



Transitive groups of axial homologies of hyperbola structures and Minkowski planes

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Abstract

In this paper a typification of the automorphism groups of hyperbola structures based on the notion of axial homologies (i.e. automorphisms fixing two generators of the same kind) is given. For the class of hyperbola structures over half-ordered fields (cf. [6, 12]) the types of the full automorphism groups are determined.

Keywords: axial homology, hyperbola structure, Minkowski plane

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1 Introduction

Analogous to the well known Lenz-Barlotti typification of projective planes there are typifications for Minkowski planes based on the notion of G - and p -translations and (p, q) -homologies (cf. [9, 8]). The typification with respect to transitive groups of G -translations was done actually for the more extensive class of hyperbola structures (cf. [9]). Beside the G -translations axial homologies can be defined for hyperbola structures. An automorphism α of a hyperbola structure \mathcal{H} is called a (G, H) -homology if α fixes pointwise the generators G, H (where $G \cap H = \emptyset$). Let $A(G, H)$ denote the group of all (G, H) -homologies. For a given subgroup Γ of the automorphism group of \mathcal{H} we are interested in the configuration $\mathfrak{A}(\Gamma)$ of all 2-sets $\{G, H\}$ of generators such that the group $A(G, H) \cap \Gamma$ acts transitively on $X \setminus (G \cup H)$ for a generator X . By determining all possible configurations $\mathfrak{A}(\Gamma)$ we obtain a typification for the automorphism groups of hyperbola structures (cf. Theorem 3.3). In the case of a hyperbola

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structure defined by a sharply 3-transitive permutation group Lemma 3.5 yields examples of automorphism groups of every type. In particular we deal with characterizations of the hyperbola structures defined by sharply 3-transitive permutation groups (cf. Theorem 3.9).

The type of the full automorphism group of a hyperbola structure \mathcal{H} will be called the type of \mathcal{H} . Just as in the theory of projective planes the typification causes the problem to determine the hyperbola structures of each type, or at least to give examples. In the last section of this paper the types of the hyperbola structures over half-ordered fields constructed by J. Jakóbowski [6] will be determined (cf. Theorem 4.12, Lemma 4.6, Theorem 4.9).

2 Hyperbola structures: definitions and notations

Hyperbola structures can be defined in different ways (cf. [5, 1, 7]). For our purpose the representation by permutation sets is most suitable. Let (M, Σ) be a permutation set¹. Putting $P := M \times M$, $\mathfrak{K} := \{(x, \gamma(x)) \mid x \in M\} \mid \gamma \in \Sigma\}$ and $\mathfrak{G}_1 := \{\{a\} \times M \mid a \in M\}$, $\mathfrak{G}_2 := \{M \times \{a\} \mid a \in M\}$ we obtain an incidence structure $\mathcal{H}(M, \Sigma) = \mathcal{H}(\Sigma) := (P, \mathfrak{K}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$; the elements of P , \mathfrak{K} and $\mathfrak{G} := \mathfrak{G}_1 \cup \mathfrak{G}_2$ are called *points*, *chains* and *generators* respectively, and $n := |M| - 1$ is called the *order* of $\mathcal{H}(M, \Sigma)$. For every point $x \in P$ and $i \in \{1, 2\}$ there is exactly one generator $X \in \mathfrak{G}_i$ with $x \in X$ (notation: $[x]_i := X$). Then $\mathcal{H}(M, \Sigma)$ is called a *hyperbola structure*, if Σ is acting sharply 3-transitively on M . The chains of a hyperbola structure are called *circles*.

Let $\mathcal{H} = (P, \mathfrak{K}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ be a hyperbola structure. For every point $w \in P$ we define: $[w] := [w]_1 \cup [w]_2$, $P^w := P \setminus [w]$, $\mathfrak{K}(w) := \{X \in \mathfrak{K} \mid w \in X\}$, $\mathfrak{K}^w := \{X \setminus \{w\} \mid X \in \mathfrak{K}(w)\}$, $\mathfrak{G}_i^w := \{X \cap P^w \mid X \in \mathfrak{G}_i, w \notin X\}$ for $i \in \{1, 2\}$, and $\mathcal{H}^w := (P^w, \mathfrak{K}^w, \mathfrak{G}_1^w \cup \mathfrak{G}_2^w)$, $\mathcal{A}(w) := (P^w, \mathfrak{K}^w \cup \mathfrak{G}_1^w \cup \mathfrak{G}_2^w)$. Then \mathcal{H}^w is called the *derivation of \mathcal{H} in the point w* . Moreover, \mathcal{H} is called a *Minkowski plane*, if $\mathcal{A}(w)$ is an affine plane for all $w \in P$; in this case $\mathcal{A}(w)$ is called the *affine derivation of \mathcal{H} in w* .

Let $\mathcal{H}(M, \Sigma)$ and $\mathcal{H}(M', \Sigma')$ be two hyperbola structures and $\varphi: P \rightarrow P'$ a bijection. The bijection φ is called an *isomorphism from $\mathcal{H}(M, \Sigma)$ onto $\mathcal{H}(M', \Sigma')$* if $\varphi(\mathfrak{K}) = \mathfrak{K}'$.

Let φ be an isomorphism. Then $\varphi(\mathfrak{G}_1) = \mathfrak{G}'_1$, $\varphi(\mathfrak{G}_2) = \mathfrak{G}'_2$ or $\varphi(\mathfrak{G}_1) = \mathfrak{G}'_2$, $\varphi(\mathfrak{G}_2) = \mathfrak{G}'_1$.

For every isomorphism φ there exist two bijections $\alpha, \beta: M \rightarrow M'$, such that $\varphi(x, y) = (\alpha(x), \beta(y))$ for all $(x, y) \in P$ or $\varphi(x, y) = (\alpha(y), \beta(x))$ for all

¹i.e. Σ is a set of permutations of the set M

$(x, y) \in P$.

Lemma 2.1 (Wefelscheid [14]). *Let $\mathcal{H}(M, \Sigma)$ and $\mathcal{H}(M', \Sigma')$ be two hyperbola structures and $\alpha, \beta: M \rightarrow M'$ two bijections.*

- (1) $\varphi: (x, y) \mapsto (\alpha(x), \beta(y))$ is an isomorphism iff $\Sigma' = \beta \Sigma \alpha^{-1}$.
- (2) $\varphi: (x, y) \mapsto (\alpha(y), \beta(x))$ is an isomorphism iff $\Sigma' = \beta \Sigma^{-1} \alpha^{-1}$.

An automorphism φ of a hyperbola structure \mathcal{H} is called *proper*, if $\varphi(\mathfrak{G}_1) = \mathfrak{G}_1$.

The following notation is convenient to describe the proper automorphisms of $\mathcal{H}(M, \Sigma)$. For $\alpha \in \text{Sym } M$ let denote $\acute{\alpha}: M \times M \rightarrow M \times M$, $(x, y) \mapsto (\alpha(x), y)$ and $\grave{\alpha}: M \times M \rightarrow M \times M$, $(x, y) \mapsto (x, \alpha(y))$. For every proper automorphism φ there exist $\alpha, \beta \in \text{Sym } M$ with $\varphi = \acute{\alpha} \circ \beta$.

The classical model of a Minkowski plane is $\mathcal{H}(\text{PGL}(2, \mathbb{F}))$, where \mathbb{F} is a commutative field. Every Minkowski plane isomorphic to a plane $\mathcal{H}(\text{PGL}(2, \mathbb{F}))$ is called a *Miquelian* Minkowski plane.

Modifying the classical model J. Jakobowski [6] constructed a class of non-Miquelian Minkowski planes. Let (\mathbb{F}, P) be a half-ordered field, i.e. P is a subgroup of index 2 of the multiplicative group (\mathbb{F}^*, \cdot) of the field $(\mathbb{F}, +, \cdot)$. Let $\chi: \mathbb{F}^* \rightarrow \{-1, 1\}$ denote the character belonging to P , $\overline{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$ the projective line over \mathbb{F} and

$$\overline{\mathbb{F}}^{(3)} := \{(x_1, x_2, x_3) \in \overline{\mathbb{F}}^3 \mid |\{x_1, x_2, x_3\}| = 3\}.$$

For $(x_1, x_2, x_3) \in \overline{\mathbb{F}}^{(3)}$ we define $\omega(x_1, x_2, x_3) = \chi(x_1 - x_2)\chi(x_2 - x_3)\chi(x_3 - x_1)$ where $\chi(x - y) := \begin{cases} 1 & \text{for } x = \infty \\ -1 & \text{for } y = \infty \end{cases}$.

A permutation $f \in \text{Sym } \overline{\mathbb{F}}$ is called *order-preserving* (*order-reversing*) if for all $(x_1, x_2, x_3) \in \overline{\mathbb{F}}^{(3)}$,

$$\omega(f(x_1), f(x_2), f(x_3)) = \omega(x_1, x_2, x_3) \quad (= -\omega(x_1, x_2, x_3))$$

holds.

We define Π^+ and Π^- to be the set of all order-preserving and all order-reversing permutations of $\overline{\mathbb{F}}$ respectively. Finally we put $\text{PGL}^+ := \text{PGL}(2, \mathbb{F}) \cap \Pi^+$ and $\text{PGL}^- := \text{PGL}(2, \mathbb{F}) \cap \Pi^-$. Then $\text{PGL}(2, \mathbb{F}) = \text{PGL}^+ \dot{\cup} \text{PGL}^-$.

Theorem 2.2 (Jakóbowski 1993). *Let (\mathbb{F}, P) be an half-ordered field and let $f, g \in \Pi^+$. Then $\Sigma(f, g) := \text{PGL}^+ \cup g^{-1}\text{PGL}^-f$ acts sharply 3-transitively on $\overline{\mathbb{F}}$.*

Theorem 2.2 is a generalization of a result of C. Pedrini [10].

Let (\mathbb{F}, P) be a half-ordered field and $f, g \in \Pi^+$. Then $\mathcal{H}(\mathbb{F}, f, g) := \mathcal{H}(\Sigma(f, g))$ is a hyperbola structure by Theorem 2.2.

For $a, b \in \mathbb{F}$, $a \neq 0$ let $a^\bullet: \overline{\mathbb{F}} \rightarrow \overline{\mathbb{F}}$ and $b^+: \overline{\mathbb{F}} \rightarrow \overline{\mathbb{F}}$ denote the canonic extension of the left multiplication $x \mapsto ax$ and translation $x \mapsto b + x$ respectively.

We note that $b^+ \in \text{PGL}^+$ for all $b \in \mathbb{F}$ (cf. [13, 2.4]).

3 Transitive groups of axial homologies of hyperbola structures

Let $\mathcal{H} = (P, \mathfrak{K}, \mathfrak{G}_1 \cup \mathfrak{G}_2)$ be a hyperbola structure. Let $\alpha \in \text{Aut } \mathcal{H}$ and $G, H \in \mathfrak{G}$ with $G \cap H = \emptyset$. The automorphism α is called an *axial homology with axis G, H* (shortly *(G, H) -homology*), if $G \cup H \subset \text{Fix } \alpha$. The automorphism α is called a *G -translation*, if $\alpha = \text{id}$ or $\text{Fix } \alpha = G$.

If α is a (G, H) -homology or a G -translation with $G \in \mathfrak{G}_i$ then $\alpha(X) = X$ for all $X \in \mathfrak{G} \setminus \mathfrak{G}_i$. Note that every (G, H) -homology and every G -translation is a proper automorphism, i.e. maps \mathfrak{G}_i onto \mathfrak{G}_i .

All (G, H) -homologies form a group $A(G, H)$. The set of all G -translations is denoted by $T(G)$.

Lemma 3.1. *Let $\gamma \in A(G, H)$ and $p \in P \setminus (G \cup H)$ with $\gamma(p) = p$. Then $\gamma = \text{id}$.*

Proof. We may assume $G, H \in \mathfrak{G}_1$. Consider $X := [p]_1$. Then $X \neq G, H$. For $x \in X$ we have $\gamma(x) = \gamma([p]_1 \cap [x]_2) = [\gamma(p)]_1 \cap [x]_2 = [p]_1 \cap [x]_2 = x$. Hence, for all $C \in \mathfrak{K}$ we have $\gamma(C) = C$, because $g := C \cap G$, $h := C \cap H$, $x := C \cap X$ are three fixed points. Let $z \in P \setminus (G \cup H)$. Then there is a circle $C \in \mathfrak{K}$ with $z \in C$, and $\gamma(z) = \gamma(C \cap [z]_2) = \gamma(C) \cap [z]_2 = C \cap [z]_2 = z$. \square

Let Γ be a subgroup of $\text{Aut } \mathcal{H}$. Then Γ is called *(G, H) -transitive*, if there is an $X \in \mathfrak{G}$ with $|G \cap X| = 1$ such that $\Gamma \cap A(G, H)$ acts transitively on $X \setminus (G \cup H)$. Moreover, Γ is called *G -transitive*, if there is an $X \in \mathfrak{G}$ with $|G \cap X| = 1$ such that $\Gamma \cap T(G)$ acts transitively on $X \setminus G$. The hyperbola structure is called *(G, H) -transitive* if $\text{Aut } \mathcal{H}$ is (G, H) -transitive.

If Γ is (G, H) -transitive and $G \in \mathfrak{G}_i$ then $\Gamma \cap A(G, H)$ acts transitively on $\mathfrak{G}_i \setminus \{G, H\}$, hence transitively on $Y \setminus (G \cup H)$ for every $Y \in \mathfrak{G} \setminus \mathfrak{G}_i$.

Looking for all imaginable configurations $\mathfrak{J} := \{G \in \mathfrak{G} \mid \Gamma \text{ is } G\text{-transitive}\}$ the subgroups Γ of the automorphism group of a hyperbola structure were typified in [9]. Here we consider for $i = 1, 2$ the configurations

$$\mathfrak{A}_i(\Gamma) := \{\{G, H\} \mid G, H \in \mathfrak{G}_i, \Gamma \text{ is } (G, H)\text{-transitive}\}.$$

Describing all possible configurations $\mathfrak{A}_1(\Gamma)$ and $\mathfrak{A}_2(\Gamma)$ we obtain a further typification of the subgroups Γ of $\text{Aut } \mathcal{H}$. For this purpose we first notice the following.

Lemma 3.2. *Let $\gamma \in \Gamma$ and $G, H \in \mathfrak{G}$ with $G \cap H = \emptyset$. Then we have:*

- (1) $A(\gamma(G), \gamma(H)) = \gamma A(G, H) \gamma^{-1}$;
- (2) *if Γ is (G, H) -transitive, then Γ is $(\gamma(G), \gamma(H))$ -transitive.*

From Lemma 3.2, we get the following result.

Theorem 3.3. *Let \mathcal{H} be a hyperbola structure and Γ a subgroup of $\text{Aut } \mathcal{H}$. For $i \in \{1, 2\}$ exactly one of the following statements is valid:*

- (0) $\mathfrak{A}_i(\Gamma) = \emptyset$;
- (1) $|\mathfrak{A}_i(\Gamma)| = 1$;
- (2) *there is an involution $\alpha: \mathfrak{G}_i \rightarrow \mathfrak{G}_i$ such that*

$$\mathfrak{A}_i(\Gamma) = \{\{X, \alpha(X)\} \mid X \in \mathfrak{G}_i\};$$

- (3) *there exists a generator $E \in \mathfrak{G}_i$ such that*

$$\mathfrak{A}_i(\Gamma) = \{\{E, X\} \mid X \in \mathfrak{G}_i, X \neq E\};$$

- (4) $\mathfrak{A}_i(\Gamma) = \{\{X, Y\} \mid X, Y \in \mathfrak{G}_i, X \neq Y\}$.

Proof. Let $\{E, F\} \in \mathfrak{A}_i(\Gamma)$. If $\gamma(\{E, F\}) = \{E, F\}$ for all $\gamma \in \Gamma$ then $|\mathfrak{A}_i(\Gamma)| = 1$. Now, let $\{A, B\} \in \mathfrak{A}_i(\Gamma)$ with $\{E, F\} \neq \{A, B\}$, say $B \neq E, F$. Then for $X \in \mathfrak{G}_i, X \neq E, F$ there is an $\eta \in \Gamma \cap A(E, F)$ with $\eta(B) = X$, hence $\{\eta(A), X\} = \{\eta(A), \eta(B)\} \in \mathfrak{A}_i(\Gamma)$ by Lemma 3.2(2). Thus every $X \in \mathfrak{G}_i$ is contained in at least one element of $\mathfrak{A}_i(\Gamma)$. If $E = A$, then $\{E, X\} \in \mathfrak{A}_i(\Gamma)$ for all $X \in \mathfrak{G}_i \setminus \{E\}$. Hence, every $X \in \mathfrak{G}_i$ is contained in exactly one element of $\mathfrak{A}_i(\Gamma)$ or $\mathfrak{A}_i(\Gamma) = \{\{E, X\} \mid X \in \mathfrak{G}_i, X \neq E\}$ or $\mathfrak{A}_i(\Gamma) = \{\{X, Y\} \mid X, Y \in \mathfrak{G}_i, X \neq Y\}$. \square

Let $j, k \in \{0, 1, 2, 3, 4\}$, $j \leq k$. We will say that Γ is of type $(j; k)$, if $\mathfrak{A}_1(\Gamma)$ and $\mathfrak{A}_2(\Gamma)$ are of the form (j) and (k) in Theorem 3.3, respectively, or if $\mathfrak{A}_1(\Gamma)$ and $\mathfrak{A}_2(\Gamma)$ are of the form (k) and (j) , respectively. For Γ there are 15 possible types.

In the following we consider a hyperbola structure $\mathcal{H} = \mathcal{H}(M, \Sigma)$ with $\text{id} \in \Sigma$. To avoid brackets, for $x \in M$ we identify the point (x, x) with x .

In the following lemma we characterize those permutations α, β of M yielding an axial homology $\alpha \circ \beta$.

Lemma 3.4. *Let $g, h \in M$, $g \neq h$ and $G_i := [g]_i$, $H_i := [h]_i$ ($i = 1, 2$). For all bijections $\alpha, \beta: M \rightarrow M$ we have:*

- (1) $\acute{\alpha} \circ \grave{\beta}$ is an axial homology $\iff \beta \Sigma \alpha^{-1} = \Sigma$, $|\text{Fix } \alpha| \geq 2$, $|\text{Fix } \beta| \geq 2$ and $\beta = \text{id}$ or $\alpha = \text{id}$.
- (2) $\acute{\alpha}$ is a (G_1, H_1) -homology $\iff g, h \in \text{Fix } \alpha$ and $\Sigma \alpha = \Sigma$.
In particular we have: $\acute{\alpha} \in A(G_1, H_1) \implies \alpha \in \Sigma_{g,h}$ ².
- (3) $\grave{\beta}$ is a (G_2, H_2) -homology $\iff g, h \in \text{Fix } \beta$ and $\Sigma = \beta \Sigma$.
In particular we have: $\grave{\beta} \in A(G_2, H_2) \implies \beta \in \Sigma_{g,h}$.

Proof. Because of Lemma 2.1 we may assume $\beta \Sigma \alpha^{-1} = \Sigma$, i.e. $\beta \Sigma = \Sigma \alpha$. We have $G_1 = \{g\} \times M$, $H_1 = \{h\} \times M$. Hence

$$\begin{aligned} \acute{\alpha} \circ \grave{\beta} \in A(G_1, H_1) &\iff \{g\} \times M, \{h\} \times M \subset \text{Fix } \acute{\alpha} \circ \grave{\beta} \\ &\iff g, h \in \text{Fix } \alpha \text{ and } \beta = \text{id}. \end{aligned}$$

Hence $\acute{\alpha} \in A(G_1, H_1)$ implies $\Sigma \alpha^{-1} = \Sigma$, thus $\Sigma \alpha = \Sigma$, and $\alpha = \text{id} \circ \alpha \in \Sigma_{g,h}$. This proves (2) and the first part of (1). The remaining parts follow in the same way. \square

The meaning of the (G, H) -transitivity for the permutation set (M, Σ) is given in the following lemma.

Lemma 3.5. *Let $g, h \in M$, $g \neq h$ and $G_i := [g]_i$, $H_i := [h]_i$ ($i = 1, 2$) and*

$$\begin{aligned} A &:= \{\alpha \in \text{Sym } M \mid \acute{\alpha} \text{ is a } (G_1, H_1)\text{-homology}\}, \\ B &:= \{\alpha \in \text{Sym } M \mid \grave{\alpha} \text{ is a } (G_2, H_2)\text{-homology}\}. \end{aligned}$$

Then we have:

- (1) $A = A^{-1} \subset \Sigma_{g,h}$ and $\Sigma A = \Sigma$;
- (2) $\text{Aut } \mathcal{H}$ is (G_1, H_1) -transitive $\iff A = \Sigma_{g,h}$;
- (3) $B = B^{-1} \subset \Sigma_{g,h}$ and $B \Sigma = \Sigma$;
- (4) $\text{Aut } \mathcal{H}$ is (G_2, H_2) -transitive $\iff B = \Sigma_{g,h}$.

Proof. (1) For $\alpha \in A$ we have $\Sigma \alpha = \Sigma$ and $\alpha \in \Sigma_{g,h}$ by Lemma 3.4. Furthermore, $(\acute{\alpha})^{-1} = \acute{\alpha}'^{-1}$, hence $\alpha^{-1} \in A$.

² $\Sigma_{g,h}$ denotes the stabilizer of g and h in Σ .

(2) “ \Rightarrow ”. Let $\gamma \in \Sigma_{g,h}$. Consider $a \in M$, $a \neq g, h$. By assumption there is an $\alpha \in \text{Sym } M$ such that $\acute{\alpha}$ is a (G_1, H_1) -homology with $\acute{\alpha}(a, a) = (\gamma(a), a)$, hence $\alpha(a) = \gamma(a)$. Since $\alpha \in \Sigma_{g,h}$ by (1) and Σ is sharply 3-transitive, we get $\alpha = \gamma$. Hence $\Sigma_{g,h} \subset A$, and therefore $A = \Sigma_{g,h}$ by (1).

“ \Leftarrow ”. Consider $(a, c), (b, c) \in M \times M \setminus (G \cup H)$. There is a $\gamma \in \Sigma_{g,h} = A$ with $\gamma(a) = b$, hence $\acute{\gamma} \in A(G, H)$ and $\acute{\gamma}(a, c) = (b, c)$.

The claims in (3) and (4) are proved in the same way as (1) and (2). \square

With respect to Lemma 3.5 it is easy to give examples of groups Γ for every type if Σ is a sharply 3-transitive group. We are satisfied with the following example.

Example 3.6. Let \mathbb{F} be a commutative field and $\mathcal{H} = \mathcal{H}(\text{PGL}(2, \mathbb{F}))$ the Miquelian Minkowski plane over \mathbb{F} . Consider the subgroup $\Gamma := \{a^\bullet \mid a \in \mathbb{F}^*\}$ and the involution $\sigma \in \text{PGL}(2, \mathbb{F})$ with $\sigma(x) = \frac{x+1}{x-1}$ for $x \in \mathbb{F}$, $x \neq 1$, and define $\Gamma_1 := \acute{\Gamma}$ and $\Gamma_2 := \langle \Gamma_1 \cup \{\acute{\sigma}\} \rangle$. Then Γ_1 is of type $(0; 1)$, and if $|\mathbb{F}| \in \{3, 5\}$ the group Γ_2 is of type $(0; 2)$.

Conversely we have the following result.

Theorem 3.7. Let $\mathcal{H} = \mathcal{H}(M, \Sigma)$ be a hyperbola structure, $\Gamma < \text{Aut } \mathcal{H}$ an automorphism group, and let $\alpha: \mathfrak{G}_1 \rightarrow \mathfrak{G}_1$ be an involution such that $\mathfrak{A}_1 = \{\{X, \alpha(X)\} \mid X \in \mathfrak{G}_1\}$. If there is a $G \in \mathfrak{G}_1$ such that $A(G, \alpha(G))$ contains at most one involution then the order of \mathcal{H} is 3 or 5.

Proof. Since $\{X, \alpha(X)\}$ is a 2-set we have $X \neq \alpha(X)$ for all $X \in \mathfrak{G}_1$. Hence, if \mathcal{H} is finite then $|\mathfrak{G}_1|$ is even, because α is involutory. Hence:

(1) The order of \mathcal{H} is at least 3.

Consider the following mapping $\bar{\alpha}: P \rightarrow P$, $x \mapsto \alpha([x]_1) \cap [x]_2$. $\bar{\alpha}$ has no fixed points since $X \neq \alpha(X)$ for every $X \in \mathfrak{G}_1$. Because of $[\bar{\alpha}([x]_1 \cap [x]_2)]_1 = \alpha([x]_1)$ we have $\bar{\alpha}^2(x) = \alpha(\alpha([x]_1)) \cap [x]_2 = [x]_1 \cap [x]_2 = x$, i.e. $\bar{\alpha}$ is an involution.

(2) If $\gamma \in \Gamma$ is a proper automorphism then $\gamma\bar{\alpha} = \bar{\alpha}\gamma$.

Because for $X \in \mathfrak{G}_1$ we have $\gamma(\{X, \alpha(X)\}) = \{\gamma(X), \gamma\alpha(X)\} = \{\gamma(X), \alpha\gamma(X)\}$, hence $\gamma\alpha(X) = \alpha\gamma(X)$ implying $\gamma\bar{\alpha}(x) = \gamma(\alpha([x]_1) \cap [x]_2) = \alpha([\gamma(x)]_1) \cap [\gamma(x)]_2 = \bar{\alpha}\gamma(x)$ for all $x \in P$.

Because of (1) we may assume that the order of \mathcal{H} is at least 5; i.e. $|\mathfrak{G}_1| \geq 6$.

(3) For all $A \in \mathfrak{G}_1$ there exists a unique involution $\gamma_A \in A(A, \alpha(A))$. For all $x \in P \setminus (A \cup \alpha(A))$ we have $\gamma_A(x) = \bar{\alpha}(x)$.

Because $|\mathfrak{G}_1| \geq 6$ there is an $S \in \mathfrak{G}_1$, $S \neq A, \alpha(A), G, \alpha(G)$, and there is a $\sigma \in A(A, \alpha(A))$ with $\sigma(A) = G$ and thus $A(A, \alpha(A)) = \sigma^{-1}A(G, \alpha(G))\sigma$. Hence, by assumption

$$A(A, \alpha(A)) \text{ contains at most one involution.} \quad (*)$$

For $x \in P \setminus (A \cup \alpha(A))$ we have $\bar{\alpha}(x) \in [x]_2$. Since Γ is $(A, \alpha(A))$ -transitive there is a $\gamma \in A(A, \alpha(A))$ with $\gamma(x) = \bar{\alpha}(x)$, and by (2) we have $\gamma^2(x) = \gamma\bar{\alpha}(x) = \bar{\alpha}\gamma(x) = \bar{\alpha}^2(x) = x$, hence $\gamma^2 = \text{id}$ by Lemma 3.1, and thus γ is an involution. By (*) the involution $\gamma =: \gamma_A$ is unique.

Now let us assume that the order of \mathcal{H} is greater than 5, i.e. $|\mathfrak{G}_1| \geq 8$. Then there exist $A, B, C \in \mathfrak{G}_1$ with $B, C \neq A, \alpha(A)$, $C \neq B, \alpha(B)$, and there is a $p \in P$ with $p \notin A \cup \alpha(A) \cup B \cup \alpha(B) \cup C \cup \alpha(C)$. By (3) there is $\gamma_A \in A(A, \alpha(A))$, $\gamma_B \in A(B, \alpha(B))$ with $\gamma_A(x) = \bar{\alpha}(x) = \gamma_B(x)$ for all $x \in P \setminus (A \cup \alpha(A) \cup B \cup \alpha(B))$ and $\gamma_A(b) = \bar{\alpha}(b) \neq b$ for $b \in B$. Hence $\gamma_A\gamma_B(p) = p$ and $\gamma_A\gamma_B(c) = \bar{\alpha}^2(c) = c$ for $c \in C \cup \alpha(C)$, i.e. $\gamma_A\gamma_B \in A(C, \alpha(C))$, hence $\gamma_A\gamma_B = \text{id}$ by Lemma 3.1, but $\gamma_A\gamma_B(b) = \bar{\alpha}(b)$ for $b \in B$, and so we have a contradiction. \square

Remark 3.8. The assumption that there is at most one involution in $A(G, \alpha(G))$ is fulfilled if one derivation $\mathcal{A}(w)$ is a desarguesian affine plane.

Theorem 3.9. For a sharply 3-transitive permutation set (M, Σ) with $\text{id} \in \Sigma$ the following statements are equivalent:

- (i) Σ is a group;
- (ii) there is a subgroup Γ of $\text{Aut } \mathcal{H}(M, \pm)$ of type $(0; 4)$;
- (iii) $\text{Aut } \mathcal{H}(M, \Sigma)$ is of type $(4; 4)$.

Proof. (i) \Rightarrow (iii). Because of Lemma 3.5 the group $\Gamma = \dot{\Sigma} \times \dot{\Sigma}$ is of type $(4; 4)$.

(iii) \Rightarrow (ii). This is obvious.

- (ii) \Rightarrow (i). If the order of $\mathcal{H}(M, \Sigma)$ is less than 5 the hyperbola structure is a Miquelian Minkowski plane (cf. [5] and [2]), hence Σ is a group. Let the order be at least 5. We may assume that

$$\mathfrak{A}_1(\Gamma) = \{\{X, Y\} \mid X, Y \in \mathfrak{G}_1, X \neq Y\}.$$

For all $g, h \in M$, $g \neq h$ we have $\Sigma_{g,h} = \Sigma_{g,h}^{-1}$ and $\Sigma\Sigma_{g,h} = \Sigma$ by Lemma 3.5. Hence

$$S := \left\langle \bigcup_{g,h \in M, g \neq h} \Sigma_{g,h} \right\rangle \subset \Sigma. \quad (**)$$

Let $\sigma \in \Sigma$. Consider $a, b, c \in M$ with $a \neq b \neq c \neq a$, $b, c \neq \sigma(a)$. Then there exist $\alpha, \beta, \gamma \in \Sigma$ with $\alpha \in \Sigma_{b,c}$, $\alpha(a) = \sigma(a)$, $\beta \in \Sigma_{\sigma(a),c}$, $\beta(b) = \sigma(b)$ and

$\gamma \in \Sigma_{\sigma(a), \sigma(b)}$, $\gamma(c) = \sigma(c)$. Then we get $\gamma\beta\alpha(a) = \sigma(a)$, $\gamma\beta\alpha(b) = \sigma(b)$ and $\gamma\beta\alpha(c) = \sigma(c)$ and $\gamma\beta\alpha \in \Sigma$ by (**), hence $\gamma\beta\alpha = \sigma$. Thus we have $\Sigma \subset S$ and consequently $\Sigma = S$ by (**). \square

Defining the type of a hyperbola structure \mathcal{H} as the type of $\text{Aut } \mathcal{H}$ we have the following corollaries to Theorem 3.9.

Corollary 3.10. *There is no hyperbola structure of type (0; 4), (1; 4), (2; 4), or (3; 4).*

Corollary 3.11. *The hyperbola structures of type (4; 4) are exactly the hyperbola structures which can be represented as $\mathcal{H} = \mathcal{H}(M, \Sigma)$ with a sharply 3-transitive group Σ .*

Example 3.12. In 1981 E. Hartmann [3] constructed a family of Minkowski planes over the reals \mathbb{R} depending on two real parameters $r_1, r_2 > 0$ using the bijection

$$f: \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \rightarrow \overline{\mathbb{R}}, x \mapsto \begin{cases} x^{-r_1} & \text{for } x > 0 \\ -|x|^{-r_2} & \text{for } x < 0 \\ \infty & \text{for } x = 0 \\ 0 & \text{for } x = \infty. \end{cases}$$

With

$$\Sigma := \{b^+ a^\bullet \mid a, b \in \mathbb{R}, a \neq 0\} \cup \{c^+ a^\bullet f b^+ \mid a, b, c \in \mathbb{R}, a \neq 0\},$$

the incidence structure $\mathcal{M}(r_1, r_2) := \mathcal{H}(\Sigma)$ is a Minkowski plane (cf. [3]).

For $g \in \mathbb{R}$ we have $\Sigma_{g, \infty} = \{g^+ m^\bullet (-g)^+ \mid m \in \mathbb{R}^*\} = (\Sigma_{g, \infty})^{-1}$. For $\gamma = c^+ a^\bullet f b^+ \in \Sigma$ and $\alpha = g^+ m^\bullet (-g)^+ = g(1-m)^+ m^\bullet \in \Sigma_{g, \infty}$ we have $\alpha\gamma = (g(1-m) + mc)^+ (ma)^\bullet f b^+ \in \Sigma$, hence α is a $([g]_2, [\infty]_2)$ -homology by Lemma 3.4. Because of Lemma 3.5 the Minkowski plane $\mathcal{M}(r_1, r_2)$ is $(G, [\infty]_2)$ -transitive for every $G \in \mathfrak{G}_2 \setminus \{[\infty]_2\}$.

For $m \in \mathbb{R}$, $m \neq 0, 1$ we have

$$\begin{aligned} f m^\bullet \in \Sigma &\iff \exists a \in \mathbb{R}^* \text{ with } a^\bullet f = f m^\bullet \\ &\iff |m|^{-r_1} = |m|^{-r_2} \iff r_1 = r_2. \end{aligned}$$

By Lemma 3.4 and Lemma 3.5 we obtain that $\mathcal{M}(r_1, r_2)$ is $(G, [\infty]_1)$ -transitive for all $G \in \mathfrak{G}_1 \setminus \{[\infty]_1\}$ if and only if $r_1 = r_2$.

If $(r_1, r_2) \neq (1, 1)$ then every automorphism of $\mathcal{M}(r_1, r_2)$ fixes the point $\infty = (\infty, \infty)$. Hence the type of $\mathcal{M}(r_1, r_2)$ is (0; 3) or (3; 3) if $r_1 \neq r_2$ or $r_1 = r_2 \neq 1$, respectively.

4 Hyperbola structures $\mathcal{H}(\mathbb{F}, f, g)$ with transitive groups of axial homologies

Let $\mathcal{H}(\mathbb{F}, f, g)$ be a hyperbola structure over a half-ordered field (\mathbb{F}, P) . We consider the following sets of bijections:

$$\begin{aligned} M &:= \{f \in \text{Sym } \mathbb{F} \mid f(0) = 0, f(\infty) = \infty, \\ &\quad \forall a \in P, \forall x \in \mathbb{F}: f(ax) = f(a)f(x)\}, \\ I &:= \{f \in \text{Sym } \mathbb{F} \mid f(0) = 0, f(\infty) = \infty, \\ &\quad \forall d \in \mathbb{F} \setminus P, \forall x \in \mathbb{F}: f(df(x)) = f(d)x\}, \\ \Lambda^+ &:= M \cap I \cap \Pi^+. \end{aligned}$$

A permutation $f \in \text{Sym } \mathbb{F}$ with $f(0) = 0$, $f(\infty) = \infty$ is called a *Moulton mapping* if $f|_P = \text{id}|_P$ and if there is a $k \in P$ such that for all $x \in \mathbb{F} \setminus P$, the equality $f(x) = kx$ holds.

Lemma 4.1. *For every Moulton mapping f we have:*

- (1) $f \in M \cap I$;
- (2) $f \in \Pi^+ \iff \forall x \in \mathbb{F} \setminus P: \chi(1-x) = \chi(1-f(x))$.

Proof. (1) For $a \in P$, $d, x \in \mathbb{F} \setminus P$ we have $f(df(a)) = f(da) = kda = f(d)a$ and $f(df(x)) = f(kdx) = kdx = f(d)x$.

- (2) “ \Rightarrow ”. For $x \in \mathbb{F} \setminus P$ we have $\chi(1-x) = -\omega(1, x, \infty) = -\omega(1, f(x), \infty) = \chi(1-f(x))$.

“ \Leftarrow ”. For $x, y \in \mathbb{F} \setminus P$ we have $\chi(x-y) = \chi(f(x)-f(y))$ because of $\chi(k) = 1$. For $x \in P$, $y \in \mathbb{F}$ we have $\chi(x-y) = \chi(x)\chi(1-x^{-1}y) = \chi(f(x))\chi(1-f(x^{-1}y)) = \chi(f(x)-f(x)f(x^{-1}y)) = \chi(f(x)-f(y))$ by (1). Hence, for all $x, y \in \mathbb{F}$, $x \neq y$ holds $\chi(x-y) = \chi(f(x)-f(y))$. Thus $f \in \Pi^+$. \square

Examples 4.2. (1) Let (\mathbb{F}, P) be an ordered field, i.e. (\mathbb{F}, P) is a half-ordered field with $P + P \subset P$. Then every Moulton mapping f is order-preserving by Lemma 4.1(2), hence $f \in \Lambda^+$.

- (2) Let $\mathbb{F} = \mathbb{Q}$ be the field of rational numbers. There is exactly one subgroup $P < \mathbb{Q}^*$ of index 2 containing all primes different from 3 with $-1 \in P$ (cf. [11]). For $x \in \mathbb{Q}^*$ there exists $z \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ such that $x = 3^\nu \frac{z}{n}$ and $3 \nmid z, n$, i.e. $z, n \in P$. Therefore $x = 3^\nu \frac{z}{n} \notin P$ iff $\nu \in 2\mathbb{Z} + 1$. Take a $k \in P \cap \mathbb{N}$ with $3 \nmid k$ and consider the Moulton mapping f defined

by k . For $x = 3^\nu \frac{z}{n}$ with $\nu \neq 0$ and $3 \nmid z, n$ we have

$$\frac{1-x}{1-kx} = \frac{n-3^\nu z}{n-3^\nu zk} = \frac{3^{-\nu}n-z}{3^{-\nu}n-zk} \in P.$$

Therefore f is order-preserving by Lemma 4.1(2), thus $f \in \Lambda^+$.

Lemma 4.3. *Let (\mathbb{F}, P) be a half-ordered field and $f \in \Lambda^+$. Then we have:*

- (1) For all $x \in P$: $f^2(x) = x$;
- (2) if there is an $x \in P$ with $f(x) \neq x$ then $-1 \in P$;
- (3) if $f|_P = \text{id}|_P$ then f is a Moulton mapping;
- (4) if $-1 \notin P$ then f is a Moulton mapping;
- (5) for $\Sigma = \text{PGL}^+ \cup \text{PGL}^- f$ the stabilizer $\Sigma_{\infty,0}$ contains at most one involution.

Proof. Since f is order-preserving $f(P) = P$.

- (1) Let $d \in \mathbb{F}^* \setminus P$. For all $x \in P$ we have $f(x)f(d) = f(xd) = f(d \cdot f(f^{-1}(x))) = f(d) \cdot f^{-1}(x)$, hence $f(x) = f^{-1}(x)$, i.e. $f^2(x) = x$.
- (2) Let $x \in P$, $f(x) \neq x$. Then $\omega(x, f(x), \infty) = \omega(f(x), x, \infty)$ by (1), thus $\chi(x - f(x)) = -\omega(x, f(x), \infty) = -\omega(f(x), x, \infty) = \chi(f(x) - x)$, hence $\chi(-1) = 1$, i.e. $-1 \in P$.
- (3) Consider a fixed $d \in \mathbb{F}^* \setminus P$. For every $x \in \mathbb{F}^* \setminus P$ there exists a $y \in P$ with $x = dy$, in particular there is a $k \in P$ with $f(d) = kd$. Hence $f(x) = f(dy) = f(df(y)) = f(d)y = kdy = kx$.
- (4) This follows by (2) and (3).
- (5) For $\gamma \in \text{PGL}(2, \mathbb{F})$ we have $\infty, 0 \in \text{Fix } \gamma f$ if and only if $\infty, 0 \in \text{Fix } \gamma$, hence $\Sigma_{\infty,0} = \{a^\bullet \mid a \in P\} \cup \{d^\bullet f \mid d \in \mathbb{F}^* \setminus P\}$. First we characterize the involutions of $\Sigma_{\infty,0}$.

$$\text{For } a \in P \text{ we have: } a^\bullet \text{ is an involution} \iff a = -1 \neq 1. \quad (\star)$$

Indeed, $(a^\bullet)^2 = \text{id}$ implies $1 = a^\bullet(a^\bullet(1)) = a^2$, hence $a \in \{-1, 1\}$, and conversely $1^\bullet = \text{id}$ and $-1 \neq 1$ implies that $(-1)^\bullet$ is involutory.

$$\text{For } d \in \mathbb{F}^* \setminus P \text{ we have: } d^\bullet f \text{ is an involution} \iff df(d) = 1. \quad (\star\star)$$

Indeed, $d^\bullet f \neq \text{id}$ and $(d^\bullet f)^2 = \text{id}$ implies $1 = df(df(1)) = df(d)$, and conversely $df(d) = 1$ implies for all $x \in \mathbb{F}$: $(d^\bullet f)^2(x) = df(df(x)) = df(d)x = x$.

We have to distinguish two cases: $-1 \notin P$ and $-1 \in P$.

First case: $-1 \notin P$. Because of (\star) the subset $\{a^\bullet \mid a \in P\}$ of $\Sigma_{\infty,0}$ contains no involutions. By (4), f is a Moulton mapping, $f(-1) = -k$ with $k \in P$.

Now, let $d \in \mathbb{F}^* \setminus P$ such that $d^\bullet f$ is involutory. Then $kd^2 = df(d) = 1$ by $(\star\star)$, hence $d^2 = \frac{1}{k}$. Since $-1 \notin P$ there is at most one $d \in \mathbb{F}^* \setminus P$ with $d^2 = \frac{1}{k}$. Hence $\{d^\bullet f \mid d \in \mathbb{F}^* \setminus P\}$ contains at most one involution.

Second case: $-1 \in P$. If $-1 \neq 1$ then $\{a^\bullet \mid a \in P\}$ contains by (\star) exactly one involution. Now let us assume that there is an involution $d^\bullet f$ with $d \in \mathbb{F}^* \setminus P$. Then $df(d) = 1$ by $(\star\star)$, hence $f(d) = \frac{1}{d}$. Since f is order-preserving we have

$$\begin{aligned} \chi(1-d) &= -\omega(1, d, \infty) = -\omega\left(1, \frac{1}{d}, \infty\right) = \chi\left(1 - \frac{1}{d}\right) = \chi\left(\frac{d-1}{d}\right) \\ &= \chi(-1)\chi(1-d)\chi(d), \end{aligned}$$

hence $1 = \chi(d)$ contradicting $d \notin P$. \square

Lemma 4.4. *Let $f \in \Pi^+$ with $\{\infty, 0, 1\} \subset \text{Fix } f$. Then $\mathcal{H} = \mathcal{H}(\mathbb{F}, f, \text{id})$ is $([\infty]_1, [0]_1)$ -transitive if and only if $f \in \Lambda^+$.*

Proof. We have $\Sigma = \text{PGL}^+ \cup \text{PGL}^- f$.

1. Let us assume that \mathcal{H} is $([\infty]_1, [0]_1)$ -transitive. By Lemma 3.5 the group of all $([\infty]_1, [0]_1)$ -homologies is

$$A([\infty]_1, [0]_1) = \dot{\Sigma}_{\infty,0} = \{\acute{\alpha} \mid \alpha \in \Sigma_{\infty,0}\}.$$

As shown in the proof of Lemma 4.3 we have:

$$\Sigma_{\infty,0} = \{a^\bullet \mid a \in P\} \cup \{d^\bullet f \mid d \in \mathbb{F}^* \setminus P\}.$$

Let $a \in P$. By Lemma 3.4 we have $\Sigma = \Sigma a^\bullet = \text{PGL}^+ \cup \text{PGL}^- f a^\bullet$, and therefore $\text{PGL}^- f = \text{PGL}^- f a^\bullet$, hence $f a^\bullet f^{-1} \in \text{PGL}^+$. Since $\infty, 0 \in \text{Fix } f a^\bullet f^{-1}$ there is a $b \in P$ with $f a^\bullet f^{-1} = b^\bullet$. Because of $f(1) = 1$ we get $b = b^\bullet(1) = f(a)$ and thus $f(ax) = f(a)f(x)$ for all $x \in \mathbb{F}$. Let $d \in \mathbb{F}^* \setminus P$. By Lemma 3.4 we have $\Sigma = \Sigma d^\bullet f = \text{PGL}^+ d^\bullet f \cup \text{PGL}^- f d^\bullet f$ and therefore $\text{PGL}^+ = \text{PGL}^- f d^\bullet f$, hence $f d^\bullet f \in \text{PGL}^-$. Because of $\infty, 0 \in \text{Fix } f d^\bullet f$ and $f(1) = 1$ we obtain $f d^\bullet f = f(d)^\bullet$, and thus $f(df(x)) = f(d)x$ for all $x \in \mathbb{F}$.

2. Now let us assume that $f \in \Lambda^+$. For $a \in P$ we have $f(a) \in P$, $a^\bullet, f(a)^\bullet \in \text{PGL}^+$, $\infty, 0 \in \text{Fix } a^\bullet$, and $f a^\bullet = f(a)^\bullet f$ because of $f \in M^+$, and therefore $\Sigma a^\bullet = \text{PGL}^+ \cup \text{PGL}^- f a^\bullet = \text{PGL}^+ \cup \text{PGL}^- f = \Sigma$. Thus Lemma 3.4 yields $\acute{a}^\bullet \in A([\infty]_1, [0]_1)$. For $d \in \mathbb{F}^* \setminus P$ we have $f(d) \in \mathbb{F}^* \setminus P$, $d^\bullet, f(d)^\bullet \in \text{PGL}^-$, $\infty, 0 \in \text{Fix } d^\bullet f$ and $f d^\bullet f = f(d)^\bullet$ because of $f \in I^+$, hence $\Sigma d^\bullet f = \text{PGL}^+ d^\bullet f \cup \text{PGL}^- f d^\bullet f = \text{PGL}^- f \cup \text{PGL}^- f(d)^\bullet = \Sigma$. Thus Lemma 3.4 yields $\acute{d}^\bullet f \in A([\infty]_1, [0]_1)$. Since $\acute{a}^\bullet(1, 0) = (a, 0)$ and $\acute{d}^\bullet f(1, 0) = (d, 0)$ we obtain that \mathcal{H} is $([\infty]_1, [0]_1)$ -transitive. \square

Lemma 4.5. *Let P be a subgroup of (\mathbb{F}^*, \cdot) of index 2 and let $f: \mathbb{F} \rightarrow \mathbb{F}$ be an additive bijection satisfying $f(ax) = f(a)f(x)$ for all $a \in P$, $x \in \mathbb{F}$. Then f is an automorphism of $(\mathbb{F}, +, \cdot)$.*

Proof. Since f is additive we have $f(-x) = -f(x)$, in particular $f(-1) = -1$, hence $f((-1)x) = f(-1)f(x)$. If $|\mathbb{F}| = 3$ we have $f = \text{id}$. Let $|\mathbb{F}| > 4$. Consider $d \in \mathbb{F}^* \setminus P$, $d \neq 0, 1$. Then $d + 1 \in P$ or $d^2 + d \in P$. Hence there is a $p \in P$ with $d + p \in P$. For all $x \in \mathbb{F}$ we have $f(dx) + f(p)f(x) = f(dx) + f(px) = f((d+p)x) = f(d+p)f(x) = f(d)f(x) + f(p)f(x)$, hence $f(dx) = f(d)f(x)$. \square

Lemma 4.6. *There exist $A, B \in \mathfrak{G}_1$, such that $\mathcal{H}(\mathbb{F}, f, g)$ is (A, B) -transitive if and only if there is an automorphism ϕ of $(\mathbb{F}, +, \cdot)$ such that $f \in \phi\text{PGL}^+\Lambda^+\text{PGL}^+$, $g \in \phi\text{PGL}^+$.*

Proof. (a) Let $A =: \{a\} \times \overline{\mathbb{F}}$ and $B =: \{b\} \times \overline{\mathbb{F}}$ with $a \neq b$. Then there is an $\alpha \in \text{PGL}^+$ with $\alpha(a) = \infty$, $\alpha(b) = 0$ and $\acute{\alpha}$ is an isomorphism from $\mathcal{H}(\mathbb{F}, f, g)$ to $\mathcal{H}(\mathbb{F}, f\alpha^{-1}, g)$ (cf. [13, 2.1]). By Lemma 3.2 the hyperbola structure $\mathcal{H}(\mathbb{F}, f, g)$ is (A, B) -transitive if and only if $\mathcal{H}(\mathbb{F}, f\alpha^{-1}, g)$ is $([\infty]_1, [0]_1)$ -transitive. By [13, Remark 2.5] there are $\rho, \sigma \in \text{PGL}^+$ such that $\{\infty, 0, 1\} \subset \text{Fix } \rho f \alpha^{-1} \cap \text{Fix } \sigma g$, and we have $\mathcal{H}(\mathbb{F}, f\alpha^{-1}, g) = \mathcal{H}(\mathbb{F}, \rho f \alpha^{-1}, \sigma g)$.

(b) Let $\mathcal{H}(\mathbb{F}, f, g)$ be (A, B) -transitive. Put $\phi := \sigma g \in \Pi^+$. By (a) the hyperbola structure $\mathcal{H}(\mathbb{F}, \rho f \alpha^{-1}, \phi)$ is $([\infty]_1, [0]_1)$ -transitive, and $\{\infty, 0, 1\} \subset \text{Fix } \phi$.

Consider $x \in \mathbb{F}^* \setminus P$. Then there exists an order-reversing bijection $\xi \in \phi^{-1}\text{PGL}^-\rho f \alpha^{-1} = \phi^{-1}\text{PGL}^-f \alpha^{-1}$ with $\xi(0) = 0$, $\xi(\infty) = \infty$, $\xi(1) = x$. By Lemma 3.5(2), ξ is a $([\infty]_1, [0]_1)$ -homology, hence $\phi^{-1}\text{PGL}^-f \alpha^{-1}\xi^{-1} = \text{PGL}^+$ because of $\text{PGL}^+\xi^{-1} \subset \Pi^-$. Thus there is a $\nu \in \text{PGL}^-$ such that $\xi = \phi^{-1}\nu f \alpha^{-1}$, hence $\text{PGL}^+ = \phi^{-1}\text{PGL}^-\nu^{-1}\phi = \phi^{-1}\text{PGL}^+\phi$.

Let $a, b \in \mathbb{F}$. Because of $a^+, b^+ \in \text{PGL}^+$ we have $\phi^{-1}a^+\phi, \phi^{-1}b^+\phi \in \text{PGL}^+$, and for $a \neq 0$ the mapping $\phi^{-1}a^+\phi$ has exactly ∞ as fixed point, hence $\phi^{-1}a^+\phi = \phi^{-1}(a)^+$ because of $\phi^{-1}a^+\phi(0) = \phi^{-1}(a)$. Therefore,

$$\begin{aligned} \phi^{-1}(a+b) &= \phi^{-1}(a^+(b)) = \phi^{-1}(a^+\phi\phi^{-1}(b)) = \phi^{-1}(a)^+(\phi^{-1}(b)) \\ &= \phi^{-1}(a) + \phi^{-1}(b). \end{aligned}$$

Hence ϕ^{-1} is additive.

For $a \in P$ we have $a^\bullet, \phi^{-1}a^\bullet\phi \in \text{PGL}^+$, hence $\phi^{-1}a^\bullet\phi = \phi^{-1}(a)^\bullet$. Therefore, for all $z \in \mathbb{F}$ we have

$$\begin{aligned} \phi^{-1}(az) &= \phi^{-1}(a^\bullet(z)) = \phi^{-1}a^\bullet(\phi^{-1}(z)) = \phi^{-1}(a)^\bullet(\phi^{-1}(z)) \\ &= (\phi^{-1}(a)) \cdot (\phi^{-1}(z)). \end{aligned}$$

Hence ϕ is an automorphism of the field \mathbb{F} by Lemma 4.5.

With $\phi^{-1}\sigma g = \text{id}$ we have $\mathcal{H}(\mathbb{F}, \rho f \alpha^{-1}, \sigma g) = \mathcal{H}(\mathbb{F}, \phi^{-1} \rho f \alpha^{-1}, \text{id})$ by [13, Remark 2.5], therefore $\phi^{-1} \rho f \alpha^{-1} \in \Lambda^+$ by Lemma 4.4, hence $f \in \phi \rho \Lambda^+ \alpha \subset \phi \text{PGL}^+ \Lambda^+ \text{PGL}^+$ and $g \in \phi \text{PGL}^+$.

- (c) For $\alpha \in \text{PGL}^+$ and $f \in \phi \text{PGL}^+ \Lambda^+ \alpha$, $g \in \phi \text{PGL}^+$ it follows with (a) by Lemma 4.4 that $\mathcal{H}(\mathbb{F}, f, g)$ is $([\alpha^{-1}(\infty)]_1, [\alpha^{-1}(0)]_1)$ -transitive. \square

Since $\mathcal{H}(\mathbb{F}, f, g)$ and $\mathcal{H}(\mathbb{F}, g, f)$ are isomorphic, Lemma 4.6 yields immediately the following corollaries.

Corollary 4.7. *There exist $A, B \in \mathfrak{G}$ such that $\mathcal{H}(\mathbb{F}, f, g)$ is (A, B) -transitive if and only if there is an automorphism ϕ of $(\mathbb{F}, +, \cdot)$ such that $f \in \phi \text{PGL}^+ \Lambda^+ \text{PGL}^+$, $g \in \phi \text{PGL}^+$ or $f \in \phi \text{PGL}^+$, $g \in \phi \text{PGL}^+ \Lambda^+ \text{PGL}^+$.*

Corollary 4.8. *If $\mathcal{H}(\mathbb{F}, f, g)$ admits a group of type $(0; 2)$ then $\mathcal{H}(\mathbb{F}, f, g)$ is the Minkowski plane of order 3 or 5.*

Proof. We may assume that $g = \text{id}$; $\infty, 0, 1 \in \text{Fix } f$ and that $\mathcal{H}(\mathbb{F}, f, g)$ is $([\infty]_1, [0]_1)$ -transitive (cf. proof of Lemma 4.6). Hence $f \in \Lambda^+$ by Lemma 4.4. For a permutation α of $\overline{\mathbb{F}}$ the bijection $\acute{\alpha}$ is an involution if and only if α is an involution. Hence, because of Lemma 3.5 and Lemma 4.3 the group $A([\infty]_1, [0]_1)$ contains at most one involution. Now Theorem 3.7 yields the assertion. \square

Theorem 4.9. *Let $f \in \Pi^+$ with $\{\infty, 0, 1\} \subset \text{Fix } f$, and $\Sigma = \text{PGL}^+ \cup \text{PGL}^- f$. Then the following statements are equivalent:*

- (i) Σ is a group;
- (ii₁) $\mathcal{H} = \mathcal{H}(\mathbb{F}, f, \text{id})$ is $(X, [\infty]_1)$ -transitive for every $X \in \mathfrak{G}_1 \setminus \{[\infty]_1\}$;
- (ii₂) $\mathcal{H} = \mathcal{H}(\mathbb{F}, f, \text{id})$ is $(X, [\infty]_2)$ -transitive for every $X \in \mathfrak{G}_2 \setminus \{[\infty]_2\}$;
- (iii) $f|_{\mathbb{F}}$ is an automorphism with $f^2 = \text{id}$.

Proof. (i) \Rightarrow (ii₁) and (i) \Rightarrow (ii₂) follow by Theorem 3.9.

(ii₁) \Rightarrow (iii). We may assume that $|\mathbb{F}| > 4$. We have $f \in \Lambda^+$ by Lemma 4.4. For all $\alpha \in \Sigma_{\infty, 0}$ we have $\Sigma \alpha = \Sigma$ by Lemma 3.5 and Lemma 3.4. Let $b \in \mathbb{F}^*$. Consider $a \in P$, $a \neq 1$. Then $a^\bullet \in \text{PGL}^+ \cap \Sigma_{\infty, 0}$ and with $c = \frac{ab}{1-a}$ we have $a^\bullet b^+ \in \text{PGL}^+ \cap \Sigma_{\infty, c}$ and $\Sigma = \Sigma a^\bullet b^+ = \Sigma b^+ = \text{PGL}^+ b^+ \cup \text{PGL}^- f b^+ = \text{PGL}^+ \cup \text{PGL}^- f b^+$, hence $\text{PGL}^- f b^+ = \text{PGL}^- f = \text{PGL}^- f(b)^+ f$. Therefore for every $\gamma \in \text{PGL}^-$ there exists a $\delta \in \text{PGL}^-$ with $\gamma f b^+ = \delta f(b)^+ f$ or $f b^+ = \gamma^{-1} \delta f(b)^+ f$. Because of $f b^+(\infty) = \infty$ and $f(b)^+ f(\infty) = \infty$ we have $\gamma^{-1} \delta(\infty) = \infty$, and hence there is $m_b, d_b \in \mathbb{F}$ such that $\gamma^{-1} \delta = d_b^+ m_b^\bullet$. From $f(b) = f b^+(0) = \gamma^{-1} \delta f(b)^+ f(0) = m_b \cdot f(b) + d_b$ we get $d_b = (1 - m_b) f(b)$, hence

$$f(b+x) = f b^+(x) = d_b^+ m_b^\bullet f(b)^+(x) = m_b(f(x) + f(b)) + (1 - m_b)f(b)$$

for all $x \in \mathbb{F}$, in particular $0 = f(0) = m_b(f(-b) + f(b)) + (1 - m_b)f(b) = m_b f(-b) + f(b)$, thus $m_b = -\frac{f(b)}{f(-b)}$. Because of $f \in \Lambda^+$ we have

$$m_b = -\frac{f(b)}{f(b)f(-1)} = -\frac{1}{f(-1)}$$

for all $b \in P$. Hence for all $b \in P$, $x \in \mathbb{F}$ we have

$$f(b+x) = -\frac{1}{f(-1)}(f(x)+f(b)) + \left(1 + \frac{1}{f(-1)}\right)f(b) = -\frac{1}{f(-1)}f(x) + f(b).$$

If $x \in P$ we obtain $f(b+x) = -\frac{1}{f(-1)}f(b) + f(x)$ by exchanging b and x in the previous formula. Therefore we have $(1 + \frac{1}{f(-1)})f(b) = (1 + \frac{1}{f(-1)})f(x)$ for all $b, x \in P$. Because of $|P| \geq 2$ this implies $1 + \frac{1}{f(-1)} = 0$, hence $f(-1) = -1$ and $m_b = 1$ for all $b \in P$. For $b \in \mathbb{F}^* \setminus P$ we obtain

$$m_b = -\frac{f(b)}{f(-b)} = -\frac{f(b)}{f(b)f(-1)} = -\frac{f(b)}{f(b) \cdot (-1)} = 1.$$

Therefore we have for all $b, x \in \mathbb{F}$:

$$f(b+x) = m_b(f(x) + f(b)) + (1 - m_b)f(b) = f(x) + f(b).$$

Hence $f|_{\mathbb{F}}$ is an automorphism of $(\mathbb{F}, +, \cdot)$ by Lemma 4.5.

Consider $d \in \mathbb{F}^* \setminus P$. Since $f \in \text{Aut}(\mathbb{F}, +, \cdot) \cap I^+$ we have for all $x \in \mathbb{F}$: $f(dx) = f(df(x)) = f(d)f^2(x)$, hence $f^2(x) = x$.

(ii)₂) \Rightarrow (iii). This follows from the fact that $\mathcal{H}(\mathbb{F}, f, g)$ and $\mathcal{H}(\mathbb{F}, g, f)$ are isomorphic (cf. [13, 2.2]) by the implication (ii)₁) \Rightarrow (iii).

(iii) \Rightarrow (i). Since $f|_{\mathbb{F}}$ is an automorphism of $(\mathbb{F}, +, \cdot)$ we have $\text{PGL}^- f = f \text{PGL}^-$, and because of $f^2 = \text{id}$ we obtain $\text{PGL}^- f \cdot \text{PGL}^- f = \text{PGL}^+$. Therefore Σ is a group. \square

Corollary 4.10. *If $\Sigma = \text{PGL}^+ \cup \text{PGL}^- f$ is a group, then $\mathcal{H}(\mathbb{F}, f, \text{id})$ is a Minkowski plane (cf. Hartmann [4]).*

Corollary 4.11. *If $\mathcal{H}(\mathbb{F}, f, g)$ admits an automorphism group of type (1; 1) then $\mathcal{H}(\mathbb{F}, f, g)$ is of type (4; 4).*

Proof. By assumption there are subgroups Γ_1, Γ_2 of $\text{Aut } \mathcal{H}(\mathbb{F}, f, g)$ such that $\mathfrak{A}_i(\Gamma_i) \neq \emptyset$ for $i = 1, 2$. Since $\mathfrak{A}_1(\Gamma_1) \neq \emptyset$ we may assume that $g = \text{id}$ and $\{\infty, 0, 1\} \subset \text{Fix } f$ (cf. proof of Lemma 4.6); then $f \in \Lambda^+$ by Lemma 4.4. Because $\mathfrak{A}_2(\Gamma_2) \neq \emptyset$, by Lemma 4.6 (now in Lemma 4.6 we have to exchange the roles of 1 and 2 as well as those of g and f) there is an automorphism ϕ of $(\mathbb{F}, +, \cdot)$

such that $\text{id} = g \in \phi\text{PGL}^+\Lambda^+\text{PGL}^+$ and $f \in \phi\text{PGL}^+$. Hence there is $\gamma \in \text{PGL}^+$ such that $f = \phi\gamma$. Because of $\infty, 0, 1 \in \text{Fix } f \cap \text{Fix } \phi$ we obtain $\infty, 0, 1 \in \text{Fix } \gamma$, hence $\gamma = \text{id}$. Therefore $f = \phi$, hence $f^2 = \text{id}$ (cf. the last sentence of the proof of Theorem 4.9, (ii₁) \Rightarrow (iii)). Thus $\Sigma = \text{PGL}^+ \cup \text{PGL}^- f$ is a group by Theorem 4.9, and therefore $\mathcal{H}(\mathbb{F}, f, g)$ is of type (4; 4) by Theorem 3.9. \square

Theorem 4.12. *A hyperbola structure $\mathcal{H}(\mathbb{F}, f, g)$ is of type (0; 0), (0; 1), or type (4; 4).*

Proof. Let $(j; k)$ be the type of $\mathcal{H}(\mathbb{F}, f, g)$. If $j \neq 0$ then there is an automorphism group of type (1; 1) because $k \geq j$, hence $(j; k) = (4; 4)$ by Corollary 4.11. Now let us assume $j = 0$. Then $k < 3$ by Theorem 4.9 and Theorem 3.9. Since the Minkowski planes of order 3 and 5 are Miquelian (cf. [2]) and therefore of type (4; 4), we obtain $k \neq 2$. \square

Examples 4.13. (1) Let (\mathbb{F}, P) be an ordered field. Then $-1 \notin P$. For every Moulton mapping f we have $f \in \Lambda^+$. For $k = -f(-1) \neq 1$ the bijection $f =: f_k$ is not an automorphism. Hence for $k = -f(-1) \neq 1$ the hyperbola structure $\mathcal{H}(\mathbb{F}, f_k, \text{id})$ is of type (0; 1) by Lemma 4.4, Theorem 4.9 and Theorem 4.12. For an ordered field (\mathbb{F}, P) these are the only examples of hyperbola structures $\mathcal{H}(\mathbb{F}, f_k, \text{id})$ of type (0; 1) (cf. Lemma 4.3(4)).

(2) Every Miquelian Minkowski plane is of type (4; 4). If (\mathbb{F}, P) is a half-ordered field and f an order-preserving involutory automorphism of the field (extended canonically onto $\overline{\mathbb{F}}$) then $\mathcal{H}(\mathbb{F}, f, \text{id})$ is of type (4; 4) (cf. Theorem 4.9).

(3) If (\mathbb{F}, P) is a half-ordered field and $f \in \Pi^+ \setminus \text{PGL}^+\Lambda^+\text{PGL}^+$ then $\mathcal{H}(\mathbb{F}, f, \text{id})$ is of type (0; 0) (cf. Corollary 4.7 and Theorem 4.12). For example take $\mathbb{F} = \mathbb{R}$ and the order-preserving bijection f with $f(x) = x^3$ for all $x \in \mathbb{R}$. Then $f \notin \text{PGL}^+\Lambda^+\text{PGL}^+$, because otherwise there would be $\delta, \gamma \in \text{PGL}^+$ with $\delta f \gamma \in \Lambda^+$, hence by Lemma 4.3(3) for all $x \in P$ would hold $x = \delta(\gamma(x)^3)$, or $\delta^{-1}(x) = \gamma(x)^3$, which is impossible.

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