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Developments in finite Phan theory

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1 Introduction

Geometric methods in the theory of Chevalley groups and their generalisations have made tremendous advances during the last few decades. Among the most noteworthy and influential of these advances are the systematic application of the concept of amalgams based on [49, 59, 117], the local-to-global approach [136], ingenious applications of combinatorial topology and geometric group theory (as in [2, 91, 92, 102, 137]), the theory of abstract root groups [127, 129, 130], and the interaction of Kac-Moody groups and twin buildings [34, 35, 36, 39, 38, 138, 140]. These methods have proven fruitful over and over again in proving, simplifying and generalising several results in group theory and have had their impact in other areas of mathematics.

The present survey attempts to give a report on the results and on the developments in recent years and to serve as a guide to the literature for the project called Curtis-Phan-Tits Theory (or, short, Phan Theory). This project has been initiated in [19] with the goal to revise Phan's results [106, 107] on presentations of twisted forms of finite Chevalley groups via rank one and rank two groups in order to make them accessible for the ongoing revision of the classification of the finite simple groups [60, 61, 62, 63, 64, 65].

The main impact of Phan's results [106, 107] in the classification can be seen in [14] side by side with the famous Curtis-Tits Theorem established in [45, 134], [135, Theorem 13.32]; see also [62, Section 2.9] and Section 4 of this survey. As in this survey my main concern is the revision of Phan's results,

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I refer to [17] regarding the current overall state of the classification of the finite simple groups.

Phan Theory enters the stage in what is called Step 2 in [17], the identification of the *minimal counterexample* G as one of the known simple groups. By Step 1, the local analysis, inside the minimal counterexample G one reconstructs one or more of the proper subgroups using the inductive assumption and available techniques. Thus, the initial point of the identification, Step 2, is a set of subgroups of G that resemble the subgroups of a central extension $\widehat{G}$ of some known simple group, referred to as the *target group*; the output of the identification step is the statement that G is isomorphic to a central quotient of $\widehat{G}$. Two of the most widely used identification tools in this step are the Curtis-Tits Theorem and Phan's theorems. I have already mentioned [14] as one of the main occurrences of these tools in the classification of the finite simple groups and refer the reader to [65, Section 7.5] for an occurrence of Phan's revised results in the revision of the classification. In Section 7, I describe a possible setup for an application of Phan Theory in the revision of the classification via centralisers of involutions.

The Curtis-Tits Theorem allows the identification of G with a quotient of a universal Chevalley group $\widehat{G}$ of twisted or untwisted type provided that G contains a generating system of subgroups identical to the system of appropriately chosen rank two Levi factors of $\widehat{G}$. For instance, in the case of the Dynkin diagram A_n, the system in question consists of all the groups $\mathsf{SL}_2(\mathbb{F})$, $\mathsf{SL}_3(\mathbb{F})$, and $\mathsf{SL}_2(\mathbb{F}) \times \mathsf{SL}_2(\mathbb{F})$ lying block-diagonally in $\widehat{G} \cong \mathsf{SL}_{n+1}(\mathbb{F})$, considered as a matrix group with respect to a suitable basis of its natural module. Phan's first theorem [106] on the other hand deals with the case $\widehat{G} \cong \mathsf{SU}_{n+1}(q^2)$, considered as a matrix group with respect to an orthonormal basis of its natural module, and the system of block-diagonal subgroups $\mathsf{SU}_2(q^2)$, $\mathsf{SU}_3(q^2)$, and $\mathsf{SU}_2(q^2) \times \mathsf{SU}_2(q^2)$.

In this survey I will describe how a systematic geometric approach making serious use of buildings and twin buildings yields a Phan-type theorem for Chevalley groups of each irreducible spherical type of rank at least three. The complete result is stated in Section 6.

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Contents

2 Geometries and amalgams

In this section I give a quick overview over the basic geometric notions and results used in the present survey. Most of these notions are due to Tits [136]. I refer the reader to [136] and also to [30, 31, 103] for a well-founded introduction to synthetic geometry including proofs and many helpful examples.

2.1 Geometries

2.1.1 Pregeometries and geometries

A *pregeometry* $\mathcal{G} = (X, *, \mathrm{typ})$ over a finite set I is a set of elements X together with a *type function* $\mathrm{typ}\colon X \to I$ and a reflexive and symmetric *incidence relation* $* \subseteq X \times X$ such that for any two elements $x, y \in \mathcal{G}$ with $x * y$ and $\mathrm{typ}(x) = \mathrm{typ}(y)$ we have $x = y$. The *rank* of a pregeometry $\mathcal{G}$ is the cardinality of its type set I. A *flag* in $\mathcal{G}$ is a set of pairwise incident elements. Hence the type function injects any flag into the type set, this image is called the *type* of the flag. A *geometry* is a pregeometry with the property that typ induces a bijection between any maximal flag of $\mathcal{G}$ and I. Flags of type I are called *chambers*.

2.1.2 Residues

The *residue* $\mathcal{G}_F$ of a flag F in a pregeometry $\mathcal{G}$ consists of the set of elements from $\mathcal{G} \setminus F$ that are incident to all elements of F with the restricted incidence and type functions, the latter co-restricted to $I \setminus \mathrm{typ}(F)$, turning the residue $\mathcal{G}_F$ into a pregeometry over $I \setminus \mathrm{typ}(F)$. If $\mathcal{G}$ is a geometry over I, then any of its residues $\mathcal{G}_F$ is a geometry over $I \setminus \mathrm{typ}(F)$. The rank of the residue of a flag F is called the *co-rank* of F. A non-empty pregeometry $\mathcal{G}$ is said to be *connected*, if the graph $(X, *)$ is connected. Following [136, Section 1.2], a pregeometry $\mathcal{G}$

is *residually connected*, if the residue in $\mathcal{G}$ of any flag of co-rank at least two is connected and the residue of any flag of co-rank one is non-empty. A residually connected pregeometry is automatically a geometry.

2.1.3 Automorphisms

An *automorphism* of a pregeometry $\mathcal{G}$ over I is a permutation of its elements that preserves type and incidence and whose inverse permutation also preserves incidence. The group of all automorphisms of $\mathcal{G}$ will be denoted by $\mathrm{Aut}\,\mathcal{G}$. A subgroup $G \leq \mathrm{Aut}\,\mathcal{G}$ acts *flag-transitively* on $\mathcal{G}$ if, for each $J \subseteq I$, it is transitive on the set of all flags of type J. A group G of automorphisms of a geometry $\mathcal{G}$ over I is flag-transitive if and only if G is transitive on the set of maximal flags of $\mathcal{G}$, because each flag of $\mathcal{G}$ can be extended to a flag of type I of $\mathcal{G}$. A pregeometry that admits a flag-transitive automorphism group is called flag-transitive. A *parabolic subgroup* (or simply a *parabolic*) of G is the stabiliser in G of a non-empty flag F of $\mathcal{G}$. The *rank* of the parabolic is defined as the co-rank of F.

The term parabolic subgroup is inspired by the parabolic subgroups of algebraic groups which occur as stabilisers of residues of buildings.

2.2 Simplicial complexes

There exist very many good books dealing with the theory of simplicial complexes, many of them with very different flavours, ranging from combinatorics and graph theory [80] to differential geometry [26] and to topology [120]. Other classical references are [116, 142].

2.2.1 Complexes

A *simplicial complex* $\mathcal{S}$ is a pair (X, Δ) where X is a set and Δ is a collection of non-empty finite subsets of X containing each subset of X of cardinality one such that $A \in \Delta$ and $\emptyset \neq B \subseteq A$ implies $B \in \Delta$. The elements of Δ are called *simplices*. A simplicial complex in which each chain of simplices is finite is called *pure*, if all of its maximal simplices have the same cardinality.

A *morphism* from a complex $\mathcal{S} = (X, \Delta)$ to a complex $\mathcal{S}' = (X', \Delta')$ is a map between X and X' which takes simplices to simplices. The *star* of a simplex $A \in \Delta$ is the set of subsets $B \in \Delta$ such that $A \subseteq B$. A *covering* is a surjective morphism ϕ from $\mathcal{S}$ to $\mathcal{S}'$ such that for every $A \in \Delta$, the function ϕ maps the star of A bijectively onto the star of $\phi(A)$. A *path* on a complex $\mathcal{S}$ is a finite sequence $x_0, x_1, \ldots, x_n$ of elements of X such that x_{i-1} and x_i are contained in a simplex for all $i = 1, \ldots, n$. The complex $\mathcal{S}$ is *connected*, if every two elements of X can be connected by a path.

2.2.2 Homotopy

The following three operations are called *elementary homotopies* of paths: substituting a subsequence x, x (a *repetition*) by x, or vice versa, substituting a subsequence x, y, x (a *return*) by x, or vice versa, or substituting a subsequence x, y, z, x (a *triangle*) by x or vice versa, provided that x, y, z form a simplex. Two paths are *homotopically equivalent* if they can be obtained from one another in a finite sequence of elementary homotopies. A *cycle*, that is, a path with $x_0 = x_n$, is called *null-homotopic*, if it is homotopically equivalent to the trivial path x_0. The *fundamental group* $\pi_1(\mathcal{S}, x)$, where $x \in X$, is the set of homotopy classes of cycles based at x where the product is defined to be concatenation of cycles. The fundamental group is independent of the choice of the base vertex x inside a fixed connected component, while it may vary for base vertices in distinct connected components. When considering connected complexes only, the coverings of $\mathcal{S}$, taken up to a certain natural equivalence, correspond bijectively to the subgroups of $\pi_1(\mathcal{S}, x)$, cf. [116, §55]. A connected complex $\mathcal{S}$ is called *simply connected*, if each covering $\mathcal{S}' \to \mathcal{S}$ with connected $\mathcal{S}'$ is an isomorphism, or equivalently ([116, §56], [120, Section 2.5]), if $\pi_1(\mathcal{S}, x) = 1$.

2.2.3 Flag complexes and realisations

With every pregeometry $\mathcal{G} = (X, \mathrm{typ}, *)$ one can associate its *flag complex* which is a simplicial complex defined on the set X whose simplices are the flags of $\mathcal{G}$. The flag complex of a pregeometry $\mathcal{G}$ is pure if and only if $\mathcal{G}$ is a geometry. A pregeometry $\mathcal{G}$ is *simply connected*, if such is its flag complex.

For a simplicial complex $\mathcal{S} = (X, \Delta)$ denote by $|\mathcal{S}|$ the set of all functions α from X to the real unit interval I satisfying that the set $\{v \in X \mid \alpha(v) \neq 0\}$ is contained in Δ and that $\sum_{v \in X} \alpha(v) = 1$, i.e., $|\mathcal{S}|$ is obtained from $\mathcal{S}$ via barycentric coordinates. In this survey I consider the weak (coherent) topology on $|\mathcal{S}|$, cf. [120, 3.1.14], and call it the *realisation* of $\mathcal{S}$. With respect to this topology, the fundamental group $\pi_1(\mathcal{S}, x)$ defined combinatorially in Section 2.2.2 coincides with the usual fundamental group defined topologically, see [120, 3.6.17].

2.2.4 Wedges of spheres

Let X and Y be pointed spaces, i.e., topological spaces with distinguished base points x_0 and y_0. Then the *wedge sum* $X \vee Y$ of X and Y is the quotient of the disjoint union $X \sqcup Y$ by the identification $x_0 \sim y_0$, i.e.,

$$X \vee Y := (X \sqcup Y)/\{x_0 \sim y_0\}.$$

In general, if $(X_i)_{i\in I}$ is a family of pointed spaces with base points $(x_i)_{i\in I}$, then the *wedge sum* of this family is given by

$$\bigvee_{i\in I} X_i := \bigsqcup_{i\in I} X_i/\{x_i \sim x_j \mid i,j \in I\}.$$

The wedge sum of a family of spheres of the same dimension n is called a *wedge of spheres* or, if one wants to specify the dimension, a *wedge of n-spheres*.

2.3 Chamber systems

Chamber systems and their interaction with pregeometries and simplicial complexes as introduced in [136] play a crucial role in this survey. Details on chamber systems can also be found in [3, 29, 31, 103, 111, 146]. In this section I sketch the most fundamental information and try to highlight some interaction with the objects introduced before.

2.3.1 Chambers

A *chamber system* $\mathcal{C} = (C, (\sim_i)_{i\in I})$ over a type set I is a set C, called the set of *chambers*, together with equivalence relations $\sim_i$, $i \in I$. For $i \in I$ and chambers $c, d \in \mathcal{C}$, the chambers c and d are called *i-adjacent* if $c \sim_i d$. The chambers c, d are *adjacent* if they are i-adjacent for some $i \in I$.

A chamber system $\mathcal{C}$ is called *thick* if for every $i \in I$ and every chamber $c \in \mathcal{C}$, there are at least three chambers (c and two other chambers) i-adjacent to c; it is called *thin* if for every $i \in I$ and every chamber $c \in \mathcal{C}$, there are exactly two chambers (c and one other chamber) i-adjacent to c.

A *gallery* in $\mathcal{C}$ is a finite sequence $(c_0, c_1, \ldots, c_t)$ such that $c_k \in C$ for all $0 \leq k \leq t$ and such that c_{k-1} is adjacent to c_k for all $1 \leq k \leq t$. The number t is called the *length* of the gallery. The chamber system $\mathcal{C}$ is said to be *connected*, if for any two chambers there exists a gallery joining them.

For $J \subseteq I$, the *J-residue* of a chamber c is the chamber system $\mathcal{R}_J(c) = (R_J(c), (\sim_j)_{j\in J})$ consisting of those chambers of $\mathcal{C}$ that can be connected to c via a gallery using j-adjacencies ($j \in J$) only; such galleries are called *J-galleries*. A J-residue with $|J| = 1$ is called a *panel*.

2.3.2 Chamber systems and pregeometries

If $\mathcal{G}$ is a pregeometry with type set I, then one can construct a chamber system $\mathcal{C} = \mathcal{C}(\mathcal{G})$ over I as follows. The chambers are the flags of $\mathcal{G}$ of type I and two such flags are i-adjacent if and only if they contain the same element of type j for all $j \in I \setminus \{i\}$. A chamber system is called *geometric*, if it can be obtained in this way.

Conversely, if $\mathcal{C}$ is a chamber system over I, the *pregeometry of $\mathcal{C}$* (denoted by $\mathcal{G}(\mathcal{C})$) is the pregeometry over I whose elements of type i are the pairs (x, i) with x an $I\backslash\{i\}$-residue of $\mathcal{C}$ in which two elements (x, k), (y, l) of $\mathcal{G}(\mathcal{C})$ are incident if and only if $x \cap y \neq \emptyset$ in $\mathcal{C}$, cf. [136]. If $\psi_{\mathcal{C}}(c)$, for $c \in \mathcal{C}$, denotes the set of all $I\backslash\{i\}$-residues, $i \in I$, containing c, then the map

$$\mathcal{C} \to \mathcal{C}(\mathcal{G}(\mathcal{C}))\colon c \mapsto \psi_{\mathcal{C}}(c)$$

is a homomorphism of chamber systems, by [31, Proposition 3.5.6].

In general, $\mathcal{G} \ncong \mathcal{G}(\mathcal{C}(\mathcal{G}))$ and $\mathcal{C} \ncong \mathcal{C}(\mathcal{G}(\mathcal{C}))$, see [136, Section 2.2]. However, if $\mathcal{G}$ is residually connected, then $\mathcal{G} \cong \mathcal{G}(\mathcal{C}(\mathcal{G}))$, cf. [31, Section 3.5], [136, Section 2.2]. Moreover, by [31, Theorem 3.5.7], the homomorphism

$$\mathcal{C} \to \mathcal{C}(\mathcal{G}(\mathcal{C}))\colon c \mapsto \psi_{\mathcal{C}}(c)$$

is an isomorphism if and only if for any set $\{(x_i, i) \mid i \in I\}$ (where x_i is an $I\backslash\{i\}$-residue of $\mathcal{C}$) such that $x_i \cap x_j \neq \emptyset$ for all $i, j \in I$, the intersection $\bigcap_{i \in I} x_i$ is non-empty, and for distinct chambers c, d of $\mathcal{C}$ there is some $I\backslash\{i\}$-residue of $\mathcal{C}$ containing c but not d.

2.3.3 Homotopy

The concept of homotopy introduced for simplicial complexes, cf. Section 2.2, can also be defined for chamber systems. Excellent sources are [103, 136].

Let $m \geq 1$ be an integer and let $(\mathcal{C}, (\sim_i)_{i \in I})$ be a chamber system over a set I. Two galleries $G = (c_0, \ldots, c_k)$ and $H = (c'_0, \ldots, c'_{k'})$ are called *elementarily m-homotopic*, if there exist two galleries X, Y and two J-galleries G_0, H_0 for some $J \subset I$ of cardinality at most m such that $G = XG_0Y$, $H = XH_0Y$. Two galleries G, H are said to be *m-homotopic* if there exists a finite sequence $G_0, G_1, \ldots, G_l$ of galleries such that $G_0 = G$, $G_l = H$ and such that G_{k-1} is elementarily m-homotopic to G_k for all $1 \leq k \leq l$. A closed gallery G is called *null-m-homotopic* if it is m-homotopic to the gallery consisting of the initial chamber of G.

The chamber system $\mathcal{C}$ is called *simply m-connected*, if it is connected and if each closed gallery is null-m-homotopic. Given a gallery G, then GG^{-1} is null-m-homotopic. Furthermore, two galleries H, G are m-homotopic if and only if the gallery GH^{-1} is null-m-homotopic.

If $\mathcal{C}$ is a chamber system over a finite set I such that the map $\mathcal{C} \to \mathcal{C}(\mathcal{G}(\mathcal{C}))\colon c \mapsto \psi_{\mathcal{C}}(c)$ from Section 2.3.2 is an isomorphism, then $\mathcal{C}$ is simply $(|I|-1)$-connected if and only if $\mathcal{G}(\mathcal{C})$ is simply connected. For $m < |I| - 1$ it is unknown to me what it means for a geometry, if its chamber system is simply m-connected. Note that there exists a rank four geometry (cf. [125]) for McLaughlin's sporadic simple group McL whose chamber system is simply 2-connected and which admits residues of rank three which are not simply connected.

2.4 Coset pregeometries and reconstruction

2.4.1 Coset pregeometries

Let I be a set, let G be a group and let $(G_i)_{i\in I}$ be a family of subgroups of G. Then

$$\Big(\bigsqcup_{i\in I} G/G_i, *, \mathrm{typ}\Big)$$

with $\mathrm{typ}(gG_i) = i$ and $gG_i * hG_j$ if and only if $gG_i \cap hG_j \neq \emptyset$ is a pregeometry of type I, the *coset pregeometry of G* with respect to $(G_i)_{i\in I}$. The groups G_i are called the *maximal parabolic subgroups* of the coset pregeometry. Since the type function is completely determined by the indices, we denote the coset pregeometry of G with respect to $(G_i)_{i\in I}$ by $((G/G_i)_{i\in I}, *)$. The family $(G_i)_{i\in I}$ forms a chamber of the coset pregeometry, called the *base chamber*. For $J \subseteq I$ define $G_J := \bigcap_{j\in J} G_j$.

2.4.2 Reconstruction

Certainly any coset pregeometry is *incidence-transitive*, i.e., for any two flags c and d with $|\mathrm{typ}(c)| = 2 = |\mathrm{typ}(d)|$ and $\mathrm{typ}(c) = \mathrm{typ}(d)$ there exists an element $g \in G$ that maps c onto d. Indeed, if $gG_i \cap hG_j \neq \emptyset$, then choose $a \in gG_i \cap hG_j$. It follows $aG_i = gG_i$ and $aG_j = hG_j$ and therefore the automorphism a^{-1} maps the incident pair gG_i, hG_j onto the incident pair G_i, G_j. Conversely, any incidence-transitive pregeometry can be described as a coset pregeometry via its parabolic subgroups.

If $\mathcal{G} = (X, *, \mathrm{typ})$ is a pregeometry over I with an incidence-transitive group G of automorphisms of $\mathcal{G}$ and a maximal flag $F = (x_i)_{i\in I}$ of $\mathcal{G}$, then the bijection

$$\big((G/G_{x_i})_{i\in I}, *'\big) \to \mathcal{G} \colon gG_{x_i} \mapsto gx_i$$

is an isomorphism between pregeometries and between G-sets. (Recall here that two actions $\phi\colon G \to \mathrm{Sym}\ M$ and $\phi'\colon G \to \mathrm{Sym}\ M'$ are called *isomorphic*, if there is a bijection $\psi\colon M \to M'$ such that $\psi \circ \phi(g) \circ \psi^{-1} = \phi'(g)$ for each $g \in G$ or, equivalently, $\psi \circ \phi(g) = \phi'(g) \circ \psi$ for all $g \in G$; in this case, we also say that M and M' are *isomorphic G-sets*.) The observation of this isomorphism $\big((G/G_{x_i})_{i\in I}, *'\big) \to \mathcal{G} \colon gG_{x_i} \mapsto gx_i$ goes back to [93] and has been proved formally in [56, 57].

It happens quite frequently that interesting geometries are not incidence-transitive. This is also the case in Phan theory, see e.g. Section 5.1.4, so that often a more general definition of a coset pregeometry is necessary. I refer the reader to [76, 78, 123, 124, 147] for details. The most general concept of reconstruction in this context known to me are complexes of groups as treated in [27, Chapter III.$\mathcal{C}$].

2.5 Geometric covering theory and Tits' Lemma

2.5.1 Amalgams

An *amalgam* $\mathcal{A}$ of groups is a set with a partial operation of multiplication and a collection of subsets $\{G_i\}_{i\in I}$, for some index set I, such that (i) $\mathcal{A} = \bigcup_{i\in I} G_i$, (ii) for each $i \in I$, the restriction of the multiplication to G_i turns G_i into a group, (iii) the product ab is defined if and only if $a, b \in G_i$ for some $i \in I$, and (iv) $G_i \cap G_j$ is a subgroup of G_i and G_j for all $i, j \in I$.

An *enveloping group* of an amalgam $\mathcal{A}$ is a group G together with a mapping ϕ from $\mathcal{A}$ to G such that the restriction of ϕ to every G_i is a homomorphism and $\phi(\mathcal{A})$ generates G. The *universal enveloping group* of $\mathcal{A}$ is isomorphic to the group $U(\mathcal{A})$ with generators $\{t_s \mid s \in \mathcal{A}\}$ and relations $t_x t_y = t_{xy}$ whenever $x, y \in G_i$ for some i; the corresponding mapping is given by $x \mapsto t_x$, see [118, Chapter I, Section 1.1, Proposition 1]. Every enveloping group is isomorphic to a quotient of the universal enveloping group $U(\mathcal{A})$. For more details we refer the reader to [117, 118]. Intransitive geometries may lead to *fused amalgams* as defined and studied in [76]. Again, I also refer to the concept of complexes of groups [27, Chapter III.$\mathcal{C}$].

In some references an enveloping group of an amalgam is called a completion of this amalgam.

2.5.2 The fundamental theorem of geometric covering theory

Many identification problems in group theory amount to finding the universal enveloping groups of certain amalgams arising inside some abstract group, for instance as stabilisers of some group action on some simplicial complex with a fundamental domain. The result that connects such amalgams and their enveloping groups with combinatorial-topological properties of the set acted on is a lemma proved in [102, 137], cf. Section 2.5.3 known as Tits' lemma. It can be obtained as a corollary of the Fundamental Theorem of geometric covering theory discussed in this section.

Suppose $\mathcal{G}$ is a geometry and $G \leq \mathrm{Aut}\,\mathcal{G}$ is an incidence-transitive group of automorphisms. Corresponding to $\mathcal{G}$ and G and some maximal flag F, there is an amalgam $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, F)$, the *amalgam of parabolics with respect to $\mathcal{G}$, G, F*, defined as the family $(G_E)_{\emptyset \neq E \subseteq F}$, where G_E denotes the stabiliser of $\emptyset \neq E \subseteq F$ in G.

In case G is flag-transitive, the amalgam $\mathcal{A}$ is independent up to conjugation of the choice of F. If $\mathcal{G}$ is connected, then $\mathcal{A}$ generates G ([91, Lemma 1.4.2]), so that G is an enveloping group of $\mathcal{A}$. One of the main tools for geometric proofs of group-theoretic identification theorems is the Fundamental Theorem of geometric covering theory, see [92].

Fundamental Theorem of geometric covering theory ([92, Theorem 1.4.5]). *Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a connected geometry over I of rank at least three, and let G be a flag-transitive group of automorphisms of $\mathcal{G}$. Moreover, let F be a maximal flag and let $\mathcal{A} = \mathcal{A}(\mathcal{G}, G, F)$ be the corresponding amalgam of parabolics. Then the coset pregeometry $\widehat{\mathcal{G}} = ((\mathcal{U}(\mathcal{A})/G_x)_{x\in F}, *)$ is a simply connected geometry that admits a covering $\pi\colon \widehat{\mathcal{G}} \to \mathcal{G}$ induced by the natural epimorphism $\mathcal{U}(\mathcal{A}) \to G$. Moreover, $\mathcal{U}(\mathcal{A})$ is of the form $\pi_1(\mathcal{G}).G$, i.e., $\mathcal{U}(\mathcal{A})/\pi_1(\mathcal{G}) \cong G$.*

2.5.3 Tits' Lemma

An immediate consequence of the Fundamental Theorem is Tits' Lemma, cf. [92, Corollary 1.4.6], [102, Lemma 5], [103, Theorem 12.28], [137, Corollary 1].

Tits' Lemma ([92, Corollary 1.4.6]). *Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a connected geometry over I with a flag-transitive group G of automorphisms of $\mathcal{G}$, let F be a maximal flag $\mathcal{G}$, and let $\mathcal{A}(\mathcal{G}, G, F)$ be the corresponding amalgam of parabolics. Then the geometry $\mathcal{G}$ is simply connected if and only if the canonical epimorphism $\mathcal{U}(\mathcal{A}(\mathcal{G}, G, W)) \to G$ is an isomorphism.*

This result reduces the problem of identifying the universal enveloping group of a certain amalgam to proving that the corresponding geometry is simply connected, i.e., proving that the fundamental group of its flag complex is trivial. Geometric covering theory has been extended to certain classes of intransitive geometries, leading to more general concepts of amalgams and different versions of the Fundamental Theorem and Tits' Lemma. I refer the reader to [76, 78] for details. Again also the concept of complexes of groups [27, Chapter III.$\mathcal{C}$] should be mentioned.

2.5.4 Shapes

For a maximal flag F a *shape* is a subset $\mathcal{W}$ of 2^F such that $2^F \ni U' \supset U \in \mathcal{W}$ implies $U' \in \mathcal{W}$, i.e., $\mathcal{W}$ is a subset of the power set of F that is closed under passing to supersets. The *amalgam of shape $\mathcal{W}$ with respect to $\mathcal{G}$, G, F* is the family $(G_U)_{U \in \mathcal{W}}$, where G_U is the stabiliser of $U \in \mathcal{W}$ in G. It is denoted by $\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, F)$. Shapes allow for a neat explanation why many presentations of groups based on amalgams of parabolics are redundant.

Redundancy Theorem ([78, Theorem 3.3]). *Let $\mathcal{G} = (X, *, \mathrm{typ})$ be a geometry over some finite set I, let G be a flag-transitive group of automorphisms of $\mathcal{G}$, and let F be a maximal flag of $\mathcal{G}$. Moreover, let $\mathcal{W} \subseteq 2^F$ be a shape, assume that for each flag $U \in 2^F \backslash \mathcal{W}$ the residue $\mathcal{G}_U$ is simply connected, and let $\mathcal{A}(\mathcal{G}, G, F)$ and $\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, F)$ be the amalgam of maximal parabolics, resp. the amalgam of shape $\mathcal{W}$ of $\mathcal{G}$ with respect to G and F. Then $G = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, F))$ and, if $\emptyset \notin \mathcal{W}$, furthermore $G = \mathcal{U}(\mathcal{A}(\mathcal{G}, G, F)) = \mathcal{U}(\mathcal{A}_{\mathcal{W}}(\mathcal{G}, G, F))$.*

3 Phan's Theorems

3.1 Phan's first theorem

The first of the group-theoretic identification theorems I discuss in this survey is Phan's first theorem. In 1977 Kok-Wee Phan [106] —the namesake of the theory reported on in this survey— described a method of identification of a group G as a quotient of the unitary group $\mathsf{SU}_{n+1}(q^2)$ via a generating configuration consisting of subgroups $\mathsf{SU}_2(q^2)$ and $\mathsf{SU}_3(q^2)$ and $\mathsf{SU}_2(q^2) \times \mathsf{SU}_2(q^2)$ in G.

3.1.1 Phan systems

It is helpful to begin by looking at this configuration of subgroups inside the group $\mathsf{SU}_{n+1}(q^2)$ in order to motivate the forthcoming definitions. For $n \geq 2$ and q a prime power, consider $G = \mathsf{SU}_{n+1}(q^2)$ acting as a matrix group with respect to an orthonormal basis on a unitary $(n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$, and let $U_i \cong \mathsf{SU}_2(q^2)$, $i = 1, 2, \dots, n$, be the subgroups of G corresponding to the (2×2)-blocks along the main diagonal represented as matrix groups with respect to the chosen orthonormal basis. Let T_i be the diagonal subgroup in U_i with respect to this basis, which is a maximal torus of U_i of size $q+1$. For $q \geq 3$ and $1 \leq i, j \leq n$ the subgroups U_i and T_i satisfy the following axioms:

(P1) if $|i - j| > 1$, then $[x, y] = 1$ for all $x \in U_i$ and $y \in U_j$,

(P2) if $|i - j| = 1$, then $\langle U_i, U_j \rangle$ is isomorphic to $\mathsf{SU}_3(q^2)$; moreover $[x, y] = 1$ for all $x \in T_i$ and $y \in T_j$, and

(P3) $G = \langle U_i \mid 1 \leq i \leq n \rangle$.

3.1.2 Phan's Theorem

If G is an arbitrary group containing a system of subgroups $U_i \cong \mathsf{SU}_2(q^2)$ with a particular maximal torus T_i of size $q+1$ chosen in each U_i such that the conditions (P1), (P2), (P3) hold for G, then one says that G admits a *Phan system of type A_n over $\mathbb{F}_{q^2}$*. In [14] this configuration is called a *generating system* of type I, the groups U_i are called *fundamental subgroups*. In that paper the following theorem, Phan's Theorem, is applied to obtain a characterisation of Chevalley groups over finite fields of odd order; note the additions made in [13] and [15].

Phan's Theorem 1 (Phan [106]). *Let $q \geq 5$, let $n \geq 3$, and let G be a group admitting a Phan system of type A_n over $\mathbb{F}_{q^2}$. Then G is isomorphic to a quotient of $\mathsf{SU}_{n+1}(q^2)$.*

My favourite way of proving a result like Phan's Theorem 1 is to translate the statement into an amalgamation problem. This means that one first constructs an abstract amalgam from the Phan system and proves that up to central extensions and isomorphisms any such amalgam is unique. Second one proves that the group admitting the Phan system is a central quotient of the universal enveloping group of the constructed unique amalgam. The first step has been well understood by now, cf. [21, 54], also [100]. Therefore in this survey I will only concern myself with the second step.

3.2 Aschbacher's geometry and its simple connectedness

3.2.1 Weak Phan systems of type A_n

I will describe the proof of a slightly more general statement than Phan's Theorem 1. Following [21], a group G admits a *weak Phan system of type A_n over $\mathbb{F}_{q^2}$*, if G contains subgroups $U_i \cong \mathsf{SU}_2(q^2)$, $i = 1, 2, \ldots, n$, and $U_{i,j}$, $1 \leq i < j \leq n$, so that the following hold:

(wP1) if $|i - j| > 1$, then $[x, y] = 1$ for all $x \in U_i$ and $y \in U_j$,

(wP2) if $|i - j| = 1$, the groups U_i and U_j are contained in $U_{i,j}$, which is isomorphic to a central quotient of $\mathsf{SU}_3(q^2)$; moreover, U_i and U_j form a standard pair (see below) in $U_{i,j}$, and

(wP3) $G = \langle U_{i,j} \mid 1 \leq i < j \leq n \rangle$.

Here a *standard pair* in the matrix group $\mathsf{SU}_3(q^2)$ is a pair of subgroups isomorphic to $\mathsf{SU}_2(q^2)$ conjugate as a pair to the two block-diagonal groups isomorphic to $\mathsf{SU}_2(q^2)$, i.e., these two groups centralise a pair of orthonormal vectors of the natural module of $\mathsf{SU}_3(q^2)$. Standard pairs in central quotients of $\mathsf{SU}_3(q^2)$ are defined as the images under the canonical homomorphism of standard pairs of $\mathsf{SU}_3(q^2)$.

3.2.2 Non-degenerate unitary space

Consider $G \cong \mathsf{SU}_{n+1}(q^2)$ as a matrix group with respect to an orthonormal basis of its natural module and let $\mathcal{A}$ be the amalgam consisting of the block-diagonal subgroups $\mathsf{SU}_2(q^2)$ and $\mathsf{SU}_3(q^2)$ and $\mathsf{SU}_2(q^2) \times \mathsf{SU}_2(q^2)$. One has to prove that the universal enveloping group of the amalgam $\mathcal{A}$ coincides with G. A natural way to show this is via Tits' Lemma, cf. Section 2.5.3, once one knows a geometry with G as a sufficiently transitive group of automorphisms such that $\mathcal{A}$ is related to the amalgam of maximal parabolics induced by the action of G.

Such a geometry $\mathcal{G}_{A_n}$ has been identified in [13, 16, 47] to be an $(n+1)$-dimensional non-degenerate unitary space V over $\mathbb{F}_{q^2}$. The elements of $\mathcal{G}_{A_n}$ are the non-trivial proper non-degenerate subspaces U of V, the type of a space U being its dimension, incidence being defined by symmetrised containment. Using standard terminology from incidence geometry, one-dimensional elements of $\mathcal{G}_{A_n}$ are called *points*, two-dimensional elements *lines*. Fixing an orthonormal basis $e_1, \ldots, e_{n+1}$ of V, we consider the action of G as a matrix group on $\mathcal{G}_{A_n}$ with respect to that basis. By Witt's Theorem, see [115], this action is flag-transitive, so that we can choose an arbitrary flag F in order to describe the amalgam of parabolics.

This amalgam $\mathcal{A}(\mathcal{G}_{A_n}, G, F)$ of parabolics, cf. Section 2.5.2, turns out to have the same universal enveloping group as the amalgam $\mathcal{A}$ consisting of the block-diagonal subgroups $\mathsf{SU}_2(q^2)$ and $\mathsf{SU}_3(q^2)$ and $\mathsf{SU}_2(q^2) \times \mathsf{SU}_2(q^2)$ of G by the Redundancy Theorem from Section 2.5.4 and by [61, Lemma 29.3].

3.2.3 Decomposing cycles

The crucial observation for applying Tits' Lemma (Section 2.5.3) and the Redundancy Theorem (Section 2.5.4) is that $\mathcal{G}_{A_n}$ is almost always simply connected and has many simply connected residues. In [21] this simple connectedness is shown by proving that every cycle of the flag complex of $\mathcal{G}_{A_n}$ is null-homotopic, while in [47] it is proved in odd characteristic by studying certain subgroup complexes of $\mathsf{SU}_{n+1}(q^2)$.

In this survey I will sketch the proof given in [21]. Fixing the base element x to be a point, a standard technique based on residual connectedness allows to reduce every cycle of $\mathcal{G}_{A_n}$ to a cycle in the point-line incidence graph, i.e., the graph on the elements of dimension one and two with incidence as adjacency. Furthermore, every cycle in the point-line incidence graph can be understood as a cycle in the *collinearity graph* Γ of $\mathcal{G}_{A_n}$, i.e., the graph consisting of the points of $\mathcal{G}_{A_n}$ as vertices in which two vertices are adjacent if and only if they lie on a common line of $\mathcal{G}_{A_n}$. A cycle in Γ that is contained entirely within the residue of an element of $\mathcal{G}_{A_n}$ is called *geometric* and, being contained in a cone, is null-homotopic. Thus, simple connectedness of $\mathcal{G}_{A_n}$ follows, if one can prove that every cycle in Γ can be decomposed into a product of geometric cycles.

A key fact exploited in [21] is that up to a few exceptions Γ has diameter two. This implies that every cycle in Γ is a product of cycles of length up to five and, thus, it suffices to show that every cycle of length three, four, and five is null-homotopic. When the dimension is large, one can always find a point that is perpendicular to all points on a fixed cycle, producing a decomposition of that cycle into geometric triangles. Hence proving simple connectedness is more or less trivial for large dimension. The difficulty of the proof lies in the case of small dimension, where [21] resorts to a case-by-case analysis.

To give the precise statement, let $n \geq 3$ and let q be any prime power. Then the geometry $\mathcal{G}_{A_n}$ is simply connected, if (n, q) is not one of $(3, 2)$ and $(3, 3)$. Since neither of these exceptions is simply connected, cf. Section 3.2.5, the result in [21] is optimal.

3.2.4 Phan-type theorem of type A_n

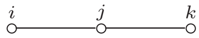
Altogether the Phan-type theorem of type A_n follows:

Phan-type Theorem 1 (Bennett, Shpectorov [21]). *Let q be a prime power, let $n \geq 3$, and let G be a group admitting a weak Phan system of type A_n over $\mathbb{F}_{q^2}$.*

(i) *If $q \geq 4$, then G is isomorphic to a central quotient of $\mathsf{SU}_{n+1}(q^2)$.*

(ii) *If $q = 2$, 3 and $n \geq 4$ and if, furthermore,*

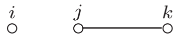
(a) *for any triple i, j, k of nodes of the Dynkin diagram A_n that form a subdiagram*



of type A_3, the subgroup $\langle U_{i,j}, U_{j,k} \rangle$ is isomorphic to a central quotient of $\mathsf{SU}_4(q^2)$;

(b) *in case $q = 2$*

- *for any triple i, j, k of nodes of the Dynkin diagram A_n that form a subdiagram*

  of type $A_1 \oplus A_2$ the groups U_i and $U_{j,k}$ commute elementwise; and
- *for any quadruple i, j, k, l of nodes of the Dynkin diagram Δ that form a subdiagram*

  of type $A_2 \oplus A_2$ the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise;

then G is isomorphic to a central quotient of $\mathsf{SU}_{n+1}(q^2)$.

3.2.5 The group $\mathsf{SU}_4(3^2)$

The extra conditions in Phan-type Theorem 1 (ii) are due to the fact that for small q and n the geometry $\mathcal{G}_{A_n}$ is not simply connected. For example, [86] describes a group H admitting a weak Phan system of type A_3 over $\mathbb{F}_{3^2}$ that is isomorphic to a non-split central extension of $\mathsf{SU}_4(3^2)$ by a group $K \cong (\mathbb{Z}/3\mathbb{Z})^2$, i.e. the sequence $1 \to K \to H \to \mathsf{SU}_4(3^2) \to 1$ is exact and non-split; in fact, H is isomorphic to the Schur cover of $\mathsf{SU}_4(3^2)$. From there it is deduced in [86] that the geometry $\mathcal{G}_{A_3}$ admits a 9-fold universal cover in case $q = 3$.

4 The Curtis-Tits Theorem

Phan's theorems can be considered as a twisted version of the Curtis-Tits Theorem. Therefore by explaining the general setup of Phan-type theorems one naturally also describes a setup of the Curtis-Tits Theorem. In this section I will give many different (sometimes inequivalent) ways how to state the Curtis-Tits Theorem. Some versions deal with determining a Chevalley group (or even a Kac-Moody group) as the universal enveloping group of a certain amalgam, others with characterisations of these groups from purely local data. One version is merely concerned with the simple connectedness of a suitable chamber system. Each version has its advantages and disadvantages. While it may be easier for the geometric group theorist to prove the simple connectedness of some complex, a local group theorist may prefer to apply a version requiring only knowledge about local data. The transition from the former point of view to the latter requires a certain amount of rigidity of the complex on which the group of interest acts. This can be exploited to obtain a classification of amalgams as achieved in [21, 54] or, more ambitiously, to obtain a classification of groups generated by a class of abstract root groups as sketched in Section 4.4.

4.1 The result

4.1.1 Chevalley groups and the Steinberg presentation

Chevalley groups can be defined by their Steinberg presentation, cf. [122, Theorem 8], the approach I decided to take in this survey. For additional background information and terminology see [25, 62, 121, 122].

Let Σ be an indecomposable root system of rank at least two and let $\mathbb{F}$ be a field. Consider the group G generated by the collection of elements

$$\{x_r(t) \mid r \in \Sigma, t \in \mathbb{F}\}$$

subject to the following relations:

(i) $x_r(t)$ is additive in t.

(ii) If r and s are roots and $r + s \neq 0$, then

$$[x_r(t), x_s(u)] = \prod_{h,k>0} x_{hr+ks}(C_{hkrs} t^h u^k)$$

where the product is taken over all $h, k > 0$ such that $hr + ks \in \Sigma$ (if there are no such numbers, then $[x_r(t), x_s(u)] = 1$), and with certain structure constants $C_{hkrs} \in \{\pm 1, \pm 2, \pm 3\}$.

(iii) $h_r(t)$ is multiplicative in t, where $h_r(t)$ equals $w_r(t)w_r(-1)$ and $w_r(t)$ equals $x_r(t)x_{-r}(-t^{-1})x_r(t)$ for $t \in \mathbb{F}^*$.

With the correct choice of the structure constants C_{hkrs} (see for example [62, Theorem 1.12.1], [122]) the group G is called the *universal Steinberg-Chevalley group* constructed from Σ and $\mathbb{F}$. For $r \in \Sigma$ the group

$$X_r = \{x_r(t) \mid t \in \mathbb{F}\} = (\mathbb{F}, +) ,$$

and any conjugate of X_r in G, is called a *root (sub)group*. By [122, Theorem 9], if Σ is an indecomposable root system of rank at least two and $\mathbb{F}$ an algebraic extension of a finite field, then the above relations (i) and (ii) suffice to define the corresponding universal Chevalley group, i.e., they imply the relations (iii).

4.1.2 Redundancy of the Steinberg presentation

The Curtis-Tits Theorem states that Steinberg's presentation of Chevalley groups in Section 4.1.1 is highly redundant and that the amalgam consisting of rank one and rank two subgroups with respect to a system of fundamental roots of a maximal torus of a Chevalley group suffices to present this Chevalley group, cf. [45, 134], [135, Theorem 13.32].

The following version of the Curtis-Tits Theorem refers directly to the Steinberg presentation.

Curtis-Tits Theorem Version 1 (Curtis [45, Corollary 1.8]). *Let Σ be an indecomposable root system of rank at least two, let Π be a fundamental system of Σ, and let $\mathbb{F}$ be an arbitrary field with five distinct elements. Define G to be the abstract group with generators $\{x_r(t) \mid r \in \Sigma, t \in \mathbb{F}\}$ and defining relations*

$$x_r(t)x_r(u) = x_r(t+u), r \in \Sigma, t, u \in \mathbb{F}\,, \tag{4.1}$$

and for independent roots r, s,

$$[x_r(t), x_s(u)] = \prod_{h,k} x_{hr+ks}(C_{hkrs}t^h u^k)\,, \tag{4.2}$$

with $h, k > 0$, $hr + ks \in \Sigma$ (if there are no such numbers, then $[x_r(t), x_s(u)] = 1$), and structure constants $C_{hkrs} \in \{\pm 1, \pm 2, \pm 3\}$.

Let $A = \bigcup A_{ij}$, where A_{ij} is the set of all roots which are linear combinations of the fundamental roots $r_i, r_j \in \Pi$. Let G^* be the abstract group with generators $\{x_r(t) \mid r \in \Sigma, t \in \mathbb{F}\}$ and defining relations (4.1)*, for $r \in A$, and* (4.2) *for independent roots r, s belonging to some A_{ij}.*

Then the natural epimorphism $G^* \to G$ is an isomorphism.

A more compact formulation (albeit without a concrete presentation) can be found in [62, 134], [135, Theorem 13.32]. Generalisations and variations on the theme are contained in [33, 132].

Curtis-Tits Theorem Version 2 (Gorenstein, Lyons, Solomon [62], Tits [135]). *Let G be the universal version of a Chevalley group of (twisted) rank at least three with root system Σ, fundamental system Π, and root groups X_α, $\alpha \in \Sigma$. For each $J \subseteq \Pi$ let G_J be the subgroup of G generated by all root subgroups X_α, $\pm\alpha \in J$.*

Let D be the set of all subsets of Π with at most two elements. Then G is the universal enveloping group of the amalgam $(G_J)_{J\in D}$.

To look at a concrete example, consider the case of the universal Steinberg-Chevalley group of type A_n, which is $G = \mathsf{SL}_{n+1}(\mathbb{F})$. With the usual choices of the root subgroups in G and of a basis of the natural module of G, the subgroups $G_{\alpha,\beta}$ generated by the *fundamental rank one subgroups* $G_\alpha := \langle X_\alpha, X_{-\alpha}\rangle$ and $G_\beta := \langle X_\beta, X_{-\beta}\rangle$ are the block-diagonal subgroups $\mathsf{SL}_3(\mathbb{F})$ and $\mathsf{SL}_2(\mathbb{F})\times\mathsf{SL}_2(\mathbb{F})$.

4.1.3 The Curtis-Tits Theorem, Phan-style

The Curtis-Tits Theorem has been extended to a result including a classification of amalgams by Phan [105] (for $\mathsf{SL}_{n+1}(q)$), by Humphreys [88] (for every finite Chevalley group with a simply laced diagram), and by Dunlap [54] (for every Chevalley group). Phan constructs a BN-pair, cf. Section 4.2.4, from the amalgam he is starting with and consequently recognises his target group as a group with a BN-pair of type A_n. Humphreys [88] gives another proof of the main result of [105] whose central idea is identical to Bennett and Shpectorov's [21] proof of uniqueness of Phan amalgams. After obtaining uniqueness Humphreys [88] simply invokes the Curtis-Tits Theorem. He mentions in passing that Curtis-Tits amalgams can be classified for the non-simply laced spherical diagrams of rank at least three, if one can control the behaviour of the root subgroups of $\mathsf{Sp}_4(q)$. Similarly, Shpectorov mentioned to me that a classification of Phan amalgams can be accomplished as soon as one can control the behaviour of the Phan amalgam in $\mathsf{Sp}_6(q)$. Both observations have been worked out in detail by now, see [54, 69].

I point out here that Timmesfeld has also obtained proofs of the Curtis-Tits Theorem. One approach is also based on the construction of BN-pairs, see [132], while an alternative approach (see [128, 131, 133]) is based on his theory of abstract root subgroups [127, 129, 130]; see Section 4.4.

Curtis-Tits Theorem Version 3 (Phan [105], Humphreys [88], Timmesfeld [132], Dunlap [54])**.** *Let Δ be a spherical Dynkin diagram of rank at least three, let $\mathbb{F}$ be a field, and let G be a group generated by subgroups G_α and $G_{\alpha,\beta}$, for all $\alpha,\beta\in\Delta$, isomorphic to Chevalley groups over $\mathbb{F}$ as indicated by the induced Dynkin diagram on the nodes α, β, with the property that in each $G_{\alpha,\beta}$ the subgroups G_α and G_β correspond to the choice of a fundamental system of roots with respect to a maximal torus of $G_{\alpha,\beta}$. Then G is a central quotient of the universal Chevalley group of type Δ over $\mathbb{F}$.*

Note that a theorem like the Curtis-Tits Theorem Version 3 is much easier to apply in local group theory than the Curtis-Tits Theorem Versions 1 and 2.

4.2 Buildings and twin buildings

4.2.1 Towards a Curtis-Tits geometry

One purpose of this survey is to point out similarities between the Curtis-Tits Theorem on one hand and Phan's theorems on the other hand by describing suitable geometries whose simple connectedness yields the respective group-theoretic identification result via Tits' Lemma [137] (Section 2.5.3). These geometries can be constructed using the opposition relation of a building or a twin building, cf. [4, 98, 139]. I start with a description of the ideas of proof of the Curtis-Tits Theorem given in [4, 98], whose generality actually implies that result for any two-spherical diagram (i.e., each sub-diagram of cardinality at most two is spherical), except that one has to exclude some small cases covered by the original Curtis-Tits Theorem. These exceptions arise from exactly those rank two diagrams and fields for which the geometry opposite a chamber inside the corresponding Moufang polygon is not connected, see [6, 28]. Before I am able to properly explain this geometric approach to the Curtis-Tits Theorem, I need to introduce the concepts of a building, a twin building, and of the opposite geometry.

A Chevalley group G acts on its natural geometry, called a building. Buildings have been developed by Tits in numerous articles since the mid-1950's. The standard reference are Tits' lecture notes [135]. Other references are [3, 29, 81, 111, 115, 146]. For Coxeter groups and root systems see [25, 48, 90].

4.2.2 Coxeter systems

For a *Coxeter matrix* $M = (m_{ij})_{i,j\in I}$ over some finite set I, i.e., a symmetric $|I| \times |I|$-matrix over $\mathbb{N} \cup \{\infty\}$ whose diagonal entries equal to one and whose off-diagonal entries are greater than or equal to two, the *Coxeter diagram* of M is the complete labelled graph with vertex set I and labels m_{ij} on the edge $\{i, j\}$. The cardinality $|I|$ is called the *rank* of the Coxeter diagram. Usually the edges with label 2 are erased, so that it is meaningful to talk about connected or disconnected Coxeter diagrams.

Let (W, S) be the *Coxeter system of type* M, i.e., $S = \{s_i \mid i \in I\}$ is a set and $W = \langle S \mid (s_i s_j)^{m_{ij}} = 1\rangle$ is the quotient of the free group generated by S and subject to the relations given by the Coxeter matrix M. The Coxeter system is *spherical*, if $|W| < \infty$, and *irreducible*, if the Coxeter diagram is connected. Irreducible spherical Coxeter diagrams have been classified, cf. [44].

Using the Bourbaki notation, the irreducible spherical Coxeter systems of rank at least three fall into the families A_n, B_n, C_n, D_n plus the exceptional diagrams E_6, E_7, E_8, F_4, H_3, H_4. If Δ is a Coxeter diagram of type M, a Coxeter system (W, S) of type M is also called a *Coxeter system of type* Δ. For $J \subseteq I$, the pair (W_J, S_J) consisting of $S_J = \{s_i \in S \mid i \in J\}$ and $W_J = \langle S_J \rangle$ is also a Coxeter system satisfying $W_J = \langle S_J \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ by [25, Section IV.1.8, Theorem 2]. The group W of a Coxter system (W, S) is called a *Coxeter group*. It is in general not possible to reconstruct the Coxeter system from the abstract group W, see [18, 37, 40].

4.2.3 Buildings

A *building of type* (W, S) (where (W, S) is a Coxeter system) is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a set and the *distance function* $\delta : \mathcal{C} \times \mathcal{C} \to W$ satisfies the following axioms for $x, y \in \mathcal{C}$ and $w = \delta(x, y)$.

(B1) $w = 1$ if and only if $x = y$,

(B2) if $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws; furthermore if $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$, and

(B3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The group W is called the *Weyl group* of the building $\mathcal{B}$. The building $\mathcal{B}$ is called *spherical*, if its Weyl group W is finite. Given a building $\mathcal{B} = (\mathcal{C}, \delta)$ one can define a chamber system on $\mathcal{C}$ in which two chambers c and d are i-adjacent, in symbols $c \sim_i d$, if and only if $\delta(c, d) = s_i$ or $\delta(c, d) = 1$. The chamber system $(\mathcal{C}, (\sim_i)_{i \in I})$ uniquely determines $\mathcal{B}$, i.e., the i-adjacency relations on $\mathcal{C}$ determine the distance function δ; cf. [136].

In this survey we only consider buildings $\mathcal{B}$ for which the chamber system $\mathcal{C}$ is thick. All thick spherical buildings with a connected Coxeter diagram Δ of rank at least three ($|\Delta|$ is also called the *rank* of the building) are known, e.g., by a local to global approach using the classification of Moufang buildings of rank two, see [141, Chapter 40], also [146]. This local to global approach is possible, because all thick spherical buildings of rank at least three with a connected Coxeter diagram are Moufang (see [135, Addendum]), whence their rank two residues are Moufang. Buildings of rank two are called *generalised polygons*, and are studied —Moufang or not— in [87, 104, 108, 143].

If $\mathcal{B}$ is a building, its chamber system contains a class of thin sub-chamber systems called *apartments*, each of which forms a building of the same type as $\mathcal{B}$. In an apartment Σ, for any $c \in \Sigma$ and $w \in W$, there is a unique chamber $d \in \Sigma$ such that $\delta(c, d) = w$. Every pair of chambers of $\mathcal{C}$ is contained in an apartment, cf. [146, Corollary 8.6]. The chamber system $\mathcal{C}$ defined by a building is always geometric; indeed buildings have the property, cf. Section 2.3.2, that for any set $\{(x_i, i) \mid i \in I\}$, with x_i an $I\backslash\{i\}$-residue of $\mathcal{C}$, such that $x_i \cap x_j \neq \emptyset$ for all

$i, j \in I$, the intersection $\bigcap_{i \in I} x_i$ is non-empty, and that for distinct chambers c, d of $\mathcal{C}$ there is some $I \backslash \{i\}$-residue of $\mathcal{C}$ containing c but not d, see [136, Section 1.6]. The geometry $\mathcal{G}(\mathcal{B}) = \mathcal{G}(\mathcal{C})$ is called the *building geometry*.

In the language of algebraic groups the following examples for buildings can be given, see [135, Theorem 5.2], also [22, Section 6.8] and [58, Section 2]. Starting with a reductive algebraic group G defined over a field $\mathbb{F}$, the *Tits building $\mathcal{G}(G, \mathbb{F})$ of G over $\mathbb{F}$* consists of the simplicial complex whose simplices are indexed by the parabolic $\mathbb{F}$-subgroups of G ordered by the reversed inclusion relation on the parabolic subgroups. The Steinberg functor and Chevalley-Demazure group schemes, see [41, 50, 138], allow to construct a vast amount of groups yielding a rich supply of buildings.

A key property of buildings is the *gate property*, see [97, 111], and [146]: For a chamber $c \in C$ and a J-residue

$$R_J(d) := \{z \in C \mid \delta(d, z) \in W_J\} \subset \mathcal{C} \qquad \text{(cf. Section 2.3.1)}$$

there exists a unique chamber $x \in R_J(d)$ such that for all $y \in R_J(d)$ one has $\delta(c, y) = \delta(c, x)\delta(x, y)$ and, in particular, $l(c, y) = l(c, x) + l(x, y)$, where l denotes the length function of W with respect to the generating system S. This chamber x is called the *projection of c onto $R_J(d)$* and is denoted by $\mathrm{proj}_{R_J(d)}\, c$.

Any building $\mathcal{B}$ (and hence its geometry $\mathcal{G}(\mathcal{B})$) of rank at least three is simply connected. In fact, more is known about the homotopy type of a building.

Solomon-Tits Theorem (Solomon, Tits [119]). *A spherical Tits building of rank n is homotopy equivalent to a wedge of spheres of dimension $n - 1$. A spherical Tits building of rank n over a field of q elements is homotopy equivalent to a wedge of q^m spheres of dimension $n - 1$, where m is the number of positive roots.*

The Solomon-Tits Theorem has numerous applications in representation theory. I refer the interested reader to [89] for an excellent survey and guide to the literature.

4.2.4 Tits systems

Let G be a group and B, N be subgroups of G. The tuple (G, B, N, S) is called a *Tits system*, if the following conditions are satisfied:

(i) G is generated by B and N;

(ii) $H = B \cap N$ is normal in N;

(iii) $W = N/H$ admits a finite system $S = \{w_i \mid i \in I\}$ of generators making (W, S) a Coxeter system;

(iv) For any $w_i \in S$ we have $w_i B w_i^{-1} \neq B$;

(v) For any $w_i \in S$ and all $w \in W$ we have $w_i B w \subseteq (BwB) \cup (Bw_i wB)$.

The pair of subgroups B, N of G is also called a *BN-pair* of G, see [25, 135]. A group G admitting a BN-pair satisfies

$$G = \bigsqcup_{w \in W} BwB.$$

For each $i \in I$ the set $P_i := B \cup Bw_iB$ is a subgroup of G. A Tits system (G, B, N, S) leads to a building whose set of chambers equals G/B and whose distance function

$$\delta \colon G/B \times G/B \to W$$

is given by $\delta(gB, hB) = w$ if and only if $Bg^{-1}hB = BwB$. In the corresponding chamber system gB and hB are i-adjacent if and only if $Bg^{-1}hB \subseteq B \cup Bw_iB$.

4.2.5 Twin buildings

The simple connectedness of a building does not imply the Curtis-Tits Theorem, since the action of a Chevalley group on its building does not yield the correct amalgam. A class of geometries that yields the correct amalgams is best described using twin buildings. Twin buildings are obtained by relating two Tits buildings via a co-distance function, see [2, 3, 101, 113, 114, 140]. Given two buildings $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ and $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ of the same type (W, S), a *co-distance*, also called *twinning*, is a map

$$\delta_* \colon (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$$

such that the following axioms hold where $\epsilon = \pm$, $x \in \mathcal{C}_\epsilon$, $y \in \mathcal{C}_{-\epsilon}$, $w = \delta_*(x, y)$:

(T1) $\delta_*(y, x) = w^{-1}$,

(T2) if $z \in \mathcal{C}_{-\epsilon}$ with $\delta_{-\epsilon}(y, z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$, and

(T3) if $s \in S$, there exists $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s \in S$ and $\delta_*(x, z) = ws$.

A *twin building* of type (W, S) is a triple $(\mathcal{B}_+, \mathcal{B}_-, \delta_*)$, where $\mathcal{B}_+$ and $\mathcal{B}_-$ are buildings of type (W, S) and δ_* is a twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$.

Every spherical twin building can be obtained in a unique way from some building $\mathcal{B} = (\mathcal{C}, \delta)$ of the same type (W, S), cf. [140, Proposition 1]. Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ be a copy of $\mathcal{B}$, let $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be $(\mathcal{C}, w_0 \delta w_0)$, and let δ_* be $w_0 \delta$

on $\mathcal{C}_+ \times \mathcal{C}_-$ and δw_0 on $\mathcal{C}_- \times \mathcal{C}_+$, where w_0 is the longest element of the Weyl group W.

If R is an arbitrary spherical residue of type J in a twin building, then by [112, 4.1] there is a unique chamber $z \in R$ with $(\delta_*(c,z))_{W_J} = \delta_*(c,z)$ in analogy to the gate property of a building. Moreover, by [112, 4.3], for all $y \in R$ we have $\delta_*(c,y) = \delta_*(c,z)\delta_-(z,y)$ and in particular $l_*(c,y) = l_*(c,z) - l(z,y)$. As for buildings, this chamber z is called the *projection* of c onto R and is denoted by $\operatorname{proj}_R c$. Furthermore, if J is a spherical subset of S, then any two J-residues of $\mathcal{B}_\epsilon$ are isomorphic for each $\epsilon \in \{+,-\}$. Additionally, there exists a twin version of the main result in [53], as observed in [52], stating that, if R, Q are spherical residues of a twin building, then $\operatorname{proj}_R Q := \{\operatorname{proj}_R x \mid x \in Q\}$ is a spherical residue contained in R. Moreover, for $R' := \operatorname{proj}_R Q$ and $Q' := \operatorname{proj}_Q R$, the maps $\operatorname{proj}_{R'}^{Q'} := \operatorname{proj}_{R'|Q'}$ from Q' to R' and $\operatorname{proj}_{Q'}^{R'} := \operatorname{proj}_{Q'|R'}$ from R' to Q' are adjacency-preserving bijections inverse to each other.

4.3 The opposite geometry and its simple connectedness

4.3.1 Opposition

The concept of the opposite geometry can be traced to Tits [139]. The opposition relation is an important concept in the theory of buildings and plays a crucial role in [2, 5, 6, 7, 8, 9, 98, 99]. Given a twin building $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$, one can define the chamber system $\operatorname{Opp}(\mathcal{T})$ on the set

$$\{(c_+, c_-) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(c_+, c_-) = 1\}$$

in which $(c_+, c_-) \sim_i (d_+, d_-)$ if and only if $c_+ \sim_i d_+$ and $c_- \sim_i d_-$. Chambers $x \in \mathcal{C}_+$ and $y \in \mathcal{C}_-$ with $\delta_*(x,y) = 1$ are called *opposite*, hence the notation.

Denote the corresponding pregeometry by $\mathcal{G}_{\mathrm{op}}$. For $\mathcal{G}_+$ and $\mathcal{G}_-$ the building geometries that correspond to $\mathcal{B}_+$ and $\mathcal{B}_-$, elements $x_+ \in \mathcal{G}_+$ and $x_- \in \mathcal{G}_-$ of the same type $i \in I$ are called *opposite*, if they are contained in opposite chambers. The elements of the pregeometry $\mathcal{G}_{\mathrm{op}}$ of type i are the pairs (x_+, x_-) of opposite elements of type i. Two pairs (x_+, x_-) and (x'_+, x'_-) are incident in $\mathcal{G}_{\mathrm{op}}$, if x_+ and x'_+ are incident in $\mathcal{G}_+$ and x_- and x'_- are incident in $\mathcal{G}_-$. Clearly, a pair $(c_+, c_-) \in \operatorname{Opp}(\mathcal{T})$ produces a maximal flag in $\mathcal{G}_{\mathrm{op}}$, and it can be shown that every maximal flag is obtained in this way. Hence the pregeometry $\mathcal{G}_{\mathrm{op}}$ is a geometry, called the *opposite geometry*. Moreover, the chamber system $\operatorname{Opp}(\mathcal{T})$ is geometric which follows by a building-theoretic argument proving that the map $c \mapsto \psi_{\mathcal{C}}(c)$ in Section 2.3 is an isomorphism.

4.3.2 Examples of classical opposite geometries

The following examples are descriptions of the opposite geometries for the four classical series of spherical buildings.

Example 1a. Let $\mathbb{F}$ be an arbitrary field and consider the universal Steinberg-Chevalley group $G \cong \mathsf{SL}_{n+1}(\mathbb{F})$ of type A_n over $\mathbb{F}$. It corresponds to the building geometry $\mathcal{G}$ of type A_n, better known as the projective geometry, whose elements of type i, $1 \leq i \leq n$, are the i-dimensional subspaces in an $(n+1)$-dimensional $\mathbb{F}$-vector space V. The geometries $\mathcal{G}_+$ and $\mathcal{G}_-$ are isomorphic to the projective geometry $\mathcal{G}$ and its dual, respectively. The latter is identical to $\mathcal{G}$ except that the types are interchanged by the map $i \mapsto n+1-i$. Elements $x_+ \in \mathcal{G}_+$ and $x_- \in \mathcal{G}_-$ of type i are opposite if they intersect trivially or, equivalently, form a direct sum decomposition $V = x_+ \oplus x_-$, cf. [2, II, §4, Lemma 23]. These decompositions are the elements of $\mathcal{G}_{\text{op}}$, where $x_+ \oplus x_-$ is incident to $x'_+ \oplus x'_-$ if and only if x_ϵ is incident to x'_ϵ for $\epsilon \in \{+,-\}$.

Example 2a. Let $G \cong \mathsf{Spin}_{2n+1}(\mathbb{F})$ be the universal Steinberg-Chevalley group corresponding to the building geometry $\mathcal{G}$ of type B_n. The geometry $\mathcal{G}$ is the geometry of all totally isotropic subspaces of a non-degenerate $(2n+1)$-dimensional orthogonal space V over $\mathbb{F}$. In this case, both $\mathcal{G}_+$ and $\mathcal{G}_-$ are isomorphic to $\mathcal{G}$ and two i-dimensional totally isotropic subspaces x_+ and x_- are opposite if x_- intersects the orthogonal complement of x_+ trivially, i.e., $x_+^\perp \cap x_- = \{0\}$ or, equivalently, $x_+^\perp \oplus x_- = V$, [2, II, §6, Lemma 29]. Such pairs (x_+, x_-) are the elements of $\mathcal{G}_{\text{op}}$, where (x_+, x_-) is incident to (x'_+, x'_-) if and only if x_ϵ is incident to x'_ϵ for $\epsilon \in \{+,-\}$.

Example 3a. Consider the universal Steinberg-Chevalley group $G \cong \mathsf{Sp}_{2n}(\mathbb{F})$ of type C_n. In this case the corresponding building geometry $\mathcal{G}$ is the geometry of all totally isotropic subspaces of a non-degenerate $2n$-dimensional symplectic space V over $\mathbb{F}$. Both $\mathcal{G}_+$ and $\mathcal{G}_-$ are isomorphic to $\mathcal{G}$. Two i-dimensional totally isotropic subspaces x_+ and x_- again are opposite if x_- intersects the orthogonal complement of x_+ trivially, i.e., $x_+^\perp \cap x_- = \{0\}$ or, equivalently, $x_+^\perp \oplus x_- = V$, [2, II, §6, Lemma 29]. The pairs (x_+, x_-) are the elements of $\mathcal{G}_{\text{op}}$, where (x_+, x_-) is incident to (x'_+, x'_-) if and only if x_ϵ is incident to x'_ϵ for $\epsilon \in \{+,-\}$.

Example 4a. Let $G \cong \mathsf{Spin}^+_{2n}(\mathbb{F})$ be the universal Steinberg-Chevalley group of type D_n, to which corresponds the building geometry $\mathcal{G}$ of totally isotropic subspaces of a non-degenerate $2n$-dimensional orthogonal space V over $\mathbb{F}$ of Witt index n. In this case, both $\mathcal{G}_+$ and $\mathcal{G}_-$ are isomorphic to $\mathcal{G}$ up to interchanging the elements of types $n-1$ and n in case n odd. Two totally isotropic subspaces x_+ and x_- of type i are opposite, if x_- intersects the orthogonal complement of x_+ trivially, i.e., $x_+^\perp \cap x_- = \{0\}$ or, equivalently, $x_+^\perp \oplus x_- = V$, [2, II, §7, Lemma 31]. Such pairs (x_+, x_-) are the elements of $\mathcal{G}_{\mathrm{op}}$, where (x_+, x_-) is incident to (x'_+, x'_-) if and only if x_ϵ is incident to x'_ϵ for $\epsilon \in \{+, -\}$.

4.3.3 The Curtis-Tits Theorem via geometric group theory

If a twin building admits a strongly transitive group of automorphisms, i.e., a group acting transitively on the pairs of opposite chambers, the group acts flag-transitively on $\mathcal{G}_{\mathrm{op}}$. In case the acting group is semisimple split algebraic or Kac-Moody, the stabilisers of the elements of a maximal flag of $\mathcal{G}_{\mathrm{op}}$ are products of the type $G_{\Pi\backslash\{\alpha\}}Z(T)$, where T is a maximal torus and Π is a system of fundamental roots with respect to T. This setup together with Tits' Lemma (Section 2.5.3) and the Redundancy Theorem [78, Theorem 3.3] (Section 2.5.4) implies that the Curtis-Tits Theorem stated as in [62] (Section 4.1.2) follows from the following simple-connectedness result.

Curtis-Tits Theorem Version 4 (Abramenko, Mühlherr [4], Mühlherr [98]). *Let $\mathcal{T}$ be a thick twin building with two-spherical diagram of rank at least three such that there is no rank two residue in $\mathcal{B}_+$ or $\mathcal{B}_-$ which is isomorphic to the buildings associated to $B_2(2)$, $G_2(2)$, $G_2(3)$ or ${}^2F_4(2)$. Then $\mathrm{Opp}(\mathcal{T})$ is simply connected.*

The proof of this theorem in [98] is derived directly from the axioms of twin buildings, properties of apartments in buildings, and certain connectedness properties of buildings like their simple connectedness. The exceptions in this approach come from the fact that the geometry opposite to a chamber in an arbitrary Moufang polygon is connected except in the cases $B_2(2)$, $G_2(2)$, $G_2(3)$ or ${}^2F_4(2)$, cf. [6, 28]. Note in passing that in [2, II, §2, Proposition 9] it is shown that there is no hope for general connectedness results in the non-Moufang case. Of course, the Curtis-Tits Theorem for Steinberg-Chevalley groups does not have any exceptions by [45, 62, 131, 134, 135].

In [4] the logic of proof is turned around. The authors prove the combinatorial Curtis-Tits Theorem Version 4 by directly proving the following generalisation of [135, Theorem 13.32]. The key is to construct an RGD system for G, cf. [140], also [109, 1.5], [3, Chapter 8].

Curtis-Tits Theorem Version 5 (Abramenko, Mühlherr [4], Mühlherr [98]). *Let $\mathcal{T}$ be a thick twin building with two-spherical diagram Δ of rank at least three such that there is no rank two residue in $\mathcal{B}_+$ or $\mathcal{B}_-$ which is isomorphic to the buildings associated to $B_2(2)$, $G_2(2)$, $G_2(3)$ or ${}^2F_4(2)$, let G be a group acting transitively on the pairs of opposite chambers of $\mathcal{T}$, and let (c_+, c_-) be a pair of opposite chambers in $\mathcal{T}$. For each $J \subseteq \Delta$ let G_J be the subgroup of G stabilising the J-residue of c_+ and the J-residue of c_-. Let D be the set of all subsets of Δ with* *at most two elements. Then G is the universal enveloping group of the amalgam $(G_J)_{J \in D}$.*

A variation on this theme can also be found in [33]. The classification of amalgams in [54] allows to formulate this Curtis-Tits Theorem in Phan-style, cf. [88, 105], also Section 4.1.3.

4.4 Abstract root subgroups

A completely different and independent approach to the Curtis-Tits Theorem is based on the classification of groups generated by a class of abstract root subgroups [127, 129, 130]. This wonderful classification result makes it possible to prove all sorts of generalisations of Steinberg-presentation-type results and the Curtis-Tits Theorem, see [128, 131, 133]. The case of simply laced diagrams is stated more easily than the general case, cf. [128]. Hence I restrict myself to presenting that case here. The article [131] deals with every spherical diagram.

Curtis-Tits Theorem Version 6 (Timmesfeld [128])**.** *Let Δ be a spherical simply laced diagram of rank at least three and let G be a group generated by subgroups X_i, $i \in \Delta$, satisfying*

(i) *X_i is a perfect central extension of $\mathsf{PSL}_2(\mathbb{F})$, $\mathbb{F}$ a division ring,*

(ii) *in each X_i there exists a non-trivial diagonal subgroup H_i normalising all X_j, $j \in \Delta$,*

(iii) *for $i \neq j$ one of the following holds:*

   (a) *$[X_i, X_j] = 1$;*

   (b) *for $X_{ij} := \langle X_i, X_j \rangle$, the quotient $X_{ij}/Z(X_{ij})$ is isomorphic to $\mathsf{PSL}_3(\mathbb{F})$, where $Z(X_{ij}) \subseteq X'_{ij}$; moreover, the unipotent subgroups of X_i, X_j are mapped onto elation subgroups, corresponding to point-line pairs, of $X_{ij}/Z(X_{ij})$.*

Suppose further that, if $|\mathbb{F}| = 4$, then $|Z(X_{ij})| < 12$ for some connected pair i, j of nodes of Δ.

Then G is a perfect central extension of $\mathsf{PSL}_{n+1}(\mathbb{F})$ ($\mathbb{F}$ a division ring), $\mathsf{PS\Omega}_{2n}(\mathbb{F})$, or the adjoint Steinberg-Chevalley group $E_n(\mathbb{F})$ ($\mathbb{F}$ a commutative field) and there exists a homomorphism mapping the X_i onto the fundamental subgroups. Furthermore, if each X_i is a factor group of $\mathsf{SL}_2(\mathbb{F})$, $\mathbb{F}$ a commutative field, then G is a factor group of the universal Steinberg-Chevalley group of type A_n, D_n, or E_n over $\mathbb{F}$.

I would like to point out that the paper [132] contains another proof of the Curtis-Tits Theorem, one that is independent of the classification of groups generated by a class of abstract root subgroups. Instead it relies on a construction of BN-pairs and can be considered as a direct generalisation of [105]. For a generalisation to Kac-Moody groups see [33].

5 Phan-type theorems for finite Chevalley groups

5.1 From Aschbacher's geometry to the general construction

In this section we discuss how Aschbacher's geometry [13] and its simple connectedness initiated Phan theory.

5.1.1 Non-degenerate unitary space, revisited

Example 1b. Consider the situation of Example 1a from Section 4.3.2, but change the field of definition to $\mathbb{F}_{q^2}$, so that $G \cong \mathsf{SL}_{n+1}(q^2)$. Consider a unitary polarity τ, that is, an involutory isomorphism from $\mathcal{G}$ onto the dual of $\mathcal{G}$ which is defined by a non-degenerate hermitian form Φ on V. The map τ sends every subspace of V to its orthogonal complement with respect to Φ and produces an involutory automorphism of the twin building $\mathcal{T}$ that switches $\mathcal{C}_+$ and $\mathcal{C}_-$ and, thus, $\mathcal{G}_+$ and $\mathcal{G}_-$. It is an automorphism in the sense that it transforms δ_+ into δ_- (and vice versa), and preserves δ_*. Corresponding to τ, there is an automorphism of G, which is also denoted by τ. The group

$$G_\tau = C_G(\tau) \cong \mathsf{SU}_{n+1}(q^2)$$

acts on

$$\mathcal{G}_\tau = \{(x_+, x_-) \in \mathcal{G}_{\mathrm{op}} \mid x_+^\tau = x_-\}\,.$$

The elements of $\mathcal{G}_\tau$ are of the form (x_+, x_-) where $x_- = x_+^\tau = x_+^\perp$ and $V = x_+ \oplus x_- = x_+ \oplus x_+^\perp$, cf. Example 1a in Section 4.3.2. Thus, the mapping $(x_+, x_-) \mapsto x_+$ establishes an isomorphism between $\mathcal{G}_\tau$ and the geometry $\mathcal{G}_{A_n}$ of all proper non-degenerate subspaces of the unitary space (V, Φ). This geometry $\mathcal{G}_{A_n}$ is exactly Aschbacher's geometry from Section 3.2.2.

5.1.2 Flips and Phan involutions

Section 5.1.1 suggests the following general construction introduced in [19]. Let $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building as defined in Section 4.2.5. Then an involutory automorphism τ of $\mathcal{T}$ satisfying

(F1) $\mathcal{C}_+^\tau = \mathcal{C}_-$,

(F2) τ flips the distances, i.e., $\delta_\epsilon(x, y) = \delta_{-\epsilon}(x^\tau, y^\tau)$ for $\epsilon = \pm$, and

(F3) τ preserves the co-distance, i.e., $\delta_*(x, y) = \delta_*(x^\tau, y^\tau)$

is called a *flip*. Notice that by (T1) of Section 4.2.5 the element $\delta_*(x, x^\tau)$ is always an involution.

A flip satisfying the additional condition

(F4) there exists a chamber $c \in \mathcal{C}_\pm$ with $\delta_*(c, c^\tau) = 1$

is called a *Phan involution*.

In case τ is a Phan involution the chamber system $\mathcal{C}_\tau$ whose chambers are pairs (c, c^τ) that belong to $\mathrm{Opp}(\mathcal{T})$, i.e.,

$$\mathcal{C}_\tau = \big\{(c_+, c_-) \in \mathrm{Opp}(\mathcal{T}) \mid \{c_+, c_-\} = \{c_+^\tau, c_-^\tau\}\big\},$$

is called the *flipflop system* of τ. By (F4) the chamber system $\mathcal{C}_\tau$ is non-empty. By [73, 85] very many $\mathcal{C}_\tau$ are geometric, in particular all flipflop systems encountered in this survey. However, it is not known to me whether $\mathcal{C}_\tau$ is geometric in general. For a geometric flipflop system $\mathcal{C}_\tau$ denote by $\mathcal{G}_\tau$ the corresponding geometry, the *flipflop geometry*.

Following [52] one can alternatively define a Phan involution to be a flip of a twin building satisfying

(F4)' $\mathrm{proj}_P\, \tau \neq P$ for each panel P of $\mathcal{T}$

where $\mathrm{proj}_R\, \tau := \{x \in R \mid \mathrm{proj}_R\, \tau(x) = x\}$ for a spherical residue R of $\mathcal{T}$ (Section 4.2.5). It is easily seen that a flip satisfying (F4)' also satisfies (F4). When talking about Phan involutions, we will generally only assume the validity of axioms (F1), (F2), (F3), (F4), unless explicitly stated otherwise.

5.1.3 Flips of spherical twin buildings

For a spherical twin building one can compute the action of τ on the Dynkin diagram of the building, see [66, Section 3.3]. Indeed, by Tits' characterisation each spherical twin building arises from a spherical building $\mathcal{B} = (\mathcal{C}, \delta)$ (cf. [140, Proposition 1] and also Section 4.2.5 of this survey) and we have

$$\delta(c, d) = \delta_+(c, d) = \delta_-(c^\tau, d^\tau) = w_0 \delta(c^\tau, d^\tau) w_0$$

for $c, d \in \mathcal{C}$. Therefore, the flip τ acts on the Dynkin diagram via conjugation with the longest word w_0 of the Weyl group. Hence, if $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ is a spherical twin building, then any adjacency-preserving involution τ that interchanges $\mathcal{B}_+$ and $\mathcal{B}_-$ and maps some chamber onto an opposite chamber is a flip if and only if the induced map $\widehat{\tau}$ on the building $\mathcal{B} = (\mathcal{C}, \delta)$ satisfies $\delta(c, d) = w_0\delta(c^{\widehat{\tau}}, d^{\widehat{\tau}})w_0$ for all chambers $c, d \in \mathcal{C}$ where w_0 is the longest word in the Weyl group W.

5.1.4 Flips and polarities

For a flip τ of a spherical twin building of type A_n considered as the building geometry, i.e., the projective geometry, the induced map $\widehat{\tau}$ (see Section 5.1.3) is an incidence-preserving involution that maps points onto hyperplanes such that for points p, q one has an incidence between p and $q^{\widehat{\tau}}$ if and only if q and $p^{\widehat{\tau}}$ are incident. Hence $\widehat{\tau}$ is a polarity of the projective geometry, cf. [32]. This means, by [32, 135, 144, 145], that $\widehat{\tau}$ is induced by a pseudo-quadratic or an alternating form, if $n \geq 4$, see also [42]. Therefore a flip is a natural generalisation of a polarity, and we are on our way towards a generalisation of Aschbacher's geometry for arbitrary twin buildings.

I mention in passing that flipflop geometries coming from flips inducing non-degenerate symmetric bilinear forms have been studied in [10, 12, 78, 110]. Although a flip inducing a non-degenerate alternating form cannot be a Phan involution, one can still study the geometry of chambers having minimal codistance from their image under that flip. This yields the geometry on hyperbolic lines of a symplectic polar space, which has been studied in contexts different from Phan theory in [46, 68, 82, 83]. In [23] this geometry has finally been investigated from the point of view of Phan theory, yielding interesting presentations of symplectic groups.

5.2 Phan's second theorem and the classical Phan-type theorem

5.2.1 Weak Phan systems of arbitrary spherical type

Let Δ be an irreducible spherical Coxeter diagram of rank at least three. A group G admits a *weak Phan system of type Δ over $\mathbb{F}_{q^2}$*, if G contains subgroups $U_\alpha \cong \mathsf{SL}_2(q^2) \cong \mathsf{SU}_2(q^2)$, $\alpha \in \Delta$, and $U_{\alpha,\beta}$, $\alpha, \beta \in \Delta$, so that the following hold:

(wP1) if $\underset{\alpha}{\circ} \qquad \underset{\beta}{\circ}$, then $[x, y] = 1$ for all $x \in U_\alpha$ and $y \in U_\beta$,

(wP2) $U_{\alpha,\beta} \cong \begin{cases} (\mathsf{P})\mathsf{SU}_3(q^2)\,, & \text{in case } \underset{\alpha}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\beta}{\circ}\,, \\ (\mathsf{P})\mathsf{Sp}_4(q)\,, & \text{in case } \underset{\alpha}{\circ}\!\!=\!\!\!=\!\!\!=\!\!\underset{\beta}{\circ}\,; \end{cases}$

moreover, U_α and U_β form a standard pair (see below) in $U_{\alpha,\beta}$, and

(wP3) $G = \langle U_{\alpha,\beta} \mid \alpha, \beta \in \Delta \rangle$.

For $U_{\alpha,\beta} \in \{\mathsf{SU}_3(q^2), \mathsf{Sp}_4(q)\}$ define $G_{\alpha,\beta} := \mathsf{SL}_3(q^2), \mathsf{Sp}_4(q^2)$ accordingly. A *standard pair* in $U_{\alpha,\beta}$ is a pair of subgroups isomorphic to $\mathsf{SU}_2(q^2) \cong \mathsf{SL}_2(q)$ conjugate as a pair to the intersections $G_\alpha \cap U_{\alpha,\beta}$ and $G_\beta \cap U_{\alpha,\beta}$, where G_α, G_β form a pair of fundamental rank one subgroups of $G_{\alpha,\beta}$ (Section 4.1.2). Standard pairs in central quotients are defined as the images under the canonical homomorphism of standard pairs of the simply connected group. A concrete description of standard pairs of $\mathsf{SU}_3(q^2)$ can be found in Section 3.2.1. For a concrete description of standard pairs of $\mathsf{Sp}_4(q)$ see [72, 74].

5.2.2 Phan-type theorem of type B_n

The analogue of Aschbacher's geometry can be constructed from Example 2a (Section 4.3.2) as Example 1b (Section 5.1.1) has been deduced from Example 1a (Section 4.3.2).

Example 2b. Consider the situation of Example 2a, but with $\mathbb{F} = \mathbb{F}_{q^2}$, let $G \cong \Omega_{2n+1}(q^2)$, i.e. the commutator subgroup of $\mathsf{GO}_{2n+1}(q^2)$, and denote the form on V by $(\cdot,\cdot)$. Since the case of even q will be covered in Section 5.2.4 via the isomorphism $\mathsf{Spin}_{2n+1}(2^e) \cong \mathsf{Sp}_{2n}(2^e)$, it suffices to study the case of q odd. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n, x\}$ be a hyperbolic basis of the orthogonal space V, so that $(e_i, f_j) = \delta_{ij}$, while x with $(x, x) = 1$ is orthogonal to each basis vector except itself. Consider the semilinear transformation τ of V which is the composition of the linear transformation given by the Gram matrix of $(\cdot,\cdot)$ with respect to the above basis and the involutory field automorphism applied to the coordinates.

It can be shown, cf. [20], that τ produces a Phan involution of $\mathcal{T}$. Furthermore, $\mathcal{C}_\tau$ is geometric and $G_\tau \cong \mathsf{SO}_{2n}(q)$ (cf. [20, Proposition 2.10]) acts flag-transitively on the corresponding flipflop geometry $\mathcal{G}_\tau$. The geometry $\mathcal{G}_\tau$ can be described as follows. For $u, v \in V$ let $((u, v)) = (u, v^\tau)$. Then $((\cdot,\cdot))$ is a non-degenerate hermitian form. The flipflop geometry $\mathcal{G}_\tau$ can be identified via $(x_+, x_-) \mapsto x_+$ with the geometry $\mathcal{G}_{B_n}$ of all subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ and, at the same time, non-degenerate with respect to $((\cdot,\cdot))$.

In [20, 75] the simple connectedness of $\mathcal{G}_{B_n}$ is proved, leading to the following result.

Phan-type Theorem 2 (Bennett, Gramlich, Hoffman, Shpectorov [20], Gramlich, Horn, Nickel [75])**.** *Let q be an odd prime power, let $n \geq 3$, and let G be a group admitting a weak Phan system of type B_n over $\mathbb{F}_{q^2}$.*

(i) *If $q \geq 5$, then G is isomorphic to a quotient of $\mathsf{Spin}(2n+1, q)$.*

(ii) *For $n \geq 4$, let G be a group admitting a weak Phan system of type B_n over $\mathbb{F}_9$. In addition, assume that $\langle U_{i-1}, U_i, U_{i+1}\rangle$ is isomorphic to a central quotient of $\mathsf{SU}(4, 9)$ (if $2 \leq i \leq n-2$) or $\mathsf{Spin}(7, 3)$ (if $i = n-1$). Then G is isomorphic to $\mathsf{Spin}(2n+1, 3)$ or a central quotient thereof.*

5.2.3 The group $\Omega(7, 3)$

In [75] a group H admitting a weak Phan system of type B_3 over $\mathbb{F}_{3^2}$ is constructed which is a 2187-fold extension of $\Omega_7(3)$. To be precise, see [75], the group H is isomorphic to a non-split extension of $\Omega(7, 3)$ by $K := (\mathbb{Z}/3\mathbb{Z})^7$, i.e. the sequence $1 \to K \to H \to \Omega(7, 3) \to 1$ is exact and non-split. This extension of $\Omega_7(3)$ has been studied in [55, 94, 95]. Altogether we can conclude that for $q = 3$ the geometry $\mathcal{G}_{B_3}$ admits a 2187-fold covering, whence is not simply connected. It is shown in [75] that this covering is universal.

5.2.4 Phan-type theorem of type C_n

The geometry needed to prove the Phan-type theorem of type C_n looks very much like the one of type B_n.

Example 3b. Consider the situation of Example 3a, but with the field of definition of order q^2, so that $G \cong \mathsf{Sp}_{2n}(q^2)$. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a hyperbolic basis of the symplectic space V, i.e., $(e_i, f_j) = \delta_{ij}$, where $(\cdot, \cdot)$ is the alternating form on V, and consider the semilinear transformation τ of V which is the composition of the linear transformation given by the Gram matrix of $(\cdot, \cdot)$ with respect to the above basis and the involutory field automorphism applied to the coordinates.

It can be shown, cf. [72], that τ produces a Phan involution of $\mathcal{T}$. Furthermore, $\mathcal{C}_\tau$ is geometric and $G_\tau \cong \mathsf{Sp}_{2n}(q)$ acts flag-transitively on the corresponding flipflop geometry $\mathcal{G}_\tau$. By [72] the geometry $\mathcal{G}_\tau$ has the following alternative description. For $u, v \in V$ let $((u, v)) = (u, v^\tau)$, so that $((\cdot, \cdot))$ is a non-degenerate hermitian form. The flipflop geometry $\mathcal{G}_\tau$ can be identified via $(x_+, x_-) \mapsto x_+$

with the geometry $\mathcal{G}_{C_n}$ of all subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ and, at the same time, non-degenerate with respect to $((\cdot,\cdot))$.

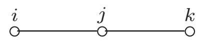
By [72] (with the missing cases dealt with in [74, 84]) the geometry $\mathcal{G}_\tau$ is almost always simply connected, resulting in the following Phan-type theorem.

Phan-type Theorem 3 (Gramlich, Hoffman, Shpectorov [72], Gramlich, Horn, Nickel [74], Horn [84]). *Let q be a prime power, let $n \geq 3$, and let G be a group admitting a weak Phan system of type C_n over $\mathbb{F}_{q^2}$.*

(i) *If $q \geq 3$, then G is isomorphic to a central quotient of $\mathsf{Sp}_{2n}(q)$.*

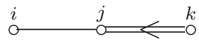
(ii) *If $q = 2$ and $n \geq 4$ and if*

(a) *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*



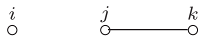
of type A_3, the subgroup $\langle U_{i,j}, U_{j,k}\rangle$ is isomorphic to a central quotient of $\mathsf{SU}_4(2^2)$;

(b) *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*



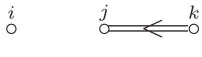
of type C_3, the subgroup $\langle U_{i,j}, U_{j,k}\rangle$ is isomorphic to a central quotient of $\mathsf{Sp}_6(2)$;

(c)
- *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*



of type $A_1 \oplus A_2$, the groups U_i and $U_{j,k}$ commute elementwise; and
- *for any quadruple of nodes of the Dynkin diagram C_n that form a subdiagram*

of type $A_2 \oplus A_2$, the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise; and
- *for any triple i, j, k of nodes of the Dynkin diagram C_n that form a subdiagram*



of type $A_1 \oplus C_2$, the groups U_i and $U_{j,k}$ commute elementwise; and

- *for any quadruple of nodes of the Dynkin diagram C_n that form a subdiagram*

of type $A_2 \oplus C_2$, the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise;

then G is isomorphic to a central quotient of $\mathsf{Sp}_{2n}(2)$.

Unlike the cases A_3 and B_3, the geometry $\mathcal{G}_{C_3}$ is simply connected even for $q = 3$, cf. [74, 84].

5.2.5 Phan's second theorem and Phan-type theorem of type D_n

The last series D_n of finite classical groups, the even-dimensional orthogonal groups, again belongs to a simply laced diagram, which has already been treated in [107]. The arguments in [107] are based on the construction of a presentation that identifies the target group as an orthogonal group via [148]. Here is the main result of [107] concerning D_n.

Phan's Theorem 2 ([107])**.** *Let $q \geq 5$ be odd and let $n \geq 4$. If G admits a Phan system of type D_n over $\mathbb{F}_{q^2}$, then G is isomorphic to a factor group of* $\mathsf{Spin}^+_{2n}(q^2)$, *if n is even, and isomorphic to a factor group of* $\mathsf{Spin}^-_{2n}(q^2)$, *if n is odd.*

This result has been revised in [71] using the following geometry.

Example 4b. Consider the situation as in Example 4a, but over the field $\mathbb{F}_{q^2}$, and let $G = \Omega^+_{2n}(q^2)$. For sake of simplicity of the exposition we assume here that q is odd, although in [71] also the case of even characteristic is dealt with. The Phan involution τ can again be defined as the composition of the linear transformation given by the Gram matrix of the bilinear form $(\cdot, \cdot)$ with respect to a hyperbolic basis and coordinate-wise application of the involutory field automorphism. This τ produces a flipflop geometry on which $G_\tau \cong \Omega^\pm_{2n}(q)$ acts flag-transitively, cf. [71, Proposition 3.10]. The geometry $\mathcal{G}_\tau$ can be described as follows. For $u, v \in V$ let $((u, v)) = (u, v^\tau)$, where $(\cdot, \cdot)$ is the orthogonal form on V, so that $((\cdot, \cdot))$ is a non-degenerate hermitian form. The flipflop geometry $\mathcal{G}_\tau$ can be identified via $(x_+, x_-) \mapsto x_+$ with the geometry $\mathcal{G}_{D_n}$ of all subspaces of V which are totally isotropic with respect to $(\cdot, \cdot)$ and, at the same time, non-degenerate with respect to $((\cdot, \cdot))$. See [71] for more details and a description of the geometry for even q.

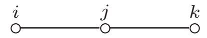
Phan-type Theorem 4 (Gramlich, Hoffman, Nickel, Shpectorov [71])**.** *Let q be a prime power, let $n \geq 3$, and let G be a group admitting a weak Phan system of type D_n over $\mathbb{F}_{q^2}$.*

(i) *If $q \geq 4$, then G is isomorphic to a central quotient of*

- $\mathsf{Spin}^{+}_{2n}(q)$*, if n even; and*
- $\mathsf{Spin}^{-}_{2n}(q)$*, if n odd.*

(ii) *If $q = 2$, 3 and $n \geq 4$ and if, furthermore,*

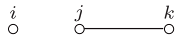
(a) *for any triple i, j, k of nodes of the Dynkin diagram D_n that form a subdiagram*



of type A_3, the subgroup $\langle U_{i,j}, U_{j,k}\rangle$ is isomorphic to a central quotient of $\mathsf{SU}_4(q^2)$;

(b) *in case $q = 2$*

- *for any triple i, j, k of nodes of the Dynkin diagram D_n that form a subdiagram*



of type $A_1 \oplus A_2$ the groups U_i and $U_{j,k}$ commute elementwise; and

- *for any quadruple i, j, k, l of nodes of the Dynkin diagram Δ that form a subdiagram*

of type $A_2 \oplus A_2$ the groups $U_{i,j}$ and $U_{k,l}$ commute elementwise;

then G is isomorphic to a central quotient of

- $\mathsf{Spin}^{+}_{2n}(q)$*, if n even; and*
- $\mathsf{Spin}^{-}_{2n}(q)$*, if n odd.*

5.3 The Devillers-Mühlherr filtration

5.3.1 Filtrations of chamber systems

Lacking concrete easy models of the flipflop geometries of exceptional type some extra theory is necessary in order to be able to extend the Phan-type theorems to exceptional type groups.

Such a theory has been developed in [52]: A *filtration* of a chamber system $\mathcal{C} = (C, (\sim_i)_{i\in I})$ over I is a family $\mathcal{F} = (C_n)_{n\in\mathbb{N}}$ of subsets of C such that

(i) $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$,

(ii) $\bigcup_{n\in\mathbb{N}} C_n = C$, and

(iii) for each $n > 0$ with $C_{n-1} \neq \emptyset$ there exists $i \in I$ such that for each chamber $c \in C_n$ there exists a chamber $c' \in C_{n-1}$ which is i-adjacent to c.

A filtration $\mathcal{F} = (C_n)_{n\in\mathbb{N}}$ is called *residual*, if for each $\emptyset \neq J \subset I$ and each $c \in C$ the family $(C_n \cap R_J(c))_{n\in\mathbb{N}}$ is a filtration of the chamber system $\mathcal{R}_J(c) = (R_J(c), (\sim_j)_{j\in J})$. For $x \in C$ define $|x| := \min\{\lambda \in \mathbb{N} \mid x \in C_\lambda\}$ and, for $X \subseteq C$, define $|X| := \min\{|x| \mid x \in X\}$ and $\mathrm{aff}(X) := \{x \in X \mid |x| = |X|\}$. Note that $\mathrm{aff}(C) = C_m$ where $m = \min\{n \in \mathbb{N} \mid C_n \neq \emptyset\}$.

5.3.2 Filtering flipflop systems inside buildings

Let $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building of type (W, S) with a flip τ satisfying axiom (F4)' from Section 5.1.2. Then there exists a filtration

$$\mathcal{F}_\tau = (C_n)_{n\in\mathbb{N}}$$

of the building $\mathcal{B}_+$ so that C_0 equals the set of chambers of the flipflop system $\mathcal{C}_\tau$ defined as follows.

For a residue R of $\mathcal{B}_+$ put $l_*(\tau, R) := \min\{l(\delta_*(x, \tau(x))) \mid x \in R\}$ and $A_\tau(R) := \{x \in R \mid l(\delta_*(x, \tau(x))) = l_*(\tau, R)\}$, where l denotes the length function of the group W with respect to the generating set S. Since S is finite, there exists an injective map $\mathrm{Inv}(W) \to \mathbb{N}\colon x \mapsto |x|$ from the involutions of W to the non-negative integers with $|1_W| = 0$ such that $l(x) < l(y)$ implies $|x| < |y|$. Defining

$$C_n := \{c \in C_+ \mid |\delta_*(c, \tau(c))| \leq n\},$$

the family $\mathcal{F}_\tau = (C_n)_{n\in\mathbb{N}}$ is a residual filtration of $\mathcal{C}(\mathcal{B}_+)$ by [52].

5.3.3 A criterion for simple connectedness of a flipflop system

The setup from Section 5.3.2 and the simple 2-connectedness of buildings (cf. the Solomon-Tits Theorem in Section 4.2.3) yield the following criterion of simple connectedness of flipflop systems established in [52].

If τ is a flip satisfying axiom (F4)' of a three-spherical twin building $\mathcal{T} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ of finite rank (i.e., a twin building of finite rank whose residues of rank three are spherical) such that

(i) the chamber system $(A_\tau(R), (\sim_t)_{t\in J})$ is connected for each J-residue R of rank two, and

(ii) the chamber system $(A_\tau(R), (\sim_t)_{t\in J})$ is simply 2-connected for each J-residue R of rank three,

then the flipflop system $\mathcal{C}_\tau$ is simply 2-connected in the sense of 2.3.3.

5.4 Wedges of spheres and the Abels-Abramenko filtration

5.4.1 Generalised flipflop geometries of type A_n

In view of Section 5.3.3 it remains to study the chamber systems

$$(A_\tau(R), (\sim_t)_{t\in J})$$

for residues R of rank two and three in order to prove the simple connectedness of the exceptional flipflop geometries. In case the diagram of the twin building $\mathcal{T}$ is simply laced, these chambers systems can be described by so-called generalised flipflop geometries of type A_n, defined in this section, cf. [24, 51, 79].

Two subspaces A and B of a vector space V are *opposite* when $V = A \oplus B$. A subspace A is *transversal* or *in general position* to a flag F, i.e., a chain of incident subspaces of V, if for any subspace B of F we have $A \cap B = \{0\}$ or $V = A + B$. In other words, A is transversal to F, in symbols $A \pitchfork_V F$, if and only if there is a subspace C of V incident with F such that A and C are opposite.

For a field $\mathbb{F}$ with an involution σ and an $(n+1)$-dimensional $\mathbb{F}$-vector space V containing a flag F equal to $0 = V_0 \lneqq V_1 \lneqq \cdots \lneqq V_t \lneqq V_{t+1} = V$ of subspaces of V endowed with σ-hermitian forms $\omega_i \colon V_{i+1} \times V_{i+1} \to \mathbb{F}$, $0 \le i \le t$, satisfying $\mathrm{Rad}\,(\omega_i) = V_i$, the *generalised flipflop geometry of type* A_n (modelled in V with respect to the flag F and the forms ω_i) consists of all proper non-trivial vector subspaces U of V transversal to F with $U \cap V_{k_U+1}$ non-degenerate with respect to ω_{k_U} where $k_U = \min\bigl\{i \in \{0,\ldots,t\} \mid U \cap V_{i+1} \neq \{0\}\bigr\}$.

In the simply laced three-spherical case over $\mathbb{F} = \mathbb{F}_{q^2}$ a geometry arising from a chamber system $(A_\tau(R), (\sim_t)_{t\in J})$, $|J| \in \{2,3\}$ (defined in Section 5.3.2), is isomorphic to a generalised flipflop geometry for $n \in \{2,3\}$ by [24], [79, Proposition 6.6].

For $t = 0$ and $\mathbb{F} = \mathbb{F}_{q^2}$, the generalised flipflop geometry on V equals Aschbacher's geometry on V, i.e., the flipflop geometry of type A_n over $\mathbb{F}_{q^2}$, cf. [19, 21] and Section 5.1.1.

For $t = n$, the generalised flipflop geometry on V equals the geometry opposite the chamber F. This follows from the fact that each ω_i has rank one with radical V_i. Therefore any vector $v \in V_{i+1} \backslash V_i$ is non-degenerate with respect to ω_i, so that any subspace U of V with $U \oplus V_i = V$ intersects V_{i+1} in a non-degenerate (with respect to ω_i) one-dimensional subspace.

5.4.2 A Solomon-Tits-type theorem

It turns out that generalised flipflop geometries of type A_n are not only a useful tool in order to prove Phan-type theorems for groups with simply laced diagrams, but are also interesting in their own right. Indeed, via the Abels-Abramenko filtration [1] it can be shown that a generalised flipflop geometry $\mathcal{G}$ of type A_n is homotopy equivalent to a wedge of $(n-1)$-spheres provided the field $\mathbb{F}$ contains sufficiently many elements.

In order to describe this filtration let p be a one-dimensional subspace of V which is non-degenerate with respect to the hermitian form ω_t and define $Y_0 := \{W \in \mathcal{G} \mid \langle p, W \rangle \in \mathcal{G}\}$ and $Y_i := Y_{i-1} \cup \{W \in \mathcal{G} \mid \dim W = n+1-i\}$ for $1 \leq i \leq n$. The strategy from [1] can be transferred literally to obtain the following Solomon-Tits-type result, which (similar to what is surveyed in [89]) gives rise to a representation of $\mathsf{SU}_{n+1}(\mathbb{F}_{q^2})$ on the integral homology group $H_{r-1}(\mathcal{G})$ tensored with $\mathbb{Q}$, which may be an interesting object to study.

Solomon-Tits-type Theorem 1 (Devillers, Gramlich, Mühlherr [51]). *Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{F}$ with an involution, let $(\mathcal{G}_j)_{1 \leq j \leq m}$ be a finite family of generalised flipflop geometries of type A_n modelled in V, and let $\mathcal{G} = \bigcap_j \mathcal{G}_j$. In case $\mathbb{F} = \mathbb{F}_{q^2}$ assume $2^{n-1}(q+1)m < q^2$. Then $|\mathcal{G}|$ is homotopy equivalent to a wedge of $(n-1)$-spheres.*

Notice in passing that this result once again proves simple connectedness of Aschbacher's geometry, at least for large fields.

Moreover, this result can be used to deduce finiteness properties of the group $\mathsf{SU}_{n+1}(\mathbb{F}_{q^2}[t, t^{-1}], \theta)$ in the spirit of [2], [3, Chapter 13], where θ is the involution of $\mathsf{SL}_{n+1}(\mathbb{F}_{q^2}[t, t^{-1}])$ which acts as the Chevalley involution on SL_{n+1}, as the Frobenius involution on $\mathbb{F}_{q^2}$, and interchanges t and t^{-1}. In fact, this group $\mathsf{SU}_{n+1}(\mathbb{F}_{q^2}[t, t^{-1}], \theta)$ is a lattice in $\mathsf{SL}_{n+1}(\mathbb{F}_{q^2}((t)))$ and in $\mathsf{SL}_{n+1}(\mathbb{F}_{q^2}((t^{-1})))$, cf. [77], whence an arithmetic group by [96, Chapter IX]. See [24] for a concrete description of the group $\mathsf{SU}_{n+1}(\mathbb{F}_{q^2}[t, t^{-1}], \theta)$ and related groups.

5.5 Phan's third theorem and the Phan-type theorem of type E_n

5.5.1 Phan's Theorem

The article [107] also contains a theorem concerning the diagrams E_6, E_7, and E_8. Phan's Theorem 2 (Section 5.2.5) plus [106] are used in order to construct a system of subgroups satisfying the hypotheses of the Curtis-Tits Theorem Version 3, which then is invoked.

Phan's Theorem 3 (Phan [107])**.** *Let $q \geq 5$ be odd. If G admits a Phan system of type E_6, E_7, or E_8 over $\mathbb{F}_{q^2}$, then G is isomorphic to a factor group of the universal Chevalley group ${}^2E_6(q^2)$, $E_7(q)$, or $E_8(q)$, respectively.*

5.5.2 Exploiting the filtrations

By the Solomon-Tits-type Theorem 1 (Section 5.4.2) a generalised flipflop geometry of type A_3 over $\mathbb{F}_{q^2}$ is simply connected, provided $2^2(q+1) < q^2$, which is the case for $q \geq 5$, while a generalised flipflop geometry of type A_2 over $\mathbb{F}_{q^2}$ is connected, if $2(q+1) < q^2$, which is the case for $q \geq 3$. Together with the criterion for simple connectedness of a flipflop system from [52] (Section 5.3.3) this implies that the flipflop geometries of type E_6, E_7, E_8 over $\mathbb{F}_{q^2}$ are simply connected provided $q \geq 5$.

For completeness I should point out here that the chamber systems

$$(A_\tau(R), (\sim_t)_{t\in J})$$

for residues J of type $A_1 \oplus A_1$, $A_1 \oplus A_2$, $A_1 \oplus A_1 \oplus A_1$ are automatically (simply) connected by the following standard argument. Assuming that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ with $\mathcal{G}_1$ connected of rank at least two and $\mathcal{G}_2$ non-empty, the geometry $\mathcal{G}$ is simply connected. Indeed, the geometry $\mathcal{G}$ is certainly connected, and choosing a base point $x \in \mathcal{G}_1$ one can prove that any cycle originating at x is homotopic to a cycle fully contained in $\mathcal{G}_1$. Such a cycle then is null homotopic because it forms a cone together with any element $z \in \mathcal{G}_2$.

5.5.3 The Phan-type theorem of type E_n

Alternatively —and this had already been done by Hoffman, Mühlherr, Shpectorov and the author roughly one year before the Solomon-Tits-type Theorem 1 was proved— one can directly compute the fundamental group for generalised flipflop geometries of type A_3. It turns out that via direct computation it is possible to show that the fundamental groups are trivial for $q \geq 4$. Together with the classification of amalgams [21, 54] and the criterion for simple connectedness of a flipflop system in [52] (Section 5.3.3) this implies the following Phan-type theorem.

Phan-type Theorem 5 (Gramlich, Hoffman, Mühlherr, Shpectorov 2005). *Let $q \geq 4$ be a prime power and let G be a group containing a weak Phan system of type E_6, E_7, or E_8 over $\mathbb{F}_{q^2}$. Then G is isomorphic to a group of type ${}^2E_6(q^2)$, $E_7(q)$, or $E_8(q)$.*

5.6 The Abramenko filtration and the Phan-type theorem of type F_4

5.6.1 Generalised flipflop geometries of type B_n and C_n

The criterion from Section 5.3.3 allows three-spherical diagrams. In view of the method of proof of the Phan-type theorem of type E_n via generalised flipflop geometries of type A_3 it is natural to ask for the definition of generalised flipflop geometries of type B_3 and C_3.

Let V be a vector space over a field $\mathbb{F}$ with an involution σ. For $U, W \leq V$, we say that U is *transversal* to W and write $U \pitchfork W$, if $U \cap W = 0$ or $\langle U, W \rangle = V$. Note that $U \pitchfork W$ if and only if $\dim(U \cap W) = \max\{0, \dim U + \dim W - \dim V\}$. For a flag $F = (0 = V_0 \leq \ldots \leq V_k = V)$ and a subspace $U \leq V$ we say that U is *transversal* to F and write $U \pitchfork F$, if $U \pitchfork V_i$ for $0 \leq i \leq k$. This is the case if and only if $\langle U, V_{k_U} \rangle = V$ where $k_U = \min\{i \mid U \cap V_j \neq \{0\}\}$.

Given a flag $F = (0 = V_0 \leq \cdots \leq V_k = V)$ we call a family $(\omega_i)_{1 \leq i \leq k}$ of σ-hermitian forms $\omega_i \colon V_i \times V_i \to \mathbb{F}$ *compatible with* F if $\mathrm{Rad}(\omega_i) = V_{i-1}$.

Let F be as above and let $\omega = (\omega_i)_i$ be a family of compatible σ-hermitian forms. For $U \leq V$ we say that U is *transversal* to (F, ω), if U is transversal to F and $U \cap V_{k_U}$ is ω_{k_U}-non-degenerate. In this case we write $U \pitchfork (F, \omega)$.

Let Δ be the building geometry of type $B_n(\mathbb{F})$ or $C_n(\mathbb{F})$ embedded in a $\mathbb{F}$-vector space V of dimension $2n+1$, resp. $2n$, and let $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, x be a standard hyperbolic basis of V (where the vector x, of course, only occurs in case B_n). Let $F = (0 = V_0 \leq \cdots \leq V_k = V)$ be a flag satisfying $F^\perp = F$. Let ω be a family of σ-hermitian forms compatible with F and assume that there is an ω_k-non-isotropic vector that is $(\cdot, \cdot)$-isotropic. The *generalised flipflop geometry* of type Δ over $\mathbb{F}$ defined by (F, ω) consists of all subspaces U of V that are totally $(\cdot, \cdot)$-isotropic and transversal to (F, ω).

A closer look reveals that half of the forms ω_i actually do not play any role, because a totally isotropic subspace U that is transversal to F cannot meet any of the V_i with $\dim V_i \leq n$. However, taking this into account would not simplify anything, but would instead make the definition of a generalised Phan geometry even more cumbersome.

5.6.2 Another Solomon-Tits-type theorem and the Phan-type theorem of type F_4

The concept of transversality introducted in Section 5.6.1 makes the Abramenko filtration from [2] accessible. This filtration has been used in [79] in order to prove the following theorem.

Solomon-Tits-type Theorem 2 (Gramlich, Witzel [79]). *Let $\mathbb{F}$ be a field with an involution σ, let $(\mathcal{G}_j)_{1\leq j\leq m}$ be a finite family of generalised flipflop geometries of type B_n or C_n embedded in some $(2n+1)$- or $2n$-dimensional $\mathbb{F}$-vector space V, and let $\mathcal{G} = \bigcap_j \mathcal{G}_j$. In case $\mathbb{F} = \mathbb{F}_{q^2}$ assume $4^{n-1}(q+1)m < q^2$. Then $|\mathcal{G}|$ is homotopy equivalent to a wedge of $(n-1)$-spheres.*

Similar to the case A_3, Hoffman, Mühlherr, Shpectorov and the author proved by direct computation that generalised flipflop geometries of type B_3 or C_3 are simply connected provided the underlying field contains at least 13 elements. Again using the simple connectedness criterion from [52] (Section 5.3.3), the final Phan-type theorem follows. Note that the generalised flipflop geometries are the correct objects in order to describe the chamber system $(A_\tau(R), (\sim_t)_{t\in J})$ by [79, Proposition 6.6].

Phan-type Theorem 6 (Gramlich, Hoffman, Mühlherr, Shpectorov 2007). *Let $q \geq 13$ be a prime power and let G be a group containing a weak Phan system of type F_4 over $\mathbb{F}_{q^2}$. Then G is isomorphic to a group of type $F_4(q)$.*

It remains to study the cases of small q.

6 Statement of the Phan-type theorem over finite fields

We have reached one of the main purposes of this survey, the statement of the Phan-type theorem over finite fields. From Section 5.1.3 we know for which groups of Lie type the Phan-type theorem can make a statement, namely ${}^2A_n$, B_n, C_n, D_{2n}, ${}^2D_{2n+1}$, ${}^2E_6$, E_7, E_8, F_4.

The Phan-type theorem for finite fields. *Let $q \geq 3$, let Δ be a spherical Dynkin diagram of rank at least three, and let G be a group with a weak Phan system of type Δ over $\mathbb{F}_{q^2}$. Then G is isomorphic to a quotient of*

- $\mathsf{SU}_{n+1}(q^2)$, *if* $\Delta = A_n$ *and* $q \geq 4$
(Bennett, Shpectorov [21], Phan [106])*;*
- $\mathsf{Spin}_{2n+1}(q)$, *if* $\Delta = B_n$ *and* $q \geq 4$
(Bennett, Gramlich, Hoffman, Shpectorov [20], Gramlich, Horn, Nickel [75])*;*
- $\mathsf{Sp}_{2n}(q)$, *if* $\Delta = C_n$
(Gramlich, Hoffman, Shpectorov [72], Gramlich, Horn, Nickel [74], Horn [84])*;*
- $\mathsf{Spin}^{\pm}_{2n}$, *if* $\Delta = D_n$ *and* $q \geq 4$*, of plus type if* n *even, of minus type if* n *odd*
(Gramlich, Hoffman, Nickel, Shpectorov [71], Phan [107])*;*
- *the universal Steinberg-Chevalley group of type* ${}^2E_6(q^2)$*, if* $\Delta = E_6$ *and* $q \geq 4$
(Devillers, Gramlich, Mühlherr [51], Gramlich, Hoffman, Mühlherr, Shpectorov 2005, Phan [107])*;*
- *the universal Steinberg-Chevalley group of type* $E_7(q)$*, if* $\Delta = E_7$ *and* $q \geq 4$
(Devillers, Gramlich, Mühlherr [51], Gramlich, Hoffman, Mühlherr, Shpectorov 2005, Phan [107])*;*
- *the universal Steinberg-Chevalley group of type* $E_8(q)$*, if* $\Delta = E_8$ *and* $q \geq 4$
(Devillers, Gramlich, Mühlherr [51], Gramlich, Hoffman, Mühlherr, Shpectorov 2005, Phan [107])*;*
- *the universal Steinberg-Chevalley group of type* $F_4(q)$*, if* $\Delta = F_4$ *and* $q \geq 13$
(Gramlich, Hoffman, Mühlherr, Shpectorov 2007, Gramlich, Witzel [79]).

7 Curtis-Tits theory, Phan theory, and the revision of the classification of the finite simple groups

In this section I briefly mention by way of example how to prove a local recognition result for Chevalley groups of simply laced type. I currently do not know how to deal with the non-simply laced case.

Following [14] (Sections 3.1.2, 4.1.2) a *fundamental (rank one) subgroup* of a (twisted) Chevalley group G is a group generated by two root subgroups X_α, $X_{-\alpha}$, respectively the subgroup of fixed points of $\langle X_\alpha, X_{-\alpha}\rangle$ with respect to an involution of G interchanging X_α and $X_{-\alpha}$.

In the revision of the classification of the finite simple groups [60, 61, 62, 63, 64, 65] one is interested in proving local recognition results of the following type.

Local Recognition Theorem 1 (Altmann, Gramlich 2007). *Let q be an odd prime power and let G be a group containing an involution x and a subgroup $K \trianglelefteq C_G(x)$ such that*

(i) $K \cong \begin{cases} \mathsf{SL}_6(q) & \text{(CT)} \\ \mathsf{SU}_6(q^2) & \text{(P)} \end{cases}$;

(ii) *$C_G(K)$ contains a subgroup $X \cong \mathsf{SL}_2(q) \cong \mathsf{SU}_2(q^2)$ with $\langle x \rangle = Z(X)$;*

(iii) *there exists an involution $g \in G$ such that $Y := gXg$ is contained in K;*

(iv) *if V is a natural module for K, then the commutator*

$$[Y, V] = \{yv - v \in V \mid y \in Y, v \in V\}$$

is a subspace of V of $\left.\begin{array}{ll} \mathbb{F}_q & \text{(CT)} \\ \mathbb{F}_{q^2} & \text{(P)} \end{array}\right\}$*-dimension two;*

(v) *$G = \langle K, gKg \rangle$; moreover, there exists $z \in K \cap gKg$ which is a gKg-conjugate of x and a K-conjugate of gxg.*

Then (up to isomorphism)

$$G/Z(G) \cong \mathsf{PSL}_8(q) \text{ or } G/Z(G) \cong E_6(q)\,, \qquad \text{(CT)}$$
$$G/Z(G) \cong \mathsf{PSU}_8(q^2) \text{ or } G/Z(G) \cong {}^2E_6(q^2)\,. \qquad \text{(P)}$$

Using ideas developed in [43, 66, 67] the above theorem is implied by a graph-theoretical local recognition theorem. From the hypotheses of the theorem one constructs a connected locally line-hyperline graph (cf. [67]), resp. a connected locally unitary line graph (cf. [11]) with G as a group of automorphisms and an induced subgraph Σ isomorphic to the commuting reflection graphs $\mathbf{W}(A_7)$ or $\mathbf{W}(E_6)$ (see [70]). This information then implies the existence of a Curtis-Tits, resp. Phan amalgam inside G from which the theorem follows.

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