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A non-abelian representation of the dual polar space $DQ(2n, 2)$

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Abstract

We prove that the dual polar space $DQ(2n, 2)$, $n \geq 3$, of rank n associated with a non-singular parabolic quadric in $\mathsf{PG}(2n, 2)$ admits a faithful non-abelian representation in the extraspecial 2-group $2_+^{1+2^n}$. The near $2n$-gon $\mathbb{I}_n$ (section 2.4) is a geometric hyperplane of $DQ(2n, 2)$. For $n \geq 3$, we first construct a faithful non-abelian representation of $\mathbb{I}_n$ in $2_+^{1+2^n}$ and subsequently extend it to a faithful non-abelian representation of $DQ(2n, 2)$ in $2_+^{1+2^n}$.

Keywords: dual polar space, non-abelian representation, extraspecial 2-group

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1 Introduction

Let p be a fixed prime number. In [12], Ivanov introduced the notion of representations in groups of point-line geometries $S = (P, L)$ of order p, that is, lines of size $p + 1$.

Definition 1.1 ([12, p. 305]). A representation of S in a group R is a mapping ψ from the point set P of S into the set of subgroups of order p in R such that the following hold:

(i) R is generated by the subgroups $\psi(x), x \in P$;

(ii) For each line $l \in L$, the subgroups $\psi(x)$, $x \in l$, are pairwise distinct and generate an elementary abelian p-subgroup of R of order p^2.

This concept of representations in groups of geometries with lines of size $p + 1$ is similar to the definition of the root group geometries of groups of Lie type defined over a field $\mathbb{F}_p$ with p elements studied by Cohen and Cooperstein [6, p. 75]. This definition of representations of geometries led to a new research area in the theory of groups and geometries [12]. For example, the knowledge of the representations is crucial for the construction of affine and c-extensions of geometries and non-split extensions of groups and modules [13, sections 2.7 and 2.8].

We write (R, ψ) to mean that ψ is a representation of S in R. The group R is then called a *representation group* of S. A representation (R, ψ) of S is *faithful* if ψ is injective, is *abelian* or *non-abelian* according as R is abelian or not. Note that, in [12], 'non-abelian representation' means that 'the corresponding representation group is not necessarily abelian'.

We indicate various possibilities for a representation of a point-line geometry of prime order and the corresponding representation group.

(1) Every representation of a projective space (as a point-line geometry) is faithful and abelian.

(2) A representation of a point-line geometry need not be faithful. For example: let $S = (P, L)$ be a $(2, 1)$-GQ and $R = \{1, r_1, r_2, r_3\}$ be the Klein 4-group. A *triad* of S is a triple of pairwise non-collinear points of S. Let P_1, P_2, P_3 be the three triads of S partitioning the point set P of S. Define a map ψ from P to the set of subgroups of order 2 in R by $\psi(x) = \langle r_i \rangle$ if $x \in P_i$. Then (R, ψ) is an abelian representation of S which is not faithful.

(3) The representation group for an abelian representation is an elementary abelian p-group. So it could be considered as a vector space over $\mathbb{F}_p$ and the corresponding representation is a full projective embedding which need not be faithful.

(4) There are point-line geometries, different from the projective spaces, whose representations are always abelian. In [14, Theorems 1.5 and 1.6] it is proved that this is the case for every finite polar space which is not of symplectic type of odd prime order.

(5) The representation group for a non-abelian representation of a finite point-line geometry could be infinite.

[Let $S = (P, L)$ be a point-line geometry of order 2 admitting at least one representation. The *universal representation group* $U(S)$ of S has the presentation:

$$U(S) = \left\langle u_x : x \in P, u_x^2 = 1, u_x u_y u_z = 1 \text{ for every } \{x, y, z\} \in L \right\rangle.$$

Let ψ_S be the map from P to the set of subgroups of order 2 in $U(S)$ defined by $x \mapsto \langle u_x \rangle$ for $x \in P$. Then $(U(S), \psi_S)$ is a representation of S, called the *universal representation* of S. Now, let $S = (P, L)$ be a generalized hexagon with parameters $(2, 2)$. Then S is isomorphic to $H(2)$ or its dual $H(2)^*$ [16, Theorem 4, p. 402]. For each $x \in P$, consider the geometric hyperplane $H(x)$ of S consisting of points at non-maximal distance from x. The subgraph of the collinearity graph of S (see section 2 for the definition) induced on the complement of $H(x)$ in P is connected if $S \simeq H(2)$ and has two connected components if $S \simeq H(2)^*$ [10]. By [12, Lemma 3.6, p. 310], the universal representation group of S is infinite when $S \simeq H(2)^*$.]

In this paper, we prove the following:

Theorem 1.2. *Let $DQ(2n, 2)$, $n \geq 2$, be the dual polar space of rank n associated with a non-singular parabolic quadric in* $\mathsf{PG}(2n, 2)$. *The following hold:*

(i) *If $DQ(2n, 2)$ admits a non-abelian representation, then $n \geq 3$.*

(ii) *$DQ(2n, 2)$, $n \geq 3$, admits a faithful non-abelian representation in the extraspecial 2-group $2_+^{1+2^n}$.*

2 Basic definitions

Let $S = (P, L)$ be a *partial linear space*, that is, a point-line geometry with a 'point-set' P and a 'line set' L of subsets of P of size at least two such that any two distinct points of S are contained in at most one line of S. If each line of S contains exactly three points, then S is called *slim*. For distinct points $x, y \in P$, we write $x \sim y$ if there is a line of S containing them (we then say that x and y are *collinear*). For $x \in P$ and $A \subseteq P$, we define

$$x^\perp = \{x\} \cup \{y \in P : x \sim y\} \text{ and } A^\perp = \bigcap_{x \in A} x^\perp.$$

If $P^\perp$ is empty, then S is called *non-degenerate*. The graph $\Gamma(P)$ with vertex set P, in which two distinct vertices are adjacent whenever they are collinear in S, is called the *collinearity graph* of S. If $\Gamma(P)$ is connected, then S is a *connected* partial linear space. A subset X of P is a *subspace* of S if any line of S containing at least two points of X is entirely contained in X. A subspace X of S is *singular* if $x \sim y$ for every pair of distinct points $x, y \in X$, that is, the induced subgraph $\Gamma(X)$ of $\Gamma(P)$ is a clique. A *geometric hyperplane* of S is a subspace of S different from P, that meets each line of S non-trivially. Two partial linear spaces $S = (P, L)$ and $S' = (P', L')$ are *isomorphic*, written as $S \simeq S'$, if there exists a bijection $\alpha \colon P \to P'$ such that $\alpha(x) \sim \alpha(y)$ in S' whenever $x \sim y$ in S and it induces a bijection from L to L'. Such a map α is called an *isomorphism* from S to S'.

2.1 Near polygons

A *near polygon* [15] is a partial linear space $S = (P, L)$ of finite diameter (that is, the diameter of $\Gamma(P)$ is finite) such that the following *'near polygon'* property holds:

> *For each point-line pair $(x, l) \in P \times L$, there exists a unique point in l which is nearest to x.*

Here, the distance $d(x, y)$ between two points x and y of S is measured in the graph $\Gamma(P)$. If the diameter of S is n, then the near polygon S is called a *near $2n$-gon*. For $x \in P$, we define

$$\Gamma_n(x) = \{y \in P \mid d(x, y) = n\}\,;$$
$$\Gamma_{<n}(x) = \{y \in P \mid d(x, y) < n\}\,.$$

For every $x \in P$ with $\Gamma_n(x) \neq 0$, $\Gamma_{<n}(x)$ is a geometric hyperplane of S. If $n = 2$ and S is non-degenerate, then S is a *generalized quadrangle* (GQ, for short). If a finite generalized quadrangle has a line containing at least three points and a point contained in at least three lines, then there exist integers s and t such that each line contains $s + 1$ points and each point is contained in $t + 1$ lines [3, Theorem 7.1, p. 98]. In that case, we say that it is an (s, t)-GQ.

Let $S = (P, L)$ be a near polygon. If every line of S contains at least three points and if every two points of S at distance 2 have at least two common neighbours, then S is called a *dense* near polygon. A subspace C of S is *convex* if every geodesic in $\Gamma(P)$ between two points of C is entirely contained in C. A *quad* is a convex subspace of S of diameter 2 such that no point of it is adjacent to all other points of it. The points and the lines contained in a quad define a generalized quadrangle. If x and y are two points of a dense near polygon at distance 2 from each other, then there is a unique quad containing x and y [15, Proposition 2.5, p. 10].

Let $S = (P, L)$ be a slim dense near $2n$-gon. If $n = 1$, then $S \simeq \mathbb{L}_3$, a line of size 3. If $n = 2$, then S is a $(2, t)$-GQ. In that case, P is finite, $t = 1, 2$ or 4 and for each such value of t there exists a unique $(2, t)$-GQ, up to isomorphism [3, Theorem 7.3, p. 99]. Thus, S is isomorphic to one of the classical generalized quadrangles $Q^+(3, 2)$, $W(2) \simeq Q(4, 2)$ and $Q^-(5, 2)$ for $t = 1, 2$ and 4, respectively. We refer to [7] for the classification of all slim dense near $2n$-gons when $n \in \{3, 4\}$.

2.2 Dual polar spaces

Here, a *polar space* is a non-degenerate point-line geometry $S = (P, L)$ satisfying the following *'one or all'* axiom (see [2, Theorem 4, p. 161] and [17, 7.1, p. 102]):

> *For each point-line pair $(x, l) \in P \times L$ with $x \notin l$, x is collinear with one or all points of l.*

A polar space is a partial linear space [2, Theorem 3]. The *rank* of a polar space S is the supremum of the lengths m of chains $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_m$ of singular subspaces in S. A polar space of rank 2 is a generalized quadrangle.

Let $S = (P, L)$ be a finite polar space of rank $n \geq 2$. Every singular subspace of S is isomorphic to a projective space. The *dimension* of a singular subspace of S is the dimension of the associated projective space. Each maximal singular subspace of S has dimension $n - 1$ [2, Proposition 11]. For singular subspaces X and Y of S with $Y \subset X$, the *co-dimension* of Y in X is the dimension of X minus the dimension of Y. Consider the point-line geometry $DS = (P', L')$, where

- P' is the collection of all maximal singular subspaces of S;
- an element of L' is the collection of all maximal singular subspaces of S containing a specific singular subspace of S of co-dimension 1 in each of them.

Then DS is a partial linear space, called the *dual polar space of rank n* associated with S. Cameron characterized these geometries in terms of points and lines and proved that dual polar spaces of rank n are dense near $2n$-gons [4, Theorem 1, p. 75].

2.3 The dual polar space $DQ(2n, 2)$

Let $Q(2n, 2)$, $n \geq 2$, be a non-singular parabolic quadric in PG$(2n, 2)$. Then the points together with the lines of $Q(2n, 2)$ is a polar space of rank n and $DQ(2n, 2)$ is the associated dual polar space of rank n. Thus, the points of $DQ(2n, 2)$ are the generators (that is, subspaces of maximal dimension $n - 1$) of $Q(2n, 2)$ and a line of $DQ(2n, 2)$ is a triple of generators containing a specific $(n - 2)$-dimensional subspace of $Q(2n, 2)$. The dual polar space $DQ(2n, 2)$ is a slim dense near $2n$-gon. The quads of $DQ(2n, 2)$ are isomorphic to $W(2)$, the unique $(2, 2)$-GQ. For each point x of $DQ(2n, 2)$, the set $\Gamma_{<n}(x)$ is a maximal subspace of $DQ(2n, 2)$.

2.4 The near $2n$-gon $\mathbb{I}_n$

Again, consider a non-singular parabolic quadric $Q(2n, 2)$, $n \geq 2$, in $\mathsf{PG}(2n, 2)$ and a hyperplane of $\mathsf{PG}(2n, 2)$ which intersects $Q(2n, 2)$ in a non-singular hyperbolic quadric $Q^+(2n-1, 2)$. The set X of all generators of $Q(2n, 2)$ which are not contained in $Q^+(2n-1, 2)$ is a subspace of $DQ(2n, 2)$ [7, Theorem 6.46, p. 140]. The points and the lines of $DQ(2n, 2)$ contained in X define a slim dense near $2n$-gon [7, Theorem 6.48, p. 141], denoted by $\mathbb{I}_n$. For $n = 2$, the generalized quadrangle $\mathbb{I}_2$ is isomorphic to $Q^+(3, 2)$. For $n \geq 3$, each quad of $\mathbb{I}_n$ is either a $(2, 1)$-GQ or a $(2, 2)$-GQ.

2.5 Extraspecial 2-groups

A finite 2-group G is *extraspecial* if the Frattini subgroup $\Phi(G)$, the commutator subgroup $G' = [G, G]$ and the center $Z(G)$ of G coincide and have order 2. We refer to [9, section 20, p. 78,79] —see also [11, chapter 5, section 5]— for the following properties of an extraspecial 2-group.

An extraspecial 2-group is of order 2^{1+2m} for some integer $m \geq 1$. Let D_8 and Q_8, respectively, denote the *dihedral* and *quaternion groups* of order 8. A non-abelian 2-group of order 8 is extraspecial and is isomorphic to D_8 or Q_8. Let G be an extraspecial 2-group of order 2^{1+2m}. Then the exponent of G is 4 and either

(i) G is a central product of m copies of D_8, or

(ii) G is a central product of $m-1$ copies of D_8 and a copy of Q_8.

So, the maximum of the orders of its abelian subgroups is 2^{m+1}. In case (i), G possesses a maximal abelian subgroup of order 2^{m+1} which is elementary abelian. In case (ii), each maximal abelian subgroup of G is isomorphic to $C_2^{m-1} \times C_4$. Here, C_k denotes the cyclic group of order k. We denote an extraspecial 2-group of order 2^{1+2m} by 2_+^{1+2m} if (i) holds, and by 2_-^{1+2m} if (ii) holds.

3 Proof of Theorem 1.2

Let $S = (P, L)$ be a slim partial linear space and (R, ψ) be a representation of S. For each $x \in P$, we identify the subgroup $\psi(x) = \langle r_x \rangle$ of R with its non-trivial element r_x. If $x, y \in P$ and $x \sim y$, then we denote by xy the unique line of S containing x and y, and define $x * y$ by $xy = \{x, y, x * y\}$. So, $r_{x*y} = r_x r_y$ for every line $\{x, y, x * y\}$ of S, by condition (ii) of Definition 1.1. The following lemma is a particular case of [14, Proposition 3.1, p. 59]. We write down the proof here for the sake of completeness of this paper. (In the statement of [14, Proposition 3.1, p. 59], the polar space should be of order 2.)

Lemma 3.1. *Let $S = (P, L)$ be a $(2, t)$-GQ and (R, ψ) be a representation of S. Then R is abelian.*

Proof. We show that $[r_x, r_y] = 1$ for all $x, y \in P$ with $x \nsim y$. Let Q be a $(2, 1)$-subGQ in S containing x and y. Such a Q exists, follows from the fact that each line contains exactly 3 points. Let $\{x, y\}^{\perp} \cap Q = \{a, b\}$. In Q, $[r_b, r_y] = [r_b, r_x] = 1$ and $r_{(a*x)*(b*y)} = r_{(a*y)*(b*x)}$, implies that $r_x r_y = r_y r_x$. □

The dual polar space $DQ(4, 2)$ is a $(2, 2)$-GQ which is isomorphic to $W(2)$. By Lemma 3.1, every representation of it is abelian. This proves Theorem 1.2(i).

We next construct non-abelian representations for the near $2(n+1)$-gon $\mathbb{I}_{n+1}$ and the dual polar space $DQ(2n + 2, 2)$ in the group $2_+^{1+2.2^n}$ when $n \geq 2$. We make use of the following recursive constructions of $\mathbb{I}_{n+1}$ and $DQ(2n + 2, 2)$ given by De Bruyn [8].

Let $\mathbb{S}_n = (\mathbb{P}_n, \mathbb{L}_n)$ denote the dual polar space $DQ(2n, 2)$ of rank $n \geq 2$. The quads of $\mathbb{S}_n$ are $(2, 2)$-GQs. Every triad $\{a, b, c\}$ of points contained in a quad of $\mathbb{S}_n$ has the property that $\{a, b, c\}^{\perp}$ contains one or three points. In the latter case, such a triad $\{a, b, c\}$ is called a *hyperbolic line* of $\mathbb{S}_n$. Thus, $\{a, b, c\}$ is a hyperbolic line of $\mathbb{S}_n$ if and only if $\{a, b, c\}^{\perp}$ is so. Now, consider the point-line geometries $\mathcal{S}_{n+1} = (\mathcal{P}_{n+1}, \mathcal{L}_{n+1})$ and $\mathbb{S}_{n+1} = (\mathbb{P}_{n+1}, \mathbb{L}_{n+1})$ constructed from $\mathbb{S}_n$, where

$$\begin{aligned}
&\mathcal{P}_{n+1} = \{(x, y) \in \mathbb{P}_n \times \mathbb{P}_n : y \in x^{\perp}\}\,;\\
&\mathcal{L}_{n+1} = \big\{\{(x, u), (y, v), (z, w)\} : \{x, y, z\} \text{ is a line or a hyperbolic line of } \mathbb{S}_n\\
&\qquad\qquad \text{and } \{x, y, z\}^{\perp} = \{u, v, w\}\big\}\,;\\
&\mathbb{P}_{n+1} = \mathcal{P}_{n+1} \cup \mathbb{P}_n \cup \bar{\mathbb{P}}_n, \text{ where } \bar{\mathbb{P}}_n = \{\bar{x} : x \in \mathbb{P}_n\}\,;\\
&\mathbb{L}_{n+1} = \mathcal{L}_{n+1} \cup \mathcal{L}^1, \text{ where } \mathcal{L}^1 = \big\{\{x, (x, u), \bar{u}\} : (x, u) \in \mathcal{P}_{n+1}\big\}\,.
\end{aligned}$$

Then $\mathcal{S}_{n+1}$ is isomorphic to the near $2(n + 1)$-gon $\mathbb{I}_{n+1}$ and $\mathbb{S}_{n+1}$ is isomorphic to the dual polar space $DQ(2n + 2, 2)$ [8, section 1.5, Corollary 1.3 and Theorem 1.4].

Now, let $R = 2_+^{1+2.2^n}, n \geq 2$. The quotient group R/R' is an elementary abelian 2-group. Set $R' = \langle \theta \rangle$ and $V = R/R'$. Consider V as a vector space of dimension 2^{n+1} over $\mathbb{F}_2$. The map

$$f \colon V \times V \to \mathbb{F}_2$$

defined by

$$f(xR', yR') = i,$$

where $[x,y]=\theta^i$, $i\in\{0,1\}$, is a non-degenerate symplectic bilinear form on V [9, Theorem 20.4, p. 78]. We write V as an orthogonal direct sum of 2^n hyperbolic planes K_i, $1\le i\le 2^n$, in V with respect to f. Let H_i be the inverse image of K_i in R under the natural surjective homomorphism from R to V. Then H_i is generated by two elements x_i and y_i such that $[x_i,y_i]=\theta$. We set

$$M=\langle x_i : 1\le i\le 2^n\rangle\,;\quad \bar{M}=\langle y_i : 1\le i\le 2^n\rangle\,.$$

Then M and $\bar{M}$ are elementary abelian 2-subgroups of R each of order 2^{2^n}. The groups M, $\bar{M}$ and R' pairwise intersect trivially and $R=M\bar{M}R'$. Further, $C_{\bar{M}}(M)$ and $C_M(\bar{M})$ are trivial.

Let (M,τ) be the faithful abelian representation of $\mathbb{S}_n$ arising from the spin-embedding of $DQ(2n,2)$ in a vector space of dimension 2^n. We refer to [1] for a description of the spin-embedding. Then the following property $(\star)$ is satisfied:

> $(\star)$ *For every point x of $\mathbb{S}_n$, the subgroup $\langle m_y : y\in\Gamma_{<n}(x)\rangle$ is of index 2 in M.*

This embedding of $DQ(2n,2)$ is the so-called minimal full polarized embedding of $DQ(2n,2)$ in the sense of [5] and the property $(\star)$ is the condition of polarization for a projective full embedding.

Let Q be a quad in $DQ(2n,2)$. Let $G=\langle\tau(Q)\rangle$. Then (G,τ) is a faithful abelian representation of Q. Since Q is a $(2,2)$-GQ, G is of order 2^4 or 2^5. Since (M,τ) is minimal and polarized and Q is a convex subspace of $DQ(2n,2)$, it follows from [5, Theorem 1.6, p. 10] that (G,τ) is also minimal and polarized. This implies that G is of order 2^4.

Lemma 3.2. *Let a,b,c be three pairwise distinct points of Q. Then $T=\{a,b,c\}$ is a line or a hyperbolic line of Q if and only if $g_ag_bg_c=1$.*

Proof. First, assume that T is a hyperbolic line of Q. Let Q' be a $(2,1)$-subGQ of Q containing a and b. Then $c\notin Q'$ and $Q=\langle Q',c\rangle$. Let $\{x,y\}=\{a,b\}^\perp\cap Q'$. Then $x,y\in T^\perp$, since T is a hyperbolic line. Let z be the unique point in Q' such that $\{x,y,z\}$ is a triad of Q'. Then $c\sim z$ and $g_z=g_{a*x}g_{b*y}=(g_ag_x)(g_bg_y)$. Since the subgroup $H=\langle g_y : y\in x^\perp\cap Q\rangle$ is of index 2 in G, $|H|=2^3$ and $H=\langle g_x,g_a,g_b\rangle$. So g_c is equal to either g_ag_b or $g_ag_bg_x$, since τ is faithful. If the latter holds, then $g_{c*z}=g_cg_z=g_y$. But this is not possible, since $y\neq c*z$ and τ is faithful. Thus $g_c=g_ag_b$ and so $g_ag_bg_c=1$.

Now assume that $g_ag_bg_c=1$ and that T is not a line. Then T is a triad, since τ is faithful. We show that T is a hyperbolic line. Suppose that T is not a hyperbolic line. Then $|T^\perp|=1$. Let $\{a,b\}^{\perp\perp}=\{a,b,d\}$. Since $\{a,b,d\}$ is a hyperbolic line, $g_ag_bg_d=1$ by the first part. Since $|T^\perp|=1$, $c\neq d$ and

$g_c = g_a g_b = g_d$, a contradiction to that τ is faithful. Hence T is a hyperbolic line of Q. $\square$

For each point x of $\mathbb{S}_n$, set $H_x = \langle m_y : y \in \Gamma_{<n}(x)\rangle$. Since H_x is a maximal subgroup of M, the centralizer of H_x in $\bar{M}$ is a subgroup $\langle \bar{m}_x\rangle$ of order 2. Since $\Gamma_{<n}(x)$ is a maximal subspace of $\mathbb{S}_n$, $\mathbb{P}_n = \langle \Gamma_{<n}(x) \cup \{w\}\rangle$ and $M = \langle H_x, m_w\rangle$ for $w \in \Gamma_n(x)$. The triviality of $C_{\bar{M}}(M)$ implies that $[\bar{m}_x, m_w] = \theta$ for every $w \in \Gamma_n(x)$.

Recall that $\bar{\mathbb{P}}_n = \{\bar{x} : x \in \mathbb{P}_n\}$. Let $\bar{\mathbb{L}}_n = \big\{\{\bar{x}, \bar{y}, \bar{z}\} : \{x, y, z\} \in \mathbb{L}_n\big\}$. Then $\bar{\mathbb{S}}_n = (\bar{\mathbb{P}}_n, \bar{\mathbb{L}}_n) \simeq DQ(2n, 2)$. Let $\bar{\tau}$ be the map from the point set $\bar{\mathbb{P}}_n$ of $\bar{\mathbb{S}}_n$ to $\bar{M}$ defined by $\bar{\tau}(\bar{x}) = \bar{m}_x$.

Proposition 3.3. *$(\bar{M}, \bar{\tau})$ is a faithful abelian representation of $\bar{\mathbb{S}}_n$ satisfying the property $(\star)$.*

Proof. For $x \neq y$ in $\mathbb{P}_n$, $\Gamma_{<n}(x) \neq \Gamma_{<n}(y)$. So $H_x \neq H_y$ and $C_{\bar{M}}(H_x) \neq C_{\bar{M}}(H_y)$. This implies that $\bar{m}_x \neq \bar{m}_y$ and hence $\bar{\tau}$ is injective.

Let $\{x, y, z\}$ be a line of $\mathbb{S}_n$. Let $w \in \Gamma_{<n}(z)$. Then $d(w, x) \leq n - 1$ if and only if $d(w, y) \leq n - 1$, by the 'near polygon' property. So $([\bar{m}_x, m_w], [\bar{m}_y, m_w])$ is equal to either $(1, 1)$ or (θ, θ). Then

$$[\bar{m}_x\bar{m}_y, m_w] = [\bar{m}_x, m_w][\bar{m}_y, m_w] = 1.$$

The first equality holds, since R has nilpotent class 2. Thus, $1 \neq \bar{m}_x\bar{m}_y \in C_{\bar{M}}(H_z)$. Since $\bar{m}_z$ is the unique non-trivial element in $C_{\bar{M}}(H_z)$, it follows that $\bar{m}_z = \bar{m}_x\bar{m}_y$. So, $\bar{m}_x\bar{m}_y\bar{m}_z = 1$ for every line $\{\bar{x}, \bar{y}, \bar{z}\}$ of $\bar{\mathbb{S}}_n$. This verifies condition (ii) of Definition 1.1.

Now, let $K = \langle \bar{\tau}(\bar{\mathbb{P}}_n)\rangle$. Then $(K, \bar{\tau})$ is a faithful abelian representation of $\bar{\mathbb{S}}_n$. For each $\bar{x} \in \bar{\mathbb{P}}_n$, $H_{\bar{x}} = \langle \bar{m}_y : \bar{y} \in \Gamma_{<n}(\bar{x})\rangle$ is equal to K or is of index 2 in K. Since m_x commutes with each element of $H_{\bar{x}}$ and m_x does not commute with $\bar{m}_w$ for $w \in \Gamma_n(x)$, the first possibility does not occur. This implies that the property $(\star)$ holds.

Since $\bar{\mathbb{S}}_n \simeq DQ(2n, 2)$ does not possess a faithful polarized projective embedding in a vector space of dimension less than 2^n [5], it follows that $K = \bar{M}$. So condition (i) of Definition 1.1 holds, thus completing the proof. $\square$

By Proposition 3.3, a similar statement in Lemma 3.2 holds for the restriction of $\bar{\tau}$ to a quad of $\bar{\mathbb{S}}_n$. Now, let $\beta\colon \mathcal{P}_{n+1} \to R$ be defined by

$$\beta((x, y)) = m_x\bar{m}_y,$$

for $(x, y) \in \mathcal{P}_{n+1}$. Since $[m_x, \bar{m}_y] = 1$ for $x, y \in \mathbb{P}_n$ with $y \in x^{\perp}$, $\beta((x, y)) = m_x\bar{m}_y$ is of order 2 in R for every point (x, y) of $\mathcal{S}_{n+1}$.

Proposition 3.4. *(R, β) is a faithful non-abelian representation of $\mathcal{S}_{n+1} \simeq \mathbb{I}_{n+1}$.*

Proof. If $\beta((x, u)) = \beta((y, v))$, then $m_x \bar{m}_u = m_y \bar{m}_v$ implies that $m_y m_x = \bar{m}_v \bar{m}_u$. Since $M \cap \bar{M}$ is trivial, it follows that $m_x = m_y$ and $\bar{m}_u = \bar{m}_v$. This implies that β is one-one.

We now verify conditions (i) and (ii) of Definition 1.1. Let $x \in \mathbb{P}_n$. Let $\{x, y, z\}$ be a hyperbolic line of $\mathbb{S}_n$ containing x and let $u \in \{x, y, z\}^{\perp}$. Then

$$\beta((y, u))\, \beta((z, u)) = m_y \bar{m}_u m_z \bar{m}_u = m_y m_z = m_x\,.$$

The last equality follows from Lemma 3.2. Thus, $m_x \in \langle \beta(\mathcal{P}_{n+1}) \rangle$ for every $x \in \mathbb{P}_n$. This also implies that $\bar{m}_x \in \langle \beta(\mathcal{P}_{n+1}) \rangle$ for every $x \in \mathbb{P}_n$. In particular, M and $\bar{M}$ are contained in $\langle \beta(\mathcal{P}_{n+1}) \rangle$. Since R is generated by M and $\bar{M}$, we get $R = \langle \beta(\mathcal{P}_{n+1}) \rangle$. Now, let $\{(x, u), (y, v), (z, w)\}$ be a line of $\mathcal{S}_{n+1}$. We have

$$\beta((x, u))\, \beta((y, v)) = (m_x \bar{m}_u)(m_y \bar{m}_v) = m_x m_y \bar{m}_u \bar{m}_v r' = m_z \bar{m}_w r'\,,$$

where $r' = [\bar{m}_u, m_y]$. The last equality holds by Lemma 3.2, since $\{x, y, z\}$ and $\{\bar{u}, \bar{v}, \bar{w}\}$ are lines or hyperbolic lines of $\mathbb{S}_n$ and $\bar{\mathbb{S}}_n$ respectively. Since $y \in u^{\perp}$ in $\mathbb{S}_n$, we get $r' = 1$. So, $\beta((x, u))\beta((y, v)) = m_z \bar{m}_w = \beta((z, w))$. □

Proof of Theorem 1.2. Let $R, M, \bar{M}, \tau, \bar{\tau}$ and β be as in the above. Let ρ be the map from $\mathbb{P}_{n+1}$ to R defined by

$$\rho = \begin{cases} \tau & \text{on } \mathbb{P}_n\,; \\ \bar{\tau} & \text{on } \bar{\mathbb{P}}_n\,; \\ \beta & \text{on } \mathcal{P}_{n+1}\,. \end{cases}$$

Then $R = \langle \rho(\mathbb{P}_{n+1}) \rangle$. Also, condition (ii) of Definition 1.1 holds for every line in $\mathcal{L}^1$. As a consequence of Proposition 3.4, (R, ρ) is a faithful non-abelian representation of $\mathbb{S}_{n+1} \simeq DQ(2n+2, 2)$. This completes the proof. □

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