Innovations in Incidence Geometry
Volume 9 (2009), Pages 189–202
ISSN 1781-6475

Finite BN-pairs of rank 2, I

Koen Thas*

Abstract

One of the fundamental problems in Incidence Geometry is the classification of finite BN-pairs of rank 2 (most notably those of type B_2), without the use of the classification theorem for finite simple groups. In this paper, which is the first in a series, we classify finite BN-pairs of rank 2 (and the buildings that arise) for which the associated parameters (s,t) are powers of 2, and such that the associated polygon has no proper thick ideal or full subpolygons. As a corollary, we obtain the complete classification of generalized octagons of order (s,t) with st a power of 2, admitting a BN-pair. (For quadrangles and hexagons, this result will be obtained in part II.)

`Keywords:` BN-pair, generalized polygon, classification

`MSC 2000:` 51E12, 20B25, 20E42

1 BN-Pairs, associated buildings and Tits' conjecture

A group G is said to have a *BN-pair* (B,N), where B,N are subgroups of G, if the following properties are satisfied:

(BN1) $\langle B,N\rangle = G$;

(BN2) $H = B\cap N \lhd N$ and $N/H = W$ is a Coxeter group with distinct generators $s_1, s_2, \dots, s_n$;

(BN3) $Bs_iBwB \subseteq BwB \cup Bs_iwB$ whenever $w \in W$ and $i \in \{1,2,\dots,n\}$;

(BN4) $s_iBs_i \neq B$ for all $i \in \{1,2,\dots,n\}$.

*The author is a Postdoctoral Fellow of the Research Foundation, Flanders (FWO), Belgium.

The subgroup B, respectively W, is a *Borel subgroup*, respectively the *Weyl group*, of G. The natural number n is called the *rank* of the BN-pair. Throughout this paper, we only consider finite BN-pairs (the group G is assumed to be finite). To each finite BN-pair one can associate a synthetic combinatorial structure, called "building", and a vast literature is available on this subject. If the rank of an abstract spherical building is at least 3, J. Tits showed in the celebrated work [15], that it is always associated to a BN-pair, and this deep observation led him to classify all finite BN-pairs of rank ≥ 3 (cf. §11.7 of *op. cit.*). When the rank of a building $\mathfrak{B}$ is 2, it is, in general, not possibly to associate to $\mathfrak{B}$ a BN-pair in a "natural way"; this is because when the type of $\mathfrak{B}$ is $\mathsf{I}_{2(3)} = \mathsf{A}_2$ or B_2, non-classical examples exist that do not admit Chevalley groups as flag-transitive automorphism groups; see, e.g., [12] for several examples. For the types $\mathsf{I}_{2(6)} = \mathsf{G}_2$ and ${}^2\mathsf{F}_4 = \mathsf{I}_{2(8)}$ (which are the only other types possible — see the result of Feit-Higman quoted below), no such examples are known.

> **Conjecture** (J. Tits, [15, §11.5.1]). *If a finite building Δ of irreducible type and rank 2 is such that $\mathrm{Aut}(\Delta)$ permutes transitively the pairs consisting of a chamber and an apartment containing it (that is, if Δ is associated with a BN-pair), then Δ is isomorphic with the building of an absolutely simple algebraic group over a finite field, or with the building of a Ree group of type ${}^2\mathsf{F}_4$ over a finite field.*

Remark 1.1. Using the classification of finite simple groups (CFSG), F. Buekenhout and H. Van Maldeghem answered Tits' question affirmatively in [1]. However, Tits had a classification free proof in mind.

A famous result of W. Feit and D. Higman [4] states that *a finite generalized n-gon only exists if and only if $n \in \{3, 4, 6, 8\}$*. The classification of finite BN-pairs of rank 2 in automorphism groups of generalized 3-gons (projective planes) is a classical result [9].

From now on, the rank of a BN-pair (which corresponds to the rank of the associated building) is always 2. In that case, the Weyl group W is a dihedral group D_m (of size $2m$) for some natural number m. We say that the BN-pair (B, N) is *of type* B_2 if W is a dihedral group $\mathbf{D}_4$, and of type A_2 if it is dihedral of order 6. It is of type G_2, respectively ${}^2\mathsf{F}_4$, if W is dihedral of order 12, respectively 16.

One can associate a building $\mathfrak{B}(G)$ to the group G —where the BN-pair is still supposed to have rank 2— in the following way. For this purpose, define $P_1 = \langle B, B^{s_1} \rangle$ and $P_2 = \langle B, B^{s_2} \rangle$.

- Call the right cosets of P_1 “points”.
- Call the right cosets of P_2 “lines”.
- Call two such (distinct) cosets “incident” if their intersection is non-empty (so P_1g is incident with P_2h, $g, h \in G$, if $P_1g \cap P_2h \neq \emptyset$).

Then $\mathfrak{B}(G)$ is a “generalized n-gon” on which G acts by right multiplication, transitively permuting the ordered n-tuples of points which form an ordinary n-gon in the building. Combinatorially, a (finite) *generalized n-gon* ($n \geq 3$) is a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ for which the following axioms are satisfied:

(i) Γ contains no ordinary k-gon (as a subgeometry), for $2 \leq k < n$;

(ii) any two elements $x, y \in \mathcal{P} \cup \mathcal{B}$ are contained in some ordinary n-gon (as a subgeometry) in Γ;

(iii) there exists an ordinary $(n+1)$-gon (as a subgeometry) in Γ.

A *generalized polygon* (GP) is a generalized n-gon for some n. Detailed information about the combinatorial theory and characterizations can be found in [10, 19], see also the recent monographs [12, 11].

The (dual) classical examples of finite generalized polygons are the buildings which can be associated to the standard BN-pair in any of the groups listed below in Theorem 2.1, cf. §4 of the present paper.

Acknowledgment. I would like to thank Pierre Deligne and Hendrik Van Maldeghem for their useful remarks on a first set of drafts of this paper.

2 Split BN-pairs (of type B_2) and Condition (†)

Let G be a group with a BN-pair (B, N) of type B_2, and let P_1, P_2 be the two maximal parabolic subgroups containing B. For $i = 1, 2$, let $s_i \in G$ normalize N and P_i, but not P_{3-i}. Put $H = B \cap N$, as before. Then we define Condition $(\dagger)$ as follows.

$(\dagger)$ For some $i \in \{1, 2\}$, there exists a subgroup U of B with $U \trianglelefteq P_i$ such that $UH = B \cap B^{s_i}$.

A (general) BN-pair (B, N) is called *split* if Property (‡) below holds:

(‡) There exists a normal nilpotent subgroup U of B such that $B = UH$.

In a celebrated work [6, 7], P. Fong and G. M. Seitz determined all finite split BN-pairs of rank 2 (the B_2-case being, by far, the most complicated type to handle in [6, 7]):

Theorem 2.1 (P. Fong and G. M. Seitz [6, 7]). *Let G be a finite group with a split BN-pair of rank 2. Then G is an almost simple group related to one of the following classical Chevalley/Ree groups:*

(1) $\mathsf{PSL}_3(q)$;

(2) $\mathsf{PSp}_4(q) \cong \mathsf{O}_5(q)$;

(3) $\mathsf{PSU}_4(q) \cong \mathsf{O}_6^-(q)$;

(4) $\mathsf{PSU}_5(q)$;

(5) $\mathsf{G}_2(q)$;

(6) ${}^3\mathsf{D}_4(q)$, *or*

(7) *a Ree group of type* ${}^2\mathsf{F}_4$.

So, a thick finite generalized polygon is isomorphic, up to duality, to one of the classical examples if and only if it verifies the Moufang Condition (see [17] *for a definition of the latter notion).*

Clearly, for BN-pairs of type B_2 we have $(\ddagger) \Rightarrow (\dagger)$. In [14], K. Thas and H. Van Maldeghem proved that a finite group with a BN-pair of type B_2 satisfying $(\dagger)$ is essentially isomorphic to a classical group:

Theorem 2.2 (K. Thas and H. Van Maldeghem [14]). *Let G be a finite group with a BN-pair of type B_2, and suppose that G, B, N satisfies the condition $(\dagger)$. Then the BN-pair is split, hence G is an almost simple group related to one of the Chevalley groups listed in Theorem 2.1.*

Equivalently, a thick finite generalized quadrangle is isomorphic, up to duality, to one of the classical examples if and only if for each point there exists an automorphism group fixing it linewise and acting transitively on the set of its opposite points.

This theorem was obtained independently of Theorem 2.1.

3 BN-Pairs of rank 2 in even characteristic

We will here prove the following result for finite BN-pairs of rank 2 (up to a technical lemma which will be obtained in part II):

Theorem 3.1. *Let G be a finite group with an irreducible effective BN-pair of rank 2, and suppose that the parameters of the associated building are powers of 2. Suppose that the associated generalized polygon has no proper thick ideal or full subpolygons. Then the BN-pair is split, hence G is an almost simple group related to one of the following classical Chevalley/Ree groups:*

(1) $\mathsf{PSL}_3(q)$;

(2) $\mathsf{PSp}_4(q) \cong \mathsf{O}_5(q)$;

(3) $\mathsf{PSU}_4(q) \cong \mathsf{O}_6^-(q)$;

(4) $\mathsf{PSU}_5(q)$;

(5) $\mathsf{G}_2(q)$;

(6) ${}^3\mathsf{D}_4(q)$, *or*

(7) *a Ree group of type* ${}^2\mathsf{F}_4$.

Equivalently, a thick finite generalized n-gon of order (s,t) with st a power of 2 is isomorphic, up to duality, to one of the classical examples if $\mathrm{Aut}(\mathcal{S})$ acts transitively on its ordered ordinary n-gons, and if there are no proper thick ideal or full subpolygons.

For BN-pairs of type A_2, that is, when the associated building is a projective plane, the outcome of Theorem 3.1 is of course well-known.

Note again that by [4], only generalized n-gons with $n \in \{3,4,6,8\}$ have to be considered. For generalized 8-gons, the result is particularly interesting because for any *known* finite example the parameters are powers of 2, and for any thick finite generalized 8-gon the product of the parameters always is even [19]. Besides that, finite generalized octagons do not admit proper full or ideal suboctagons [19, 1.8.8].

On part II. In part II, we will prove the technical Proposition 4.2 below, which will be used in the proof of the main result. We will also obtain the results described in §5 (with particular emphasis on the quadrangles, and examples), and further study GPs with a BN-pair, while containing a proper thick full or ideal subpolygon. When st is a power of 2, we will complete the classification started in this paper.

4 BN-Pairs of type B_2, G_2 and ${}^2\mathsf{F}_4$

We now sketch the proof of Theorem 3.1 for BN-pairs of type B_2, G_2 and ${}^2\mathsf{F}_4$.

4.1 Setting and notation

We assume that the reader is well acqainted with the standard terminology (such as “collinearity”, “concurrency”, “opposition”, “automorphism groups”, etc.) on incidence geometries — for what is needed in this paper, the references [10, 19] are sufficient.

The Ree-Tits octagon. We do not aim to precisely define the only known class, up to duality, of finite generalized octagons, namely the Ree-Tits octagons. The reader is referred to J. Tits [16] for a detailed account, and also the book of H. Van Maldeghem [19]. We only give a very short indirect description. Let $\mathcal{M}$ be a finite metasymplectic space, that is, a building of type F_4, of which the planes are defined over the finite field $\mathbb{F}_q$, q a power of 2. Suppose also that $\mathcal{M}$ admits a polarity — so that q is an odd power of 2. Then the absolute points and lines of this polarity form, together with the natural incidence, a generalized octagon of order (q, q^2) which we denote by $\mathbf{O}(q)$, and which admits a Ree group of type ${}^2\mathsf{F}_4$ as an automorphism group which acts transitively on the ordered ordinary 8-gons. Together with its point-line dual $\mathbf{O}(q)^D$, the Ree-Tits octagons are the only class of generalized octagons known presently.

The classical generalized hexagons. These objects are much better understood and investigated than the Ree-Tits octagons, and the reader is referred to [19] for details. Let us just mention that they are related to the simple groups G_2 and ${}^3\mathsf{D}_4(q)$ via the standard BN-pairs in these groups.

The classical generalized quadrangles. Similarly as for the hexagons, we refer to [10] for more information.

Setting. We set some more notation.

- Γ is supposed to be a finite generalized quadrangle (4-gon), hexagon (6-gon) or octagon (8-gon) of order (s, t), where st is a power of 2, with $\{s, t\} \cap \{0, 1\} = \emptyset$.
- $\mathrm{Aut}(\Gamma)$ is the automorphism group of Γ, while $\mathrm{Aut}(K)$ is the automorphism group of a group K.

- The group $G^\dagger$ is a subgroup of $\mathrm{Aut}(\Gamma)$ which admits an irreducible, effective BN-pair (B,N) of type B_2, G_2 or ${}^2\mathsf{F}_4$. So $G^\dagger$ acts faithfully on the points (and lines) of Γ.
- L is an arbitrary but fixed line of Γ. One of the purposes is to derive the main result from a more general local result, so one may want to keep the local arguments in mind. Often, we will identify L, and any other line, with the set of points incident with it.
- $G = G^\dagger_L$.

4.2 Statement and proof of the theorem

We are ready to obtain the main result of the present paper.

Let M be any line opposite L (the lines are at maximal distance in the line graph). Then G_M induces a 2-transitive group on (the points incident with) L and M of degree $s+1$, and its isomorphism class is independent of M. Let K be the kernel of the action of G on the points incident with L; the Levi-factor corresponding with L is

$$\mathfrak{L} = G/K\,. \tag{1}$$

Let θ be a nontrival 2-central involution in $G_M/(G_M \cap K)$. If θ would fix more than one point incident with L, all points of L (and M) are fixed, since s is a power of 2, contradiction. (Note that a Sylow 2-subgroup of $G_M/(G_M \cap K)$ fixes some point r of L, and is transitive on $L \setminus \{r\}$.) So θ fixes precisely one point incident with L.

We now address the following result of D. Holt (which is stated less generally here than in [8]):

Theorem 4.1 (D. Holt [8]). *Let (T,X) be a finite 2-transitive permutation group of odd degree k. Suppose T contains a 2-central involution ϕ that fixes exactly one letter, and let $\mathcal{J}$ be the set of involutions that also fix exactly one letter. Put $\langle \mathcal{J} \rangle = H$. Then one of the following cases occurs:*

(i) $H = \langle \mathbf{O}(H), \phi \rangle$; [1]

(ii) $|X| \equiv 1 \mod 4$ *and* $H \cong \mathbf{A}_k$ *in its action on* X *or* $|X| \equiv 3 \mod 4$ *and* $H \cong \mathbf{S}_k$ *in its action on* X;

(iii) (H,X) *is isomorphic to one of the simple groups* $\mathsf{PSL}_2(2^n)$, $n>1$, $\mathsf{PSU}_3(2^n)$, $n>1$, *or* $\mathsf{Sz}(2^n)$, $n = 2m+1 \geq 3$, *in its natural* 2*-transitive representation.*

[1] $\mathbf{O}(H)$ is the unique "largest" normal subgroup of H of odd order.

Put $\mathcal{J} = \{$involutions in $G_M/(G_M \cap K)$ which fix precisely one point incident with $L\}$.

We first consider Theorem 4.1, case (ii), with $H = \langle \mathcal{J} \rangle$. Note that $|G| = |\{\text{lines opposite } L\}| \times |G_M|$, and that $|K| = |K_M| \times |M^K|$. Let $G_M/(G_M \cap K) \cong G_M K/K = \mathfrak{L} = G/K$; then $|G| = |G_M/(G_M \cap K)| \times |K_M| \times |M^K|$. Whence $|G_M| = |G_M/(G_M \cap K)| \times |K_M|$ leads to $|M^K| = |\{\text{lines opposite } L\}|$ — that is, K acts transitively on the lines opposite L. In other words

$$\Gamma \text{ is half 1-Moufang (w.r.t. the lines of } \Gamma\text{) in the terms of [18, 19].} \qquad (*)$$

This situation is handled further on.

Suppose $G_M/(G_M \cap K) \neq \mathfrak{L}$; then clearly

$$\begin{aligned} G_M/(G_M \cap K) &\cong \mathbf{A}_{s+1} \\ &\cong \langle G_U \parallel U \text{ opposite } L \rangle \,/\, (\langle G_U \parallel U \text{ opposite } L \rangle \cap K)\,, \end{aligned} \qquad (2)$$

and $\mathfrak{L} \cong \mathbf{S}_{s+1}$.

Let Λ be any ordinary n-gon containing L as a side, let $x \,\mathbf{I}\, L$, $x \in \Lambda$, Y be the line in Λ which is opposite L, and y be the point in Λ opposite x. Let $P_1 := G_Y/(G_Y \cap K) \cong \mathbf{A}_{s+1}$. Consider the action of $(G_\Lambda)_x$ on the lines incident with x (or y) not contained in Λ; then just by comparing sizes, it is not hard to see that w.l.o.g. we can suppose that $(G_x^\dagger)_y/((G_x^\dagger)_y \cap K') =: P_2$, where K' is the kernel of the action of $G_x^\dagger$ on the lines through x, contains $\mathbf{A}_{t+1}$. (By for instance counting $|G_x|$ in two ways, one obtains that

$$(t+1)t|K \cap G_Y| \cdot |P_1| = (s+1)s|K' \cap (G_x^\dagger)_y| \cdot |P_2|\,. \qquad (3)$$

Put $|P_1| = (s+1)!/2$ in this equation, and observe that, if there are no proper thick full or ideal subpolygons, then $|K' \cap (G_x^\dagger)_y| \cdot |K \cap G_Y|$ is odd. Case (i) of Theorem 4.1 (in this dual situation) is immediate — cf. the relevant part of the proof below. Now consider the largest powers of 2 dividing both sides of (3) to exclude case (iii) (minus case (ii)).)

So without loss of generality, we may assume that $t \leq s$ when handling case (ii).

In this paragraph, we will also suppose that $s > 4$. (The cases $s = 2, 4$ are left to the reader, but can equally be found in [1], since CFSG is not needed for these values of s.) Now choose an involution α in G_M which fixes precisely $s - 3$ points of M (note that such an involution must exist![2]); then α fixes a

[2] Let $x_1, \ldots, x_{s-3}$ be distinct points on L, and let D be the subgroup of G_M fixing these points. Then $|D|$ is even, so D contains an involution, say β. Since β induces an element of $\mathbf{A}_{s+1}$, it either fixes precisely $x_1, \ldots, x_{s-3}$ on L, or it fixes all points incident with L. But then, since $t - 1$ is odd, Γ would have a proper thick full subGP, contradiction.

thick proper subpolygon of Γ elementwise, say Γ', of order $(s-3,t')$, since $t-1$ is odd. The standard inequalities $s \geq s't'$ for $n=4$ and $st \geq {s'}^2{t'}^2$ for $n=6$ [19, 1.8.12] now lead to a contradiction. For $n=8$, we have the inequality

$$s'{t'}^2 \leq st\,, \tag{4}$$

see [20]. Substituting $s' = s-3$ in (4) and taking the fact that $s \geq t$ into account (so that $2t \leq s$ since $2st$ is a perfect square for octagons), together with Higman's inequality for octagons [19, 1.7.2], we obtain a contradiction.

We now suppose to be in Theorem 4.1, case (i), that is, we assume

$$H = \langle \mathcal{J} \rangle = \langle \mathbf{O}(H), \phi \rangle\,, \tag{5}$$

with ϕ as in the statement of Theorem 4.1. Then $|H| = 2|\mathbf{O}(H)|$, as $\mathbf{O}(H)\langle\phi\rangle = \langle\phi\rangle\mathbf{O}(H)$, while H must act transitively on L. Recall the classical result of Burnside saying that a finite 2-transitive permutation group contains a unique minimal normal subgroup, which is either elementary abelian or nonabelian simple. For $G_M/(G_M \cap K)$, we will denote this subgroup by $N(G_M)$.

Suppose that $N(G_M)$ is simple. By W. Feit and J. Thompson [5], $N(G_M)$ either has even size, or its order is a prime. Let us first assume that the size is even. As $N(G_M) \cap \mathbf{O}(H) \trianglelefteq N(G_M)$, this intersection must be trivial. As both groups normalize each other, they commute. Since $|N(G_M)| > s+1$ (as a transitive group of even size), we can choose a nonidentity element α of $N(G_M)$ which fixes a point of L. Suppose $\mathbf{O}(H)$ is not transitive on L. It is then clear, since $H = \mathbf{O}(H)\langle\phi\rangle$ and H acts transitively on L, that $\mathbf{O}(H)$ must have (precisely) two distinct orbits on L. This contradicts the fact that $s+1 = |L|$ is odd. So $\mathbf{O}(H)$ is transitive. The fact that $\mathbf{O}(H)$ acts transitively on L leads to the conclusion that α must act as the identity on L, so we have a contradiction.

Now suppose that $N(G_M)$ is an elementary abelian ℓ-group for the odd prime ℓ. Then $N(G_M)$ acts sharply transitively on L, so that $s+1 = \ell^n$ for the positive integer n. Put $s = 2^h$. An elementary arithmetic exercise leads to the fact that either $n=1$ or $\ell = 3, n = 2, h = 3$.

We first handle the case $n = 1$. Choose a point $x\,\mathbf{I}\,L$. Then $(G_M)_xK/K$ can be naturally (and faithfully) interpreted as an automorphism group of the affine 1-space $\mathsf{AG}(1,\ell)$ (by its action by conjugation on $N(G_M)$). So $(G_M)_xK/K \leq \mathsf{AGL}_1(\ell) \cong \mathbb{F}_\ell^\times$, and G_MK/K is a sharply 2-transitive group on L. But then each element of $(G_M)_x$ is 2-central while fixing only one point, which contradicts $|H| = 2\ell$ unless $(s,p) = (2,3)$ (a case which is known without CFSG).

Let $(n,\ell,h) = (2,3,3)$. One can fairly easy handle this case using the list of 2-transitive groups of degree 9 [2], but since this is done in [13], the reader is referred to *op. cit.* for the details.

Now suppose that $\langle \mathcal{J} \rangle / (\langle \mathcal{J} \rangle \cap K) =: P_1$ is isomorphic to one of the simple groups $\mathsf{PSL}_2(2^n)$, $n > 1$, $\mathsf{PSU}_3(2^n)$, $n > 1$, or $\mathsf{Sz}(2^n)$, $n = 2m+1 \geq 3$, in its natural 2-transitive representation of degree $s+1$. First note that, since G/K also satisfies the conditions of Theorem 4.1, we know that G/K also falls in one of the three possibilities of Theorem 4.1. It follows that if $\mathcal{J}'$ is the set of involutions of G that fix precisely one point of L, either $\langle \mathcal{J}' \rangle / (K \cap \langle \mathcal{J}' \rangle) =: P_2$ is isomorphic to one of the simple groups $\mathsf{PSL}_2(2^n)$, $n > 1$, $\mathsf{PSU}_3(2^n)$, $n > 1$, or $\mathsf{Sz}(2^n)$, $n = 2m+1 \geq 3$, or Property $(*)$ is satisfied. The only possible obstruction would be case (ii). (Case (i) can be handled as before.) Suppose by way of contradiction that $P_2 \cong \mathbf{A}_{s+1}$. Then

$$|\langle \mathcal{J}' \rangle| = |M^{\langle \mathcal{J}' \rangle}|\, |\langle \mathcal{J}' \rangle_M|\,. \tag{6}$$

Note that $|M^{\langle \mathcal{J}' \rangle}|$ is a power of 2 since $\langle \mathcal{J}' \rangle \trianglelefteq G$, and that $\langle \mathcal{J} \rangle \trianglelefteq \langle \mathcal{J}' \rangle_M$. So clearly we can write

$$|\mathbf{A}_{s+1}| = (s+1)!/2 = 2^m \cdot |P_1| \cdot \theta, \tag{7}$$

where $m \in \mathbb{N}$, and θ divides the size of $\mathrm{Aut}(P_1)$. For $s \geq 8$ we now easily obtain a contradiction by comparing the odd factors of both sides of (7). The cases $s = 2, 4$ are left to the reader.

Now suppose that $P_1 = P_2$. From the following observation, which we will obtain in part II, one can now decide that either (B, N) is split, or Property $(*)$ is satisfied (w.r.t. lines).

Proposition 4.2. *(We use the same notation as in the current proof.) If for every $A \in \{G_M/(G_M \cap K), \langle G_U \parallel U \text{ opposite } L\rangle / (\langle G_U \parallel U \text{ opposite } L\rangle \cap K), G/K\}$ we have that $S \leq A \leq \mathrm{Aut}(S)$, with S the little projective group of a split BN-pair of rank 1 (acting in the expected way), then either K is transitive on the lines opposite L, or (B, N) is split.*

By interchanging the role of points and lines —which we can do since Γ admits a BN-pair— Γ is half 1-Moufang w.r.t. points, so, by definition, Γ is 1-Moufang. By a result of H. Van Maldeghem [18], we can now conclude that Γ is a Moufang generalized polygon, which group theoretically translates in the fact that (B, N) is split. Theorem 2.1 leads us to the desired result. $\square$

(For terminology on split BN-pairs of rank 1, or "Moufang sets", we refer to [3].)

4.3 Implication

We briefly state another corollary of Theorem 3.1, which was open without CFSG.

Theorem 4.3. *A finite generalized octagon of order (s,t), where st is a power of 2, is half 1-Moufang if and only if it is isomorphic to a Ree-Tits octagon $\mathbf{O}(s)$, s then being an odd power of 2, if and only if it admits a BN-pair.*

Proof. The result follows from the fact that a half 1-Moufang generalized octagon admits a BN-pair, and Theorem 3.1 (noting again that finite octagons do not have proper thick full or ideal suboctagons [19, 1.8.8]). $\square$

Remark 4.4. The author is, at present, trying to show that the extra assumption on the parameters is always satisfied for a BN-pair of type ${}^2\mathsf{F}_4$.

5 From "local" to "global" BN-pairs (of type B_2)

In this final section, we describe, without proof, some of the results of [13] on BN-pairs of type B_2. The starting point of this section is the following observation [13]:

> *Let Γ^* be any of the (dual) classical generalized quadrangles in even characteristic, and say its order is (s,t), and hence st is a power of 2. Let (B^*,N^*) be an irreducible effective BN-pair in $\langle B^*,N^*\rangle \leq \mathrm{Aut}(\Gamma^*)$ for which $\mathfrak{B}(\langle B^*,N^*\rangle) \cong \Gamma^*$, and let $\{L^*,M^*\}$ be any pair of non-concurrent lines. Then no central Baer quadrangle exists "in" $\langle B^*,N^*\rangle_{L,M}$.*

The "Baer quadrangle" alluded to in the previous statement is, by definition, a thick sub generalized quadrangle which is fixed pointwise (i.e. elementwise) by a 2-central involution of some Sylow 2-subgroup in $\langle B^*,N^*\rangle_{L,M}$.

Let $\mathfrak{B}(\langle B,N\rangle) =: \Gamma$ be the building which is associated to the irreducible effective BN-pair (B,N) of type B_2. Then the following property is easy to prove for any line L:

> (BN)$_L$: $\mathrm{Aut}(\Gamma)_L$ *acts transitively on the ordered pairs (x,y) of points for which $x \sim y \neq x$ and $xy \not\sim L$.*

The first main result of [13] is essentially the following, which describes a classification of generalized quadrangles which satisfy the local property (BN)$_L$ for at least one line L, together with an assumption motivated by the first observation of this section.

Theorem 5.1. *Let Γ be a thick finite generalized quadrangle of order (s,t), and suppose G is a subgroup of $\mathrm{Aut}(\Gamma)$ that fixes some line L, and acts transitively* *on the ordered pairs (x,y) for which $x \sim y \neq x$, and $xy \not\sim L$. Suppose furthermore that s and t are even, and that not every 2-central involution in G_M fixes a Baer subquadrangle, for some $M \not\sim L$, pointwise. Then we have the following possibilities.*

(a) *Let K be the subgroup of G that fixes L pointwise. Then s is a power of 2, and $H \trianglelefteq G/K \leq \mathrm{Aut}(H)$, where H is isomorphic to one of $\mathsf{PSL}_2(s)$, $\mathsf{Sz}(\sqrt{s})$, $\mathsf{PSU}_3(\sqrt[3]{s^2})$. Furthermore, if t is a power of 2, and $\mathfrak{O} = \{\mathbb{1}\}$, where*

$$\mathfrak{O} := (G/Z) \,/\, \big(\langle G_R \parallel R \not\sim L\rangle K/K\big)\,, \tag{8}$$

then for any point $x \,\mathbf{I}\, L$, G_x contains a normal nilpotent subgroup U such that $G_x = U(G_x \cap N)$, with N the stabilizer in G of any ordinary 4-gon containing L as a side and x as an edge.

(b) *There is an odd prime ℓ such that, when denoting the elementary abelian ℓ-group of order ℓ^m by E_{ℓ^m}, we have*

$$G/K = \mathsf{E}_{\ell^n} \rtimes (G/K)_x\,, \tag{9}$$

with x any point incident with L, such that $s+1 = \ell^n$ for some integer n. The group $(G/K)_x$ is isomorphic to some subgroup of $\mathsf{A\Gamma L}_n(\ell)$, and st divides $|K|$.

Finally, we mention that similar local results as those described in Theorem 5.1 can be found for the cases $n = 6$ and $n = 8$ — see part II. However, for BN-pairs of type B_2, this local theory seems to make more sense, because there, effectively, nonclassical finite generalized quadrangles exist which admit “local BN-pairs”. There even exist nonisomorphic finite generalized quadrangles which admit isomorphic “local BN-pairs” (cf. part II).

References

[1] **F. Buekenhout** and **H. Van Maldeghem**, Finite distance-transitive generalized polygons, *Geom. Dedicata* **52** (1994), 41–51.

[2] **J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker** and **R. A. Wilson**, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups.* With computational assistance from J. G. Thackray, Oxford University Press, Eynsham, 1985.

[3] **T. De Medts** and **Y. Segev**, A course on Moufang sets, *Innov. Incidence Geom.* **9** (2009), 79–122.

[4] **W. Feit** and **D. Higman**, The nonexistence of certain generalized polygons, *J. Algebra* **1** (1964), 114–131.

[5] **W. Feit** and **J. Thompson**, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 755–1029.

[6] **P. Fong** and **G. M. Seitz**, Groups with a (B, N)-pair of rank 2. I, *Invent. Math.* **21** (1973), 1–57.

[7] ______, Groups with a (B, N)-pair of rank 2. II, *Invent. Math.* **24** (1974), 191–239.

[8] **D. F. Holt**, Transitive permutation groups in which an involution central in a Sylow 2-subgroup fixes a unique point, *Proc. London Math. Soc.* **37** (1978), 165–192.

[9] **T. G. Ostrom** and **A. Wagner**, On projective and affine planes with transitive collineation groups, *Math. Z.* **71** (1959), 186–199.

[10] **S. E. Payne** and **J. A. Thas**, *Finite generalized quadrangles*, Research Notes Math. **110**, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984.

[11] **J. A. Thas, K. Thas** and **H. van Maldeghem**, *Translation generalized quadrangles*, Series Pure Math. **26**, World Scientific, Singapore, 2006.

[12] **K. Thas**, *Symmetry in finite generalized quadrangles*, Front. Math. **1**, Birkhäuser-Verlag, Basel, Boston, Berlin, 2004.

[13] ______, Finite BN-pairs of rank 2, II, *in preparation*.

[14] **K. Thas** and **H. Van Maldeghem**, Geometric characterizations of Chevalley groups of Type B_2, *Trans. Amer. Math. Soc* **360** (2008), 2327–2357.

[15] **J. Tits**, *Buildings of spherical type and finite BN-Pairs*, Lecture Notes in Math. **386**, Springer, Berlin, 1974.

[16] ______, Moufang octagons and the Ree groups of type ${}^2F_4$, *Amer. J. Math.* **105** (1983), 539–594.

[17] **J. Tits** and **R. Weiss**, *Moufang polygons*, Springer Monogr. Math., Springer-Verlag, Berlin, 2002.

[18] **H. Van Maldeghem**, Some consequences of a result of Brouwer, *Ars Combin.* **48** (1998), 185–190.

[19] ______, *Generalized polygons*, Monogr. Math. **93**, Birkhäuser-Verlag, Basel, Boston, Berlin, 1998.

[20] **A. Yanushka**, A restriction on the parameters of a suboctagon, *J. Combin. Theory Ser. A* **26** (1979), 193–196.

Koen Thas

Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281, S22, B-9000 Ghent, Belgium

e-mail: kthas@cage.UGent.be