(4, q^n)$, then \mathcal{O}' is called a *pointed generalized conic* or a *generalized pointed conic*.

The following result generalizes another recent result of Brown and Lavrauw [3] from TGQs to EGQs.

Theorem 6.4 (Thas [26]). *Let $S^{(x)}$ be an EGQ of order (q^n, q^{2n}) , q even, containing a subGQ $T(\mathcal{O})$, where \mathcal{O} is a pointed generalized conic in $PG(3n - 1, q)$, having x as translation point. Then either $q^n = 4$ and $S^{(x)} \cong Q(5, 4)$, or $q^n = 8$ and $S^{(x)} \cong T_3(\mathcal{O}')$ with \mathcal{O}' the Suzuki-Tits ovoid of $PG(3, 8)$.*

Both Theorem 6.3 and Theorem 6.4 rely on the following observation, which works in any characteristic (part (ii) being a direct corollary of part (i)).

Theorem 6.5 (Thas [26]). (i) *Let $S^{(x)}$ be an EGQ of order (s, s^2) , $s > 1$, containing a subGQ S' of order s which has at least one axis of symmetry L incident with x . Then $S^{(x)}$ is a TGQ for the translation point x*

(ii) *Let $S^{(x)}$ be an EGQ of order (s, s^2) , $s > 1$, containing a subGQ S' of order s which is a TGQ with translation point x . Then $S^{(x)}$ is a TGQ with translation point x .*

7. Subquadrangles of flock GQs $\mathcal{S}(\mathcal{F})$

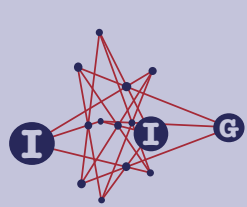
Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q\}$ be a flock of the quadratic cone \mathcal{K} of $PG(3, q)$, with q even. If the planes $\xi_1, \xi_2, \dots, \xi_q$ of the respective conics $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q$ all contain a common point, then the flock is linear; see Thas [15].

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q\}$ be a flock of the quadratic cone \mathcal{K} of $PG(3, q)$, with q odd. If the planes $\xi_1, \xi_2, \dots, \xi_q$ of the respective conics $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q$ all contain a common interior point of \mathcal{K} , then the flock is linear; see Thas [15].

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_q\}$ be a flock of the quadratic cone \mathcal{K} of $PG(3, q)$, with q odd, and suppose that the planes $\xi_1, \xi_2, \dots, \xi_q$, with $\mathcal{C}_i \subseteq \xi_i$, all contain a common exterior point of \mathcal{K} . If m is any given non-square of $GF(q)$, then coordinates

ACADEMIA
PRESS





page 9 / 13

go back

full screen

close

quit

can be chosen in such a way that \mathcal{K} has equation $X_0X_1 = X_2^2$ and the planes ξ_i have equation

$$a_i X_0 - m a_i^\sigma X_1 + X_3 = 0, \quad (1)$$

with $\{a_1, a_2, \dots, a_q\} = \text{GF}(q)$ and σ an automorphism of $\text{GF}(q)$. Conversely, given any non-square m of $\text{GF}(q)$, the planes ξ_i with equation (1), where $\{a_1, a_2, \dots, a_q\} = \text{GF}(q)$ and where σ is any automorphism of $\text{GF}(q)$, define a flock \mathcal{F} of the cone \mathcal{K} with equation $X_0X_1 = X_2^2$. Also, the planes ξ_i all contain the exterior point $(0, 0, 1, 0)$ of \mathcal{K} . This flock \mathcal{F} is linear if and only if $\sigma = 1$. The flocks \mathcal{F} are called *Kantor-Knuth flocks*. For proofs we refer to Thas [15]. The point-line dual of the GQ which corresponds to a Kantor-Knuth flock is a TGQ $T(\mathcal{O})$, with \mathcal{O} isomorphic to its translation dual \mathcal{O}^* , with \mathcal{O} good at some element π , and with \mathcal{O}^* good at the tangent space τ of \mathcal{O} at π ; see Payne [11]. Such an egg is called a *Kantor-Knuth egg*.

In the next theorems all subquadrangles of order s of flock GQs $\mathcal{S}(\mathcal{F})$ of order (s^2, s) are determined.

Theorem 7.1 (O’Keefe and Penttila [9]). *If $\mathcal{S}(\mathcal{F})$ is a flock GQ of order (s^2, s) , s even, with center (∞) , then $\mathcal{S}(\mathcal{F})$ contains exactly $s^3 + s^2$ subquadrangles of order s containing the point (∞) .*

An easy way to see this goes as follows. Payne proved in [11] that for all sets $\{L, M, N\}$ of two by two nonconcurrent lines for which there is a line $U \perp I(\infty) \perp L$ so that $\{L, M, N\} \subseteq U^\perp$, we have that $|\{L, M, N\}^{\perp\perp}| = s + 1$. That is, the set $\{L, M, N\}$ is 3-regular. Now let $\{L, M, N\}$ be such a set of lines. If B' is the set of lines of $\mathcal{S}(\mathcal{F})$ incident with points of the form $X \cap Y$, with $X \in \{L, M, N\}^\perp$ and $Y \in \{L, M, N\}^{\perp\perp}$, and P' is the set of points which are incident with at least two distinct lines of B' , then endowed with the induced incidence \mathcal{I}' , $\mathcal{S}' = (P', B', \mathcal{I}')$ is a subGQ of order s of $\mathcal{S}(\mathcal{F})$, see Payne and Thas [13, §2.6.2]. Now a standard counting argument yields $s^3 + s^2$ of such subGQs (notice that any point of a GQ of order (s^2, s) is contained in at most $s^3 + s^2$ subGQs of order s , see e.g. [4]).

Remark 7.2. Payne [10] proves that each of these $s^3 + s^2$ subquadrangles is a $T_2(\mathcal{O})$ of Tits, for some oval \mathcal{O} of $\text{PG}(2, q)$.

Theorem 7.3 (Brown and Thas [5]). *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order (s^2, s) , s odd, with center (∞) . If \mathcal{S}' is a subquadrangle of order s of $\mathcal{S}(\mathcal{F})$ containing the point (∞) , then \mathcal{F} is a Kantor-Knuth flock. Hence either $\mathcal{S} \cong H(3, s^2)$ or \mathcal{S}' is one of the $s^3 + s^2$ subquadrangles of order s containing the point (∞) . In all cases $\mathcal{S}' \cong W(s)$.*

An application of Theorem 7.3 is the next theorem.

ACADEMIA
PRESS





page 10 / 13

go back

full screen

close

quit

Theorem 7.4 (Thas [27]). *Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$ be a good egg in $\text{PG}(4n - 1, q)$, q odd. If $T(\mathcal{O})$ contains a subGQ of order q^n that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong \mathcal{S}(\mathcal{F})^D$ with \mathcal{F} a Kantor-Knuth flock.*

For q even, one can use Theorem 6.2 and [25, Chapter 7] to show that if $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is an egg in $\text{PG}(4n - 1, q)$ and $T(\mathcal{O})$ contains a subGQ of order q^n that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong \mathcal{Q}(5, q^n)$, see [27].

Theorem 7.5 (Brown and Thas [4], Thas and Thas [20]). *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order (s^2, s) , with center (∞) . If $\mathcal{S}(\mathcal{F})$ contains a subquadrangle of order s which does not contain (∞) , then $\mathcal{S} \cong H(3, s^2)$ and $\mathcal{S}' \cong W(s)$.*

Remark 7.6. The even case of Theorem 7.5 is due to Brown and Thas, the odd case to Thas and Thas.

8. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with \mathcal{O}^* good

As for q even the egg $\mathcal{O}(n, 2n, q)$ is good if and only if $\mathcal{O}^*(n, 2n, q)$ is good, we may assume that q is odd.

If $\mathcal{O}^*(n, 2n, q)$, with q odd, is good, then by Thas [17] the TGQ $T(\mathcal{O})$ with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is isomorphic to the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. If $\mathcal{O}^*(n, 2n, q)$ is good at the tangent space τ of $\mathcal{O}(n, 2n, q)$ at π , then π corresponds to the point (∞) of $\mathcal{S}(\mathcal{F})$. So by Theorems 7.3 and 7.5 we have the following result.

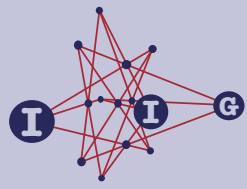
Theorem 8.1 (Brown and Thas [4, 5], Thas and Thas [20]). *Let $\mathcal{S} = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and q odd, be a TGQ such that the translation dual \mathcal{O}^* of \mathcal{O} is good at the tangent space τ of \mathcal{O} at π . If \mathcal{S} contains a subquadrangle \mathcal{S}' of order q^n , then either $\mathcal{S} \cong \mathcal{Q}(5, q^n)$ (and then $\mathcal{S}' \cong \mathcal{Q}(4, q^n)$) or \mathcal{O} is a nonclassical Kantor-Knuth egg and \mathcal{S}' is one of the $q^{3n} + q^{2n}$ subquadrangles of order q^n (isomorphic to $\mathcal{Q}(4, q^n)$) containing the line π of $T(\mathcal{O})$.*

9. Byproduct: A characterization of TGQs

Part of the proof of Theorem 7.5 (odd case) was also used by Thas and Thas [20] to prove the following characterization theorem of TGQs whose point-line duals arise from flocks.

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page 11 / 13

go back

full screen

close

quit

Theorem 9.1 (Thas and Thas [20]). *If $\mathcal{S} = (P, B, \mathbb{I})$ is a flock GQ of order (s^2, s) , with s odd, then the point-line dual of \mathcal{S} is a TGQ if and only if \mathcal{S} has a regular point x , with $(\infty) \neq x \sim (\infty)$, where (∞) is the elation point of \mathcal{S} .*

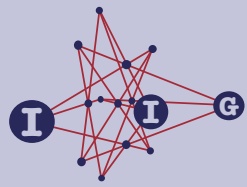
Remark 9.2. Thas [18] proved the following result for q even. Let \mathcal{S} be a flock GQ of order (s^2, s) , with s even. Then the point-line dual of \mathcal{S} is isomorphic to the classical GQ $H(3, s^2)$ if and only if \mathcal{S} has a regular point x , with $(\infty) \neq x \sim (\infty)$, where (∞) is the elation point of \mathcal{S} .

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page 12 / 13

go back

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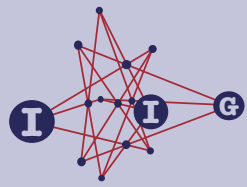
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page 13 / 13

go back

full screen

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