A non-abelian representation of the dual polar space $DQ(2n, 2)$

Kamal Lochan Patra Binod Kumar Sahoo

Abstract

We prove that the dual polar space $DQ(2n, 2)$, $n \geq 3$, of rank $n$ associated with a non-singular parabolic quadric in $PG(2n, 2)$ admits a faithful non-abelian representation in the extraspecial $2$-group $2^{1+2n}$. The near $2n$-gon $I_n$ (section 2.4) is a geometric hyperplane of $DQ(2n, 2)$. For $n \geq 3$, we first construct a faithful non-abelian representation of $I_n$ in $2^{1+2n}$ and subsequently extend it to a faithful non-abelian representation of $DQ(2n, 2)$ in $2^{1+2n}$.

Keywords: dual polar space, non-abelian representation, extraspecial 2-group

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1 Introduction

Let $p$ be a fixed prime number. In [12], Ivanov introduced the notion of representations in groups of point-line geometries $S = (P, L)$ of order $p$, that is, lines of size $p + 1$.

Definition 1.1 ([12, p. 305]). A representation of $S$ in a group $R$ is a mapping $\psi$ from the point set $P$ of $S$ into the set of subgroups of order $p$ in $R$ such that the following hold:

(i) $R$ is generated by the subgroups $\psi(x), x \in P$;

(ii) For each line $l \in L$, the subgroups $\psi(x), x \in l$, are pairwise distinct and generate an elementary abelian $p$-subgroup of $R$ of order $p^2$.

This concept of representations in groups of geometries with lines of size $p + 1$ is similar to the definition of the root group geometries of groups of Lie
type defined over a field $\mathbb{F}_p$ with $p$ elements studied by Cohen and Cooperstein [6, p. 75]. This definition of representations of geometries led to a new research area in the theory of groups and geometries [12]. For example, the knowledge of the representations is crucial for the construction of affine and $\sigma$-extensions of geometries and non-split extensions of groups and modules [13, sections 2.7 and 2.8].

We write $(R, \psi)$ to mean that $\psi$ is a representation of $S$ in $R$. The group $R$ is then called a representation group of $S$. A representation $(R, \psi)$ of $S$ is faithful if $\psi$ is injective, is abelian or non-abelian according as $R$ is abelian or not. Note that, in [12], ‘non-abelian representation’ means that ‘the corresponding representation group is not necessarily abelian’.

We indicate various possibilities for a representation of a point-line geometry of prime order and the corresponding representation group.

(1) Every representation of a projective space (as a point-line geometry) is faithful and abelian.

(2) A representation of a point-line geometry need not be faithful. For example: let $S = (P, L)$ be a $(2, 1)$-GQ and $R = \{1, r_1, r_2, r_3\}$ be the Klein 4-group. A triad of $S$ is a triple of pairwise non-collinear points of $S$. Let $P_1, P_2, P_3$ be the three triads of $S$ partitioning the point set $P$ of $S$. Define a map $\psi$ from $P$ to the set of subgroups of order 2 in $R$ by $\psi(x) = \langle r_i \rangle$ if $x \in P_i$. Then $(R, \psi)$ is an abelian representation of $S$ which is not faithful.

(3) The representation group for an abelian representation is an elementary abelian $p$-group. So it could be considered as a vector space over $\mathbb{F}_p$, and the corresponding representation is a full projective embedding which need not be faithful.

(4) There are point-line geometries, different from the projective spaces, whose representations are always abelian. In [14, Theorems 1.5 and 1.6] it is proved that this is the case for every finite polar space which is not of symplectic type of odd prime order.

(5) The representation group for a non-abelian representation of a finite point-line geometry could be infinite.

[Let $S = (P, L)$ be a point-line geometry of order 2 admitting at least one representation. The universal representation group $U(S)$ of $S$ has the presentation:

$$U(S) = \langle u_x : x \in P, u_x^2 = 1, u_x u_y u_z = 1 \text{ for every } \{x, y, z\} \in L \rangle.$$]

Let $\psi_S$ be the map from $P$ to the set of subgroups of order 2 in $U(S)$ defined by $x \mapsto \langle u_x \rangle$ for $x \in P$. Then $(U(S), \psi_S)$ is a representation
of \( S \), called the universal representation of \( S \). Now, let \( S = (P, L) \) be a generalized hexagon with parameters \((2,2)\). Then \( S \) is isomorphic to \( H(2) \) or its dual \( H(2)^* \) [16, Theorem 4, p. 402]. For each \( x \in P \), consider the geometric hyperplane \( H(x) \) of \( S \) consisting of points at non-maximal distance from \( x \). The subgraph of the collinearity graph of \( S \) (see section 2 for the definition) induced on the complement of \( H(x) \) in \( P \) is connected if \( S \cong H(2) \) and has two connected components if \( S \cong H(2)^* \) [10]. By [12, Lemma 3.6, p. 310], the universal representation group of \( S \) is infinite when \( S \cong H(2)^* \).

In this paper, we prove the following:

**Theorem 1.2.** Let \( DQ(2n, 2) \), \( n \geq 2 \), be the dual polar space of rank \( n \) associated with a non-singular parabolic quadric in \( PG(2n, 2) \). The following hold:

(i) If \( DQ(2n, 2) \) admits a non-abelian representation, then \( n \geq 3 \).

(ii) \( DQ(2n, 2) \), \( n \geq 3 \), admits a faithful non-abelian representation in the extraspecial 2-group \( 2^{1+2n} \).

**2 Basic definitions**

Let \( S = (P, L) \) be a partial linear space, that is, a point-line geometry with a ‘point-set’ \( P \) and a ‘line set’ \( L \) of subsets of \( P \) of size at least two such that any two distinct points of \( S \) are contained in at most one line of \( S \). If each line of \( S \) contains exactly three points, then \( S \) is called slim. For distinct points \( x, y \in P \), we write \( x \sim y \) if there is a line of \( S \) containing them (we then say that \( x \) and \( y \) are collinear). For \( x \in P \) and \( A \subseteq P \), we define

\[
x^\perp = \{ x \} \cup \{ y \in P : x \sim y \} \quad \text{and} \quad A^\perp = \bigcap_{x \in A} x^\perp.
\]

If \( P^\perp \) is empty, then \( S \) is called non-degenerate. The graph \( \Gamma(P) \) with vertex set \( P \), in which two distinct vertices are adjacent whenever they are collinear in \( S \), is called the collinearity graph of \( S \). If \( \Gamma(P) \) is connected, then \( S \) is a connected partial linear space. A subset \( X \) of \( P \) is a subspace of \( S \) if any line of \( S \) containing at least two points of \( X \) is entirely contained in \( X \). A subspace \( X \) of \( S \) is singular if \( x \sim y \) for every pair of distinct points \( x, y \in X \), that is, the induced subgraph \( \Gamma(X) \) of \( \Gamma(P) \) is a clique. A geometric hyperplane of \( S \) is a subspace of \( S \) different from \( P \), that meets each line of \( S \) non-trivially. Two partial linear spaces \( S = (P, L) \) and \( S' = (P', L') \) are isomorphic, written as \( S \cong S' \), if there exists a bijection \( \alpha : P \to P' \) such that \( \alpha(x) \sim \alpha(y) \) in \( S' \) whenever \( x \sim y \) in \( S \) and it induces a bijection from \( L \) to \( L' \). Such a map \( \alpha \) is called an isomorphism from \( S \) to \( S' \).
2.1 Near polygons

A near polygon [15] is a partial linear space \( S = (P, L) \) of finite diameter (that is, the diameter of \( \Gamma(P) \) is finite) such that the following 'near polygon' property holds:

\[
\text{For each point-line pair } (x, l) \in P \times L, \text{ there exists a unique point in } l \text{ which is nearest to } x.
\]

Here, the distance \( d(x, y) \) between two points \( x \) and \( y \) of \( S \) is measured in the graph \( \Gamma(P) \). If the diameter of \( S \) is \( n \), then the near polygon \( S \) is called a near \( 2n \)-gon. For \( x \in P \), we define

\[
\Gamma_n(x) = \{ y \in P \mid d(x, y) = n \}; \\
\Gamma_{<n}(x) = \{ y \in P \mid d(x, y) < n \}.
\]

For every \( x \in P \) with \( \Gamma_n(x) \neq 0 \), \( \Gamma_{<n}(x) \) is a geometric hyperplane of \( S \). If \( n = 2 \) and \( S \) is non-degenerate, then \( S \) is a generalized quadrangle (GQ, for short). If a finite generalized quadrangle has a line containing at least three points and a point contained in at least three lines, then there exist integers \( s \) and \( t \) such that each line contains \( s + 1 \) points and each point is contained in \( t + 1 \) lines [3, Theorem 7.1, p. 98]. In that case, we say that it is an \((s, t)\)-GQ.

Let \( S = (P, L) \) be a near polygon. If every line of \( S \) contains at least three points and if every two points of \( S \) at distance 2 have at least two common neighbours, then \( S \) is called a dense near polygon. A subspace \( C \) of \( S \) is convex if every geodesic in \( \Gamma(P) \) between two points of \( C \) is entirely contained in \( C \). A quad is a convex subspace of \( S \) of diameter 2 such that no point of it is adjacent to all other points of it. The points and the lines contained in a quad define a generalized quadrangle. If \( x \) and \( y \) are two points of a dense near polygon at distance 2 from each other, then there is a unique quad containing \( x \) and \( y \) [15, Proposition 2.5, p. 10].

Let \( S = (P, L) \) be a slim dense near \( 2n \)-gon. If \( n = 1 \), then \( S \simeq \mathbb{L}_3 \), a line of size 3. If \( n = 2 \), then \( S \) is a \((2, t)\)-GQ. In that case, \( P \) is finite, \( t = 1, 2 \) or 4 and for each such value of \( t \) there exists a unique \((2, t)\)-GQ, up to isomorphism [3, Theorem 7.3, p. 99]. Thus, \( S \) is isomorphic to one of the classical generalized quadrangles \( Q^+(3, 2) \), \( W(2) \simeq Q(4, 2) \) and \( Q^-(5, 2) \) for \( t = 1, 2 \) and 4, respectively. We refer to [7] for the classification of all slim dense near \( 2n \)-gons when \( n \in \{3, 4\} \).
2.2 Dual polar spaces

Here, a polar space is a non-degenerate point-line geometry $S = (P, L)$ satisfying the following ‘one or all’ axiom (see [2, Theorem 4, p. 161] and [17, 7.1, p. 102]):

For each point-line pair $(x, l) \in P \times L$ with $x \notin l$, $x$ is collinear with one or all points of $l$.

A polar space is a partial linear space [2, Theorem 3]. The rank of a polar space $S$ is the supremum of the lengths $m$ of chains $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_m$ of singular subspaces in $S$. A polar space of rank 2 is a generalized quadrangle.

Let $S = (P, L)$ be a finite polar space of rank $n \geq 2$. Every singular subspace of $S$ is isomorphic to a projective space. The dimension of a singular subspace of $S$ is the dimension of the associated projective space. Each maximal singular subspace of $S$ has dimension $n - 1$ [2, Proposition 11]. For singular subspaces $X$ and $Y$ of $S$ with $Y \subset X$, the co-dimension of $Y$ in $X$ is the dimension of $X$ minus the dimension of $Y$. Consider the point-line geometry $DS = (P', L')$, where

- $P'$ is the collection of all maximal singular subspaces of $S$;
- an element of $L'$ is the collection of all maximal singular subspaces of $S$ containing a specific singular subspace of $S$ of co-dimension 1 in each of them.

Then $DS$ is a partial linear space, called the dual polar space of rank $n$ associated with $S$. Cameron characterized these geometries in terms of points and lines and proved that dual polar spaces of rank $n$ are dense near $2n$-gons [4, Theorem 1, p. 75].

2.3 The dual polar space $DQ(2n, 2)$

Let $Q(2n, 2)$, $n \geq 2$, be a non-singular parabolic quadric in $PG(2n, 2)$. Then the points together with the lines of $Q(2n, 2)$ is a polar space of rank $n$ and $DQ(2n, 2)$ is the associated dual polar space of rank $n$. Thus, the points of $DQ(2n, 2)$ are the generators (that is, subspaces of maximal dimension $n - 1$) of $Q(2n, 2)$ and a line of $DQ(2n, 2)$ is a triple of generators containing a specific $(n - 2)$-dimensional subspace of $Q(2n, 2)$. The dual polar space $DQ(2n, 2)$ is a slim dense near $2n$-gon. The quads of $DQ(2n, 2)$ are isomorphic to $W(2)$, the unique $(2, 2)$-GQ. For each point $x$ of $DQ(2n, 2)$, the set $\Gamma_{<n}(x)$ is a maximal subspace of $DQ(2n, 2)$. 
2.4 The near $2n$-gon $I_n$

Again, consider a non-singular parabolic quadric $Q(2n, 2)$, $n \geq 2$, in $\text{PG}(2n, 2)$ and a hyperplane of $\text{PG}(2n, 2)$ which intersects $Q(2n, 2)$ in a non-singular hyperbolic quadric $Q^+(2n-1, 2)$. The set $X$ of all generators of $Q(2n, 2)$ which are not contained in $Q^+(2n-1, 2)$ is a subspace of $\text{DQ}(2n, 2)$ [7, Theorem 6.46, p. 140]. The points and the lines of $DQ(2n, 2)$ contained in $X$ define a slim dense near $2n$-gon [7, Theorem 6.48, p. 141]. For $n = 2$, the generalized quadrangle $I_2$ is isomorphic to $Q^+(3, 2)$. For $n \geq 3$, each quad of $I_n$ is either a $(2, 1)$-GQ or a $(2, 2)$-GQ.

2.5 Extraspecial $2$-groups

A finite $2$-group $G$ is extraspecial if the Frattini subgroup $\Phi(G)$, the commutator subgroup $G' = [G, G]$ and the center $Z(G)$ of $G$ coincide and have order 2. We refer to [9, section 20, p. 78,79] —see also [11, chapter 5, section 5]— for the following properties of an extraspecial $2$-group.

An extraspecial $2$-group is of order $2^{1+2m}$ for some integer $m \geq 1$. Let $D_8$ and $Q_8$, respectively, denote the dihedral and quaternion groups of order 8. A non-abelian $2$-group of order 8 is extraspecial and is isomorphic to $D_8$ or $Q_8$. Let $G$ be an extraspecial $2$-group of order $2^{1+2m}$. Then the exponent of $G$ is 4 and either

(i) $G$ is a central product of $m$ copies of $D_8$, or

(ii) $G$ is a central product of $m - 1$ copies of $D_8$ and a copy of $Q_8$.

So, the maximum of the orders of its abelian subgroups is $2^{m+1}$. In case (i), $G$ possesses a maximal abelian subgroup of order $2^{m+1}$ which is elementary abelian. In case (ii), each maximal abelian subgroup of $G$ is isomorphic to $C_2^{m-1} \times C_4$. Here, $C_k$ denotes the cyclic group of order $k$. We denote an extraspecial $2$-group of order $2^{1+2m}$ by $2_{1+2m}$ if (i) holds, and by $2^{1+2m}$ if (ii) holds.

3 Proof of Theorem 1.2

Let $S = (P, L)$ be a slim partial linear space and $(R, \psi)$ be a representation of $S$. For each $x \in P$, we identify the subgroup $\psi(x) = \langle r_x \rangle$ of $R$ with its non-trivial element $r_x$. If $x, y \in P$ and $x \sim y$, then we denote by $xy$ the unique line of $S$ containing $x$ and $y$, and define $x * y$ by $xy = \{x, y, x * y\}$. So, $r_{x*y} = r_x r_y$ for every line $\{x, y, x * y\}$ of $S$, by condition (ii) of Definition 1.1. The following lemma is a particular case of [14, Proposition 3.1, p. 59]. We write down the
proof here for the sake of completeness of this paper. (In the statement of [14, Proposition 3.1, p. 59], the polar space should be of order 2.)

**Lemma 3.1.** Let $S = (P, L)$ be a $(2,1)$-GQ and $(R, \psi)$ be a representation of $S$. Then $R$ is abelian.

**Proof.** We show that $[r_x, r_y] = 1$ for all $x, y \in P$ with $x \sim y$. Let $Q$ be a $(2,1)$-subGQ in $S$ containing $x$ and $y$. Such a $Q$ exists, follows from the fact that each line contains exactly 3 points. Let $\{x, y\}^\perp \cap Q = \{a, b\}$. In $Q, [r_b, r_y] = [r_b, r_x] = 1$ and $r_{(a+x)+(b+y)} = r_{(a+y)+(b+x)}$, implies that $r_x r_y = r_y r_x$. □

The dual polar space $DQ(4, 2)$ is a $(2,2)$-GQ which is isomorphic to $W(2)$. By Lemma 3.1, every representation of it is abelian. This proves Theorem 1.2(i).

We next construct non-abelian representations for the near $2(n+1)$-gon $\Pi_{n+1}$ and the dual polar space $DQ(2n + 2, 2)$ in the group $2^{1+2n}$ when $n \geq 2$. We make use of the following recursive constructions of $\Pi_{n+1}$ and $DQ(2n + 2, 2)$ given by De Bruyn [8].

Let $S_n = (\mathbb{P}_n, L_n)$ denote the dual polar space $DQ(2n, 2)$ of rank $n \geq 2$. The quads of $S_n$ are $(2,2)$-GQs. Every triad $\{a, b, c\}$ of points contained in a quad of $S_n$ has the property that $\{a, b, c\}^\perp$ contains one or three points. In the latter case, such a triad $\{a, b, c\}$ is called a hyperbolic line of $S_n$. Thus, $\{a, b, c\}$ is a hyperbolic line of $S_n$ if and only if $\{a, b, c\}^\perp$ is so. Now, consider the point-line geometries $S_{n+1} = (\mathcal{P}_{n+1}, \mathcal{L}_{n+1})$ and $S_{n+1} = (\mathcal{P}_{n+1}, \mathbb{L}_{n+1})$ constructed from $S_n$, where

\[
\mathcal{P}_{n+1} = \{(x, y) \in \mathbb{P}_n \times \mathbb{P}_n : y \in x^\perp\};
\]
\[
\mathcal{L}_{n+1} = \{(x, u), (y, v), (z, w) : (x, y, z) \text{ is a line or a hyperbolic line of } S_n, \quad \text{and } \{x, y, z\}^\perp = \{u, v, w\}\};
\]
\[
\mathbb{P}_{n+1} = \mathcal{P}_{n+1} \cup \mathbb{P}_n \cup \bar{\mathbb{P}}_n, \quad \text{where } \bar{\mathbb{P}}_n = \{\bar{x} : x \in \mathbb{P}_n\};
\]
\[
\mathbb{L}_{n+1} = \mathcal{L}_{n+1} \cup \mathcal{L}^1, \quad \text{where } \mathcal{L}^1 = \{(x, (x, u), \bar{u}) : (x, u) \in \mathcal{P}_{n+1}\}.
\]

Then $S_{n+1}$ is isomorphic to the near $2(n+1)$-gon $\Pi_{n+1}$ and $S_{n+1}$ is isomorphic to the dual polar space $DQ(2n + 2, 2)$ [8, section 1.5, Corollary 1.3 and Theorem 1.4].

Now, let $R = 2^{1+2n}, n \geq 2$. The quotient group $R/R'$ is an elementary abelian 2-group. Set $R' = (\emptyset)$ and $V = R/R'$. Consider $V$ as a vector space of dimension $2^{n+1}$ over $\mathbb{F}_2$. The map

\[ f : V \times V \rightarrow \mathbb{F}_2 \]

defined by

\[ f(x R', y R') = i, \]
where \([x, y] = \theta^i, i \in \{0, 1\}\), is a non-degenerate symplectic bilinear form on \(V\) [9, Theorem 20.4, p. 78]. We write \(V\) as an orthogonal direct sum of \(2^n\) hyperbolic planes \(K_i, 1 \leq i \leq 2^n\), in \(V\) with respect to \(f\). Let \(H_i\) be the inverse image of \(K_i\) in \(R\) under the natural surjective homomorphism from \(R\) to \(V\). Then \(H_i\) is generated by two elements \(x_i\) and \(y_i\) such that \([x_i, y_i] = \theta\). We set
\[
M = \langle x_i : 1 \leq i \leq 2^n \rangle; \quad \bar{M} = \langle y_i : 1 \leq i \leq 2^n \rangle.
\]
Then \(M\) and \(\bar{M}\) are elementary abelian 2-subgroups of \(R\) each of order \(2^{2n}\). The groups \(M, \bar{M}\) and \(R\) pairwise intersect trivially and \(R = MM\bar{M}\). Further, \(C_M(M)\) and \(C_M(\bar{M})\) are trivial.

Let \((M, \tau)\) be the faithful abelian representation of \(\mathbb{S}_n\) arising from the spin-embedding of \(DQ(2n, 2)\) in a vector space of dimension \(2^n\). We refer to [1] for a description of the spin-embedding. Then the following property \((*)\) is satisfied:

\[ (*) \text{ For every point } x \text{ of } \mathbb{S}_n, \text{ the subgroup } \langle m_y : y \in \Gamma_{<n}(x) \rangle \text{ of } \text{index 2 in } M. \]

This embedding of \(DQ(2n, 2)\) is the so-called minimal full polarized embedding of \(DQ(2n, 2)\) in the sense of [5] and the property \((*)\) is the condition of polarization for a projective full embedding.

Let \(Q\) be a quad in \(DQ(2n, 2)\). Let \(G = \langle \tau(Q) \rangle\). Then \((G, \tau)\) is a faithful abelian representation of \(Q\). Since \(Q\) is a \((2, 2)\)-GQ, \(G\) is of order \(2^4\) or \(2^5\). Since \((M, \tau)\) is minimal and polarized and \(Q\) is a convex subspace of \(DQ(2n, 2)\), it follows from [5, Theorem 1.6, p. 10] that \((G, \tau)\) is also minimal and polarized. This implies that \(G\) is of order \(2^4\).

**Lemma 3.2.** Let \(a, b, c\) be three pairwise distinct points of \(Q\). Then \(T = \{a, b, c\}\) is a line or a hyperbolic line of \(Q\) if and only if \(g_ag_bg_cg_c = 1\).

**Proof.** First, assume that \(T\) is a hyperbolic line of \(Q\). Let \(Q'\) be a \((2, 1)\)-subGQ of \(Q\) containing \(a\) and \(b\). Then \(c \notin Q'\) and \(Q = \langle Q', c \rangle\). Let \(\{x, y\} = \{a, b\}^+ \cap Q'\). Then \(x, y \in T^\perp\), since \(T\) is a hyperbolic line. Let \(z\) be the unique point in \(Q'\) such that \(\{x, y, z\}\) is a triad of \(Q'\). Then \(c \sim z\) and \(g_z = g_{ax}g_{by} = (g_ag_b)(g_yg_y)\). Since the subgroup \(H = \langle g_y : y \in x^+ \cap Q \rangle\) is of index 2 in \(G\), \(|H| = 2^3\) and \(H = \langle g_x, g_a, g_y \rangle\). So \(g_c\) is equal to either \(g_ag_b\) or \(g_ag_bg_c\), since \(\tau\) is faithful. If the latter holds, then \(g_{ax}z = g_ag_z = g_y\). But this is not possible, since \(y \neq c \sim z\) and \(\tau\) is faithful. Thus \(g_c = g_yg_y\) and so \(g_ag_bg_c = 1\).

Now assume that \(g_ag_bg_cg_c = 1\) and that \(T\) is not a line. Then \(T\) is a triad, since \(\tau\) is faithful. We show that \(T\) is a hyperbolic line. Suppose that \(T\) is not a hyperbolic line. Then \(|T^\perp| = 1\). Let \(\{a, b\}^{\perp, 1} = \{a, b, d\}\). Since \(\{a, b, d\}\) is a hyperbolic line, \(g_ag_bg_d = 1\) by the first part. Since \(|T^\perp| = 1\), \(c \neq d\) and
Non-abelian representation of $DQ(2n,2)$

For each point $x$ of $S_n$, set $H_x = \langle m_y : y \in \Gamma_{<n}(x) \rangle$. Since $H_x$ is a maximal subgroup of $M$, the centralizer of $H_x$ in $M$ is a subgroup $\langle \bar{m}_x \rangle$ of order 2. Since $\Gamma_{<n}(x)$ is a maximal subspace of $S_n$, $P_n = \langle \Gamma_{<n}(x) \cup \{w\} \rangle$ and $M = \langle H_x, m_w \rangle$ for $w \in \Gamma_n(x)$. The triviality of $C_M(M)$ implies that $[\bar{m}_x, m_w] = \theta$ for every $w \in \Gamma_n(x)$.

Recall that $P_n = \{x : x \in P_n\}$. Let $\mathbb{L}_n = \{\{\bar{x}, \bar{y}, \bar{z} : x, y, z \in \mathbb{L}_n\}$. Then $S_n = (P_n, \mathbb{L}_n) \simeq DQ(2n,2)$. Let $\tau$ be the map from the point set $P_n$ of $S_n$ to $M$ defined by $\bar{\tau}(x) = \bar{m}_x$.

**Proposition 3.3.** $(M, \tau)$ is a faithful abelian representation of $S_n$ satisfying the property $(\ast)$.

**Proof.** For $x \neq y$ in $P_n$, $\Gamma_{<n}(x) \neq \Gamma_{<n}(y)$. So $H_x \neq H_y$, and $C_M(H_x) \neq C_M(H_y)$. This implies that $\bar{m}_x \neq \bar{m}_y$, and hence $\tau$ is injective.

Let $\{x, y, z\}$ be a line of $S_n$. Let $w \in \Gamma_{<n}(z)$. Then $d(w, x) \leq n - 1$ if and only if $d(w, y) \leq n - 1$, by the ‘near polygon’ property. So $[\bar{m}_x, m_w]$ is equal to either $(1, 1)$ or $(\bar{\theta}, \bar{\theta})$. Then

$$[\bar{m}_x \bar{m}_y, m_w] = [\bar{m}_x, m_w] [\bar{m}_y, m_w] = 1.$$

The first equality holds, since $R$ has nilpotent class 2. Thus, $1 \neq \bar{m}_x \bar{m}_y \in C_M(H_x)$. Since $\bar{m}_z$ is the unique non-trivial element in $C_M(H_x)$, it follows that $\bar{m}_z = \bar{m}_x \bar{m}_y$. So $\bar{m}_x \bar{m}_y \bar{m}_z = 1$ for every line $\{\bar{x}, \bar{y}, \bar{z}\}$ of $S_n$. This verifies condition (ii) of Definition 1.1.

Now, let $K = \langle \bar{\tau}(P_n) \rangle$. Then $(K, \bar{\tau})$ is a faithful abelian representation of $S_n$. For each $\bar{x} \in \bar{P}_n$, $H_{\bar{x}} = \langle \bar{m}_y : y \in \Gamma_{<n}(\bar{x}) \rangle$ is equal to $K$ or is of index 2 in $K$. Since $m_x$ commutes with each element of $H_{\bar{x}}$ and $m_x$ does not commute with $m_w$ for $w \in \Gamma_n(x)$, the first possibility does not occur. This implies that the property $(\ast)$ holds.

Since $S_n \simeq DQ(2n,2)$ does not possess a faithful polarized projective embedding in a vector space of dimension less than $2^n$ [5], it follows that $K = M$. So condition (i) of Definition 1.1 holds, thus completing the proof.

By Proposition 3.3, a similar statement in Lemma 3.2 holds for the restriction of $\bar{\tau}$ to a quad of $S_n$. Now, let $\beta : P_{n+1} \to R$ be defined by

$$\beta((x, y)) = m_x \bar{m}_y,$$

for $(x, y) \in P_{n+1}$. Since $[m_x, \bar{m}_y] = 1$ for $x, y \in P_n$ with $y \in x^\perp$, $\beta((x, y)) = m_x \bar{m}_y$ is of order 2 in $R$ for every point $(x, y)$ of $S_{n+1}$.
Proposition 3.4. \((R, \beta)\) is a faithful non-abelian representation of \(S_{n+1} \simeq \mathbb{I}_{n+1}\).

Proof. If \(\beta((x, u)) = \beta((y, v))\), then \(m_x \bar{m}_u = m_y \bar{m}_v\) implies that \(m_y m_x = \bar{m}_u \bar{m}_v\). Since \(M \cap \bar{M}\) is trivial, it follows that \(m_x = m_y\) and \(m_u = \bar{m}_v\). This implies that \(\beta\) is one-one.

We now verify conditions (i) and (ii) of Definition 1.1. Let \(x \in \mathbb{P}_n\). Let \(\{x, y, z\}\) be a hyperbolic line of \(S_n\) containing \(x\) and let \(u \in \{x, y, z\}^\perp\). Then
\[
\beta((y, u)) \beta((z, u)) = m_y \bar{m}_u m_v \bar{m}_u = m_y m_z = m_x.
\]
The last equality follows from Lemma 3.2. Thus, \(m_x \in \langle \beta(\mathbb{P}_{n+1}) \rangle\) for every \(x \in \mathbb{P}_n\). This also implies that \(\bar{m}_x \in \langle \beta(\mathbb{P}_{n+1}) \rangle\) for every \(x \in \mathbb{P}_n\). In particular, \(M\) and \(\bar{M}\) are contained in \(\langle \beta(\mathbb{P}_{n+1}) \rangle\). Since \(R\) is generated by \(M\) and \(\bar{M}\), we get \(R = \langle \beta(\mathbb{P}_{n+1}) \rangle\).

The proof of Theorem 1.2. Let \(R, M, \bar{M}, \tau, \bar{\tau}\) and \(\beta\) be as in the above. Let \(\rho\) be the map from \(\mathbb{P}_{n+1}\) to \(R\) defined by
\[
\rho = \begin{cases} 
\tau & \text{on } \mathbb{P}_n; \\
\bar{\tau} & \text{on } \mathbb{P}_n; \\
\beta & \text{on } \mathbb{P}_{n+1}.
\end{cases}
\]
Then \(R = \langle \rho(\mathbb{P}_{n+1}) \rangle\). Also, condition (ii) of Definition 1.1 holds for every line in \(\mathcal{L}_1\). As a consequence of Proposition 3.4, \((R, \rho)\) is a faithful non-abelian representation of \(S_{n+1} \simeq DQ(2n + 2, 2)\). This completes the proof. \(\square\)

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Non-abelian representation of $DQ(2n, 2)$

References


Kamal Lochan Patra
School of Mathematical Sciences, National Institute of Science Education and Research, Institute of Physics Campus, Sainik School Post, Bhubaneswar-751005, India
e-mail: klpatra@niser.ac.in

Binod Kumar Sahoo
School of Mathematical Sciences, National Institute of Science Education and Research, Institute of Physics Campus, Sainik School Post, Bhubaneswar-751005, India
e-mail: bksahoo@niser.ac.in