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Bounds on partial ovoids and spreads in classical generalized quadrangles

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Abstract

We present an improvement on a recent bound for small maximal partial ovoids of $\mathbf{W}(q^3)$. We also classify maximal partial ovoids of size (q^2-1) of $\mathbf{Q}(4,q)$ which allow a certain large automorphism group, and discuss examples for small q.

Keywords: partial ovoid, partial spread, generalized quadrangle

MSC 2000: 51E23, 51E12

1 Introduction

Generalized polygons were introduced in 1959 by Tits in the appendix of his celebrated paper on triality [23]. Ever since they have played a key role in algebraic and combinatorial geometry. Among the generalized polygons, the finite generalized quadrangles (GQs) occupy a special place, in particular because of the existence of several classical and non-classical examples which can be studied in a geometric, algebraic as well as combinatorial way; they are connected with a broad collection of other interesting objects from Galois geometry and algebra, such as polarities, (pseudo-)ovals, (pseudo-)ovoids, q-clans, flocks, fans, fibrations, herds, 4-gonal families, BN-pairs, ... Since their introduction several authors have been interested in the study of substructures of finite GQs, in particular in the existence of *spreads* and *ovoids* (see further for a formal introduction to these objects). As there already is an extensive literature on this subject containing some good surveys, we will only tersely overview some of the most

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important (recent) results on (partial) spreads and ovoids of finite GQs and refer the interested reader to the literature. More recently people have become interested in the existence, the construction and bounds on the size of maximal partial ovoids and spreads of finite GQs. In this paper we will contribute to this theory by providing some new results. The GQs encountered will mostly be the finite classical GQs, although if possible we will try to extend results to general finite GQs of order (s,t).

2 Generalized quadrangles

A *finite GQ* $\mathbf{Q}$ *of order* (s,t) is a finite point-line incidence geometry such that

- two distinct lines intersect in at most one point;
- every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines;
- given any point p not on a given line L, there exists a unique line M containing p and intersecting L. (The unique point on L collinear with p will be denoted by $\operatorname{proj}_L p$.)

Sometimes a GQ will be denoted by $\mathbf{Q} = (P,B,I)$, when we want to specify the point set P, the line set B and the incidence relation I.

If $s > 1$ and $t > 1$, then $\mathbf{Q}$ will be called *thick*, otherwise $\mathbf{Q}$ will be called *thin*. If $s = t$, then $\mathbf{Q}$ will be said to be of order s.

3 Spreads and ovoids

In this short section we will briefly mention results concerning the (non-)existence of spreads and ovoids in finite classical GQs, and provide some references.

We first define the objects which will form the core of this article. A *spread* $\mathcal{S}$ of a GQ $\mathbf{Q}$ is a set of lines of $\mathbf{Q}$ partitioning the points of $\mathbf{Q}$. Dually an *ovoid* $\mathcal{O}$ of $\mathbf{Q}$ is a set of points such that each line of $\mathbf{Q}$ meets $\mathcal{O}$ in a unique point. A *partial spread* $\mathcal{M}$ of $\mathbf{Q}$ is a set of mutually disjoint lines of $\mathbf{Q}$. A partial spread is called *maximal* if it cannot be extended to a larger partial spread. Dually a *partial ovoid* is a set of mutually non-collinear points of $\mathbf{Q}$; it is called *maximal* if it cannot be extended to a larger partial ovoid.

A *k-arc* $\mathcal{K}$ of a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, is a set of k mutually non-collinear points, that is, a partial ovoid of size k. One easily observes that $k \leq st+1$ (see, e.g., [13]), and if $k = st+1$, then $\mathcal{K}$ is an ovoid of $\mathcal{S}$. A k-arc is *complete* if it is not contained in a k'-arc with $k' > k$. Dually, one defines *dual* k*-arcs* and *complete dual* k*-arcs*.

Throughout this paper, if $\mathcal{M}$ is a set of lines of a GQ, we will denote by $\widetilde{\mathcal{M}}$ the set of all points covered by the lines of $\mathcal{M}$.

Theorem 3.1 ([13, 1.8.3])**.** *A GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $t > s^2 - s$, has no ovoid. Dually, a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $s > t^2 - t$, has no spread.*

First note that a spread or an ovoid of a finite GQ of order (s,t) clearly contains $st + 1$ lines, respectively points.

The GQ $\mathbf{Q}(4,q)$ has spreads if and only if q is even [16] and has ovoids for all values of q (every $\mathbf{Q}^-(3,q) \subset \mathbf{Q}(4,q)$ is an ovoid of $\mathbf{Q}(4,q)$). Dualizing yields the corresponding results for the GQ $\mathbf{W}(q)$.

The GQ $\mathbf{Q}(5,q)$ has spreads for all values of q [17], but never has ovoids [18]; see also Theorem 3.1. Dualizing yields the corresponding results for the GQ $\mathbf{H}(3,q^2)$.

The (non-)existence of spreads of the GQ $\mathbf{H}(4,q^2)$ is a long-standing open problem. The only thing known so far is that $\mathbf{H}(4,4)$ does not admit a spread (by an unpublished computer result of A. Brouwer).

Finally the GQ $\mathbf{H}(4,q^2)$ does never admit an ovoid [18].

We synthesize some of these (and other) results in the following theorem.

Theorem 3.2 ([13, 3.4.1, 3.4.2 and 3.4.3]; see also [20] for (v))**.**

(i) *The GQ $\mathbf{Q}(4,q)$ always has ovoids. It has spreads if and only if q is even.*

(ii) *The GQ $T_2(\mathcal{O})$ of Tits always has ovoids.*

(iii) *The GQ $\mathbf{Q}(5,q)$ has spreads but no ovoids.*

(iv) *The GQ $T_3(\mathcal{O})$ of Tits has no ovoid but always has spreads.*

(v) *Each TGQ $T(\mathcal{O})$, where $\mathcal{O}$ is good at some element π, has spreads.*

(vi) *The GQ $\mathbf{H}(4,q^2)$ has no ovoid. For $q = 2$ it has no spread.*

(vii) *The GQ $\mathcal{P}(\mathcal{S},x)$ of Payne always has spreads. It has an ovoid if and only if $\mathcal{S}$ has an ovoid containing x.*

For a good reference on the existence question of spreads and ovoids in (not necessarily classical) finite GQs we refer to Thas and Payne [20].

4 Small maximal partial spreads and ovoids

Theorem 4.1 ([4]). (i) *The second smallest maximal partial ovoids $\mathcal{K}$ of $\mathbf{W}(q^2)$, $q = p^h$, $p > 3$ prime, $h \geq 1$, contain at least $s(q^2) + 1$ points, where $s(q^2)$ denotes the cardinality of the second smallest non-trivial minimal blocking sets in $\mathsf{PG}(2, q^2)$. If $q = p > 2$, then $\mathcal{K}$ contains at least $3(p^2 + 1)/2 + 1$ points.*

(ii) *The second smallest maximal partial ovoids $\mathcal{K}$ of $\mathbf{W}(q^3)$, $q = p^h$, $p \geq 7$ prime, $h \geq 1$, contain at least $q^3 + q^2 + q + 1$ points. If $|\mathcal{K}| = q^3 + q^2 + q + 1$, then $\mathcal{K}$ consists of the point set of a subgeometry $\mathsf{PG}(3, q)$ of $\mathsf{PG}(3, q^3)$.*

In [4] the existence of maximal partial ovoids of $\mathbf{W}(q^3)$ of size $q^3 + q^2 + q + 1$ was posed as an open problem. It was J. A. Thas who suggested to the authors to try to use the Klein correspondence in order to prove the (non-)existence of such a partial ovoid. We will now use this correspondence to prove its non-existence. We mention the following lemma which will be used without further notice.

Lemma 4.2. *Consider a hyperplane π of $\mathsf{PG}(5, q^3)$ and a subspace $\Omega := \mathsf{PG}(5, q)$ of $\mathsf{PG}(5, q^3)$. Then π intersects Ω in at least a plane $\mathsf{PG}(2, q)$.* $\square$

Theorem 4.3. *The GQ $\mathbf{W}(q^3)$, $q = p^h$, $p \geq 7$, does not admit a maximal partial ovoid of size $q^3 + q^2 + q + 1$.*

Proof. Assume that $\mathbf{W}(q^3)$ admits a maximal partial ovoid $\mathcal{K}$ of size $q^3 + q^2 + q + 1$. We consider $\mathbf{W}(q^3)$ in its natural representation in $\mathsf{PG}(3, q^3)$. Then $\mathcal{K}$ corresponds to some $\mathsf{PG}(3, q)$-subgeometry of $\mathsf{PG}(3, q^3)$. Using the Klein correspondence the lines of $\mathsf{PG}(3, q^3)$ are mapped onto the points of $\mathbf{Q}^+(5, q^3)$, the lines of $\mathsf{PG}(3, q)$ onto the points of some $\mathbf{Q}^+(5, q) \subset \mathbf{Q}^+(5, q^3)$ and the lines of $\mathbf{W}(q^3)$ onto the points of some $\mathbf{Q}(4, q^3) \subset \mathbf{Q}^+(5, q^3)$. It is easily seen, since $\mathcal{K}$ is a partial ovoid, that $\mathbf{Q}(4, q^3) \cap \mathbf{Q}^+(5, q) = \emptyset$ (as point sets). However, the $\mathsf{PG}(4, q^3)$ determined by $\mathbf{Q}(4, q^3)$ and the $\mathsf{PG}(5, q)$ determined by $\mathbf{Q}^+(5, q)$ must intersect in at least a plane $\mathsf{PG}(2, q)$. As in $\mathsf{PG}(5, q)$ every plane has nonempty intersection with $\mathbf{Q}^+(5, q)$, $\mathcal{K}$ cannot be a partial ovoid. $\square$

Concerning small maximal partial ovoids of $\mathbf{W}(q)$, it is interesting to note that computer searches [4], exhaustive for $q \in \{2, 3, 4, 5\}$ and heuristic for $q \in \{7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27\}$, suggest that the second smallest maximal partial ovoids of $\mathbf{W}(q)$ will probably have size $2q + 1$. A maximal partial ovoid of this size can easily be constructed by taking all points except one point r on a hyperbolic line H in $\mathbf{W}(q)$, together with one arbitrary point (not collinear with one of the remaining points of H) from each of the $q + 1$ lines of $\mathbf{W}(q)$

through r. In [4] also a theoretical construction of a maximal partial ovoid of size $3q-1$ is given for every $q>3$.

In view of the isomorphism relations between the GQs under consideration in this section, the only case not handled yet is the one of maximal partial spreads of $\mathbf{W}(q)$, for odd q. In [4] a counting technique first introduced in [8] is used to prove the following bound. It is also explained how a fine tuning of this technique can in some cases slightly improve this result.

Theorem 4.4 ([4]). *Suppose that $\mathcal{M}$ is a maximal partial spread of $\mathbf{W}(q)$, q odd. Then $|\mathcal{M}| \geq \lceil 1,419q \rceil$.*

Here as well, the theoretical obtained bound is rather small compared to the results obtained by computer for small q, which seem to point in the direction of a bound of order $q\sqrt{q}$; see [4].

4.1 The GQs $\mathbf{Q}(5,q)$ and $\mathrm{H}(3,q^2)$

In [7] the following lower bound on the size of maximal partial ovoids of $\mathbf{Q}(5,q)$ was proved using a counting technique analogous to the one used to prove Theorem 4.4. Recall that the trivial lower bound equals $q+1$.

Theorem 4.5 ([7]). *Let $\mathcal{K}$ be a maximal partial ovoid of $\mathbf{Q}(5,q)$. Then $|\mathcal{K}| \geq 2q+1$. If $q \geq 4$, then $|\mathcal{K}| \geq 2q+2$.*

As far as constructions of small maximal partial ovoids of $\mathbf{Q}(5,q)$ are concerned for general q, the best known construction provides such a partial ovoid of size q^2+1. This goes as follows. Consider any elliptic quadric $\mathbf{Q}^-(3,q) \subset \mathbf{Q}(5,q)$. Then it is easily seen that the q^2+1 points of $\mathbf{Q}^-(3,q)$ form a maximal partial ovoid (see also [7]). Using a recent result of Ball [3] we can see that in fact every ovoid of $\mathbf{Q}(4,q) \subset \mathbf{Q}(5,q)$ determines a maximal partial ovoid of $\mathbf{Q}(5,q)$ (Ball shows that every $\mathbf{Q}^-(3,p^h) \subset \mathbf{Q}(4,p^h)$ intersects an ovoid of $\mathbf{Q}(4,p^h)$ in $1 \pmod p$ points). Also in [1] constructions of maximal partial ovoids of $\mathbf{Q}(5,q)$ of size q^2+1 are provided. There however the authors show, using the computer, the existence of maximal partial ovoids of $\mathbf{Q}(5,q)$ of size strictly less than q^2+1, for $q=7,8$. Very recently, in [5], Cimrakova and Fack showed by computer the existence of several maximal partial ovoids of size strictly less than q^2+1 for all $q \in \{4,5,7,8,9,11,13\}$. However, these maximal partial ovoids are still "much" larger than the bound from Theorem 4.5, so there still remains a lot of work to be done.

When it comes to the existence of small maximal partial spreads of $\mathbf{Q}(5,q)$, we know from the foregoing that the existence of spreads of $\mathbf{Q}(4,q)$ implies the existence of maximal partial spreads of size q^2+1 (the trivial lower bound) of $\mathbf{Q}(5,q)$, and this bound is reached if and only if q is even. So, if q is odd, a maximal partial spread contains at least q^2+2 lines. The size of small maximal partial spreads of $\mathbf{Q}(5,q)$ is discussed in [2]. The best known result is the next one.

Theorem 4.6 ([11]). *A maximal partial spread of $\mathbf{Q}(5,q)$ has size at least equal to $q^2+\frac{4}{9}q+1$.*

In an appendix to this paper, we give a proof of the fact that $\mathbf{Q}(5,q)$ $(q \geq 5)$ does not admit maximal partial spreads of size less than q^2+4. (This was a result contained in a first version of the paper, before [11] was written.) Although it is now worse than Metsch's theorem, we hope that its (elementary) proof still might be useful.

In [5] the best known results on the size of maximal partial spreads can be found for $q \in \{3,4,5,7,8\}$ (computer results). The best theoretical construction (based on an idea of J. A. Thas) can be found in [2], where the authors prove, starting from a small maximal partial spread of $\mathsf{PG}(3,q)$, the existence for odd q of a maximal partial spread of size $(m+1)q^2+1$, with $m=\lceil 2\log_2(q)\rceil$. Another way to construct small maximal partial spreads of $\mathbf{Q}(5,q)$, might be to start with a large partial spread (which is not a spread) of $\mathbf{Q}(4,q)\subset\mathbf{Q}(5,q)$ and then add lines. For a maximal partial spread of $\mathbf{Q}(4,q)$ of size $q^2-\epsilon$ this yields a maximal partial spread of $\mathbf{Q}(5,q)$ of size at most $q^2+(\epsilon+1)(q+1)$. Unfortunately not much is known about large maximal partial spreads of $\mathbf{Q}(4,q)$, and the largest known such partial spread is still much smaller than the theoretical bound q^2-q+1, q odd. Finally it is interesting to note that in [9] Hirschfeld and Korchmáros construct maximal partial spreads of $\mathbf{Q}(5,q)$ (in fact they construct maximal partial ovoids of $\mathbf{H}(3,q^2)$) from $\mathsf{GF}(q^2)$-maximal curves for even q. However, their examples have size $q^2+1+2gq$, where g is the genus of the curve used, which is also still much larger then our obtained lower bound (except for the trivial case of rational algebraic curves which have genus 0 and yield maximal partial ovoids of size q^2+1).

4.2 The GQ $\mathbf{H}(4,q^2)$

Consider a fixed $\mathbf{H}(3,q^2)\subset\mathbf{H}(4,q^2)$ and a fixed $\mathbf{H}(2,q^2)\subset\mathbf{H}(3,q^2)$. Of course $\mathbf{H}(2,q^2)$ is an ovoid of $\mathbf{H}(3,q^2)$, and since every other $\mathbf{H}(2,q^2)\subset\mathbf{H}(3,q^2)$ contains at least 1 point of the fixed $\mathbf{H}(2,q^2)$ it is clear that $\mathbf{H}(2,q^2)$ is a maximal partial ovoid of $\mathbf{H}(4,q^2)$. It has size q^3+1, which is the trivial lower bound, and so it is a smallest maximal partial ovoid of $\mathbf{H}(4,q^2)$. To the authors' knowledge not much more is known about small maximal partial ovoids of $\mathbf{H}(4,q^2)$.

We now turn to small maximal partial spreads. Since $\mathbf{H}(3, q^2)$ is the only subGQ of order (q^2, q) of $\mathbf{H}(4, q^2)$ and $\mathbf{H}(3, q^2)$ does not admit a spread, it follows that a smallest maximal partial spread of $\mathbf{H}(4, q^2)$ contains at least $q^3 + 2$ lines. (Note that if the size would have been $q^3 + 1$, the points on the lines of the partial spread would form the point set of a subGQ of order (q^2, q).) However using an analogous technique as was used in [2] to show the non-existence of a maximal partial spread of size $q^2 + 2$ of $\mathbf{Q}(5, q)$, we can easily exclude the existence of such a maximal partial spread.

Theorem 4.7. *The GQ $\mathbf{H}(4, q^2)$ does not admit a maximal partial spread of size $q^3 + 2$.*

Proof. Suppose that $\mathcal{M}$ is a maximal partial spread of size $q^3 + 2$, and let X be the set of points of $\mathbf{H}(4, q^2)$ not covered by $\mathcal{M}$. Then $|X| = q^7 - q^3 - q^2 - 1$. Further, through every point of X there is a unique line intersecting $\widetilde{\mathcal{M}}$ in exactly 2 points (while all other lines through such a point intersect $\widetilde{\mathcal{M}}$ in 1 point). Consequently the number of lines intersecting $\widetilde{\mathcal{M}}$ in 2 points equals $|X|/(q^2 - 1)$. This quantity can never be an integer, a contradiction. □

The best known result is the following.

Theorem 4.8 ([12]). *A maximal partial spread of the GQ $\mathbf{H}(4, q^2)$ has more than $q^3 + q\sqrt{q} - q/2 - 3/8\sqrt{q} + 7/8$ lines.*

5 Partial ovoids of Q$(4, q)$ of size $q^2 - 1$ for $q \in \{2, 3, 5, 7, 11\}$, and beyond

Partial ovoids of $\mathbf{Q}(4, q)$ of size $q^2 - 1$ are only known for $q \in \{2, 3, 5, 7, 11\}$. For $q = 2$ there is, up to isomorphism, a unique 3-arc of $\mathbf{Q}(4, 2)$, see, e.g., [21]. For $q = 3$ an 8-arc can easily be constructed as follows. Consider any elliptic quadric $\mathbf{Q}^-(3, 3) \subset \mathbf{Q}(4, 3)$ and any point $p \notin \mathbf{Q}^-(3, 3)$. Then p is collinear with exactly 4 points of $\mathbf{Q}^-(3, 3)$, and these points determine a conic C. The set $C^\perp$ contains two points (among which p). It is easily seen that $(\mathbf{Q}^-(3, 3) \setminus C) \cup C^\perp$ is a complete 8-arc of $\mathbf{Q}(4, 3)$. It has longtime been thought that these were the only values of q for which $(q^2 - 1)$-arcs of $\mathbf{Q}(4, q)$ existed. However in 2003 Penttila disproved this by showing with the computer the existence of $(q^2 - 1)$-arcs of $\mathbf{Q}(4, q)$ for $q \in \{5, 7, 11\}$ [14]. In [4] Cimrakova and Fack confirmed these results, also with the use of the computer. Their searches were heuristic and so by no means exclude the existence of $(q^2 - 1)$-arcs of $\mathbf{Q}(4, q)$ for other values of odd q. It is important to note that no computer free constructions are known of $(q^2 - 1)$-arcs for $q \in \{7, 11\}$. It is however possible to describe the above construction of an 8-arc of $\mathbf{Q}(4,3)$ in a slightly different way and then generalize this construction for $q=5$. We first provide the "alternative" construction of the 8-arc.

Consider a fixed $\mathbf{Q}(3,3) \subset \mathbf{Q}(4,3)$ and consider any point $p \in \mathbf{Q}(4,3) \setminus \mathbf{Q}(3,3)$. Then the points collinear with p in $\mathbf{Q}(3,3)$ form a conic C. Clearly $C^\perp$ consists of two points, say p and p'. Now let $\mathcal{K}'$ be the set of all points of $\mathbf{Q}(4,3) \setminus \mathbf{Q}(3,3)$ not contained in one of the cones pC and $p'C$. It is easily seen that $\mathcal{K} := \mathcal{K}' \cup \{p,p'\}$ is an 8-arc of $\mathbf{Q}(4,3)$ (isomorphic to the higher described 8-arc). We now generalize this construction for $q=5$. A conic C of $\mathbf{Q}(3,q) \subset \mathbf{Q}(4,q)$ will be called *doubly subtended* if $|C^\perp| = 2$. Again consider a fixed $\mathbf{Q}(3,5) \subset \mathbf{Q}(4,5)$. Let C_i, $i=1,2,3,4$, be a collection of four doubly subtended conics of $\mathbf{Q}(3,5)$, with the property that $|C_i \cap C_j| = 2$ if $i \neq j$. Let $\{p_i, p_i'\} := C_i^\perp$, $i=1,2,3,4$.

Theorem 5.1. *With the above notation, the set*

$$\mathcal{K} := \mathbf{Q}(4,5) \setminus \big(\mathbf{Q}(3,5) \cup_i (p_iC_i \cup p_i'C_i)\big) \cup_i \{p_i, p_i'\}$$

is a 24*-arc of* $\mathbf{Q}(4,5)$.

Proof. We first show that $|\mathcal{K}| = 24$. Consider a cone $p_i^{(\prime)}C_i$ (here $p_i^{(\prime)}$ means p_i or p_i'). Then there are exactly 6 cones p_jC_j, $p_j'C_j$ intersecting $p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)$ in exactly 4 points. As $|p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)| = 24$ this implies that every point of $p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)$ is contained in an average of 2 of the 8 cones p_jC_j, $p_j'C_j$, $j=1,2,3,4$. Assume that there would exist such a point x, contained in at least 3 of the considered cones. First notice that the four conics C_1, C_2, C_3, C_4 cover exactly 12 points of $\mathbf{Q}(3,5)$, and that each of these points is contained in precisely 2 conics C_i. Further, because of our assumption, the point x is collinear with at least 3 distinct points of $\bigcup_i C_i$. This implies the existence of a conic C_k, $k \in \{1,2,3,4\}$, such that x is collinear with 2 points of C_k, while x is contained in either p_kC_k or $p_k'C_k$, a contradiction. Consequently every point of $p_i^{(\prime)}C_i \setminus (\{p_i^{(\prime)}\} \cup C_i)$, $i=1,2,3,4$, is contained in exactly 2 of the 8 considered cones. From this we now easily deduce that $|\bigcup_i (p_i^\perp \cup p_i'^\perp) \setminus \mathbf{Q}(3,5)| = 104$. It follows that $|\mathcal{K}| = 24$. We now proceed by proving that the set $\mathcal{K}$ is a partial ovoid. We already know from the construction that the 8 points p_i, p_i', $i=1,2,3,4$, are two by two non-collinear, while for every other point $p \in \mathcal{K}$ we have that $p \not\sim p_i$ and $p \not\sim p_i'$, $i=1,2,3,4$. Assume that $p \sim p'$ with $p, p' \in \mathcal{K}$ and $p \neq p'$. Then the line pp' intersects $\mathbf{Q}(3,5)$ in a point $z \notin \bigcup_i C_i$. Consequently every p_i and p_i' is collinear with one of the points of $pp' \setminus \{p,p',z\}$. As $|pp' \setminus \{p,p',z\}| = 3$ we find a point of $pp' \setminus \{p,p',z\}$ that is contained in at least 3 of the cones p_iC_i, $p_i'C_i$, a contradiction in view of the first part of the proof. Finally it is easily seen that $\mathcal{K}$ is maximal. □

The previous theorem implies that in order to construct a 24-arc in $\mathbf{Q}(4,5)$ it is sufficient to find 4 doubly subtended conics in some $\mathbf{Q}(3,5) \subset \mathbf{Q}(4,5)$ which intersect two by two in exactly 2 points. In order to do so consider $\mathbf{Q}(4,5)$ in $\mathsf{PG}(4,5)$ determined by $X_0^2+X_1X_2+X_3X_4=0$ and the $\mathbf{Q}(3,5) \subset \mathbf{Q}(4,5)$ in the hyperplane $X_0=0$. Then the conics in the hyperplane $X_0=0$ with equations

$$C_1 : \begin{cases} X_1 = X_2 \\ X_1^2 + X_3X_4 = 0 \end{cases} \qquad C_2 : \begin{cases} X_1 = -X_2 \\ -X_1^2 + X_3X_4 = 0 \end{cases}$$

$$C_3 : \begin{cases} X_3 = X_4 \\ X_3^2 + X_1X_2 = 0 \end{cases} \qquad C_4 : \begin{cases} X_3 = -X_4 \\ -X_3^2 + X_1X_2 = 0 \end{cases}$$

are quickly seen to be doubly subtended and satisfy the desired property (i.e. $|C_i \cap C_j| = 2$ if $i \neq j$). Consequently we have constructed a 24-arc in $\mathbf{Q}(4,5)$.

One might wonder whether it is possible to generalize this construction for other values of q, and at least in theory this seems to be possible for $q \in \{7,11\}$. Suppose that for $q=7$ we find a set of 12 doubly subtended conics $C_1,\ldots,C_{12}$ in $\mathbf{Q}(3,7) \subset \mathbf{Q}(4,7)$ such that $|C_i \cap C_j| \in \{0,2\}$, $C_1,\ldots,C_{12}$ cover exactly 32 points of $\mathbf{Q}(3,7)$ and such that each of these 32 points is contained in exactly 3 conics C_i. Then a construction analogous to the one for $q=5$ would yield a 48-arc of $\mathbf{Q}(4,7)$. However it is not clear whether such a set of conics exists. Finally, also for $q=11$ one could obtain a generalization (60 conics covering completely $\mathbf{Q}(3,11) \subset \mathbf{Q}(4,11)$), but here as well the existence of such a set is not known. It may be interesting to study the link between such sets of conics and (partial) flocks of $\mathbf{Q}(3,q)$ (cf. [19]).

Tim Penttila has noted to us in a private communication [14] that the examples $\mathcal{K}$ of complete (q^2-1)-arcs of $\mathbf{Q}(4,q)$ which were constructed by him for $q=5,7,11$ all satisfy the following property:

$$(q^2-1)^2 \text{ divides the size of } \mathrm{Aut}(\mathbf{Q}(4,q))_{\mathcal{K}}. \qquad (*)$$

We end our paper by showing that, conversely, if such an arc satisfies $(*)$, we necessarily have $q \in \{5,7,11\}$. We do not consider the case $q=3$, as then all maximal 8-arcs are known.

Recall Dickson's classification of the subgroups of $\mathsf{PSL}(2,q)$, with $q=p^h$, p a prime (see [10, Hauptsatz 8.27, p. 213]); we list the possible subgroups $H \leq \mathsf{PSL}(2,q)$, as follows:

(i) H is an elementary abelian p-group;

(ii) H is a cyclic group of order k, where k divides $(q \pm 1)/r$, where $r = \gcd(q-1,2)$;

(iii) H is a dihedral group of order $2k$, where k is as in (ii);

(iv) H is the alternating group A_4, where $p > 2$ or $p = 2$ and $h \equiv 0 \mod 2$;

(v) H is the symmetric group S_4, where $p^{2h} - 1 \equiv 0 \mod 16$;

(vi) H is the alternating group A_5, where $p = 5$ or $p^{2h} - 1 \equiv 0 \mod 5$;

(vii) H is a semidirect product of an elementary abelian group of order p^m with a cyclic group of order k, where k divides $p^m - 1$ and $p^h - 1$;

(viii) H is a $\mathsf{PSL}(2, p^m)$, where m divides h, or a $\mathsf{PGL}(2, p^n)$, where $2n$ divides h.

Let $\mathcal{K}$ be a complete $(q^2 - 1)$-arc of $\mathbf{Q}(4, q)$, q odd and $q > 3$. Suppose the size of $G = \mathrm{Aut}(\mathbf{Q}(4, q))_{\mathcal{K}}$ is divisible by $(q^2 - 1)^2$. Note that G fixes the grid $\mathcal{S}(\mathcal{K})$ which consists of the lines skew from $\mathcal{K}$ and the points incident with these lines. Put

$$|G| = (q^2 - 1)^2 r,$$

where r is natural. We remark that in $\mathsf{PGL}(5, q)_{\mathbf{Q}(4,q)}$, the stabilizer of $\mathcal{S}(\mathcal{K})$ has size $2(q^3 - q)^2$, and inside $\mathsf{PSL}(5, q)_{\mathbf{Q}(4,q)}$, this stabilizer *restricted to its action on* $\mathcal{S}(\mathcal{K})$ (so after modding out the kernel) has size $\frac{(q^3-q)^2}{4}$; it is isomorphic to the direct product

$$\mathsf{PSL}(2, q) \times \mathsf{PSL}(2, q).$$

There is a unique involution fixing $\mathcal{S}(\mathcal{K})$ pointwise, and we denote the group it generates by N. Suppose H is the subgroup of GN/N inside $\mathsf{PSL}(5, q)$; then

$$|H| = (q^2 - 1)^2 r',$$

where $r' \geq \frac{r}{8h}$. Then H can be written as

$$H = H_1 \times H_2 \leq \mathsf{PSL}(2, q) \times \mathsf{PSL}(2, q),$$

where H_1 is the linewise stabilizer of one regulus ($\mathcal{L}_1$) of $\mathcal{S}(\mathcal{K})$ in H, and H_2 the linewise stabilizer of the other regulus ($\mathcal{L}_2$).

Assume that p (the odd prime divisor of q) divides $|H|$; then the subgroup of $\mathsf{PSL}(5, q)_{\mathbf{Q}(4,q)}$ that induced H on $\mathcal{S}(\mathcal{K})$ has a p-element θ, and this necessarily is a symmetry about some line M of $\mathcal{S}(\mathcal{K})$ by work of W. M. Kantor and K. Thas (see Chapter 7 of [22]). Suppose $U \sim M$ is a line not in $\mathcal{S}(\mathcal{K})$; then $U \cap \mathcal{K}$ is a point u. As θ fixes U, $u^\theta \neq u$ is a point collinear with u while being in $\mathcal{K}$, a contradiction. So there are no p-elements in H, and this means that cases (i), (vii) and (viii) of the subgroup list of $\mathsf{PSL}(2, q)$ are ruled out.

For now, we suppose not to be in the cases (iv), (v), (vi), so (ii) and (iii) remain to be handled. Suppose we are in one of these cases. Clearly there is an

$i \in \{1, 2\}$ such that, if N_i is the elementwise stabilizer of $\mathcal{L}_i$ in H, we have

$$|H/N_i| \geq \frac{(q^2-1)\sqrt{r}}{2\sqrt{2h}},$$

while

$$2(q+1) \geq |H/N_i|.$$

So $p = 5$ and $h = 1$. Now we look at the cases (iv), (v), (vi). In the same way as for the cases (ii) and (iii), we obtain

$$\frac{(q^2-1)\sqrt{r}}{2\sqrt{2h}} \leq |H/N_i|,$$

where $H/N_i \in \{\mathsf{A}_4, \mathsf{A}_5, \mathsf{S}_4\}$, so $|H/N_i| \in \{12, 24, 60\}$. When $H/N_i = \mathsf{A}_4$, we easily obtain $h = 1$ and $p = 5$. When $H/N_i = \mathsf{S}_4$, we obtain $h = 1$ and $p \in \{5, 7\}$. When $H/N_i = \mathsf{A}_5$, we obtain $p \in \{5, 7, 11, 13\}$ and $h = 1$. Now suppose $p = 13$. Then

$$60 \ \text{ divides } \ (13^2-1)^2 r' = 168^2 r',$$

so r' is a multiple of 5.[1] This provides us with the desired contradiction, and ends the proof of the result.

Note that a direct corollary of the proof is the same result under the assumption of a transitive action on the arc of a group contained in the linewise stabilizer of one of the reguli in $\mathcal{S}(\mathcal{K})$.

Let us finally mention that in a recent paper [6], De Beule and Gács show the following:

Theorem 5.2 ([6])**.** *Complete (q^2-1)-arcs of $\mathbf{Q}(4, q)$ do not exist when q is not a prime.*

Appendix

In the appendix we show that $\mathbf{Q}(5, q)$, $q \geq 5$, cannot admit a maximal partial spread of size $q^2 + 3$. We first prove the following lemma.

Lemma 5.3. *If a partial spread $\mathcal{M}$ of $\mathbf{Q}(5, q)$ which does not contain a spread of some $\mathbf{Q}(4, q) \subset \mathbf{Q}(5, q)$ covers all points of some $\mathbf{Q}(4, q) \subset \mathbf{Q}(5, q)$, then $|\mathcal{M}| \geq q^2 + q + 1$ if q is even, and $|\mathcal{M}| \geq 2q^2 + q$ if q is odd.*

[1] One could also note that $13^2 - 1 = 168$ is not a multiple of 5, an observation which contradicts the description of case (vi).

Proof. For q even this is trivial. For odd q this follows immediately from the fact that a maximal partial spread of $\mathbf{Q}(4,q)$ contains at most q^2-q+1 lines if q is odd [15]. $\square$

Theorem 5.4. *The GQ $\mathbf{Q}(5,q)$ does not admit a maximal partial spread of size q^2+3 if $q \geq 5$.*

Proof. Assume that $\mathcal{M}$ is a maximal partial spread of $\mathbf{Q}(5,q)$ of size q^2+3. Consider a line L of $\mathbf{Q}(5,q)$ that contains at least 4 points of $\widetilde{\mathcal{M}}$ and suppose that there is a point p on L not belonging to $\widetilde{\mathcal{M}}$. As $|\mathcal{M}| = q^2+3$ it follows that there exists a line through p not intersecting $\widetilde{\mathcal{M}}$, a contradiction. Hence a line of $\mathbf{Q}(5,q)$ intersects $\widetilde{\mathcal{M}}$ in either $1,2,3$ or $q+1$ points. Further, if a point p does not belong to $\widetilde{\mathcal{M}}$, then there is either a unique line through p intersecting $\widetilde{\mathcal{M}}$ in 3 points, while all other lines through p intersect $\widetilde{\mathcal{M}}$ in exactly 1 point, or there are exactly 2 lines through p intersecting $\widetilde{\mathcal{M}}$ in 2 points, and all other lines through p intersect $\widetilde{\mathcal{M}}$ in a unique point. We will denote the set of points of the former type by X_1 and the set of points of the latter type by X_2. Also define $x_i := |X_i|$, $i = 1,2$. Finally denote by $\mathcal{F}$ the set of lines of $\mathbf{Q}(5,q)$ not in $\mathcal{M}$ that are completely covered by $\widetilde{\mathcal{M}}$; here we put $f := |\mathcal{F}|$. An easy counting argument shows that there are $(q^2x_1+(q^2-1)x_2)/q$ lines having exactly 1 point in $\widetilde{\mathcal{M}}$, that there are $x_1/(q-2)$ lines having exactly 3 points in $\widetilde{\mathcal{M}}$, and that there are $2x_2/(q-1)$ lines having exactly 2 points in $\widetilde{\mathcal{M}}$. Considering the points not covered by $\widetilde{\mathcal{M}}$ and the lines not in $\mathcal{M}$, we obtain:

$$\begin{cases} x_1 + x_2 = q^4 - q^2 - 2q - 2\,; \\ \Big(q + \dfrac{1}{q-2}\Big)x_1 + \Big(\dfrac{q^2-1}{q} + \dfrac{2}{q-1}\Big)x_2 + f = (q^2+1)(q^3+1) - q^2 - 3\,. \end{cases}$$

Solving for x_1 and x_2 in function of f, we obtain:

$$x_2 = \frac{1}{2}(q-1)q(2q^3+q^2-q^4+4q-6+(q-2)f).$$

As $x_2 \geq 0$ we obtain that

$$f \geq q^3 - q - 6 - \frac{6}{q-2}.$$

Next we count in two ways the ordered triples (K,L,M) with $K, M \in \mathcal{M}$, $K \neq M$, $L \in \mathcal{F}$ and $K \sim L \sim M$. We obtain

$$f(q+1)q = (q^2+3)(q^2+2)y,$$

where y is the average number of transversals in $\mathcal{F}$ of two distinct lines of $\mathcal{M}$. Using the above obtained bound for f, we deduce that $y > 2$ (recall that $q > 3$).

This implies the existence of two distinct lines K and M of $\mathcal{M}$ with the property that at least 3 of their $q+1$ transversals are lines of $\mathcal{F}$. These lines determine in a unique way some $\mathbf{Q}(3,q) \subset \mathbf{Q}(5,q)$. Assume that not all points of this $\mathbf{Q}(3,q)$ would belong to $\widetilde{\mathcal{M}}$, say $p \in \mathbf{Q}(3,q) \setminus \widetilde{\mathcal{M}}$. Then the two lines through p in $\mathbf{Q}(3,q)$ intersect $\widetilde{\mathcal{M}}$ in respectively at least 2 and at least 3 points, a contradiction. Consequently all $q+1$ transversals of K and M belong to $\mathcal{F}$ and we have shown the existence of a $\mathbf{Q}(3,q) \subset \mathbf{Q}(5,q)$ which is completely covered by $\widetilde{\mathcal{M}}$. Consider such a fixed $\mathbf{Q}(3,q)$. The $(q^2+3)(q+1)-(q+1)^2$ remaining points of $\widetilde{\mathcal{M}}$ have to be partitioned by the $q+1$ $\mathbf{Q}(4,q)$ subGQs on $\mathbf{Q}(5,q)$ which contain $\mathbf{Q}(3,q)$. This implies that there exists a $\mathbf{Q}(4,q)$ containing $\mathbf{Q}(3,q)$ which is such that the set Z of points of $\widetilde{\mathcal{M}}$ in $\mathbf{Q}(4,q)$ not in $\mathbf{Q}(3,q)$, contains at least q^2-q+2 points. Consider such a (fixed) $\mathbf{Q}(4,q)$. We count in two ways the pairs (u,v) with $u \in \mathbf{Q}(4,q) \setminus (\mathbf{Q}(3,q) \cup Z)$, $v \in Z$ and $u \sim v$. We obtain

$$\left[q(q^2-1)-|Z|\right] h = |Z| z,$$

where h is the average number of points of Z collinear with a given point of $\mathbf{Q}(4,q) \setminus (\mathbf{Q}(3,q) \cup Z)$ and where z is the average number of points of $\mathbf{Q}(4,q) \setminus (\mathbf{Q}(3,q) \cup Z)$ collinear with a given point of Z. It is clear that $h \leq 2$, and since $|Z| \geq q^2-q+2$, we deduce that

$$z \leq 2\frac{q^3-q^2-2}{q^2-q+2}\,.$$

First suppose that $q > 5$. Then the above inequality implies the existence of a point v of Z such that at least $q-1$ of the lines of $\mathbf{Q}(4,q)$ through v belong to $\mathcal{F}$. Let C be $v^\perp \cap \mathbf{Q}(3,q)$. Then $C^\perp \cap \mathbf{Q}(4,q)$ either consists of 2 points or a single point, depending on whether q is odd or even. Let w be a point of $\mathbf{Q}(4,q) \setminus (\mathbf{Q}(3,q) \cup C^\perp)$ and suppose that w does not belong to Z. As $w^\perp \cap v^\perp \cap \mathbf{Q}(3,q)$ contains at most 2 points, there are at least $q-3 > 2$ points of Z collinear with w on at least $q-3$ distinct lines through w (the $q-3$ points mentioned are points of $v^\perp$). This contradicts the fact that there are at most 2 lines through w which contain more than 1 point of $\widetilde{\mathcal{M}}$ (recall that the points of $\mathbf{Q}(3,q)$ are covered by $\widetilde{\mathcal{M}}$). Consequently $w \in Z$. One now easily shows that also all points of $C^\perp \cap \mathbf{Q}(4,q)$ must belong to Z, and finally it then follows that all points of $\mathbf{Q}(4,q)$ belong to Z, that is, $\mathbf{Q}(4,q)$ is covered by $\widetilde{\mathcal{M}}$. By the foregoing lemma it follows that $|\mathcal{M}| > q^2+3$, a contradiction.

Finally suppose that $q=5$. Then the above inequality implies the existence of a point $v \in Z$, such that v is collinear with at most 8 points which do not belong to Z. Hence, at least 4 of the lines of $\mathbf{Q}(4,q)$ through v belong to $\mathcal{F}$. With the notation of the above paragraph $C^\perp \cap \mathbf{Q}(4,q)$ consists of two points v and w. Let $u \neq w$ be any point of $\mathbf{Q}(4,q) \setminus \mathbf{Q}(3,q)$ which is collinear with w.

Then u is collinear with exactly 1 point of C. Consequently there are at least 3 points of Z collinear with u on 3 distinct lines incident with u, yielding at least 3 lines through u containing at least 2 points of $\widetilde{\mathcal{M}}$. This implies that $u \in Z$. Consequently all points of the cone wC belong to $\widetilde{\mathcal{M}}$. The proof can now be finished in a similar way as in the first part of the proof. $\square$

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