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On generalizing generalized polygons

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Abstract

The purpose of this paper is to reveal in geometric terms a decade-old construction of certain families of graphs with nice extremal properties. Construction of the graphs in question is motivated by the way in which regular generalized polygons may be embedded in their Lie algebras, so that point-line incidence corresponds to the vanishing Lie product. The only caveat is that the generalized polygons are greatly limited in number. By performing successive truncations on an infinite root system of type $\widetilde{A}_1$, we are able to obtain an infinite series of incidence structures which approximate the behavior of generalized polygons. Indeed, the first two members of the series are exactly the affine parts of the generalized polygons of type A_2 and B_2.

Keywords: Turán problem, cage, large girth, generalized polygon, affine part, Lie algebra, root system
MSC 2000: 51E12, 05C35

1 Introduction

The material in this paper is based not on recent work but rather on undisclosed features of work that is now a decade old. At that time, our objective was to construct graphs which were extremal in the sense of the Turán problem for a fixed cycle of even length (see Section 2.1). The target asymptotic for this problem is provided by the upper bound which appears in the *Even Circuit Theorem*. However, this bound is known to be sharp in only a handful of cases, each corresponding to the existence of incidence graphs of regular generalized polygons. Such considerations led F. Lazebnik, V.A. Ustimenko and me to attempt to construct graphs which in some sense approximated the structure and behavior of these incidence graphs. Ultimately, our goal was realized with the construction of the doubly infinite family of graphs $\mathrm{CD}(k, q)$.

As it turns out, the geometries which give rise to these approximating graphs depend on some rather deep and sophisticated algebra. Yet, in all of our papers on the subject one finds only meager reference to such algebraic underpinnings. The reason is two-fold. First, it happens that we were able to give a quite nice, self-contained description of the graphs $\mathrm{CD}(k,q)$, making it unnecessary to divulge information about the algebraic origins of their geometries. Second, virtually all of our articles appeared in graph theory journals, and we did not wish to burden that readership with material that might be construed as esoteric or extraneous.

The current paper is designed to fill this lacuna. In it, our premise is that it is these underlying geometries, and not their incidence graphs, which are the objects of primary interest. Accordingly, they deserve their own detailed and rigorous treatment.

The balance of the paper is organized as follows. Sections 2 and 3 are purely motivational, providing a comparison of the extremal properties of generalized polygons with those of our approximating graphs $\mathrm{CD}(k,q)$. For the reader who is interested only in the derivation of the underlying geometries, these sections may be safely skipped. Section 4 serves as a microcosm of what is to appear in later sections. There, we provide various models of the generalized 3-gon, ultimately leading to a method for embedding the 3-gon in the upper Borel subalgebra of its Lie algebra. Emphasis is placed on a certain embedded substructure called the "affine part". As we show in Section 5, this embedding procedure is most general, and can be used to construct Lie algebraic models for all generalized polygons and their affine parts. In Section 6, we introduce the affine Lie algebra of type $\widetilde{A}_1$ and derive its root system. We further justify a sense in which this root system gives us legitimate hopes for success. Our desired geometries are constructed in Section 7, via the method of "truncating" the root system of type $\widetilde{A}_1$. Finally, in Section 8 we explain how these newly constructed geometries give rise to the coordinate relations used in Section 3 to define their incidence graphs $\mathrm{CD}(k,q)$.

2 Generalized polygons as extremal structures

2.1 Turán Problem and the Even Circuit Theorem

We begin by introducing a problem from extremal graph theory which served as the initial impetus for our investigations. By "graph", we shall always mean "undirected graph without loops and multiple edges". We refer to its number of vertices as the *order* of the graph, and to its number of edges as the *size*.

Let $\mathcal{F}$ be a family of graphs, and let $\mathrm{ex}(v, \mathcal{F})$ be the greatest size e of any graph of order v which contains no subgraph isomorphic to a graph from $\mathcal{F}$. The Turán problem is to determine $\mathrm{ex}(v, \mathcal{F})$, or at least the asymptotic behavior of $\mathrm{ex}(v, \mathcal{F})$ $(v \to \infty)$, for given $\mathcal{F}$. A more ambitious objective is to classify those such graphs which are extremal, i.e., of size $e = \mathrm{ex}(v, \mathcal{F})$.

Our present interest shall be in the case $\mathcal{F} = \{C_{2t}\}$, where C_{2t} denotes the cycle of length $2t$. In what follows, we shall write $\mathrm{ex}(v, 2t)$ in place of the more cumbersome $\mathrm{ex}(v, \{C_{2t}\})$.

The best known upper bound in this case is given by the Even Circuit Theorem (see [3, 10] for proofs, although the result is almost universally attributed to Erdős). Specifically, the theorem states

$$\mathrm{ex}(v, 2t) = O\left(v^{1+\frac{1}{t}}\right).$$

The Even Circuit Theorem has led many to speculate that

$$\mathrm{ex}(v, 2t) = \Omega\left(v^{1+\frac{1}{t}}\right),$$

despite the fact that the only cases for which this bound is known to be sharp are $t \in \{2, 3, 5\}$. As previously mentioned, these are achieved by incidence graphs of regular generalized m-gons (where $m = t + 1$).

As the next well known result shows, these "best performers" are greatly limited in number.

Theorem 2.1 (Feit-Higman [11]). *Thick generalized m-gons exist only for $m \in \{3, 4, 6, 8\}$. Regular ones exist only for $m \in \{3, 4, 6\}$.*

In fact, we conjecture that there are no other values of t for which the bound in the Even Circuit Theorem is sharp. Thus, we regard these graphs as artifacts which falsely inflate one's perception of what is achievable in the Turán Problem when the family of forbidden subgraphs is limited to a single fixed cycle of even length.

2.2 Related problems in extremal graph theory

2.2.1 Cage problem

Fix integers $r \geq 2$ and $g \geq 3$. An (r, g)-*graph* is a regular graph of valency r and girth g. An (r, g)-*cage* is an (r, g)-graph of minimum order. We denote the order of an (r, g)-cage by $v(r, g)$.

The cage problem can be simply stated as follows: For given r and g, find all (r, g)-cages. A softening of this problem is to determine the orders of such cages, although this is rarely accomplished without at least one explicit construction. Another worthwhile endeavor is to provide good bounds on the order of cages, in particular to determine the asymptotic behavior of $v(r,g)$.

The following theorem of Tutte gives a lower bound on $v(r,g)$.

Theorem 2.2 (Tutte [28]). *For any (r,g)-graph of order v the following holds:*

$$v \geq \begin{cases} 1 + r + r(r-1) + \cdots + r(r-1)^{(g-3)/2} & (g \text{ odd})\,; \\ 2\left(1 + (r-1) + \cdots + (r-1)^{(g-2)/2}\right) & (g \text{ even})\,. \end{cases}$$

Corollary 2.3. *Any (r,g)-graph which achieves equality in Theorem 2.2 is automatically an (r,g)-cage.*

Discounting the trivial cases (i.e., cycles and complete graphs), it is quite astonishing how few graphs achieve equality in the Tutte bound.

In the odd girth case we are getting only the Moore graphs, which exist only when $g = 5$ and $r = 3, 7$ and possibly 57. Specifically, when $r = 3$ we get the Petersen graph, and when $r = 7$ we get the Hoffman-Singleton graph. (Both are uniquely determined, up to isomorphism, by their parameters.) At present, it is still a mystery as to whether a Moore graph with valency $r = 57$ exists.

Incidence graphs of regular generalized polygons are the only graphs which arise in the even girth case. By the Feit-Higman Theorem, this occurs only when $g \in \{6, 8, 12\}$. However, for each such value of g there are infinitely many admissible values of r, namely $r = q + 1$ where q is any prime power.

While Theorem 2.2 provides the best theoretical lower bound on the order of cages, finding suitable upper bounds proved historically to be a much more difficult problem. The earliest such bound was achieved by Sachs [24], who showed by explicit construction that (r,g)-graphs of finite order always exist. The obvious implication here is that (r,g)-cages exist for every pair r, g. We refer to [33] for more a detailed account of cages, see also [4] and the webpage [23] maintained by G. Royle.

Shortly thereafter, Erdős and Sachs [9] used probablistic methods to derive a much smaller general upper bound. Their result was later improved, though slightly, by Walther [31], [32] and by Sauer [26]. The following upper bound is due to Sauer:

$$v(r,g) \leq \begin{cases} 2(r-1)^{g-2} & (g \text{ odd})\,; \\ 4(r-1)^{g-3} & (g \text{ even})\,. \end{cases}$$

2.2.2 Dense graphs of large girth

By dense in this context, we mean "having many edges" with respect to a fixed girth. This is actually another formulation of the Turán Problem, where we choose the forbidden family $\mathcal{F} = \{C_3, C_4, \ldots, C_{g-1}\}$ to ensure that the girth be at least g.

Note that the goals here are the same as in previously stated Turán problem: to determine $\mathrm{ex}(v, \{C_3, \ldots, C_{g-1}\})$ and, more ambitiously, to find all extremal graphs.

Interestingly, there is no current evidence to substantiate that the asymptotic behavior of the function $\mathrm{ex}(v, \{C_3, \ldots, C_{g-1}\})$ differs in the slightest from that of $\mathrm{ex}(v, g-2)$ provided g is even. To clarify, this is not to say that the two corresponding "asymptotically equivalent" families of graphs coincide. Indeed, the Erdős-Rényi graphs [8] are extremal relative to avoiding a 4-cycle, however they contains 3-cycles. The point is that the number of edges in the Erdős-Rényi graphs are of the same asymptotic order as those of the (girth 6) generalized 3-gons, though for the former graphs the constant is a bit smaller. (See [15, 18] for more general occurrences of this phenomenon.)

2.2.3 Families of graphs with large girth

For $i \geq 1$, let G_i be a regular graph of valency r, girth g_i, and order v_i. Following Biggs [1], we say that $\{G_i\}$ is a *family of graphs with large girth* if

$$g_i \geq \gamma \log_{r-1}(v_i)$$

for some constant γ. It is well known that $\gamma \leq 2$ (e.g., see [2]), but no family has ever been found for which $\gamma = 2$.

For many years the only significant results in this direction were the non-constructive theorems of Erdős and Sachs with improvements by Sauer, Walther and others. These produced $\gamma = 1$ (see [2, p. 107] for more details and references). The first explicit constructions were given by Margulis [21] with $\gamma \approx 0.44$ for some infinite families with arbitrarily large valency.

At present, the largest value that has been achieved is $\gamma = 4/3$. These are realized by independent constructions of Margulis [21] and Lubotzky, Phillips, Sarnak [20, 25]. Specifically, they are Cayley graphs of the group $\mathrm{PGL}_2(\mathbb{Z}_q)$ with respect to a set of $p+1$ generators, where p, q are distinct primes congruent to $1 \bmod 4$ with the Legendre symbol $\left(\frac{p}{q}\right) = -1$.

Note that incidence graphs of regular generalized polygons fail to provide families with large girth because their girth never exceeds 12. Still, we have introduced this extremal notion for a definite reason, see Section 3.2.

3 Graphs $\mathrm{CD}(k, q)$ and their properties

The graphs $\mathrm{CD}(k, q)$ are the incidence graphs of the geometries we describe in this paper. As previously mentioned, the motivation behind their construction was to derive graphs which would have extremal properties approaching those of the incidence graphs of regular generalized polygons, yet be prolific in number. As it turns out, they are best described in terms of another family of graphs $\mathrm{D}(k, q)$ constructed in [14].

3.1 Constructions of graphs $\mathrm{D}(q)$, $\mathrm{D}(k, q)$, $\mathrm{CD}(k, q)$

Let q be a prime power, and let P and L be two copies of the countably infinite dimensional vector space V over $\mathsf{GF}(q)$. Elements of P will be called points and those of L lines. In order to distinguish points from lines we introduce the use of parentheses and brackets: If $x \in V$, then $(x) \in P$ and $[x] \in L$. We adopt the following coordinate representations for points and lines:

$$
\begin{aligned}
(p) &= (p_{01}, p_{11}, p_{12}, p_{21}, p'_{22}, p''_{22}, p_{23}, \ldots, p'_{ii}, p''_{ii}, p_{i,i+1}, p_{i+1,i}, \ldots)\,, \\
[l] &= [l_{10}, l_{11}, l_{12}, l_{21}, l'_{22}, l''_{22}, l_{23}, \ldots, l'_{ii}, l''_{ii}, l_{i,i+1}, l_{i+1,i}, \ldots)\,.
\end{aligned}
$$

We now define an incidence structure (P, L, I) as follows. We say point (p) is incident to line $[l]$ if the following coordinate relations hold:

$$
\begin{aligned}
p_{11} - l_{11} &= l_{10}p_{01} \\
p_{12} - l_{12} &= l_{11}p_{01} \\
p_{21} - l_{21} &= l_{10}p_{11} \\
&\vdots \\
p'_{ii} - l'_{ii} &= l_{10}p_{i-1,i} \\
p''_{ii} - l''_{ii} &= l_{i,i-1}p_{01} \\
p_{i,i+1} - l_{i,i+1} &= l'_{ii}p_{01} \\
p_{i+1,i} - l_{i+1,i} &= l_{10}p''_{ii}
\end{aligned}
$$

We denote the incidence graph of (P, L, I) by $\mathrm{D}(q)$.

Now, for each integer $k \geq 2$ we define the incidence structure (P_k, L_k, I_k) as follows. First, P_k and L_k are obtained from P and L, respectively, by projecting each vector (point or line) onto its initial k coordinates. Incidence I_k is then defined by imposing the first $k - 1$ coordinate relations and ignoring all others. We denote by $\mathrm{D}(k, q)$ the incidence graph of the structure (P_k, L_k, I_k).

From an extremal point of view, the only discernable weakness of the graphs $\mathrm{D}(k,q)$ is that they become disconnected when $k \geq 6$, and continue to disconnect thereafter at regular intervals, see [17]. In every such case, however, the connected components of $\mathrm{D}(k,q)$ are pairwise isomorphic. We denote by $\mathrm{CD}(k,q)$ any one of these connected components.

Proofs of the following propositions may be found in [14] and [16], respectively.

Proposition 3.1. *Let q be any prime power, and $k \geq 2$. Then*

(i) $\mathrm{D}(k,q)$ *is a regular bipartite graph of valency q and order $2q^k$;*

(ii) $\mathrm{Aut}(\mathrm{D}(k,q))$ *is both vertex- and edge-transitive;*

(iii) *for even k, the girth of $\mathrm{D}(k,q)$ is at least $k+4$;*

(iv) *for odd k, the girth of $\mathrm{D}(k,q)$ is at least $k+5$.*

Proposition 3.2. *Let $\mathrm{CD}(k,q)$ denote a connected component of $\mathrm{D}(k,q)$. Then*

(i) $\mathrm{CD}(k,q)$ *is a regular, bipartite graph of valency q and order $2q^{k-\lfloor\frac{k+2}{4}\rfloor+1}$;*

(ii) $\mathrm{Aut}(\mathrm{CD}(k,q))$ *is both vertex- and edge-transitive;*

(iii) *for even k, the girth of $\mathrm{CD}(k,q)$ is at least $k+4$;*

(iv) *for odd k, the girth of $\mathrm{CD}(k,q)$ is at least $k+5$.*

We believe the reader will concur that such manner of description belies any deep algebraic principles that may be involved in the construction of these graphs. Moreover, nondisclosure of such principles is in a sense excusable, since we are able to derive all important properties of these graphs directly from their coordinate relations.

3.2 Graphs $\mathrm{CD}(k,q)$ as near-extremal structures

We revisit the extremal problems described in Section 2, this time alluding to various improvements on asymptotics, courtesy of the graphs $\mathrm{CD}(k,q)$.

Regarding the two formulations of the Turán Problem, the best known general lower bound on each of $\mathrm{ex}(v,2t)$ and $\mathrm{ex}(v,\{C_3,C_4,\ldots,C_{2t+1}\})$ is provided, with one exception, by the graphs $\mathrm{CD}(k,q)$. Specifically, this bound is

$$c_t\left(v^{1+\frac{2}{3t-3+\epsilon}}\right),$$

where c_t is a positive constant and $\epsilon \in \{0,1\}$ depends on the parity of t. The excepted case, $t=5$, is achieved by the regular generalized hexagons.

Graphs $\mathrm{CD}(k,q)$ also lead to a rather marked improvement in bounding the order of cages. Namely, we have

$$v(r,g) \leq 2rq^{\frac{3g-13}{4}}$$

where q is the smallest prime power for which $r \leq q$.

Finally, for each fixed choice of prime power q we get that $\{\mathrm{CD}(k,q)\}_{k\geq 2}$ is a family of graphs with large girth in the sense of Biggs. Here, one achieves $\gamma = 4\log_q(q-1)/3$.

It is worth pointing out that there is a tacit distinction between the two girth-related problems of 2.2.2 and 2.2.3. For example, while the graphs of Lubotzky, Phillips, Sarnak produce the larger value $\gamma = 4/3$, making them better in the sense of 2.2.3, the graphs $\mathrm{CD}(k,q)$ have more edges for each fixed choice of order and girth, so are superior in the sense of 2.2.2.

4 Generalized 3-gons

With this section comes a shift in perspective from the graphs $\mathrm{CD}(k,q)$ to their underlying geometries. It turns out that $\mathrm{CD}(2,q)$ coincides with the incidence graph of the so-called "affine part" of the generalized 3-gon. Thus we concentrate first on generalized 3-gons, i.e., projective planes. We construct three models of the classical projective plane $\mathsf{PG}(2,q)$, the first two of which are well known. We conclude with a far less familiar model, in which the affine part is highly detectable. This latter model depends on some rather sophisticated notions from Lie theory, and will be crucial to our exposition.

4.1 Models of $\mathsf{PG}(2,q)$

We start with the most familiar model, in which points and lines correspond to subspaces embedded in $\mathsf{GF}(q)^3$.

4.1.1 Linear algebraic model of $\mathsf{PG}(2,q)$

Fix a 3-dimensional vector space V over $\mathsf{GF}(q)$. We take as points all one-dimensional subspaces of V, and as lines all two-dimensional subspaces. Incidence is containment, i.e., point V_1 is incident to line V_2 if and only if V_1 is a subspace of V_2. We refer to any point-line incident pair $\{V_1,V_2\}$ as a *flag*.

Recall that $\mathsf{P\Gamma L}(3,q)$ is the full collineation group of $\mathsf{PG}(2,q)$. We work instead with its subgroup of central-axial collineations, namely $\mathsf{PGL}(3,q)$, which is again acting transitively on points, lines and flags. Relative to this subgroup, it is possible to "internalize" this geometry in the sense of Felix Klein. This yields our second model.

4.1.2 Group geometric model of $\mathsf{PG}(2, q)$

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ denote the standard basis of the vector space V defined above. Taking $V_1 = \langle \vec{e}_1 \rangle$ and $V_2 = \langle \vec{e}_1, \vec{e}_2 \rangle$, we may identify (via left action) the stabilizers of point V_1 and line V_2 as follows:

$$\mathrm{Stab}(V_1) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$$

$$\mathrm{Stab}(V_2) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

Here, asterisks are used to denote arbitrary values from $\mathsf{GF}(q)$, subject of course to preserving nonsingularity of the matrices.

We now take as points all cosets of $\mathrm{Stab}(V_1)$ in $\mathsf{PGL}(3, q)$, and as lines all cosets of $\mathrm{Stab}(V_2)$ in $\mathsf{PGL}(3, q)$. Incidence is defined as follows: point $x\,\mathrm{Stab}(V_1)$ is incident to line $y\,\mathrm{Stab}(V_2)$ precisely when

$$x\,\mathrm{Stab}(V_1) \cap y\,\mathrm{Stab}(V_2) \neq \emptyset\,.$$

It is quite visible from our descriptions of $\mathrm{Stab}(V_1)$ and $\mathrm{Stab}(V_2)$ that the intersection of these two groups has the form

$$\mathrm{Stab}(V_1) \cap \mathrm{Stab}(V_2) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

This is nothing more than the stabilizer of the flag $\{V_1, V_2\}$.

4.1.3 Lie algebraic model of $\mathsf{PG}(2, q)$

Here, it is necessary that we presume some familiarity with certain notions from the theory of Lie groups and Lie algebras. Excellent introductory sources with differing perspectives are [7, 13, 22]. In any case, we believe that even the uninitiated reader will reap certain benefits from our presentation, hopefully coming away with a clear "local" picture of our general paradigm and methodology.

Let $\mathcal{L} = \mathcal{L}(G)$ denote the simple Lie algebra for $G = \mathsf{GL}(3, q)$ defined over the field $\mathsf{GF}(q)$. Recall that $\mathcal{L}$ is a nonassociative algebra with respect to the linear, skew-symmetric operation of Lie product, denoted $[\ ,\ ]$, and that this operation satisfies the Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

Let $\Pi = \{r_1, r_2\}$ be a fundamental basis for $\mathcal{L}$. As usual, we refer to r_1, r_2 as the fundamental roots.

The Cartan matrix $A = (A_{ij})$ of $\mathcal{L}$ is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

From this matrix, one is able to generate the entire root system of $\mathcal{L}$ via linear extension of the action of the Weyl group $W = \langle w_1, w_2 \rangle$ on the fundamental roots, specifically $r_j^{w_i} = r_j - A_{ij} r_i$. The resulting root system is given by $\Phi = \Phi^+ \cup \Phi^-$, where

$$\Phi^+ = \{r_1, r_2, r_1 + r_2\} \quad \text{and} \quad \Phi^- = \{-r_1, -r_2, -(r_1 + r_2)\}.$$

We refer to the elements of Φ^+ as the positive roots of $\mathcal{L}$ with respect to Π, and to those of Φ^- as the negative roots.

Now let $\Pi^* = \{r_1^*, r_2^*\}$ be the dual basis of Π, and define the contragredient action of W on Π^* via $r_j^{*w}(s) = r_j^*(s^{w^{-1}})$. This gives us the dual root system

$$\Phi^* = \{\pm r_1^*, \pm r_2^*, \pm(r_1^* - r_2^*)\}.$$

Recall the Cartan decomposition of a simple Lie algebra, namely

$$\mathcal{L} = \mathcal{H} \oplus \mathcal{L}^+ \oplus \mathcal{L}^-,$$

where $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{L}$ (maximal self-normalizing subalgebra), and $\mathcal{L}^+$ and $\mathcal{L}^-$ are the positive and negative root spaces of $\mathcal{L}$, respectively. Subspaces $\mathcal{L}^+$ and $\mathcal{L}^-$ are *a fortiori* ideals of $\mathcal{L}$, but they are not subalgebras. However, the direct sums $\mathcal{L}^U = \mathcal{H} \oplus \mathcal{L}^+$ and $\mathcal{L}^L = \mathcal{H} \oplus \mathcal{L}^-$ *are* subalgebras of $\mathcal{L}$, which we call the upper and lower Borel subalgebras of $\mathcal{L}$, respectively. In what follows, we shall be mainly concerned with the upper Borel subalgebra $\mathcal{L}^U$.

Under the adjoint action of $\mathcal{H}$, each of $\mathcal{L}^+$, $\mathcal{L}^-$ decomposes into a direct sum of $\mathcal{H}$-invariant subspaces, which we call root spaces. It is well known that every root space is one-dimensional. For each $s \in \Phi$, let e_s be a corresponding (nonzero) root vector. In such manner, we get a further decomposition of $\mathcal{L}$ into the direct sum of $\mathcal{H}$ with the totality of root spaces. In particular, we see that $\{r_1^*, r_2^*, e_{r_1}, e_{r_2}, e_{r_1+r_2}\}$ is a basis for $\mathcal{L}^U$, presuming once more that $\mathcal{L}$ is the Lie algebra for $\mathsf{GL}(3, q)$.

We are now ready to present our Lie algebraic model of $\mathsf{PG}(2,q)$. There are three types of points and three types of lines; in each case "type" is determined by the linear functional $s^* \in \Phi^*$ which is showing up as the lead term in each such expression.

Points: r_1^* (one element)
$-r_1^* + r_2^* + \lambda_{r_1} e_{r_1}$ (q elements)
$-r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$ (q^2 elements)

Lines: r_2^* (one element)
$r_1^* - r_2^* + \lambda_{r_2} e_{r_2}$ (q elements)
$-r_1^* + \lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{r_1+r_2}$ (q^2 elements)

Incidence is defined by vanishing of the Lie product, i.e., point p is incident to line l if and only if $[p,l] = 0$.

The reader will observe that each point and line is a specific linear combination relative to the basis $\{r_1^*, r_2^*, e_{r_1}, e_{r_2}, e_{r_1+r_2}\}$ of $\mathcal{L}^U$. However, most linear combinations have been excluded from consideration. This raises the question as to how such representative elements are chosen.

First, the lead term $s^* \in \Phi^*$ falls into one of two possible orbits under the action of the Weyl group W on Φ^*, namely $\{-r_2^*\}^W$ and $\{-r_1^*\}^W$. The choice here is a bit arbitrary, but we define points and lines in this respective order. Second, for a fixed $s^* \in \Phi^*$ an expression of the form $s^* + v$ is an admissible object (point or line) if and only if $s^* + v$ lies in the same orbit as s^* under the action of the stabilizer B in $\mathsf{GL}(3,q)$ of the flag $\{r_1^*, r_2^*\}$. From our explanation, the reader will note that there are precisely three orbits of points and three orbits of lines relative to this action (in each case, of respective sizes 1, q, q^2), which is why we chose to list them separately and distinguish them by "type".

Observe that restriction of our model to the points r_1^*, $-r_1^* + r_2^*$, $-r_2^*$ and the lines r_2^*, $r_1^* - r_2^*$, $-r_1^*$ yields the 6-cycle, i.e., the *thin* generalized 3-gon. However, it is a quite different substructure of $\mathsf{PG}(2,q)$ with which we will be concerned. We speak now of the affine part of $\mathsf{PG}(2,q)$, which we define presently.

Fix a flag $\{p,l\}$ in $\mathsf{PG}(2,q)$. We define the *affine part of* $\mathsf{PG}(2,q)$ *relative to* $\{p,l\}$ to be the substructure induced on all points of maximum distance from l, and all lines of maximum distance from p. (In order to interpret "distance" more succinctly here, it may be convenient to consider the incidence graph of $\mathsf{PG}(2,q)$, where this notion corresponds to the usual one for graphs.) Because we are working inside a flag-transitive geometry, affine parts relative to different flags are isomorphic.

4.1.4 Lie algebraic model of the affine part of $\mathsf{PG}(2,q)$

In terms of our Lie algebraic model of $\mathsf{PG}(2,q)$ given in 4.1.3, we may choose the affine part relative to the flag $\{r_1^*, r_2^*\}$. Then the points and lines at maximum distance from this flag correspond to the two largest orbits of B, each of size q^2. This gives us our desired model of the affine part:

Affine points: $-r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$
Affine lines: $-r_1^* + \lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{r_1+r_2}$

We hasten to point out that the affine part of $\mathsf{PG}(2,q)$ is not the affine plane $\mathsf{AG}(2,q)$, although both objects share the same point set. Whereas $\mathsf{AG}(2,q)$ is obtained by removing a single line from $\mathsf{PG}(2,q)$ ("line at infinity"), the affine part requires the removal of $q+1$ lines ("line at infinity" plus a parallel class). In particular, the incidence graph of this substructure is regular of valency q.

Example 4.1. We provide Lie algebraic models of the Fano plane $\mathsf{PG}(2,2)$ and its affine part. In each case, lines are represented as subsets of points (the latter appear in boldface). The two structures are depicted in Figures 1 and 2, respectively.

Lines of the Fano plane:

$$\{\boldsymbol{r_1^*},\ \boldsymbol{-r_2^*},\ \boldsymbol{-r_2^* + e_{r_1+r_2}}\} = r_1^* - r_2^*$$
$$\{\boldsymbol{r_1^*},\ \boldsymbol{-r_1^* + r_2^*},\ \boldsymbol{r_1^* + r_2^* + e_{r_1}}\} = r_2^*$$
$$\{\boldsymbol{r_1^*},\ \boldsymbol{-r_2^* + e_{r_2}},\ \boldsymbol{-r_2^* + e_{r_2} + e_{r_1+r_2}}\} = r_1^* - r_2^* + e_{r_2}$$
$$\{\boldsymbol{-r_2^*},\ \boldsymbol{-r_1^* + r_2^* + e_{r_1}},\ \boldsymbol{-r_2^* + e_{r_2} + e_{r_1+r_2}}\} = -r_1^* + e_{r_1}$$
$$\{\boldsymbol{-r_2^* + e_{r_2}},\ \boldsymbol{-r_1^* + r_2^* + e_{r_1}},\ \boldsymbol{-r_2^* + e_{r_1+r_2}}\} = -r_1^* + e_{r_1} + e_{r_1+r_2}$$
$$\{\boldsymbol{-r_1^* + r_2^*},\ \boldsymbol{-r_2^* + e_{r_2}},\ \boldsymbol{-r_2^* + e_{r_2} + e_{r_1+r_2}}\} = -r_1^* + e_{r_1+r_2}$$
$$\{\boldsymbol{-r_2^*},\ \boldsymbol{-r_1^* + r_2^*},\ \boldsymbol{-r_2^* + e_{r_2}}\} = -r_1^*$$

Lines of the affine part of the Fano plane:

$$\{\boldsymbol{-r_2^*},\ \boldsymbol{-r_1^* + r_2^* + e_{r_1}},\ \boldsymbol{-r_2^* + e_{r_2} + e_{r_1+r_2}}\} = -r_1^* + e_{r_1}$$
$$\{\boldsymbol{-r_2^* + e_{r_2}},\ \boldsymbol{-r_1^* + r_2^* + e_{r_1}},\ \boldsymbol{-r_2^* + e_{r_1+r_2}}\} = -r_1^* + e_{r_1} + e_{r_1+r_2}$$
$$\{\boldsymbol{-r_1^* + r_2^*},\ \boldsymbol{-r_2^* + e_{r_2}},\ \boldsymbol{-r_2^* + e_{r_2} + e_{r_1+r_2}}\} = -r_1^* + e_{r_1+r_2}$$
$$\{\boldsymbol{-r_2^*},\ \boldsymbol{-r_1^* + r_2^*},\ \boldsymbol{-r_2^* + e_{r_2}}\} = -r_1^*$$

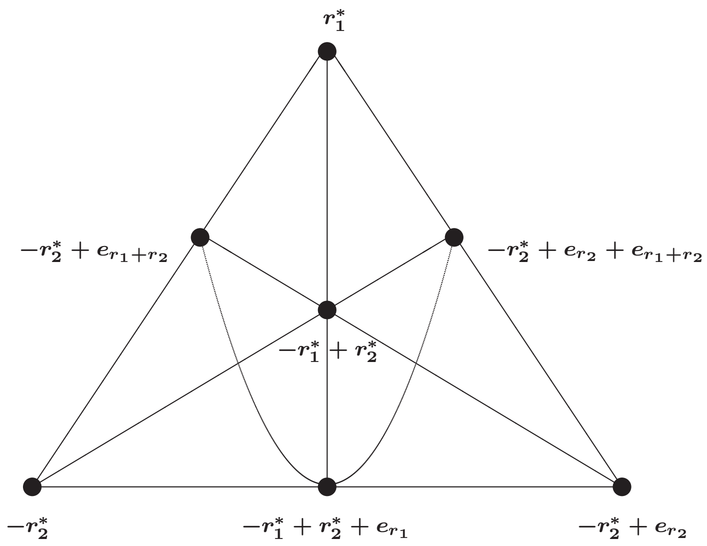


Figure 1: Fano plane

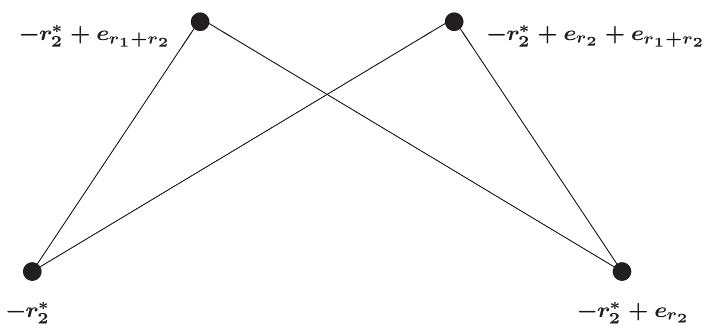


Figure 2: Affine part of Fano plane

5 Generalized m-gons

5.1 Group geometric model

Each rank two Chevalley group G gives rise to a generalized m-gon of order q, where $\mathsf{GF}(q)$ is the field over which G is defined. We have already witnessed this in the case of $\mathsf{GL}(3, q)$. Indeed, $\mathsf{GL}(3, q)$ is a rank two Chevalley group defined over $\mathsf{GF}(q)$, and its coset geometry provided us with a group geometric model of the generalized 3-gon (projective plane) of order q. It turns out that this construction is entirely general.

There are seven types of rank two Chevalley groups, viz. A_2, B_2, G_2, ${}^2A_3$, ${}^2A_4$, ${}^3D_4$, ${}^2F_4$. The first three are said to be of "normal type", the last four "twisted". Generalized 3-gons arise from A_2, generalized 4-gons from B_2, ${}^2A_3$ and ${}^2A_4$, 6-gons from G_2 and ${}^3D_4$, and 8-gons from ${}^2F_4$. The types for which the generalized m-gons turn out to be regular are A_2, B_2, G_2 and ${}^2A_4$. However, as the 4-gon coming from ${}^2A_4$ defined over $\mathsf{GF}(q)$ is isomorphic to the one arising from B_2 defined over $\mathsf{GF}(q^2)$, it suffices to restrict our attention to A_2, B_2 and G_2.

Thus let G be of type A_2, B_2 or G_2 defined over $\mathsf{GF}(q)$, where $q = p^\alpha$. Let U be a fixed Sylow p-subgroup of G, and let B be its normalizer in G. Then, as G has rank two there are just two subgroups properly situated between B and G. (In general, a rank r group of Lie type has $2^r - 2$ such intermediate subgroups, of which r are maximal.) We denote these subgroups by P_1, P_2 and call them (*maximal*) *parabolic* subgroups of G. Groups U and B are commonly referred to as *unipotent* and *Borel* subgroups of G, respectively.

We now formulate our group geometric model for generalized m-gons as follows: Let G be any rank two group of Lie type, and let P_1, P_2 be the parabolic subgroups of G relative to a fixed choice of Borel subgroup B. We take as points all cosets of P_1 in G, and as lines all cosets of P_2 in G. We define point xP_1 to be incident to line yP_2 precisely when $xP_1 \cap yP_2 \neq \emptyset$.

Example 5.1. Let us specialize this setting to the one we have already seen in 4.1.2, where the rank two group is of type A_2, that is, $\mathsf{GL}(3, q)$. A convenient choice for Sylow p-subgroup here is the group of all upper triangular matrices with 1's along the diagonal. (This group has order q^3 so is surely a Sylow p-subgroup of $\mathsf{GL}(3, q)$.) In this case, B is simply the group of all upper triangular matrices, which the reader will recall is the stabilizer of a flag. We may now identify $\mathrm{Stab}(V_1)$ and $\mathrm{Stab}(V_2)$ as the two parabolic subgroups P_1, P_2 of $\mathsf{GL}(3, q)$. Indeed, $\mathrm{Stab}(V_1)$ and $\mathrm{Stab}(V_2)$ are the only groups which lie strictly between B and $\mathsf{GL}(3, q)$.

5.2 Lie algebraic model

This is a direct analogue of the Lie algebraic model of $\mathsf{PG}(2, q)$ presented in 4.1.3. Let G be a rank two group of Lie type, and let $\mathcal{L}$ be its corresponding Lie algebra. We embed the points and lines of our generalized m-gon in the upper Borel subalgebra $\mathcal{L}^U$ as follows:

Points: $s^* + v$, where $s^* \in \{-r_2^*\}^W$ and $s^* + v \in \{s^*\}^B$

Lines: $s^* + v$, where $s^* \in \{-r_1^*\}^W$ and $s^* + v \in \{s^*\}^B$

Incidence is once again defined in terms of the vanishing Lie product, cf. 4.1.3.

5.3 Lie algebraic model of affine part

Appealing to 4.1.4, we define affine objects to be those at maximum distance from the flag $\{r_1^*, r_2^*\}$. Again, these objects comprise the two largest orbits under the action of the Borel subgroup B of G, i.e., the stabilizer of the flag $\{r_1^*, r_2^*\}$. The resulting form of affine points is thus $-r_2^* + v$, and that of affine lines is $-r_1^* + v$, where in each case $s^* + v \in \{s^*\}^B$ for $s^* \in \{-r_2^*, -r_1^*\}$.

Example 5.2. We provide Lie algebraic models for the generalized 4-gon of type B_2 and its affine part.

Points:
r_2^* (one element)
$r_1^* - r_2^* + \lambda_{r_2} e_{r_2}$ (q elements)
$-r_1^* + r_2^* + \lambda_{r_1} e_{r_1} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$ (q^2 elements)
$-r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$ (q^3 elements)

Lines:
r_1^* (one element)
$-r_1^* + 2r_2^* + \lambda_{r_1} e_{r_1}$ (q elements)
$r_1^* - 2r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$ (q^2 elements)
$-r_1^* + \lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{r_1+r_2} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$ (q^3 elements)

Affine points: $-r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$

Affine lines: $-r_1^* + \lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{r_1+r_2} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$

Note that our model for the generalized 4-gon involves four types of points and four types of lines. As alluded to above, these types correspond to Borel orbits, the two largest of which render the points and lines of the affine part of the 4-gon.

A similar model can be constructed for the generalized 6-gon of type G_2, however this exhausts all possibilities available to us. Our problem at this stage is more metaphysical than pragmatic: How to detect the essence of these structures so limited in number, and extrapolate on that essence. While we would like to claim some level of ingenuity in solving this problem, it turns out that the biggest role was played by the Beauty of Mathematics. It cried out to us where to look, and fortuitously, Ustimenko heard the message and interpreted it correctly.

6 Affine Lie algebras

Roughly speaking, the classification of simple Lie algebras runs parallel to that of Lie groups (over the complex numbers) and groups of Lie type (over finite fields). There are some curious pathologies in the latter case, where a twisted group of Lie type will emerge which has no Lie group "relative", but such occurrences are rare and have been understood for a long time.

Less studied is the class of Kac-Moody algebras, though its subclass of affine Lie algebras has many important ties with Lie algebras. Both varieties give rise to (extended) Dynkin diagrams, (affine) Weyl groups and (affine) root systems, and may be constructed from their Cartan matrices.

Some differences, however, are striking. Whereas the root system of a Lie algebra is finite, that of an affine Lie algebra is infinite. This accounts for infinitely many root vectors, making affine Lie algebras infinite dimensional graded structures. Moreover, while all root spaces of a Lie algebra are one-dimensional, in the case of an affine Lie algebra this dimension may be larger.

The most crucial difference, however, is that for affine Lie algebras the connection to groups is far more tenuous. Still, such connections have been well studied and are attractively embodied in the general framework of "groups with a twin root datum (donnée radicielle)". This notion may be regarded as the group theoretical counterpart of the Kac-Moody algebra. We refer the reader to the lecture notes of P.-E. Caprace and B. Rémy [6] as an excellent introductory source, see also [5, 27].

6.1 Affine Lie algebra of type $\widetilde{A}_1$

In our preamble, we mentioned Dynkin diagrams. Every Lie algebra and affine Lie algebra has one. In the case of rank two Lie algebras of normal type, such diagrams consist of two nodes with some adjoining edges. The number of such edges conveys immediate information about the root system and Weyl group structure; ultimately it reveals all about the corresponding Lie algebra and Lie group. It is one of the most compact modes of description in all of mathematics.

In Figure 3, we give the Dynkin diagrams of all rank two Lie algebras of normal type. Recall that each such diagram is related to the existence of a regular generalized polygon.

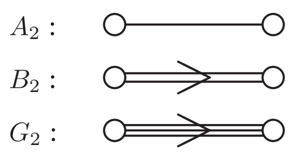


Figure 3: Dynkin diagrams of type A_2, B_2, G_2

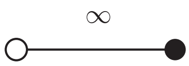
$\widetilde{A}_1$: 

Figure 4: Extended Dynkin diagram of type $\widetilde{A}_1$

The bad news is we have exhausted all Dynkin diagrams of the classical Lie algebras. The good news is there is one additional diagram to consider. It is a rank two "extended" Dynkin diagram, formed by appending an additional node (corresponding to an imaginary root) to the Dynkin diagram of a rank one Lie algebra of type A_1, see Figure 4. Note that the two nodes are joined by infinitely many edges, which, in particular, signifies that the corresponding affine Weyl group will be the infinite dihedral group. Keep in mind that there is no hope of a new generalized polygon here, as Feit and Higman have demonstrated.

Curiously, despite uniqueness of the diagram of type $\widetilde{A}_1$, there are two distinct Cartan matrices which may be associated with it, namely the one indicated in 6.2 below, and the one in which we disturb only its $(1, 2)$-entry reassigning $a_{12} = -3$. Some time ago, we convinced ourselves that both produce the same asymptotics relative to the kinds of properties we were investigating. Thus, we worked only with the former.

6.2 Affine root system of type $\widetilde{A}_1$

Here, our mode of construction will closely resemble the one given in 4.1.3, and its generalization in 5.2.

Let $\widetilde{\mathcal{L}} = \mathcal{L}(\widetilde{A}_1)$ denote the affine Lie algebra under investigation. As $\widetilde{\mathcal{L}}$ has rank two, it has a fundamental basis of two independent roots, say $\Pi = \{r_1, r_2\}$.

Consider the Cartan matrix $A = (A_{ij})$ of $\widetilde{\mathcal{L}}$ given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

From this matrix, one is able to generate the entire root system of $\widetilde{\mathcal{L}}$ via the action of the Weyl group $W = \langle w_1, w_2 \rangle$, specifically $r_j^{w_i} = r_j - A_{ij} r_i$. The resulting root system is given by $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^- = -\Phi^+$ and

$$\Phi^+ = \{r_1,\ r_2,\ r_1 + r_2,\ 2r_1 + r_2,\ r_1 + 2r_2,\ 2r_1 + 2r_2, \ldots,\\ ir_1 + (i-1)r_2,\ (i-1)r_1 + ir_2,\ ir_1 + ir_2, \ldots\}$$

Note that there are infinitely many roots, as predicted.

Let now $\Pi^* = \{r_1^*, r_2^*\}$ be the dual basis for Π, and apply the contragredient action of W on Π^* to obtain the dual root system Φ^*. This is difficult to write down, but fortunately unnecessary. As the reader will see, we shall only need that $-r_i^* \in \Phi^*$ for $i = 1, 2$, which is an obvious fact.

Roots of the form $ir_1 + ir_2$ are imaginary, i.e., isotropic with respect to the Killing form induced by the Cartan matrix A. In such case the corresponding root spaces are two-dimensional. Hence we use primes to distinguish between independent root vectors from the same root space, i.e., $\{e'_{ir_1+ir_2}, e''_{ir_1+ir_2}\}$ is a basis for the root space corresponding to $ir_1 + ir_2 \in \Phi$. All other root spaces are one-dimensional.

6.3 Crucial observation of Ustimenko

Consider the positive root system Φ^+ of $\widetilde{\mathcal{L}}$ derived above. Ustimenko noticed that subject to its natural ordering, Φ^+ had the property that its first three members form a positive root system $\{r_1, r_2, r_1 + r_2\}$ of type A_2, while its first four members yield a positive root system $\{r_1, r_2, r_1 + r_2, 2r_1 + r_2\}$ of type B_2. (Curiously, a positive root system of type G_2 does not arise in such manner.)

On the basis of Ustimenko's observation, we were now in a position to speculate how one might manufacture actual geometries from these truncated root systems. In this endeavor, we would again reap the benefits of Ustimenko's past experience, see [29]. (See also [30, 12].)

7 Arriving at our generalization

From the outset, we faced a rather difficult and perplexing problem. Although we clearly had the capacity to define an incidence structure in $\widetilde{\mathcal{L}}^U$ in which incidence would correspond to the vanishing Lie product, what would be the correct choices for points and lines? In all finite cases, these choices were dictated to us by the Weyl group W and Borel subgroup B. But in the situation in which we presently found ourselves, that of constructing geometries from truncations of an affine root system of type $\widetilde{A}_1$, these groups would vanish after the initial two truncations, never again to reappear.

7.1 Eliminating the role of groups

Recall the roles played by W and B in the finite case. Elements of the form $s^* + v \in \mathcal{L}^U$ are points of the m-gon provided $s^* \in \{-r_2^*\}^W$ and $s^* + v \in \{s^*\}^B$, and they are lines provided $s^* \in \{-r_1^*\}^W$ and $s^* + v \in \{s^*\}^B$. However, in the infinite case it is a difficult matter to describe all elements of $\{-r_2^*\}^W$ and $\{-r_1^*\}^W$. So our first decision was to restrict attention to affine parts as prospective geometries, wherein only $-r_2^*$ and $-r_1^*$ would appear as lead terms.

Now we make a crucial observation. The Borel orbits indicated above may be described in a manner which is completely void of reference to groups. Namely, for $s^* \in \Phi^*$ define the set $s^*(\downarrow) = \{e_r \mid r \in \Phi^+,\, s^*(r) < 0\}$. Then $\{s^*\}^B$ is simply the totality of expressions $s^* + v$ for which v is in the $\mathsf{GF}(q)$-span of $s^*(\downarrow)$. We encourage the reader to corroborate this fact for the already presented Lie algebraic models of the generalized 3- and 4-gon.

Obviously, $s^*(\downarrow)$ has independent meaning in $\widetilde{\mathcal{L}}^U$. In other words, we have completely extricated ourselves from our former dependence on groups, allowing us to define our new models in a purely Lie algebraic manner.

7.2 Construction of a universal affine part

We start by defining a model which uses the entire positive root space $\widetilde{\mathcal{L}}^+$ of $\widetilde{\mathcal{L}}$. *A priori*, we understand this geometry cannot be very compelling. Indeed, the affine Weyl group of $\widetilde{\mathcal{L}}$ is infinite dihedral, which forces the corresponding Coxeter (i.e., thin) geometry to have infinite girth. Nonetheless, it is not any thick geometry arising from $\widetilde{\mathcal{L}}$ which holds our interest. It was always our intention to explore "sections" of $\widetilde{\mathcal{L}}^U$ which correspond to truncated root systems. Our next model serves as a universal object for these.

Fix the field $\mathsf{GF}(q)$. Points of our geometry will be all elements of $\widetilde{\mathcal{L}}^U$ of the form $-r_2^* + v$, where v is in the $\mathsf{GF}(q)$-span of $-r_2^*(\downarrow)$. Lines will be elements of the form $-r_1^* + v$ where v is in the span of $-r_1^*(\downarrow)$. As usual, incidence corresponds to the vanishing Lie product.

As we predicted, the incidence graph of this geometry has infinite girth. As we hadn't at the time predicted, it is disconnected. It is in fact the graph $\mathrm{D}(q)$ described in Section 3.1, a regular forest of valency q.

7.3 Construction of truncated affine parts

For each $k \geq 2$, let $\Phi^+[k]$ denote the set consisting of the initial $k+1$ roots from Φ^+ subject to its natural ordering. Now define

$$s^*(\downarrow)[k] = \{e_r \mid r \in \Phi^+[k],\, s^*(r) < 0\}\,.$$

Our model of the "truncated affine part" may now be formulated as follows:

Points: $-r_2^* + v$, where v is in the $\mathsf{GF}(q)$-span of $-r_2^*(\downarrow)[k]$

Lines: $-r_1^* + v$, where v is in the $\mathsf{GF}(q)$-span of $-r_1^*(\downarrow)[k]$

As usual, incidence is defined in terms of the vanishing Lie product.

Incidence graphs of these geometries are none other than the graphs $\mathrm{D}(k,q)$ defined in Section 3.1. In fact, each $\mathrm{D}(k,q)$ is a quotient graph of $\mathrm{D}(q)$, although a lot more can be said in a quite broader setting [19].

Certainly, one could make a valid argument that this family of geometries provides a natural generalization of the affine parts of regular generalized m-gons. Indeed, its two smallest members coincide with the affine parts of the 3-gon and 4-gon. However, as we have discussed, these geometries disconnect as k grows.

Thus we turn to the family of connected component subgeometries, i.e., those for which the incidence graphs are $\mathrm{CD}(k,q)$. It turns out these component subgeometries yield an even better approximation of the behavior of the affine parts of m-gons than do the full geometries constructed above. For example, their relative girth is larger. Moreover, the affine parts of the 3-gon and 4-gon are again appearing as the two smallest members. Nothing has been sacrificed. This, at long last, is our desired generalization.

There remains but one objective to fulfill. We wish to explain how the coordinate relations of the graphs $\mathrm{D}(k,q)$ are obtained from their underlying Lie algebraic models.

8 Passage to coordinate relations

In this section, we uncloak the mystery surrounding the appearance of such terms as p_{ij}, p'_{ii}, p''_{ii}, l_{ij}, l'_{ii}, l''_{ii} as coordinates of points and lines in the description of graphs $\mathrm{D}(k,q)$ (see Section 3.1). In addition, we illustrate at the level of example how the coordinate relations for these graphs are derived.

8.1 Subscripts

Recall that points of our geometries are represented as $-r_2^* + v$, where v is a linear combination of elements from $-r_2^*(\downarrow)[k]$. The scalar coefficients in this representation, which we formerly denoted by λ_s ($s \in \Phi^+$), may also be denoted by p_{ij}, where $s = ir_1 + jr_2$. The case for lines is entirely analogous.

Example 8.1. Consider the point $(p) = (p_{01}, p_{11}, p_{12}, p_{21})$, where we assume $k = 4$. Here, $\Phi^+(4) = \{r_1, r_2, r_1 + r_2, r_1 + 2r_2, 2r_1 + r_2\}$, in which case we correspondingly obtain

$$-r_2^*(\downarrow)[4] = \{e_{r_2}, e_{r_1+r_2}, e_{r_1+2r_2}, e_{2r_1+r_2}\}\,.$$

Thus the Lie algebraic representation of (p) is given by

$$-r_2^* + p_{01}e_{r_2} + p_{11}e_{r_1+r_2} + p_{12}e_{r_1+2r_2} + p_{21}e_{2r_1+r_2} .$$

Let's now illustrate this same procedure for the line $[l] = [l_{10}, l_{11}, l_{12}, l_{21}]$. Here we have

$$-r_2^*(\downarrow)[4] = \{e_{r_1}, e_{r_1+r_2}, e_{r_1+2r_2}, e_{2r_1+r_2}\} .$$

The Lie algebraic representation for $[l]$ is thus given by

$$-r_1^* + l_{10}e_{r_1} + l_{11}e_{r_1+r_2} + l_{12}e_{r_1+2r_2} + l_{21}e_{2r_1+r_2} .$$

Subscripts of the form $\{ii\}$ which appear in such terms as p'_{ii}, p''_{ii}, l'_{ii}, l''_{ii} are performing the exact same function. However, because the root space of each imaginary root $ir_1 + ir_2$ is two-dimensional, we need two scalars for each $i \geq 2$ to fulfill this role. Thus p'_{ii} and l'_{ii} represent coefficients of $e'_{ir_1+ir_2}$, and p''_{11} and l''_{ii} represent coefficients of $e''_{ir_1+ir_2}$, presuming of course that $\{e'_{ir_1+ir_2}, e''_{ir_1+ir_2}\}$ is the chosen basis for the root space of $ir_1 + ir_2$.

8.2 Coordinate relations

The first coordinate relation which appears in the definition of all graphs $\mathrm{D}(k, q)$ is

$$p_{11} - l_{11} = l_{10}p_{01} .$$

Our goal is to derive this relation, relying solely on our Lie algebraic model. Thus we may assume $k = 2$, in which case point $(p) = (p_{01}, p_{11})$ and line $[l] = [l_{10}, l_{11}]$ have respective representations given by

$$(p) \to X_p = -r_2^* + p_{01}e_{r_2} + p_{11}e_{r_1+r_2} ;$$
$$[l] \to X_l = -r_1^* + l_{10}e_{r_1} + l_{11}e_{r_1+r_2} .$$

In terms of our model, X_p and X_l are incident if and only if $[X_p, X_l] = 0$. So let's compute this Lie product in $\widetilde{\mathcal{L}}^U$, and see what happens:

$$\begin{aligned}
[X_p, X_l] = [-r_2^*, -r_1^*] &+ [p_{01}e_{r_2}, -r_1^*] + [p_{11}e_{r_1+r_2}, -r_1^*] + [-r_2^*, l_{10}e_{r_1}] \\
&+ [p_{01}e_{r_2}, l_{10}e_{r_1}] + [p_{11}e_{r_1+r_2}, l_{10}e_{r_1}] + [-r_2^*, l_{11}e_{r_1+r_2}] \\
&+ [p_{01}e_{r_2}, l_{11}e_{r_1+r_2}] + [p_{11}e_{r_1+r_2}, l_{11}e_{r_1+r_2}] \\
= (p_{11} - p_{01}l_{10} - l_{11})e_{r_1+r_2} &
\end{aligned}$$

We conclude that $[X_p, X_l] = 0$ if and only if $p_{11} - p_{01}l_{10} - l_{11} = 0$, which is equivalent to the indicated coordinate relation.

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