page 1 / 13

go back

full screen

close

quit

Subquadrangles of order s of generalized quadrangles of order (s, s^2)

Joseph A. Thas   Koen Thas*

Abstract

In this paper we consider the subquadrangles of order s for all known classes of generalized quadrangles of order (s, s^2).

Keywords: generalized quadrangle, subquadrangle, flock quadrangle, translation quadrangle

MSC 2000: 51E12, 05B25

1. Introduction

We survey results on subquadrangles of order s for all known classes of generalized quadrangles of order (s, s^2). An interesting application of this theory goes as follows. If $\mathcal{S}$ is a generalized quadrangle of order (s, s^2), if $\mathcal{S}'$ is a subquadrangle of order s of $\mathcal{S}$, and if p is a point of $\mathcal{S}$ not in $\mathcal{S}'$, then all points of $\mathcal{S}'$ collinear with p form an *ovoid* of $\mathcal{S}'$, that is, a set $\mathbf{O}$ of s^2+1 points of $\mathcal{S}'$ such that each line of $\mathcal{S}'$ contains exactly one point of $\mathbf{O}$. If $\mathcal{S}'$ is the classical generalized quadrangle $\mathcal{Q}(4, s)$ arising from a nonsingular quadric in $\mathsf{PG}(4, s)$, then, by a standard argument, $\mathbf{O}$ defines a projective plane of order s^2. In such a way new projective planes were discovered; see e.g. Thas and Payne [19]. Also, several theorems of this survey will provide interesting and strong characterizations of the classical generalized quadrangle $\mathcal{Q}(5, s)$ arising from a nonsingular elliptic quadric in $\mathsf{PG}(5, s)$, respectively of the Kantor-Knuth generalized quadrangles. As a byproduct a nice characterization of a class of translation generalized quadrangles of order (s, s^2), with s odd, will be given.

*The second author is a Postdoctoral Fellow of the Research Foundation – Flanders (FWO), Belgium.

2. Definitions

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathtt{I})$ in which P and B are disjoint (non-empty) sets of objects called *points* and *lines* respectively, and for which $\mathtt{I} \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) each point is incident with $t + 1$ lines, $t \geq 1$, and two distinct points are incident with at most one line;

(ii) each line is incident with $s + 1$ points, $s \geq 1$, and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \,\mathtt{I}\, M \,\mathtt{I}\, y \,\mathtt{I}\, L$.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have *order* s.

A *subquadrangle* (or also *subGQ*) $\mathcal{S}' = (P', B', \mathtt{I}')$ of $\mathcal{S}$ is a GQ such that $P' \subseteq P$, $B' \subseteq B$ and $\mathtt{I}'$ is the restriction of $\mathtt{I}$ to $(P' \times B') \cup (B' \times P')$; see [13, Chapter 2].

If $\mathcal{S} = (P, B, \mathtt{I})$ is a GQ of order (s, t), then the GQ $\mathcal{S}^D = (B, P, \mathtt{I})$ of order (t, s) is called the *(point-line) dual* of $\mathcal{S}$. As there is a point-line duality for GQs of order (s, t), we assume without further notice that the dual of a given theorem or definition has also been given.

For background on GQs we refer to the monograph by Payne and Thas [13].

3. The classical examples

We give a brief description of three families of examples known as the *classical GQs*, all of which are associated with classical groups and were first recognized as GQs by Tits; see Dembowski [6].

(i) Consider a nonsingular quadric $\mathcal{Q}$ of projective index 1, that is, of Witt index 2, of the projective space $\mathsf{PG}(d, q)$, with $d = 3, 4$ of 5. Then the points of $\mathcal{Q}$ together with the lines of $\mathcal{Q}$ (which are the subspaces of maximal dimension on $\mathcal{Q}$) form a GQ $\mathcal{Q}(d, q)$ with parameters

$$\begin{aligned} s = q, t = 1, &\quad \text{when } d = 3,\\ s = t = q, &\quad \text{when } d = 4,\\ s = q, t = q^2, &\quad \text{when } d = 5. \end{aligned}$$

ACADEMIA PRESS
UNIVERSITEIT GENT

(ii) Let H be a nonsingular Hermitian variety of the projective space $\mathsf{PG}(d, q^2)$, $d = 3$ or $d = 4$. Then the points of H together with the lines on H form a GQ $H(d, q^2)$ with parameters

$$\begin{aligned} s &= q^2, t = q, \quad \text{when } d = 3, \\ s &= q^2, t = q^3, \ \text{when } d = 4. \end{aligned}$$

(iii) The points of $\mathsf{PG}(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ with parameters

$$s = t = q.$$

The GQ $\mathcal{Q}(5, q)$ is isomorphic to the dual of $H(3, q^2)$, the GQ $\mathcal{Q}(4, q)$ is isomorphic to the dual of $W(q)$, and $\mathcal{Q}(4, q)$ (and then also $W(q)$) is self-dual if and only if q is even; see [13, Section 3.2].

All subquadrangles of order q of $\mathcal{Q}(5, q)$ are the $\mathcal{Q}(4, q)$ subGQs on it.

4. Translation and flock generalized quadrangles

Let $\mathcal{S} = (P, B, \mathtt{I})$ be a GQ of order (s, t), with $s \neq 1 \neq t$. A collineation θ of $\mathcal{S}$ is an *elation* about the point p if $\theta = \mathrm{id}$ or if θ fixes all lines incident with p and no point of $P \setminus p^\perp$ ($p^\perp$ is the set of all points collinear with p). If there is a group G of elations about p acting regularly on $P \setminus p^\perp$, then we say that $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* G and *elation point* or *base point* or *center* p. Briefly, we write that $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ is an EGQ. If the group G is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, G)$ is a *translation generalized quadrangle* (TGQ) with *translation group* G and *translation point* or *base point* or *center* p.

For any TGQ $\mathcal{S}^{(p)}$ each line incident with p is an *axis of symmetry*, that is, there is a (maximal) group of s collineations of $\mathcal{S}$ fixing $L^\perp$ elementwise (see [13, Chapter 8]).

In $\mathsf{PG}(2n + m - 1, q)$ consider a set $\mathcal{O}(n, m, q)$ of $q^m + 1$ $(n - 1)$-dimensional subspaces $\pi_0, \pi_1, \ldots, \pi_{q^m}$, every three of which generate a $\mathsf{PG}(3n-1, q)$ and such that each element π_i of $\mathcal{O}(n, m, q)$ is contained in an $(n + m - 1)$-dimensional subspace τ_i having no point in common with any π_j for $j \neq i$. It is easy to check that τ_i is uniquely determined, with $i = 0, 1, \ldots, q^m$. The space τ_i is called the *tangent space* of $\mathcal{O}(n, m, q)$ at π_i. For $n = m = 1$ such a set $\mathcal{O}(1, 1, q)$ is an oval in $\mathsf{PG}(2, q)$ and more generally, for $n = m$, such a set $\mathcal{O}(n, n, q)$ is called a *pseudo-oval* or a *generalized oval* or an *$[n - 1]$-oval* of $\mathsf{PG}(3n - 1, q)$. For $m = 2n = 2$ such a set $\mathcal{O}(1, 2, q)$ is an ovoid of $\mathsf{PG}(3, q)$ and more generally, for $n \neq m$ such a set $\mathcal{O}(n,m,q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathsf{PG}(2n+m-1,q)$.

Now embed $\mathsf{PG}(2n+m-1,q)$ as a hyperplane in a $\mathsf{PG}(2n+m,q)$, and construct a point-line geometry $T(n,m,q)$ as follows.

- POINTS are of three types:
  - (i) the points of $\mathsf{PG}(2n+m,q) \setminus \mathsf{PG}(2n+m-1,q)$;
  - (ii) the $(n+m)$-dimensional subspaces of $\mathsf{PG}(2n+m,q)$ which contain a tangent space τ_i but are not contained in $\mathsf{PG}(2n+m-1,q)$;
  - (iii) a symbol (∞).
- LINES are of two types:
  - (a) the n-dimensional subspaces of $\mathsf{PG}(2n+m,q)$ which contain an element π_i but are not contained in $\mathsf{PG}(2n+m-1,q)$;
  - (b) the elements of $\mathcal{O}(n,m,q)$.
- INCIDENCE is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathsf{PG}(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{O}(n,m,q)$ contained in it. The point (∞) is incident with all the lines of type (b) and with no lines of type (a).

Payne and Thas [13] prove that $T(n,m,q)$ is a TGQ of order (q^n,q^m) with center (∞), and that conversely every TGQ is isomorphic to a $T(n,m,q)$.

In the case where $n=m=1$, so $\mathcal{O}(1,1,q)=\mathcal{O}$ is an oval of $\mathsf{PG}(2,q)$, the GQ $T(1,1,q)$ is the Tits GQ $T_2(\mathcal{O})$. When $m=2n=2$, so $\mathcal{O}(1,2,q)=\mathcal{O}$ is an ovoid of $\mathsf{PG}(3,q)$, the GQ $T(1,2,q)$ is the Tits GQ $T_3(\mathcal{O})$. Note that $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$ if and only if $\mathcal{O}$ is a conic, while $T_3(\mathcal{O}) \cong \mathcal{Q}(5,q)$ if and only if $\mathcal{O}$ is an elliptic quadric (see [13, Chapter 3]).

In the extension $\mathsf{PG}(2n+m-1,q^n)$ of $\mathsf{PG}(2n+m-1,q)$, with $m \in \{n,2n\}$, we consider n $(\frac{m}{n}+1)$-dimensional spaces $\mathsf{PG}^{(i)}(\frac{m}{n}+1,q^n)=\xi_i$, with $i=1,2,\ldots,n$, which are conjugate with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$, that is, which form an orbit of the Galois group corresponding to this extension, and which span $\mathsf{PG}(2n+m-1,q^n)$. In ξ_1 we consider an oval $\mathcal{O}_1$ for $m=n$ and an ovoid $\mathcal{O}_1$ for $m=2n$. Let $\mathcal{O}_1=\{x_0^{(1)},x_1^{(1)},\ldots,x_{q^m}^{(1)}\}$. Further, let $x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)}$, with $i=0,1,\ldots,q^m$, be conjugate with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. The points $x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)}$ generate an $(n-1)$-dimensional space π_i over $\mathsf{GF}(q)$, with $i=0,1,\ldots,q^m$. Then $\mathcal{O}=\{\pi_0,\pi_1,\ldots,\pi_{q^m}\}$ is a generalized oval of $\mathsf{PG}(3n-1,q)$ for $m=n$, and a generalized ovoid of $\mathsf{PG}(4n-1,q)$ for $m=2n$.

Here, we speak of a *regular* or *elementary pseudo-oval*, respectively a *regular* or *elementary pseudo-ovoid*. In such a case the corresponding GQ is isomorphic to a GQ of Tits ($T_2(\mathcal{O}_1)$, respectively $T_3(\mathcal{O}_1)$). If $\mathcal{O}_1$ is a conic or an elliptic quadric, then $T(n,m,q)$ is isomorphic to a classical GQ ($\mathcal{Q}(4,q^n)$, respectively $\mathcal{Q}(5,q^n)$) and $\mathcal{O}(n,m,q)$ is called *classical*, see [13, Chapter 3]; for $n=m$ the classical $\mathcal{O}(n,n,q)$ is called a *pseudo-conic* and for $m=2n$ a *pseudo-quadric*.

For $m=n$, any known $[n-1]$-oval is regular, for $m=2n$ and q even any known $[n-1]$-ovoid is regular, but for $m=2n$ and q odd there are $[n-1]$-ovoids which are not regular; see Thas [16].

One can prove that $n \leq m \leq 2n$, and that for q even $m \in \{n, 2n\}$; see [13, Chapter 8].

Let either $n=m$ with q odd, or let $n \neq m$. Then by [13, Section 8.7] the q^m+1 tangent spaces of an $\mathcal{O}(n,m,q)=\mathcal{O}$ form an $\mathcal{O}^* = \mathcal{O}^*(n,m,q)$ in the dual space of $\mathrm{PG}(2n+m-1,q)$. So in addition to $T(n,m,q)=T(\mathcal{O})$ there arises a TGQ $T(\mathcal{O}^*)$ with the same parameters. The TGQ $T(\mathcal{O}^*)$ is called the *translation dual* of $T(\mathcal{O})$, and $\mathcal{O}^*$ is called the *translation dual* of $\mathcal{O}$. For $m=2n$ and q odd there are examples for which $T(\mathcal{O}) \not\cong T(\mathcal{O}^*)$; see Payne [11].

Let $\mathcal{O}(n,2n,q)$ be an egg of $\mathrm{PG}(4n-1,q)$. We say that $\mathcal{O}(n,2n,q)$ is *good* at the element π_i of $\mathcal{O}(n,2n,q)$ if any $\mathrm{PG}(3n-1,q)$ containing π_i and at least two other elements of $\mathcal{O}(n,2n,q)$ contains exactly q^n+1 elements of $\mathcal{O}(n,2n,q)$. In such a case the corresponding TGQ $T(n,2n,q)$ contains at least $q^{3n}+q^{2n}$ subquadrangles of order q^n; for q odd the $q^{3n}+q^{2n}$ subquadrangles of order q^n defined by the element π_i at which $\mathcal{O}(n,2n,q)$ is good are isomorphic to the classical GQ $\mathcal{Q}(4,q^n)$, see Thas [16]. The TGQ $T(n,2n,q)$ is isomorphic to a GQ of Tits of order (q^n,q^{2n}) if and only if the corresponding egg $\mathcal{O}(n,2n,q)$ is good at each of its elements; see Payne and Thas [13]. If q is even and $\mathcal{O}(n,2n,q)$ is good at $\pi \in \mathcal{O}(n,2n,q)$, then the translation dual $\mathcal{O}^*(n,2n,q)$ is good at the tangent space τ of $\mathcal{O}(n,2n,q)$ at π; see Thas [16]. For each known egg $\mathcal{O}(n,2n,q)$, either $\mathcal{O}(n,2n,q)$ or $\mathcal{O}^*(n,2n,q)$ is good at one of its elements.

Let $\mathcal{K}$ be the quadratic cone with vertex x in $\mathrm{PG}(3,q)$. A partition $\mathcal{F}$ of $\mathcal{K} \setminus \{x\}$ into q disjoint conics is called a *flock* of $\mathcal{K}$. Then with $\mathcal{F}$ there corresponds a GQ $\mathcal{S}(\mathcal{F})$ of order (q^2,q); see Thas [15]. The GQ $\mathcal{S}(\mathcal{F})$ is isomorphic to the classical GQ $H(3,q^2)$ if and only if the flock is *linear*, that is, if and only if all planes of the conics of the flock contain a common line; see Payne and Thas [13] and Payne [12]. The GQ $\mathcal{S}(\mathcal{F})$ is an EGQ with center (∞). By Payne and Thas [14] the center (∞) is uniquely determined if $\mathcal{S}(\mathcal{F})$ is not classical.

If the point-line dual of $\mathcal{S}(\mathcal{F})$ is a TGQ $T(\mathcal{O})$ with center L (then $(\infty) \,\mathrm{I}\, L$), where $\mathcal{O}=\mathcal{O}(n,2n,q)$, then the translation dual $\mathcal{O}^*$ of $\mathcal{O}$ is good at the tangent space τ_i of $\mathcal{O}$ at $\pi_i = L \in \mathcal{O}$; see Thas [16]. Conversely, by Thas [17], if the translation dual $\mathcal{O}^*$ of an egg $\mathcal{O}=\mathcal{O}(n,2n,q)$, with q odd, is good at one of its elements, then $T(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. If q is even and if the TGQ $T(\mathcal{O})$ is the point-line dual of a flock GQ (then $\mathcal{O}$ and $\mathcal{O}^*$ are good), then, by Johnson [7], the GQ $T(\mathcal{O})$ is classical.

Finally, we remark that every known GQ of order (s, s^2) is either a TGQ $T(\mathcal{O})$ or the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$.

5. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with $\mathcal{O}$ good

In this section we consider the subquadrangles of order q^n of translation generalized quadrangles $T(\mathcal{O})$ of order (q^n, q^{2n}), with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and $\mathcal{O}$ good at some element π.

Theorem 5.1 (Brown and Thas [4, 5]). *Let $\mathcal{S} = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$, be a TGQ. If $\mathcal{O}$ is good at some element π and if $\mathcal{S}$ has a subquadrangle of order $s = q^n$ which does not contain the point (∞), then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

Theorem 5.2 (Brown and Thas [5]). *Let $\mathcal{S} = (P, B, \mathrm{I})$ be a TGQ of order (s, s^2), s odd, with $\mathcal{S} = T(\mathcal{O})$ and $\mathcal{O} = \mathcal{O}(n, 2n, q)$ good at some element π. If $\mathcal{S}'$ is a subquadrangle of order $s = q^n$ of $\mathcal{S}$ containing the point (∞), then $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and either $\mathcal{S} \cong \mathcal{Q}(5, s)$ or $\mathcal{S}'$ is one of the $s^3 + s^2$ subquadrangles of order s (isomorphic to $\mathcal{Q}(4, s)$) containing the line π of $\mathcal{S}$.*

Theorem 5.3 (Brown and Thas [5]). *Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$, with q odd, be an egg in $\mathsf{PG}(4n-1, q)$ which is good at π. If there is a subspace $\mathsf{PG}(3n-1, q)$ of $\mathsf{PG}(4n-1, q)$ which contains at least four elements of $\mathcal{O}$, but which does not contain π, then $\mathcal{O}$ is a Kantor-Knuth egg (see Section 7). If there is a subspace $\mathsf{PG}(3n-1, q)$ which contains at least five elements of $\mathcal{O}$, but which does not contain π, then $\mathcal{O}$ is classical.*

Remark 5.4. Independently and in a completely different way, Lavrauw [8] also proved the second part of Theorem 5.3.

Theorem 5.5 (Thas [21], see also [22, 23]). *Let $\mathcal{S} = (P, B, \mathrm{I})$ be a TGQ of order (s, s^2), $s > 1$, with $\mathcal{S} = T(\mathcal{O})$ and $\mathcal{O} = \mathcal{O}(n, 2n, q)$. If $\mathcal{S}'$ is a subquadrangle of order s of $\mathcal{S}$ containing the point (∞), then $\mathcal{S}'$ is a TGQ $T(\mathcal{O}')$ with $\mathcal{O}' = \mathcal{O}'(n, n, q)$ a pseudo-oval on $\mathcal{O}$.*

Open problem 5.6. *If in Theorem 5.5, s is even, does the existence of $\mathcal{S}'$ imply that $\mathcal{S}$ is isomorphic to a $T_3(\overline{\mathcal{O}})$ of Tits?*

We mention several recent results on this problem in the next section.

ACADEMIA PRESS
UNIVERSITEIT GENT

Let us end this section with the following results on good TGQs with a large automorphism group, followed by an application to group actions on subGQs.

Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}), where q is odd, and where $\mathcal{O}$ is good at some element $\mathsf{PG}(n-1,q)$. Let $\mathbb{K}$ be the kernel [13, Chapter 8] of $\mathcal{S}$. Then by Thas [24], $(q^n+1)(q^n-1)q^{6n}(|\mathbb{K}|-1)$ is a divisor of $|\mathrm{Aut}(\mathcal{S})|$. If $\mathcal{S}$ is classical, then for an arbitrary line L of $\mathcal{S}$, $|\mathrm{Aut}(\mathcal{S})_L| = (q^n+1)^2(q^n-1)^2q^{6n}2h$, where $q^n = p^h$ for the prime p.

Theorem 5.7 (Thas [27]). *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}), where q is an odd prime power, with $\mathcal{O}$ the generalized ovoid in $\mathsf{PG}(4n-1,q)$ corresponding to $\mathcal{S}$. Suppose that $\mathcal{O}$ is good at some element $\mathsf{PG}(n-1,q)$, and let L be the line $\mathsf{PG}(n-1,q)$ of $\mathcal{S}$. If*

$$(q^n+1)^2(q^n-1)q^{6n} \quad \textit{divides} \quad |\mathrm{Aut}(\mathcal{S})_L|\,,$$

then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(5,q^n)$.

Corollary 5.8 (Thas [27]). *Let $\mathcal{F}$ be a semifield flock of $\mathsf{PG}(3,q^n)$, q odd. If q^n+1 divides the size of $\mathrm{Aut}(\mathcal{F})_\Pi$, where Π is any flock plane, or, equivalently, if q^n+1 divides $\frac{|\mathrm{Aut}(\mathcal{F})|}{q^n}$, then $\mathcal{F}$ is linear.*

From Theorem 5.7, Thas deduced the following theorem on subGQs:

Theorem 5.9 (Thas [27]). *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order (s,s^2), s odd, so that the generalized ovoid $\mathcal{O}$ is good at some element. If $\mathrm{Aut}(\mathcal{S})$ acts transitively on its subGQs of order s, then $\mathcal{S} \cong \mathcal{Q}(5,s)$.*

6. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$ in even characteristic

Brown and Lavrauw [2] generalized the following result of Brown on ovoids of $\mathsf{PG}(3,q)$, q even, by obtaining a similar result for generalized ovoids.

Theorem 6.1 (Brown [1]). *If an ovoid of $\mathsf{PG}(3,q)$, q even, has a conic plane section, the ovoid must be an elliptic quadric. Equivalently, if a $T_3(\mathcal{O})$ of order (q,q^2), where $\mathcal{O}$ is an ovoid of $\mathsf{PG}(3,q)$, q even, has a classical subGQ of order q containing the point (∞), then $T_3(\mathcal{O}) \cong \mathcal{Q}(5,q)$.*

In terms of translation generalized quadrangles, Brown and Lavrauw prove the following.

Theorem 6.2 (Brown and Lavrauw [2]). *If a $T(\mathcal{O})$ of order (q^n, q^{2n}), where $\mathcal{O}$ is a generalized ovoid of $\mathsf{PG}(4n-1,q)$, q even, has a classical subGQ of order q containing the point (∞), then $T(\mathcal{O}) \cong \mathcal{Q}(5,q^n)$.*

In [26], Thas then generalized the latter result to elation generalized quadrangles.

Theorem 6.3 (Thas [26]). *Let $\mathcal{S}^{(x)}$ be an EGQ of order (q, q^2), q even, having a classical subGQ $\mathcal{S}'$ of order q containing x. Then $\mathcal{S}^{(x)} \cong \mathcal{Q}(5, q)$.*

Let $\mathcal{O}$ be a generalized oval in $\mathsf{PG}(3n-1, q) = \langle \mathcal{O} \rangle$, q even. Then the $q^n + 1$ tangent spaces to the elements of $\mathcal{O}$ meet in an $(n-1)$-dimensional subspace $\mathfrak{N}$ of $\mathsf{PG}(3n-1, q)$, called the *nucleus* of $\mathcal{O}$ (cf. [13, Chapter 8]). It is easily seen that for any $\pi \in \mathcal{O}$, $\mathcal{O}' = (\mathcal{O} \setminus \{\pi\}) \cup \{\mathfrak{N}\}$ is again a generalized oval. If $\mathcal{O}$ is a generalized conic, that is, if $T(\mathcal{O}) \cong \mathcal{Q}(4, q^n)$, then $\mathcal{O}'$ is called a *pointed generalized conic* or a *generalized pointed conic*.

The following result generalizes another recent result of Brown and Lavrauw [3] from TGQs to EGQs.

Theorem 6.4 (Thas [26]). *Let $\mathcal{S}^{(x)}$ be an EGQ of order (q^n, q^{2n}), q even, containing a subGQ $T(\mathcal{O})$, where $\mathcal{O}$ is a pointed generalized conic in $\mathsf{PG}(3n-1, q)$, having x as translation point. Then either $q^n = 4$ and $\mathcal{S}^{(x)} \cong \mathcal{Q}(5, 4)$, or $q^n = 8$ and $\mathcal{S}^{(x)} \cong T_3(\mathcal{O}')$ with $\mathcal{O}'$ the Suzuki-Tits ovoid of $\mathsf{PG}(3, 8)$.*

Both Theorem 6.3 and Theorem 6.4 rely on the following observation, which works in any characteristic (part (ii) being a direct corollary of part (i)).

Theorem 6.5 (Thas [26]). (i) *Let $\mathcal{S}^{(x)}$ be an EGQ of order (s, s^2), $s > 1$, containing a subGQ $\mathcal{S}'$ of order s which has at least one axis of symmetry L incident with x. Then $\mathcal{S}^{(x)}$ is a TGQ for the translation point x*

(ii) *Let $\mathcal{S}^{(x)}$ be an EGQ of order (s, s^2), $s > 1$, containing a subGQ $\mathcal{S}'$ of order s which is a TGQ with translation point x. Then $\mathcal{S}^{(x)}$ is a TGQ with translation point x.*

7. Subquadrangles of flock GQs $\mathcal{S}(\mathcal{F})$

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ of $\mathsf{PG}(3, q)$, with q even. If the planes $\xi_1, \xi_2, \ldots, \xi_q$ of the respective conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ all contain a common point, then the flock is linear; see Thas [15].

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ of $\mathsf{PG}(3, q)$, with q odd. If the planes $\xi_1, \xi_2, \ldots, \xi_q$ of the respective conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ all contain a common interior point of $\mathcal{K}$, then the flock is linear; see Thas [15].

Let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ of $\mathsf{PG}(3, q)$, with q odd, and suppose that the planes $\xi_1, \xi_2, \ldots, \xi_q$, with $\mathcal{C}_i \subseteq \xi_i$, all contain a common exterior point of $\mathcal{K}$. If m is any given non-square of $\mathsf{GF}(q)$, then coordinates can be chosen in such a way that $\mathcal{K}$ has equation $X_0X_1 = X_2^2$ and the planes ξ_i have equation

$$a_iX_0 - ma_i^\sigma X_1 + X_3 = 0\,, \tag{1}$$

with $\{a_1, a_2, \dots, a_q\} = \mathsf{GF}(q)$ and σ an automorphism of $\mathsf{GF}(q)$. Conversely, given any non-square m of $\mathsf{GF}(q)$, the planes ξ_i with equation (1), where $\{a_1, a_2, \dots, a_q\} = \mathsf{GF}(q)$ and where σ is any automorphism of $\mathsf{GF}(q)$, define a flock $\mathcal{F}$ of the cone $\mathcal{K}$ with equation $X_0X_1 = X_2^2$. Also, the planes ξ_i all contain the exterior point $(0, 0, 1, 0)$ of $\mathcal{K}$. This flock $\mathcal{F}$ is linear if and only if $\sigma = \mathbf{1}$. The flocks $\mathcal{F}$ are called *Kantor-Knuth flocks*. For proofs we refer to Thas [15]. The point-line dual of the GQ which corresponds to a Kantor-Knuth flock is a TGQ $T(\mathcal{O})$, with $\mathcal{O}$ isomorphic to its translation dual $\mathcal{O}^*$, with $\mathcal{O}$ good at some element π, and with $\mathcal{O}^*$ good at the tangent space τ of $\mathcal{O}$ at π; see Payne [11]. Such an egg is called a *Kantor-Knuth egg*.

In the next theorems all subquadrangles of order s of flock GQs $\mathcal{S}(\mathcal{F})$ of order (s^2, s) are determined.

Theorem 7.1 (O'Keefe and Penttila [9]). *If $\mathcal{S}(\mathcal{F})$ is a flock GQ of order (s^2, s), s even, with center (∞), then $\mathcal{S}(\mathcal{F})$ contains exactly $s^3 + s^2$ subquadrangles of order s containing the point (∞).*

An easy way to see this goes as follows. Payne proved in [11] that for all sets $\{L, M, N\}$ of two by two nonconcurrent lines for which there is a line $U \,\mathtt{I}\, (\infty) \,\mathtt{I}\, L$ so that $\{L, M, N\} \subseteq U^\perp$, we have that $|\{L, M, N\}^{\perp\perp}| = s + 1$. That is, the set $\{L, M, N\}$ is 3-*regular*. Now let $\{L, M, N\}$ be such a set of lines. If B' is the set of lines of $\mathcal{S}(\mathcal{F})$ incident with points of the form $X \cap Y$, with $X \in \{L, M, N\}^\perp$ and $Y \in \{L, M, N\}^{\perp\perp}$, and P' is the set of points which are incident with at least two distinct lines of B', then endowed with the induced incidence $\mathtt{I}'$, $\mathcal{S}' = (P', B', \mathtt{I}')$ is a subGQ of order s of $\mathcal{S}(\mathcal{F})$, see Payne and Thas [13, §2.6.2]. Now a standard counting argument yields $s^3 + s^2$ of such subGQs (notice that any point of a GQ of order (s^2, s) is contained in at most $s^3 + s^2$ subGQs of order s, see e.g. [4]).

Remark 7.2. Payne [10] proves that each of these $s^3 + s^2$ subquadrangles is a $T_2(\mathcal{O})$ of Tits, for some oval $\mathcal{O}$ of $\mathsf{PG}(2, q)$.

Theorem 7.3 (Brown and Thas [5]). *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order (s^2, s), s odd, with center (∞). If $\mathcal{S}'$ is a subquadrangle of order s of $\mathcal{S}(\mathcal{F})$ containing the point (∞), then $\mathcal{F}$ is a Kantor-Knuth flock. Hence either $\mathcal{S} \cong H(3, s^2)$ or $\mathcal{S}'$ is one of the $s^3 + s^2$ subquadrangles of order s containing the point (∞). In all cases $\mathcal{S}' \cong W(s)$.*

An application of Theorem 7.3 is the next theorem.

ACADEMIA PRESS
UNIVERSITEIT GENT

Theorem 7.4 (Thas [27]). *Let $\mathcal{O} = \mathcal{O}(n, 2n, q)$ be a good egg in $\mathsf{PG}(4n-1, q)$, q odd. If $T(\mathcal{O})$ contains a subGQ of order q^n that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong \mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a Kantor-Knuth flock.*

For q even, one can use Theorem 6.2 and [25, Chapter 7] to show that if $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is an egg in $\mathsf{PG}(4n-1, q)$ and $T(\mathcal{O})$ contains a subGQ of order q^n that is fixed pointwise by some nontrivial automorphism of $T(\mathcal{O})$, then $T(\mathcal{O}) \cong \mathcal{Q}(5, q^n)$, see [27].

Theorem 7.5 (Brown and Thas [4], Thas and Thas [20]). *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order (s^2, s), with center (∞). If $\mathcal{S}(\mathcal{F})$ contains a subquadrangle of order s which does not contain (∞), then $\mathcal{S} \cong H(3, s^2)$ and $\mathcal{S}' \cong W(s)$.*

Remark 7.6. The even case of Theorem 7.5 is due to Brown and Thas, the odd case to Thas and Thas.

8. Subquadrangles of translation generalized quadrangles $T(\mathcal{O})$, with $\mathcal{O}^*$ good

As for q even the egg $\mathcal{O}(n, 2n, q)$ is good if and only if $\mathcal{O}^*(n, 2n, q)$ is good, we may assume that q is odd.

If $\mathcal{O}^*(n, 2n, q)$, with q odd, is good, then by Thas [17] the TGQ $T(\mathcal{O})$ with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ is isomorphic to the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. If $\mathcal{O}^*(n, 2n, q)$ is good at the tangent space τ of $\mathcal{O}(n, 2n, q)$ at π, then π corresponds to the point (∞) of $\mathcal{S}(\mathcal{F})$. So by Theorems 7.3 and 7.5 we have the following result.

Theorem 8.1 (Brown and Thas [4, 5], Thas and Thas [20]). *Let $\mathcal{S} = T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and q odd, be a TGQ such that the translation dual $\mathcal{O}^*$ of $\mathcal{O}$ is good at the tangent space τ of $\mathcal{O}$ at π. If $\mathcal{S}$ contains a subquadrangle $\mathcal{S}'$ of order q'^n, then either $\mathcal{S} \cong \mathcal{Q}(5, q^n)$ (and then $\mathcal{S}' \cong \mathcal{Q}(4, q^n)$) or $\mathcal{O}$ is a nonclassical Kantor-Knuth egg and $\mathcal{S}'$ is one of the $q^{3n} + q^{2n}$ subquadrangles of order q^n (isomorphic to $\mathcal{Q}(4, q^n)$) containing the line π of $T(\mathcal{O})$.*

9. Byproduct: A characterization of TGQs

Part of the proof of Theorem 7.5 (odd case) was also used by Thas and Thas [20] to prove the following characterization theorem of TGQs whose point-line duals arise from flocks.

Theorem 9.1 (Thas and Thas [20]). *If $\mathcal{S} = (P, B, \mathtt{I})$ is a flock GQ of order (s^2, s), with s odd, then the point-line dual of $\mathcal{S}$ is a TGQ if and only if $\mathcal{S}$ has a regular point x, with $(\infty) \neq x \sim (\infty)$, where (∞) is the elation point of $\mathcal{S}$.*

Remark 9.2. Thas [18] proved the following result for q even. Let $\mathcal{S}$ be a flock GQ of order (s^2, s), with s even. Then the point-line dual of $\mathcal{S}$ is isomorphic to the classical GQ $H(3, s^2)$ if and only if $\mathcal{S}$ has a regular point x, with $(\infty) \neq x \sim (\infty)$, where (∞) is the elation point of $\mathcal{S}$.

References

[1] **M. R. Brown**, Ovoids of PG$(3, q)$, q even, with a conic section, *J. London Math. Soc.* **62** (2000), 569–582.

[2] **M. R. Brown** and **M. Lavrauw**, Eggs in PG$(4n-1, q)$, q even, containing a pseudo-conic, *Bull. London Math. Soc* **36** (2004), 633–639.

[3] ______, Eggs in PG$(4n-1, q)$, q even, containing a pseudo pointed conic, *European J. Combin.* **26** (2005), 117–128.

[4] **M. R. Brown** and **J. A. Thas**, Subquadrangles of order s of generalized quadrangles of order (s, s^2), I, *J. Combin. Theory Ser. A* **106** (2004), 15–32.

[5] ______, Subquadrangles of order s of generalized quadrangles of order (s, s^2), II, *J. Combin. Theory Ser. A* **106** (2004), 33–48.

[6] **P. Dembowski**, *Finite Geometries*. Reprint of the 1968 original, Classics Math., Springer-Verlag, Berlin, 1997.

[7] **N. L. Johnson**, Semifield flocks of quadratic cones, *Simon Stevin* **61** (1987), 313–326.

[8] **M. Lavrauw**, Characterisations and properties of good eggs in PG$(4n-1, q)$, q odd, *Discrete Math.* **301** (2005), 106–116.

[9] **C. M. O'Keefe** and **T. Penttila**, Subquadrangles of generalized quadrangles of order (q, q^2), q even, *J. Combin. Theory Ser. A* **94** (2001), 218–229.

[10] **S. E. Payne**, A new infinite family of generalized quadrangles, Proceedings of the Sixteenth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, Fla., 1985), *Congr. Numer.* **49** (1985), 115–128.

[11] ______, An essay on skew translation generalized quadrangles, *Geom. Dedicata* **32** (1989), 93–118.

[12] ______, The fundamental theorem of q-clan geometry, *Des. Codes Cryptogr.* **8** (1996), 181–202.

[13] **S. E. Payne** and **J. A. Thas**, *Finite Generalized Quadrangles*, Research Notes Math. **110**, Pitman Advanced Publishing Program, Boston/London/Melbourne, 1984.

[14] ______, Generalized quadrangles, BLT-sets, and Fisher flocks, Proceedings of the Twenty-second Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1991), *Congr. Numer.* **84** (1991), 161–192.

[15] **J. A. Thas**, Generalized quadrangles and flocks of cones, *European J. Combin.* **8** (1987), 441–452.

[16] ______, Generalized quadrangles of order (s, s^2), I, *J. Combin. Theory Ser. A* **67** (1994), 140–160.

[17] ______, Generalized quadrangles of order (s, s^2), III, *J. Combin. Theory Ser. A* **87** (1999), 247–272.

[18] ______, Characterizations of translation generalized quadrangles, *Des. Codes Cryptogr.* **23** (2001), 249–257.

[19] **J. A. Thas** and **S. E. Payne**, Spreads and ovoids in finite generalized quadrangles, *Geom. Dedicata* **52** (1994), 227–253.

[20] **J. A. Thas** and **K. Thas**, Subquadrangles of order s of generalized quadrangles of order (s, s^2), III, *J. Combin. Theory Ser. A* **113** (2006), 1791–1804.

[21] **K. Thas**, *Symmetrieën in Eindige Veralgemeende Vierhoeken (Symmetries in Finite Generalized Quadrangles)*, Master Thesis, Ghent University, Ghent (1999), 186 pp.

[22] ______, *Automorphisms and Combinatorics of Finite Generalized Quadrangles*, Ph.D. Thesis, Ghent University, Ghent (2002), xxviii + 412 pp.

[23] ______, On generalized quadrangles with some concurrent axes of symmetry, *Bull. Belg. Math. Soc. Simon Stevin* **9** (2002), 217–243.

[24] ______, Translation generalized quadrangles for which the translation dual arises from a flock, *Glasgow Math. J.* **45** (2003), 457–474.

[25] ______, *Symmetry in Finite Generalized Quadrangles*, Front. Math. **1**, Birkhäuser Verlag, Basel—Boston—Berlin, 2004.

[26] ______, Elation generalized quadrangles of order (q, q^2), q even, with a classical subGQ of order q, *Adv. Geom.* **6** (2006), 265–273.

[27] ______, A stabilizer lemma for translation generalized quadrangles, *European J. Combin.* **28** (2007), 1–16.

Joseph A. Thas

GHENT UNIVERSITY, DEPARTMENT OF PURE MATHEMATICS AND COMPUTER ALGEBRA, KRIJGSLAAN 281, S22, B-9000 GHENT, BELGIUM

e-mail: `jat@cage.UGent.be`

website: `http://cage.ugent.be/~jat`

Koen Thas

GHENT UNIVERSITY, DEPARTMENT OF PURE MATHEMATICS AND COMPUTER ALGEBRA, KRIJGSLAAN 281, S22, B-9000 GHENT, BELGIUM

e-mail: `kthas@cage.UGent.be`

website: `http://cage.ugent.be/~kthas`

ACADEMIA PRESS

UNIVERSITEIT GENT