A new proof for the uniqueness of Lyons’ simple group

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Abstract

In this paper we will provide a new proof for the uniqueness of the Lyons group using the 5-local building-like geometry of this group discovered by Kantor.

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1 Introduction

In 1972, R. Lyons examined groups having the property that the centralizer of an involution is isomorphic to the 2-cover of the alternating group $A_{11}$ [6, 7]. He could gather a lot of information about groups having this property, but the questions if such groups exist and if two of them are isomorphic remained open. In 1973 Charles Sims [8] provided a computer-aided proof showing that there exists exactly one such group. This group is usually called the Lyons group (short Ly) or the Lyons–Sims group (short LyS) and is one of the 26 sporadic groups. Sims’ proof has the disadvantage that it involves computations in the symmetric group of 8835156 letters which can only be done by computer. Therefore it does not provide any insight in the Lyons group. The first computer-free proof was done by Aschbacher and Segev in 1992 (see [1]). In 1981, William Kantor constructed a geometry $\Delta$ of rank three with diagram $\xrightarrow{\text{PG}(2,5)}$ having $\text{PG}(2,5)$ and the Cayley hexagon of order 5 as non-trivial residues such that Ly is a flag-transitive automorphism group of $\Delta$ (see [5]). This geometry can be regarded as the natural geometry of the Lyons group since it is extraordinary beautiful and almost classical. For this reason, one wishes to prove the existence
and uniqueness of the Lyons group by using this geometry. The greatest obstacle is that $\Delta$ is not simply connected (the universal cover $\tilde{\Delta}$ of $\Delta$ is an infinite building), hence there is no canonical way to prove existence and uniqueness of the group from this geometry. In this paper, we will show that $\Delta$ is determined by certain thin subgeometries which are covered by apartments in $\tilde{\Delta}$. We will conclude that $\text{Ly}$ is the universal completion of a certain amalgam of rank three and hence uniquely determined.

We will use the following notation:

- For a set $X$, let $\mathcal{P}(X)$ be the power set of $X$, $\mathcal{P}_n(X) := \{Y \subseteq X; |Y| = n\}$ and $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$.
- For any prime power $q$ we denote the desarguesian projective plane of order $q$ by $\text{PG}(2,q)$ and the Cayley hexagon of order $q$ by $\mathcal{H}(q)$.
- If $A$, $B$ and $G$ are groups, $G = A.B$ means that $G$ possesses a normal subgroup isomorphic to $A$ such that the factor group is isomorphic to $B$.
- We will write $G = A : B$ if we emphasize that it is a split extension and $G = A \leq B$ for a nonsplit extension.
- A cyclic group of order $n$ is denoted by $\mathbb{Z}_n$ or simply by $n$.
- If $p$ is a prime, a special group of order $p^{n+k}$ with center of order $p^n$ is denoted by $p^{n+k}$. $H$ means that there is an ascending series of normal subgroups of $G$ of order $p^{n_1}, p^{n_1+n_2}, \ldots, p^{n_1+n_2+\ldots+n_k}$ such that the last factor group is isomorphic to $H$ and all other factor groups are elementary abelian.

This work is mostly contained in the author’s PhD thesis (see [3]).

2 Preliminaries

2.1 Coverings of simplicial complexes

A simplicial complex is a pair $\Delta = (V, S)$ such that $V$ is a nonempty set with $V \cap \mathcal{P}(V) = \emptyset$ and $S$ is a subset of $\mathcal{P}^*(V)$ such that $\mathcal{P}_1(V) \subseteq S$ holds and $\sigma \in S$ and $\emptyset \neq \tau \subseteq \sigma$ implies $\tau \in S$. The set $V$ is called the set of vertices of $\Delta$ and an element of $S$ is called a simplex of $\Delta$. We will say that $x, y \in V$ are adjacent (short $x \sim y$) if $\{x, y\}$ is a simplex and $x \neq y$. If $\Delta = (V, S)$, $\Delta' = (V', S')$ are complexes, then $\Delta'$ is called a subcomplex of $\Delta$ if $V' \subseteq V$ and $S' \subseteq S$ holds. For $\sigma \in S$, the subcomplex $(\sigma, \mathcal{P}^*(\sigma))$ is simply called $\sigma$.

For a simplicial complex $\Delta = (V, S)$ let $\text{Cl}(\Delta)$ be the complex $(V, \text{Cl}(S))$,
where $\text{Cl}(S) := \{\sigma \in \mathcal{P}^*(V) ; x \sim y \text{ for all } x, y \in \sigma\}$. We will call $\Delta$ complete if $\Delta = \text{Cl}(\Delta)$.

If $\Delta = (V, S), \Delta' = (V', S')$ are complexes, then a map $\varphi : V \to V'$ is called a morphism if $\varphi(S) \subseteq S'$ holds. We will write $\varphi : \Delta \to \Delta'$ instead of $\varphi : V \to V'$. $\varphi$ is called an isomorphism if $\varphi$ is bijective and $\varphi^{-1}$ is also a morphism. Let $\text{Aut} \Delta$ be the group of all automorphisms of $\Delta$.

If $\sigma$ is in $S$, we define the subcomplexes $\Delta_\sigma = (V_\sigma, S_\sigma), \text{st}(\sigma) = (V_\sigma \cup \sigma, S'_\sigma)$, where $V_\sigma := \{v \in V \setminus \sigma ; v \cup \{v\} \in S\}, S_\sigma = \{\tau \in S ; \tau \cap \sigma = \emptyset, \sigma \cup \tau \in S\}$ and $S'_\sigma := \mathcal{P}^*(\sigma) \cup S_\sigma \cup \{\tau \cup \rho ; \tau \in S_\sigma, \rho \in \mathcal{P}^*(\sigma)\}$. The subcomplex $\Delta_\sigma$ is called the residue of $\sigma$ in $\Delta$ and $\text{st}(\sigma)$ is called the star of $\sigma$ in $\Delta$.

If $\Delta'$ is a subcomplex of $\Delta$ and $G$ a subgroup of $\text{Aut} \Delta$, let $G_{\Delta'}(\Delta)$ be the subgroup of all elements in $G_{\Delta'}$ acting trivially on $\Delta'$. The factor group $G_{\Delta'}/G_{\Delta'}$ is denoted by $G_{\Delta'}/G_{\Delta'}$. For a simplex $\sigma$, we simply write $G_{\sigma}$ instead of $G_{\sigma(\Delta)}$ and $G^\sigma$ instead of $G_{\Delta^\sigma}$.

A path of length $n$ in $\Delta$ is a sequence $\delta = (v_0, v_1, \ldots, v_n)$ with $\{v_i, v_{i+1}\} \in S$. We define $l(\delta) = n, o(\delta) = v_0, \text{end}(\delta) = v_n$ and $\delta^{-1} = (v_n, v_{n-1}, \ldots, v_0)$. If $\gamma = (v_0, \ldots, v_n), \delta = (w_0, \ldots, w_m)$ are in $P(\Delta)$ with $v_n = \text{end}(\gamma) = o(\delta) = w_0$, then set $\gamma \delta := (v_0, \ldots, v_n, w_0, w_1, \ldots, w_m), \gamma\delta$. The set of all paths in $\Delta$ is denoted by $P(\Delta)$, the set of all paths with origin $v_0$ by $P(\Delta)(v_0, *)$, the set of all paths with end $v_n$ by $P(\Delta)(*, v_n)$, and $P(\Delta)(v_0, v_n)$ is $P(\Delta)(v_0, *) \cap P(\Delta)(* , v_n)$. We say that $\Delta$ is connected if $P(\Delta)(v_0, v_1) \neq \emptyset$ for all $v_0, v_1 \in V$. The maximal connected subcomplexes of $\Delta$ are called the components of $\Delta$.

Two paths $\gamma, \delta$ are said to be elementary homotopic if there are a simplex $\sigma$ and paths $\gamma_1, \gamma_2, \gamma_3$ with $\gamma_2, \gamma_3 \in \mathcal{P}(\sigma)$ and $\gamma = \gamma_1 \gamma_2 \gamma_3$. Two paths $\gamma$ and $\delta$ are called homotopic if there is a sequence $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_k = \delta$ such that $\gamma_i$ and $\gamma_{i+1}$ are elementary-homotopic for all $i$. Homotopy is an equivalence relation; we denote by $[\gamma]$ the equivalence class of $\gamma$. It is straightforward that homotopic paths have the same origin and the same end.

For a vertex $v_0 \in V$ we define the fundamental group of $\Delta$ relative to $v_0$ in the following way: Set $\pi_1(\Delta, v_0) := \{[\gamma] ; \gamma \in P(\Delta)(v_0, v_0)\}$ and $[\gamma] : [\delta] := [\gamma \delta]$. for $\gamma, \delta \in P(\Delta)(v_0, v_0)$. It is easily seen that this multiplication defines a group structure on $\pi_1(\Delta, v_0)$ with $[v_0]$ as neutral element and $[\gamma]^{-1} = [\gamma^{-1}]$. Furthermore, if $v_1 \in V$ and $\delta \in P(v_0, v_1)$, then $[\gamma] \mapsto [\delta^{-1} \gamma \delta]$ is an isomorphism between $\pi_1(\Delta, v_0)$ and $\pi_1(\Delta, v_1)$. If $\Delta$ is connected, then the isomorphism type of the fundamental group does not depend on the choice of the vertex $v_0$. In this case we set $\pi_1(\Delta)$ as a group isomorphic to $\pi_1(\Delta_1, v_0)$ for any $v_0 \in V$.

A complex $\Delta$ is said to be simply-connected if it is connected and $\pi_1(\Delta) = 1$.

A map $\varphi : \tilde{\Delta} \to \Delta$ between simplicial complexes $\tilde{\Delta}$ and $\Delta$ is called a covering if $\varphi$ is surjective and induces an isomorphism from $\text{st}(\tilde{\Delta})$ to $\text{st}(\varphi(\tilde{\Delta}))$ for all
simplices $\tilde{\sigma}$ of $\tilde{\Delta}$.

For a covering $\varphi: \tilde{\Delta} \to \Delta$, a path $\gamma = (v_0, \ldots, v_n) \in P(\Delta)$ and a vertex $\tilde{v}_0 \in \varphi^{-1}(v_0)$ there exists exactly one path $\tilde{\gamma} \in P(\tilde{\Delta})(\tilde{v}_0, \ast)$ with $\varphi(\tilde{\gamma}) = \gamma$. We say that $\tilde{\gamma}$ is a lift of $\gamma$.

A map $g \in \text{Aut} \tilde{\Delta}$ is called a deck transformation of $\varphi$ if $\varphi \circ g = \varphi$. The deck transformations of $\varphi$ form a subgroup of $\text{Aut} \tilde{\Delta}$ denoted by $\text{Aut} \tilde{\Delta}_\varphi$. The normalizer of $\text{Aut} \tilde{\Delta}_\varphi$ in $\text{Aut} \tilde{\Delta}$ is denoted by $\text{Aut}(\tilde{\Delta}, \varphi)$. If $\tilde{\Delta}$ is connected, then it is easily seen that $\text{Aut} \tilde{\Delta}_\varphi$ operates freely on $\varphi^{-1}(v)$ for all $v \in V$. We call the covering $\varphi$ normal if this action is transitive for all $v \in V$.

If $\Delta$ is connected, then a covering $\varphi: \tilde{\Delta} \to \Delta$ is called universal if $\tilde{\Delta}$ is connected and if the following property holds: if $\psi: \tilde{\Delta} \to \Delta$ is another covering, then there exists a covering $\phi: \tilde{\Delta} \to \tilde{\Delta}$ with $\varphi = \psi \circ \phi$.

The proof of the following theorem is standard.

**Theorem 2.1.** If $\Delta$ is a connected complex, then there exists up to isomorphism exactly one universal covering $\varphi: \tilde{\Delta} \to \Delta$. This covering is normal, $\pi_1(\Delta) \cong \text{Aut} \tilde{\Delta}_\varphi$ and $\text{Aut} \Delta \cong \text{Aut}(\tilde{\Delta}, \varphi)/\text{Aut} \tilde{\Delta}_\varphi$. Furthermore, $\tilde{\Delta}$ is simply-connected.

**Definition 2.2.** Let $\Delta = (V, S)$ be a simplicial complex. Set

$$E_0(\Delta) := \{(x, y) \in V^2; \{x, y\} \in S\}$$

and let $G$ be a group. A 1-cocycle from $\Delta$ to $G$ is a map $\mu: E_0(\Delta) \to G$ such that $\mu(x, y)\mu(y, z) = \mu(x, z)$ for all $x, y, z \in V$ with $\{x, y, z\} \in S$. The set of all 1-cocycles from $\Delta$ to $G$ is denoted by $Z^1(\Delta, G)$.

If $\Delta$ is a simplicial complex and $\mu: E_0 \to G$ is a 1-cocycle, we can define a new complex $(\Delta \times G)_\mu$ by taking $\Delta \times G$ as vertex set and all nonempty sets of the form $\{(x_1, g), (x_2, \mu(x_2, x_1)g), \ldots, (x_n, \mu(x_n, x_1)g)\}$ with $g \in G$ and $\{x_1, \ldots, x_n\} \in S$ as simplices. (Note that this is independent of the choice of $x_1$.)

The proof of the following theorem is very easy.

**Theorem 2.3.** Let $\varphi: \tilde{\Delta} \to \Delta$ be a normal covering and $G := \text{Aut} \tilde{\Delta}_\varphi$. For each $x \in V$ choose an element $\tilde{x} \in \varphi^{-1}(x)$. Then, for $\{x, y\} \in S$ there exists a unique element $\mu(y, x) \in G$ such that $\tilde{x}$ and $\tilde{y}^\mu(y,x)$ are adjacent. The mapping $\mu: E_0(\Delta) \to G; (x, y) \mapsto \mu(x, y)$ is a 1-cocycle and $(x, g) \mapsto \tilde{x}^g$ defines an isomorphism between $(\Delta \times G)_\mu$ and $\tilde{\Delta}$. If $\tilde{\Delta}$ is connected, then $\varphi$ is an isomorphism iff $\mu(x, y) = 1$ for all pairs of adjacent vertices $x$ and $y$.

A geometry of rank $n$ is a pair $(\Delta, \tau)$ where $\Delta = (V, S)$ is a connected, complete simplicial complex and $\tau: V \to \{1, \ldots, n\}$ is a map such that for every
$\sigma \in S$ with $\dim \sigma < n - 2$ the residue $\Delta_{\sigma}$ is connected and the following property holds: $\tau|\sigma$ is bijective or there exist at least two different simplices $\sigma_1, \sigma_2$ such that $\sigma \subseteq \sigma_1 \cap \sigma_2$ and $\tau|\sigma_i$ is bijective for $i = 1, 2$.

In a geometry, simplices are called flags, and two adjacent but different vertices are called incident. We set $\text{type}(\sigma) := \tau(\sigma), \text{cotype}(\sigma) := \{1, \ldots, n\} \setminus \text{type}(\sigma), \text{rank}(\sigma) := |\text{type}(\sigma)|$ and $\text{corank}(\sigma) := n - \text{rank}(\sigma)$. If $\text{corank}(\sigma) \geq 2$, then the residue $\Delta_{\sigma}$ is a geometry of rank equal to $\text{corank}(\sigma)$.

### 2.2 Groups of Type Ly

**Definition 2.4.** A finite group $G$ is called a group of type Ly if there is an involution $t \in G$ not contained in $Z^*(G)$ such that $C_G(t)$ is isomorphic to $2 \leq A_{11}$ (the double cover of the alternating group on eleven letters).

We present here some facts about groups of type Ly and the Chevalley group $G_2(5)$.

**Theorem 2.5.** Let $G$ be a group of type Ly. Then the following statements hold:

(a) $G$ is simple (see [6, 2.1(e)]).

(b) The order of $G$ is $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$ ([6, 3.2]).

(c) There is a unique conjugacy class of involutions and elements of order 4 in $G$, respectively. If $t$ is an element of order 4 in $G$, then the image of $t$ in $A_{11} \cong C_G(t^2)/\langle t^2 \rangle$ is a double transposition ([6, 2.1]).

(d) There are exactly two conjugacy classes of elements of order 3 in $G$, which are called 3A and 3B. If $t$ is an involution in $G$ and $x$ is an element of order 3 in $C_G(t)$, then $x$ is in 3A if and only if $x$ corresponds to a 3-cycle in $A_{11} \cong C_G(t^2)/\langle t^2 \rangle$. In this case the normalizer of $\langle x \rangle$ in $G$ is isomorphic to $3 \cdot \text{Aut} \text{McL}$. If $x$ is in 3B, then the normalizer of $\langle x \rangle$ is a group of type $3^6 : (2 \cdot A_5, 2)$. ([6, 2.2–2.6]).

(e) There are two conjugacy classes of elements of order 5, called 5A and 5B. If $t$ is an involution and $x$ is an element of order 5 in $C_G(t)$, then $x$ is in 5A if and only if the image of $x$ in $A_{11}$ is a 5-cycle. The normalizer of an element in 5A is a group of type $5^{1+4} : (4 \cdot S_5)$, while the normalizer of an element in 5B is of type $(5 \times 5^3) : S_3$ (cf. [6, 2.9–2.16] and [7]).

(f) The group $G$ has exactly 53 conjugacy classes. The character table of $G$ is uniquely determined (see [6, Table II, pp. 557–559] with conjugacy class 25 instead of $S_9$, or [2, p. 175]).

(g) There is up to conjugation a unique subgroup $H$ of $G$ with $H \cong G_2(5)$. The group $G$ has rank 5 on the set of cosets of $H$. The non-trivial two-point
stabilizers are isomorphic to $5^{1+4} : (4 \cdot S_4)$, $\text{PSU}(3,3)$, $2 \cdot (A_5 \times A_4) : 2$ and $3 : \text{PGL}(2,7)$. If $\chi$ is the permutation character belonging to $H/G$, then $\chi = 1a + 45694a + 1534500a + 3029266a + 4226695a$ (notation as in [2]). See [6, 5.4–5.7].

(h) If $H \leq G$ is isomorphic to $G_2(5)$ and $t \in H$ is an involution, an element in $3A$ or an element in $3B$, then $t^H = t^G \cap H$.

(i) If $t$ is an involution in $H$, then $C_H(t)$ is a group of type $2 \cdot (A_5 \times A_5) : 2$. The proof of Lemma 5.3 in [6] implies that $C_H(t)/(t)$, regarded as subgroup of $A_{11} \cong C_G(t)/(t)$, has two orbits on the set $\{1, \ldots, 11\}$. These orbits have size 5 and 6. It acts as $S_5$ on the orbit of size 5 and as $\text{PGL}(2,5)$ on the orbit of size 6.

(j) If $t$ is an involution in $H$, then $t$ fixes exactly 42 points and 42 lines in $\mathbb{H}(5)$ (This can be seen by regarding the permutation characters of $G_2(5)$ on the set of points resp. lines in $\mathbb{H}(5)$. We can label these points resp. lines by $p_i$, $p_{ij}$ resp. $l_i$, $l_{ij}$ with $1 \leq i, j \leq 6$ such that $p_i$ is incident to $l_{ij}$, $l_i$ is incident to $p_{ij}$ and $p_{ij}$ is incident to $l_j$. The group $C_H(t)$ acts as $\text{PGL}(2,5)$ on both sets $\{l_i; 1 \leq i \leq 6\}$ and $\{p_i; 1 \leq i \leq 6\}$. The kernels of these two actions are isomorphic to $\text{SL}(2,5)$ and intersect in $\{t\}$.

(k) If $t \in H$ is an element in $3A$, then $N_H(\langle t \rangle) \cong 3 \cdot U_3(5).2$ and $t$ fixes exactly 126 lines and no points in $\mathbb{H}(5)$. The lines fixed by $t$ form a spread in $\mathbb{H}(5)$.

(l) If $t \in H$ is an element in $3B$, then $t$ fixes no lines and exactly 6 points in $\mathbb{H}(5)$.

(m) Up to conjugacy, there is a unique subgroup $T \leq H$ of type $4 \times 4$. If $A$ is the set of all elements in $\mathbb{H}(5)$ fixed by $T$, then $A$ is an apartment in $\mathbb{H}(5)$.

2.3 Amalgams

An amalgam of rank $n$ of groups consists of a collection of groups $(G_J)_{\emptyset \neq J \subseteq \{1, \ldots, n\}}$ and homomorphisms $(\varphi_{J,K} : G_K \to G_J)_{\emptyset \neq J \subseteq K \subseteq \{1, \ldots, n\}}$ such that $\varphi_{J,K} \circ \varphi_{K,L} = \varphi_{J,L}$ for $J \subseteq K \subseteq L$ holds. In our case, $G_K$ will always be a subgroup of $G_J$ for $J \subseteq K$, and $\varphi_{J,K}$ will always be the inclusion. For $i_1, \ldots, i_m \in \{1, \ldots, n\}$, we will write $G_{i_1 i_2 \ldots i_m}$ instead of $G_{\{i_1, i_2, \ldots, i_m\}}$.

A completion of an amalgam $A = ((G_J)_J, (\varphi_{J,K})_{J \subseteq K})$ is a group $G$ and a collection of maps $\psi_J : G_J \to G$ such that $\psi_J \circ \varphi_{J,K} = \psi_K$ for all $J \subseteq K$. A completion $(G, (\psi_J)_J)$ is called faithful if all maps $\psi_J$ are injective.

A completion $(G, (\psi_J)_J)$ of an amalgam $A = (G_J)_J$ is called universal if the following condition holds: For any other completion $(H, (\pi_J)_J)$ of $A$ there exists a homomorphism $\Phi : G \to H$ with $\Phi \circ \psi_J = \pi_J$ for all $J$. Universal
completions always exist and are unique up to isomorphism (but they are not always faithful). See for instance [10, 1.1].

If \( A = ((G_J)_J, (\varphi_{J,K})_{J \subseteq K}) \) and \( \mathcal{A} = ((G'_J)_J, (\rho_{J,K})_{J \subseteq K}) \) are amalgams of groups, then they are called isomorphic if there exist isomorphisms \( \phi_J : G_J \rightarrow G'_J \) such that \( \varphi_{J,K} = \rho_{J,K} \circ \phi_K \) for all \( J \subseteq K \). One easily sees that isomorphic amalgams have isomorphic universal completions.

## 3 The 5-local geometry of the Lyons group

### 3.1 Construction

We briefly describe the construction of the 5-local geometry of a group of type \( Ly \), for more details see [5] or [3]. Let \( G \) be a group of type \( Ly \) and \( H \) be a subgroup of \( G \) isomorphic to \( G_2(5) \). We set \( \mathfrak{P} := H/G \); this will be the set of points in our geometry. Then \( G \) has rank 5 on \( \mathfrak{P} \) with non-trivial double point stabilizers of type \( 5^{1+4} \cdot 4 \cdot S_4 \), \( PSU(3,3) \), \( 2 \cdot (A_4 \times A_5) \), and \( 3 : PGL(2,7) \). For a point \( x \), we denote the corresponding orbits \( \Gamma(x) \), \( \Gamma_2(x) \), \( \Gamma_3(x) \) and \( \Gamma_4(x) \), respectively (or just \( \Gamma_2, \Gamma_3 \) etc.). The graph having vertex set \( \mathfrak{P} \) with \( x, y \) adjacent iff \( y \in \Gamma(x) \) will also be denoted by \( \Gamma \).

Let \( y \) be in \( \Gamma(x) \) and \( R := O_3(G_{xy}) \cong 5^{1+4} \). If \( l \) is the set of fixed points of \( R \) in \( \mathfrak{P} \), then \( |l| = 6 \), \( N_G(R) = G_l \) is a group of type \( 5^{1+4} : 4 \cdot S_6 \) and \( G^l = S_6 \). We set \( \mathcal{L} := \{ [l] : y \in G \} \). This will be the set of lines in our geometry. Let \( l \) be a line and \( x \) a point in \( l \). Then there exists a unique line \( L \) in the Cayley hexagon \( \mathbb{H}(5) \) with \( G_{l,x} = G_{L,x} \). If \( P \) is a point in \( \mathbb{H}(5) \) incident to \( L \) and \( E \) is the subgroup of \( G_{x,L,P} \) fixing all points in \( \mathbb{H}(5) \) collinear to \( P \), then \( E \) is elementary abelian of order \( 5^4 \). Let \( \pi \) be set of fixed points of \( E \) in \( \mathfrak{P} \) and \( \mathfrak{F} := \{ \pi^g : y \in G \} \). The members of \( \mathfrak{F} \) will be the planes in our geometry.

Now let \( \Delta \) be the geometry with \( \mathfrak{P} \cup \mathcal{L} \cup \mathfrak{F} \) as vertex set, symmetrized inclusion as incidence relation and the natural type function. Then we have:

**Theorem 3.1.**

(a) If \( x \) is a point, then \( G_x \cong G_2(5) \) and \( \Delta_x \) is isomorphic to \( \mathbb{H}(5) \). The planes incident to \( x \) correspond to the points in \( \mathbb{H}(5) \) and the lines incident to \( x \) correspond to the lines in \( \mathbb{H}(5) \).

(b) If \( l \) is a line, then \( \Delta_l \) is a generalized digon, \( G_l = 5^{1+4} : 4 \cdot S_6 \) and \( G^l = S_6 \). The actions on the sets of points resp. planes in \( \Delta_l \) are not isomorphic.

(c) If \( \pi \) is a plane, then \( \Delta_\pi \) is a projective plane of order \( 5 \). We have \( G_\pi \cong 5^3 \cdot SL(3,5) \) and \( G^\pi \cong SL(3,5) \).
(d) If $\sigma$ is a maximal simplex in $\Delta$, then $G_{\sigma}$ is a group of type $5^{1+4+1} : (4 \times 4)$. The group $G$ acts transitively on the set of maximal simplices of $\Delta$.

The geometry $\Delta$ is a geometry with affine diagram $\tilde{\Delta}_2 = \begin{array}{ccc} & & \\ & \varphi : \tilde{\Delta} \to \Delta & \\
\end{array}$. If $\varphi : \tilde{\Delta} \to \Delta$ is the universal covering of $\Delta$, then $\tilde{\Delta}$ is a building having the same diagram as $\Delta$ by a theorem of J. Tits (see [9, Theorem 1]). Since apartments in $\Delta$ are infinite, $\tilde{\Delta}$ is infinite. Thus $\pi_1(\Delta)$ is infinite and $\Delta$ and $\tilde{\Delta}$ are not isomorphic.

Clearly, $G$ is a subgroup of the automorphism group of $\Delta$. By Theorem 2.1, there exists a subgroup $\tilde{G}$ of $\text{Aut}(\tilde{\Delta}, \varphi)$ containing $\Pi := \text{Aut} \tilde{\Delta}$ such that $\tilde{G}/\Pi \cong G$. Using the following theorem, we can deduce that $G$ and $\tilde{G}$ are the full automorphism groups of $\Delta$ and $\tilde{\Delta}$ respectively.

**Theorem 3.2.** Let $\Omega$ be a geometry having diagram $\begin{array}{ccc} & & \\ & \varphi : \Omega \to \tilde{\Omega} & \\
\end{array}$ with non-trivial residues isomorphic to $\text{PG}(2,5)$ and $\text{H}(5)$ such that for every point $p$ the planes of $\Omega_p$ correspond to the points in $\tilde{\Omega}(5)$. Suppose $X$ is a subgroup of $\text{Aut} \Omega$ acting transitively on the set of maximal simplices such that $X^p \cong G_2(5)$ for a point $p$, $X^l \cong S_0$ for a line $l$ and $X^E = \text{SL}(3,5)$ for a plane $E$. Then $\text{Aut} \Omega = X$ and $X_p = 1$ for every point $p$.

**Proof.** Set $A := \text{Aut} \Omega$ and let $p$ be a point in $\Omega$. Then, by Frattini, we have $A = XAp$. Furthermore, since $Xp$ is the full automorphism group of $\Omega_p$, we get $A_p = XpAp$, hence $\text{A}(p)$, $\text{A}_p$. We will show $U = A_1$.

If $l$ is a line in $\Omega_p$, then $U \leq A_l$, hence $A_l = UXl$. If $K$ is the kernel of the action of $A_l$ on the set of planes in $\Omega_l$, then $U \leq K$ and $X_lU = K \leq A_l$. Hence $K$ fixes $p$; since $A_l$ is transitive on the set of points in $\Omega_l$, we get $K = A_l$. So $U$ fixes every point collinear to $p$.

If $E$ is a plane in $\Omega_p$, then $U$ fixes every point in $E$, so we get $U \leq A_1$. If $l$ is a line in $E$, then every point in $\Omega_l$ is fixed by $U$, so again we get $U \leq A_1$.

Now let $l$ be a line in $\Omega_p$ and $p \neq q$ a point on $l$. We have proved that $U$ fixes every element in $\Omega_q$ having distance at most 3 to $l$ in $\Omega_q \cong \tilde{\Omega}(5)$. If $\alpha = (E_1, l_1, E_2, l_2, E_3, l_3, E_4)$ is an ordered root in $\Omega_4$, then the image of $U$ in $\alpha'$ is contained in the root subgroup $U_\alpha$. Hence we get $|U : U \cap A_{(q)}| \leq 5$ and $U \cap A_{(q)} = U \cap A_{(E_1)}$, since $U_\alpha$ is sharply transitive on the set of lines in $\Omega_4, E_1$ different from $l_1$.

Suppose $U \cap A_{(q)} \neq U$. Then we have $|U : U \cap A_{(q)}| = 5$. If $x$ is in $U \setminus A_{(q)}$, then the image of $x$ in $A^{E_1}$ is an elation with axis $l_1$. If $r$ is the center of $x$, then $r$ is incident to $l_1$. So every element in $U$ fixes every line in $E_1$ incident to the point $r$. But $X_{E_2, p, l_1}^{E_2} \cong \text{GL}(2,5)$ acts transitively on the point set of $\Omega_l$, and on the set of planes in $\Omega_l$ different from $E_2$. Hence, $X_{E_1, E_2, p, l_1}$ is still transitive on the set of points incident to $l_1$. Since $U$ is normalized by this group, the elements of $U$ fix every line in $E_1$, a contradiction.
We have proved \( A(p) \leq A(q) \), and by symmetry equality holds. Therefore, by induction, \( A(p) = A(q) \) for every point \( q \). Hence \( A(p) \) must be trivial. \( \square \)

**Corollary 3.3.** \( G = \text{Aut} \Delta \) and \( \tilde{G} = \text{Aut} \tilde{\Delta} \).

The crucial point in the proof of the theorem is that an element in \( \text{Aut} \Omega \) which leaves invariant all points incident to a line automatically fixes all planes incident to this line and vice versa. This is a very unusual situation. For example, if \( \Omega \) is a projective space of dimension 3 over a field \( \mathbb{K} \) and \( L \) is a line in \( \Omega \), then the stabilizer of \( L \) in \( \text{PGL}(4, \mathbb{K}) \) acts as a group of type \( \text{PGL}(2, \mathbb{K}) \times \text{PGL}(2, \mathbb{K}) \) on the generalized digon \( \Omega_L \), and one factor of this group fixes all points incident to \( L \) and the other factor fixes all planes incident to \( L \).

The automorphism group of the building \( \tilde{\Delta} \) is relatively small; it does act transitively on the set of maximal flags, but the stabilizer of a maximal flag is finite. Therefore, the automorphism group of \( \Delta \) does not possess a BN-pair and \( \tilde{\Delta} \) is not a classical building.

### 3.2 Apartments in \( \Delta \)

If \( G \) is a group of type \( \text{Ly} \), then there is exactly one conjugacy class of subgroups isomorphic to \( \mathbb{Z}_4 \times \mathbb{Z}_4 \). If \( T \) is such a group, let \( A \) be the set of fixed elements of \( T \) in \( \Delta \). Then \( A \) is a thin subgeometry of \( \Delta \) containing exactly 12 points, 24 planes and 36 lines. The points of \( A \) can be labeled by the set \( \{1, 2, 3, 4\} \times \{a, b, c\} \), two points are collinear if and only if both coordinates are different. Now lines and planes can be identified as sets of two respectively three collinear points (see Figure 1; this description of \( A \) can be found on [5, p. 246]).

We call every \( G \)-conjugate of \( A \) an **apartment** of \( \Delta \). This is justified since it is easily seen that every connected component \( \varphi^{-1}(A) \) is an apartment in \( \Delta \).

If \( N \) is the normalizer of \( T \) in \( G \), then \( N/T \cong S_4 \times S_3 \) acts transitively on the set of maximal flags of \( A \). This action can be described by the natural action of \( S_4 \times S_3 \) on the set \( \{1, 2, 3, 4\} \times \{a, b, c\} \).

Let \( \mathfrak{A} \) be set of all apartments of \( \Delta \). For any subset \( X \) of \( \Delta \) let \( \mathfrak{A}_X \) be the set of all apartments containing \( X \).
4 Closed paths of small length in $\Delta$

4.1 The diameter of $\Delta$

**Lemma 4.1.** There are three orbits of triples $(x, y, z)$ such that $y$ is collinear to $x$ and $z$ but $x, y$ are not contained in a common plane.

(I) In this case $d(yx, yz) = 4$ and $G_{x,y,z} = 5 : (4 \times 4)$.

(IIa) In this case $d(yx, yz) = 6$ and $G_{x,y,z} = 4 : S_4$.

(IIb) In this case $d(yx, yz) = 6$ and $G_{x,y,z} = 3 : 8$.

Here, $d$ refers to the distance function of $\Delta_y$ (and not the one of $\Delta$).

**Proof.** $G_y$ acts transitively on both \{(l, m); l and m lines in $\Delta_y$, $d(l, m) = 4$\} and \{(l, m); l and m lines in $\Delta_y$, $d(l, m) = 6$\}. The stabilizer of a pair of lines is in the first case a group of type $5^1 : (4 \times 4)$, in the second case a group isomorphic to $GL(2, 5)$. In the first case the stabilizer of a pair $(l, m)$ acts transitively on the set of pairs $(x, z)$, where $x$ is a point on $l$ and $z$ a point on $m$ and both are different from $y$. Hence we get our first orbit.
Now if $l$ and $m$ are lines in $\Delta_x$ having maximal distance, then it is easily seen that $G_{y,l,m}$ has two orbits on the set $\{(x,z): x \text{ point on } l, z \text{ point on } m, x \neq y \neq z\}$ and that the stabilizers are isomorphic to $4 \cdot S_4$ and $3 : 8$, respectively. Hence the claim follows. $\blacksquare$

**Lemma 4.2.** If $(x,y,z)$ is a path of type (I), then $x$ and $z$ are in relation $\Gamma_3$.

**Proof.** Since the order of $G_{x,y,z}$ is divisible by 5, either $z \in \Gamma(x)$ or $z \in \Gamma_3(x)$ holds. Now let $T$ be a subgroup of type $Z_4 \times Z_4$ in $G_{x,z}$ and $A := \text{Fix}_{\Delta}(T)$. Then $x$ and $z$ cannot be collinear since in this case the line $xz$ would also be in $A$, which can easily be recognized as impossible. $\blacksquare$

**Lemma 4.3.** Let $z$ be in $\Gamma_3(x)$.

(a) There are exactly six planes $\pi_1, \ldots, \pi_6$ in $\Delta_x$ and six planes $\pi'_1, \ldots, \pi'_6$ in $\Delta_z$ such that $\pi_i$ and $\pi'_i$ are incident to a common line $l_i$.

(b) The planes $\pi_1, \ldots, \pi_6$ and $\pi'_1, \ldots, \pi'_6$ have pairwise maximal distance in $\Delta_x$ and $\Delta_z$, respectively.

(c) Every point in $\Gamma(x) \cap \Gamma(z)$ is incident to one of the lines $l_1, \ldots, l_6$.

**Proof.**

(a) Let $(x,y,z)$ be a path of type (I) and let $(yx, \pi, l, \pi', yz)$ be the unique shortest path in $\Delta_x$ between $yx$ and $yz$. Now $G_{x,y,z} \leq G_{x,z,\pi} \leq G_{x,z}$, the first group is isomorphic to $5 : (4 \times 4)$, the second to $5 : (4 \cdot S_4)$ and the third to $2 : (A_5 \times A_4).2$. Since $G$ is transitive on the set of paths of type (I), the claim follows.

(b) $G_{x,z}$ acts as $\text{PGL}(2,5)$ and hence 2-transitively on each set $\{\pi_1, \ldots, \pi_6\}$ and $\{\pi'_1, \ldots, \pi'_6\}$. If $i$ and $j$ are different, then the order of $G_{\pi_i, \pi_j}$ is divisible by 3. Hence these two planes must have maximal distance.

(c) Suppose there is a point $y \in \Gamma(x) \cap \Gamma(z)$ such that $(x,y,z)$ is a path of type (Ia) or (Ib). Let $t$ be the central involution in $G_{x,z}$ and $s$ the central involution in $G_{x,y,z,z}$. If $s \neq t$, then $s$ corresponds in $C_G(t)/\langle t \rangle \cong A_{11}$ to a product of four disjoint transpositions (see Theorem 2.5(c)). In $G_{x,y,z,z}$, there is an element of the conjugacy class $3A$ (i.e. an element corresponding to 3-cycle in $A_{11} \cong C_G(t)/\langle t \rangle$) centralizing $s$. Now $C_G(t)$ fixes a partition of type $(5,6)$ and acts as $S_6$ on the set of 5 letters and as $\text{PGL}(2,5)$ on the set of 6 letters (see Theorem 2.5(i)). Therefore the 3 fixed letters of $s$ must be in the orbit of size 5. But this implies that $s$ cannot be a square in $G_{x,y,z,z} \leq C_G(t)$, a contradiction.

Now if $s = t$, then by Theorem 2.5(j), there is a plane $\pi_i$ having distance at most 3 to $xy$. Clearly, $d(\pi_i, xy) = 1$ is a contradiction, and if $d(xy, \pi_i) = 3$,
then there is a point \( w \) in \( l_1 \) (and hence collinear to \( z \)) such that \((w, y, z)\) is a path of type (I), also a contradiction.

\[\text{Lemma 4.4.} \quad \text{If } x, y \text{ and } z \text{ are three pairwise collinear points, then there is a plane } \pi \text{ incident to all three of them.}\]

\[\text{Proof.} \quad \text{Suppose not. If } d(yx, yz) = 4, \text{ then } (x, y, z) \text{ would be a path of type (I) and hence } z \text{ would be in } \Gamma_3(x). \text{ So we must have } d(xy, xz) = d(yx, yz) = 6 \text{ in } \Delta_x \text{ and } \Delta_y, \text{ respectively. Let } (xy, \pi_1, l_1, \pi_2, l_2, \pi_3, xz) \text{ be a path in } \Delta_x \text{ connecting } xy \text{ and } xz. \text{ Then there is a path } (xz, \pi_3, l_3, \pi_4, l_4, \pi_5, yz) \text{ connecting } xz \text{ and } yz \text{ in } \Delta_z. \text{ There is a point } w \text{ lying on both } l_2 \text{ and } l_3. \text{ We now have that } y \in \Gamma_3(w) \text{ and that } \pi_2 \text{ and } \pi_4 \text{ are two of the six planes in } \Delta_w \text{ containing a line whose points are all collinear to } y. \text{ But } \pi_2 \text{ and } \pi_4 \text{ have distance } 4, \text{ a contradiction.} \]

Let \( \Lambda := \text{Cl}(\Gamma) \) be the complex whose vertices are the points of \( \Delta \) and whose simplices are the sets of pairwise collinear points. Then \( \Lambda \) contains all information about \( \Delta \) since the planes of \( \Delta \) can be identified as the maximal simplices in \( \Lambda \) and the lines in \( \Delta \) correspond to the 5-dimensional simplices contained in exactly six maximal simplices. Now we see that every covering of \( \Delta \) corresponds to a covering of \( \Lambda \) and vice versa and that \( \pi_1(\Delta) \equiv \pi_1(\Lambda) \) holds.

\[\text{Lemma 4.5.} \quad \text{If } (x, y, z) \text{ is a path of type (IIa), then } z \in \Gamma_2(x).\]

\[\text{Proof.} \quad \text{We know that } z \text{ must be in } \Gamma_2(x) \text{ or in } \Gamma_4(x), \text{ and since } 32 \text{ is a divisor of } |G_{x,y,z}|, \text{ the latter possibility can be excluded.} \]

\[\text{Lemma 4.6.} \quad \text{If } (x, y, z) \text{ is a path of type (IIb), then } z \in \Gamma_4(x).\]

\[\text{Proof.} \quad \text{Again, } z \in \Gamma_2(x) \cup \Gamma_4(x) \text{ must hold. Suppose that } z \in \Gamma_2(x). \text{ Let } g \text{ be an element of order } 4 \text{ in } H := G_{x,y,z} \cong 3 : 8. \text{ Then there are exactly two lines } l_1 \text{ and } l_2 \text{ in } \Delta_x \text{ which are fixed pointwise by } g. \text{ With Theorem 2.5(j), it can be easily seen that these two lines are the only lines in } \Delta_x \text{ fixed by } H. \text{ Hence } y \text{ must be incident to one of these lines, say to } l_1. \text{ Set } J := N_{G_{x,y,z}}(\langle g \rangle). \text{ Since } g \text{ commutes with an element of order } 3 \text{ in } G_{x,y,z} \leq G_{x,z} \cong \text{PSU}(3, 3), \text{ we have } H \leq J \cong 4 \cdot S_4 \text{ (see [2]). So there is an element } a \in J \setminus H \text{ which fixes } l_1. \text{ The point } y^a \neq y \text{ is incident to } l_1 \text{ and belongs to } \Gamma(x) \cap \Gamma(z). \text{ But now } y, y^a \text{ and } z \text{ are pairwise collinear and therefore must be incident to a common plane } \pi. \text{ Since } x \text{ lies on } yy^a = l_1, \text{ and } x \text{ and } z \text{ are collinear, a contradiction. Thus } z \in \Gamma_4(x). \]

Since \( G \) acts transitively on the sets of paths of type (I), (IIa) and (IIb), we have shown:
Theorem 4.7. For all points \( x \) and \( y \) there is point collinear to both of them. In particular, the diameter of \( \Gamma \) is 2. If \( d(x, y) = 2 \), then \( G_{xy} \) acts transitively on \( \Gamma(x) \cap \Gamma(y) \).

Lemma 4.8. Let \( x \) be a point and \( y \in \Gamma_2(x) \). Let \( \Psi_{xy} \) be the geometry of rank 2 having \( \Gamma(x) \cap \Gamma(y) \) as set of points and \( \mathcal{A}_{xy} \) as set of lines and canonical incidence relation. Then \( \Psi_{xy} \) is \( G_{xy} \)-isomorph to the Cayley hexagon \( \mathbb{H}(2) \).

Proof. Let \( z \) be a point \( \Gamma(x) \cap \Gamma(y) \) and \( A \) an apartment containing \( x \), \( y \) and \( z \). Then \( G_{x,y} \cong \text{PSU}(3, 3) \cong G_2(2)' \), \( G_{x,y,A} = (4 \times 4).S_3 \), \( G_{x,y} = 4 \cdot S_4 \) and \( G_{x,y,z,A} = (4 \times 4).2 \). Since \( G_{xy} \) acts transitively on the set of maximal flags in \( \Psi_{xy} \), the claim follows. \( \square \)

Lemma 4.9. Let \( x \) and \( y \) be two points in relation \( \Gamma_4 \), \( H := G_{xy} \cong 3 : \text{PGL}(2, 7) \) and \( g \) be an element of order 3 in \( O_3(H) \). Then \( g \) is a 3A-element and \( O_3(H) = \langle g \rangle \) is the kernel of the action of \( G_{xy} \) on \( \Gamma(x) \cap \Gamma(y) \). If \( z \) and \( w \) are two different points in \( \Gamma(x) \cap \Gamma(y) \), then \( xw \) and \( xz \) have maximal distance in \( \Delta_z \).

Proof. Let \( S \) be a Sylow 7-subgroup of \( H \). Then \( N_A(S) \) contains a 3-sylow subgroup \( P \) of \( H \). By [6, Proposition 2.8] all 3A-Elements in \( P \) are contained in \( C_H(S) \cap P = O_3(H) \). For \( z \in \Gamma(x) \cap \Gamma(y) \), \( H_z \) fixes a line in \( \Delta_z \), hence a 3-element in \( H_z \) must be a 3A-element, and therefore \( O_3(H) \leq H_z \). Thus the first claim follows.

Now if \( w \) and \( z \) are distinct points in \( \Gamma(x) \cap \Gamma(y) \), then \( w \) and \( z \) cannot be collinear nor in relation \( \Gamma_3 \) (since 3 divides \( |G_{x,y,z,w}| \)), hence \( d(xw, xz) = 6 \) must hold. \( \square \)

Lemma 4.10. Let \( x \) be a point and \( y \in \Gamma_3(x) \).

(a) If \( z \) is in \( \Gamma(x) \cap \Gamma(y) \), then \( \Gamma(x) \cap \Gamma(y) \) contains five points from each \( \Gamma(z) \) and \( \Gamma_3(z) \) and 25 points from \( \Gamma_2(z) \).

(b) Define a graph structure on \( \Gamma(x) \cap \Gamma(y) \) such that \( z \) and \( w \) are adjacent if and only if \( x, y, z \) and \( w \) are contained in a common apartment or \( w \) and \( z \) are collinear. Then this graph is connected.

Proof. Let \( \pi_1, \ldots, \pi_6 \) be the six planes in \( \Delta_z \) as in Lemma 4.3.

(a) Suppose \( z \) is contained in \( \pi_1 \). If \( w \) is another point in \( \Gamma(x) \cap \Gamma(y) \), then \( z \) and \( w \) are collinear exactly if \( w \) is also in \( \pi_1 \). Hence we have 5 points in \( \Gamma(x) \cap \Gamma(y) \cap \Gamma(z) \).

In each plane \( \pi_2, \ldots, \pi_6 \) there exists exactly one point in \( \Gamma(x) \cap \Gamma(y) \) which is in relation \( \Gamma_3 \) to \( z \), so we have \( |\Gamma(x) \cap \Gamma(y) \cap \Gamma_3(z)| = 5 \). The other points must be in relation \( \Gamma_2 \) to \( z \) by Lemma 4.9.
(b) For all $i \neq j$ there is an apartment containing $x, y, \pi_i$ and $\pi_j$. Hence the claim follows. \hfill \qed

**Lemma 4.11.** Let $y$ be in $\Gamma_4(x)$.

(a) If $z$ is in $\Gamma(x) \cap \Gamma(y)$, then there is a point $w \in \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)$.

(b) The graph having $\Gamma(x) \cap \Gamma(y)$ as vertex set with $z$ and $w$ adjacent if and only if they are in relation $\Gamma_3$ is connected.

**Proof.** (a) Let $H := G_{x,y}$ and $g$ be an element of order 3 in $O_3(H)$. Then $g$ is a 3A-element by Lemma 4.9, hence $g$ fixes exactly 750 points in $\Gamma_4(x)$ since $N_{G_x}(\langle g \rangle) \cong 3 \cdot U_3(5).2$. Let $t$ be an involution in $H$ and $l_1, \ldots, l_6$ the 6 lines in $\Delta_x$ fixed by $t$ pointwise. By Theorem 2.5(j), we have $C_{G_x}(t) \langle t \rangle \cong (A_5 \times \text{PGL}(2, 5)).2$, and the points on the $l_i$ different from $x$ can be labeled by the set $\{1, \ldots, 5\} \times \{1, \ldots, 6\}$ where $(i, j)$ is incident to $l_j$ such that the action of $C_{G_x}(t) \langle t \rangle$ on the set of these points corresponds to the natural action of this group on the set $\{1, \ldots, 5\} \times \{1, \ldots, 6\}$.

In $C_{G_x}(t) \langle t \rangle$, $g$ corresponds to an element $(s, 1)$ with $s$ a 3-cycle. We can assume $s = (123)$. Then $g$ fixes all of the lines $l_1, \ldots, l_6$ and every point with first coordinate 4 or 5. For $1 \leq i \leq 6$ set $y_i := (4, i)$. Now $g$ fixes 126 lines in $\Delta_y$, $l_i$, and 125 others. There are exactly 3 fixed points of $g$ on each of these 125 lines. These points are $y_i$, the unique point in $\Gamma_2(x)$ on this line and one other point in $\Gamma_4(x)$. So we get up to $6 \cdot 125 = 750$ fixed points of $g$ in $\Gamma_4(x)$.

Suppose that these points are all different. Then these points are all the 750 fixed points of $g$ in $\Gamma_4(x)$. We can assume that $y$ is collinear to $y_1$. But if $h$ is a 3B-element in $C_{G_{x,y}}(t)$, then $h$ fixes none of the lines $l_1, \ldots, l_6$, and since $h$ centralizes $g$, $y_1^h$ must be a point with first coordinate 4. Hence $y_1^h$ cannot be $y$, a contradiction.

We conclude that there must be a point $z \in \Gamma_4(x)$ and $1 \leq i < j \leq 6$ such that $z$ is collinear to $y_i$ and $y_j$. Now $C_{G_{x,y}}(t) \cap G_{y_i,y_j} = 2 \cdot (A_4 \times 2).2$. Since 32 is a divisor of $|G_{y_i,y_j}|$, the points $y_i$ and $y_j$ are in relation $\Gamma_2$. Since $G_x$ is transitive on $\Gamma_4(x)$, the claim follows.

(b) Now by Lemma 4.8, if $w, z \in \Gamma(x) \cap \Gamma(y)$ with $w \in \Gamma_2(z)$, then $H_z = 3 : 8$ and $H_z,w = 3$, hence $z$ has exactly 8 neighbours in $\Gamma(x) \cap \Gamma(y)$. If $H_z < J < H$, then $|J : H_z|$ must be 2. Since the connected component of $z$ in $\Gamma(x) \cap \Gamma(y)$ is a block of size at least 9, we see that $\Gamma(x) \cap \Gamma(y)$ must be connected. \hfill \qed
4.2 The action of a point stabilizer on the sets of planes and lines in $\Delta$

Lemma 4.12. Let $J_1, J_2 \leq \text{SL}(3, 5)$ be isomorphic to $S_4$ and $A_4$, respectively. Then the following statements hold.

(a) Both $J_1$ and $J_2$ have exactly 5 orbits on the point set of $\text{PG}(2, 5)$:

<table>
<thead>
<tr>
<th>Orbit</th>
<th>$\mathcal{D}$</th>
<th>$\text{Ex}_1$</th>
<th>$\text{Ex}_2$</th>
<th>$\text{Inn}_1$</th>
<th>$\text{Inn}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of points</td>
<td>6</td>
<td>3</td>
<td>12</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Stabilizer of a representative in $J_1$</td>
<td>$Z_4$</td>
<td>$D_4$</td>
<td>$Z_2$</td>
<td>$S_3$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>Stabilizer of a representative in $J_2$</td>
<td>$Z_2$</td>
<td>$V_4$</td>
<td>1</td>
<td>$Z_3$</td>
<td>$Z_2$</td>
</tr>
</tbody>
</table>

Table 1: Action of $J_1$ and $J_2$ on the points of $\text{PG}(2, 5)$

(b) Both $J_1$ and $J_2$ have exactly 5 orbits on the set of lines of $\text{PG}(2, 5)$:

<table>
<thead>
<tr>
<th>Orbit</th>
<th>$\Xi$</th>
<th>$\text{Sec}_1$</th>
<th>$\text{Sec}_2$</th>
<th>$\text{Pas}_1$</th>
<th>$\text{Pas}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of lines</td>
<td>6</td>
<td>3</td>
<td>12</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Stabilizer of a representative in $J_1$</td>
<td>$Z_4$</td>
<td>$D_4$</td>
<td>$Z_2$</td>
<td>$S_3$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>Stabilizer of a representative in $J_2$</td>
<td>$Z_2$</td>
<td>$V_4$</td>
<td>1</td>
<td>$Z_3$</td>
<td>$Z_2$</td>
</tr>
</tbody>
</table>

Table 2: Action of $J_1$ and $J_2$ on the lines of $\text{PG}(2, 5)$

(c) Table 3 shows to how many points/lines on each orbit a line/point is incident:

<table>
<thead>
<tr>
<th>Orbit</th>
<th>$\Xi$</th>
<th>$\text{Sec}_1$</th>
<th>$\text{Sec}_2$</th>
<th>$\text{Pas}_1$</th>
<th>$\text{Pas}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{D}$</td>
<td>1/1</td>
<td>1/2</td>
<td>4/2</td>
<td>0/0</td>
<td>0/0</td>
</tr>
<tr>
<td>$\text{Ex}_1$</td>
<td>2/1</td>
<td>2/2</td>
<td>0/0</td>
<td>0/0</td>
<td>2/1</td>
</tr>
<tr>
<td>$\text{Ex}_2$</td>
<td>2/4</td>
<td>0/0</td>
<td>2/2</td>
<td>1/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\text{Inn}_1$</td>
<td>0/0</td>
<td>0/0</td>
<td>3/1</td>
<td>0/0</td>
<td>3/2</td>
</tr>
<tr>
<td>$\text{Inn}_2$</td>
<td>0/0</td>
<td>1/2</td>
<td>2/1</td>
<td>2/3</td>
<td>1/1</td>
</tr>
</tbody>
</table>

Table 3: Incidence between points and lines in each orbit

For example, the entry in the first column of the second row means that a point of $\text{Ex}_1$ is incident to exactly two lines in $\Xi$ and that a line in $\Xi$ contains exactly one point in $\text{Ex}_1$. 
Proof. In $\text{SL}(3, 5)$, there is only one conjugacy class of subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The centralizer of such a group is a group isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$ and the normalizer a split extension of the centralizer by a group isomorphic to $S_3$. Every complement of the centralizer equals the normalizer of a Sylow 3-subgroup in the normalizer, hence there is a unique conjugacy class of complements. It follows that there is only one conjugacy class of subgroups isomorphic to $S_4$ and $A_4$, respectively. So we can assume that $J_1$ is the subgroup of monomial matrices having 1 and $-1$ as entries and that $J_2 = J'_1$. Now all the claims can be easily verified. □

For the rest of this section, let $x$ be a point and $H := G_x$. We will use the abbreviations $\Gamma, \Gamma_2, \Gamma_3$ and $\Gamma_4$ to represent $\Gamma(x), \Gamma_2(x), \Gamma_3(x)$ and $\Gamma_4(x)$ respectively.

Theorem 4.13. The group $H$ has exactly seven orbits on $\mathfrak{F}$. The stabilizer of a representative of each orbit is listed in Table 4:

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Stabilizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{F}_1(x)$</td>
<td>$5^{3+2} : \text{GL}(2, 5)$</td>
</tr>
<tr>
<td>$\mathfrak{F}_2(x)$</td>
<td>$5^{1+2} : (4 \times S_4)$</td>
</tr>
<tr>
<td>$\mathfrak{F}_3(x)$</td>
<td>$5 : (4 \times 4)$</td>
</tr>
<tr>
<td>$\mathfrak{F}_4(x)$</td>
<td>$31 : 3$</td>
</tr>
<tr>
<td>$\mathfrak{F}_5(x)$</td>
<td>$S_4$</td>
</tr>
<tr>
<td>$\mathfrak{F}_6(x)$</td>
<td>$S_4$</td>
</tr>
<tr>
<td>$\mathfrak{F}_7(x)$</td>
<td>$A_4$</td>
</tr>
</tbody>
</table>

Table 4: Stabilizers of a representative of each orbit of $H$ on $\mathfrak{F}$

Instead of $\mathfrak{F}_i(x)$ we will simply write $\mathfrak{F}_i$.

Proof. The planes in $\mathfrak{F}_1$ are the planes incident to $x$, those in $\mathfrak{F}_2$ contain exactly one line whose points are all collinear to $x$ and those in $\mathfrak{F}_3$ contain exactly one point collinear to $x$. Now 31 is a divisor of $|G|, |H|$ and $|\text{SL}_3(5)|$. A Sylow 31-subgroup has index 3 in its normalizer in $\text{SL}_3(5)$ and index 6 in its normalizer in $G$ and $H$. Looking at the permutation character of $G$ on the set of points in $\Delta$, one sees that an element of order 31 fixes exactly one point. We conclude that if $g \in G$ has order 31, then there are exactly two $\langle g \rangle$-invariant planes in $\Delta$ which are conjugate under $N_G(\langle g \rangle)$. Therefore, we have exactly one orbit of planes whose stabilizer is a group of type $31 : 3$. We call this orbit $\mathfrak{F}_4$. 

If \( g \in G \) is of order 5 or 6 and \( \pi \) is a plane fixed by \( g \), then there is a fixed point of \( g \) in \( \pi \). By counting the number of fixed points of such an element, we see that if \( \pi \) is a plane outside \( \bigcup_{i=1}^{4} \mathfrak{F}_i \), then \( G_{x,\pi} \) contains no element of order 5 or 6. Hence \( G_{x,\pi} \) is a \( \{2,3\} \)-group. Moreover, \( G_{x,\pi} \) contains no group of type \( 4 \times 4 \), because in the other case \( x \) and \( \pi \) would be contained in a common apartment. But all planes in a common apartment with \( x \) are in \( \mathfrak{F}_1(x) \cup \mathfrak{F}_2(x) \cup \mathfrak{F}_3(x) \) (this can be seen in Figure 1 on page 44). Therefore \( |G_{x,\pi}| \in \{6, 8, 12, 16, 24\} \). If \( G_{x,\pi} \) has order 12 or 24, then \( G_{x,\pi} \) is isomorphic to \( S_4 \) because these are the only groups of order 12 resp. 24 containing no element of order 6. There are exactly \( 2^5 \cdot 3^2 \cdot 5^6 \cdot 7 \cdot 31 \) planes in \( \mathfrak{F} \setminus \bigcup_{i=1}^{4} \mathfrak{F}_i \). So we see that there are five possibilities:

(a) There is only one other orbit with stabilizer of order 6.

(b) There are two other orbits with stabilizers both of order 12, hence isomorphic to \( A_4 \).

(c) There are two other orbits with stabilizers of order 24 and 8.

(d) There are three other orbits with stabilizers of order 16, 16 and 24.

(e) There are three other orbits with stabilizers isomorphic to \( S_4 \), \( S_4 \) and \( A_4 \).

Counting the number of fixed points of an element in the conjugacy class \( 3B \), we see that the cases (c) and (d) cannot hold.

Now let \( y \) be a point in \( \Gamma_4(x) \) and \( z \in \Gamma(x) \cap \Gamma(y) \). Then there is another point \( w \in \Gamma(x) \cap \Gamma(y) \) such that \( 3 : 8 \cong G_{x,y,z} = G_{x,y,w,z} \). Let \( \pi \) be a plane in \( \Delta_y \) having distance 3 to both \( yz \) and \( yw \). Then \( |G_{x,y,z,w,\pi}| = 4 \). Since there is an element in \( G_{x,y} \) which interchanges \( w \) and \( z \), we conclude that 8 is a divisor of \( |G_{x,y,\pi}| \). Since \( y \) is in \( \Gamma_4(x) \), \( \pi \) cannot be in \( \bigcup_{i=1}^{4} \mathfrak{F}_i(x) \). Hence cases (a) and (b) can be excluded and (e) must hold.

We will later see that we can distinguish the orbits \( \mathfrak{F}_5 \) and \( \mathfrak{F}_6 \) by the convention that a plane in \( \mathfrak{F}_5 \) contains \( \Gamma_2 \)-points whereas a plane in \( \mathfrak{F}_6 \) does not. Lemma 4.12 and Theorem 4.13 imply that there are 16 \( G_x \)-orbits of pairs \( (y, \pi) \) where \( \pi \) is a plane in \( \bigcup_{i=4}^{4} \mathfrak{F}_i \) and \( y \) is incident to \( \pi \).

**Lemma 4.14.** Let \( \pi \) be a plane in \( \mathfrak{F}_3 \). Then \( \pi \) contains exactly one point in \( \Gamma(x) \) and five points in both \( \Gamma_2(x) \) and \( \Gamma_3(x) \). There is a line \( l^* \) in \( \pi \) which contains all points in \( \Gamma_2(x) \). All planes incident to \( l^* \) are in \( \mathfrak{F}_3(x) \). Moreover, \( G_{x,l^*} \) is isomorphic to \( GL(2,5) \) and acts transitively on the set of these planes.

**Proof.** Let \( y \) be the unique point in \( \pi \) collinear to \( x \) and \( l = \text{proj}_{yx} \pi \) (in \( \Delta_y \)). Then all points on \( l \) different from \( y \) are in \( \Gamma_4 \). The points on \( \pi \) outside \( l \) are in
Lemma 4.15. Let \( \pi \) be a plane in \( \mathfrak{S}_4 \) and \( l \) a line incident to \( \pi \). Then there is a plane \( \pi' \) incident to \( l \) which is not in \( \mathfrak{S}_4 \).

Proof. Let \( y \) be a point in \( \pi \). Then there is a line \( m \) in \( \Delta_y \) containing a point collinear to \( x \). Now \( \pi \) cannot be incident to \( m \) because all points in \( \pi \) are \( G_x \)-conjugate.

There is a path of minimal length between \( m \) and \( \pi \) in \( \Delta_y \), and in this path there must be at least one line which is incident to a plane in \( \mathfrak{S}_4 \) and to a plane outside \( \mathfrak{S}_4 \). Thus the claim follows. \( \square \)

Lemma 4.16. For \( y \in \Gamma_2(x) \) there are exactly three \( H_y \)-orbits of planes incident to \( y \). Planes in the first orbit are in \( \mathfrak{S}_3 \), planes in second orbit are in \( \mathfrak{S}_5 \) and planes in the last orbit are in \( \mathfrak{S}_7 \). In the second case we have \( y \in \mathcal{O}(\pi) \). In the third case, \( y \in \text{Im}_1(\pi) \) holds.

Proof. Let \( \chi \) be the permutation character of \( G_y \) on the set of planes incident to \( y \). Then \( \chi \) is known by the character table of \( G_2(5) \) since this set corresponds to the point set of \( \mathbb{H}(5) \). Up to conjugacy there is a unique subgroup in \( G_2(5) \) isomorphic to \( \text{PSU}(3,3) \). For a plane \( \pi \) in \( \Delta_y \) we compute \( |(G_y)_{x} \backslash G_y / H_y| = \langle \chi, 1_{H_y} \rangle = (\chi_{H_y}, 1_{H_y}) = 3 \). Therefore we have proved the first part of the claim.
There is exactly one orbit of planes whose elements are in $F_3$. This orbit contains exactly $378$ planes. By Lemma 4.12 the other orbits have size $3024, 2016, 1512, 1008$ or $756$. However, $2016$ and $1512$ is the only possibility to choose two of these numbers such that their sum is $3906 - 378 = 3528$. Thus there is one orbit such that $H_{y,z} = Z_3$ for all planes $\pi$ in this orbit. Hence elements in this orbit are either contained in $F_4$ or $F_7$. Lemma 4.15 implies that there cannot be a line whose points are all in $\Gamma_2$. We conclude that the $\text{Inn}_1$-points in a $F_7$-plane must be points in $\Gamma_2$.

Now let $z$ be in $\Gamma(x) \cap \Gamma(y)$. Lemma 4.8 shows together with Lemma 4.10 that there are exactly $30$ points $w \in \Gamma(x) \cap \Gamma(y)$ such that $d(yz, yw) = 4$ in $\Delta_y$ holds. If $\pi$ is a plane incident to $yz$, $l$ the unique line in $\pi$ containing all $\Gamma_2$-points in $\pi$, then $G_{l,x} \cong \text{GL}(2,5)$ operates transitively on the set of planes incident to $l$ (Lemma 4.14). On each plane apart from $\pi$ there is exactly one point in $\Gamma(x)$ different from $z$. These points are all different, hence we get all the $6 \cdot 5 = 30$ points $w$ in $\Gamma(x) \cap \Gamma(y)$ for which $d(yw, yz) = 4$ in $\Delta_y$ holds.

Now let $m$ be a line which is different from $l$ and $yz$ and incident to $\pi$ and $y$. Set $J := G_{x,y,m,\pi}$. Then $J$ is cyclic of order $4$, and $J$ fixes exactly one other plane $\pi'$ incident to $m$. We see that $\pi'$ cannot contain a point $w$ in $\Gamma(x)$ because in this case we would have $d(yz, yw) = 4$ in $\Delta_y$, which would imply that the intersection of $\pi$ and $\pi'$ is either $y$ or $l$.

So $\pi'$ cannot be a plane in $F_3$, hence $G_{x,y,\pi'} = Z_4$. Therefore $\pi'$ is a plane in $F_5$ (by convention) and $y$ is in $\mathcal{O}(\pi')$. \hfill $\square$

**Lemma 4.17.** If $y$ is a point in $\Gamma_4(x)$, then there are exactly ten $H_y$-orbits on the sets of lines and planes incident to $y$, respectively.

**Proof.** Since $U_3(5)$ contains a unique conjugacy class of subgroups isomorphic to $\text{PGL}(2,7)$ (see [2]), there is up to conjugacy exactly one subgroup of type $3 : \text{PGL}(2,7)$ in $G_2(5)$, hence the claim can be verified using the character table of this group. \hfill $\square$

**Theorem 4.18.**

(a) If $\pi$ is a plane in $F_4$, then all points in $\pi$ are $\Gamma_4$-points.

(b) If $\pi \in F_5$, then the points in $\mathcal{O}$ are in $\Gamma_2$, the points in $\text{Ex}_1 \cup \text{Inn}_2$ are in $\Gamma_3$ and the points in $\text{Ex}_2 \cup \text{Inn}_1$ are in $\Gamma_4$.

(c) If $\pi \in F_6$, then the points in $\text{Ex}_1 \cup \text{Ex}_2 \cup \text{Inn}_2$ are in $\Gamma_4$ and the points in $\mathcal{O} \cup \text{Inn}_1$ are in $\Gamma_3$.

(d) If $\pi$ is in $F_7$, then the points in $\text{Inn}_1$ are in $\Gamma_2$, the points in $\text{Inn}_2$ are in $\Gamma_3$ and the points in $\text{Ex}_1 \cup \text{Ex}_2 \cup \mathcal{O}$ are in $\Gamma_4$. 


Proof. Let \( y \in \Gamma_4(x) \) and \( z \in \Gamma(x) \cap \Gamma(y) \). Since \( G_{x, y, z} \) has index 2 in its stabilizer in \( G_{xz} \), there exists another point \( w \in \Gamma(x) \cap \Gamma(z) \) such that \( H_{y, z} = H_{y, w} \cong 3 : 8 \).

If \( \pi \) is a plane in \( \Delta_y \) with \( d(\pi, yz) = d(\pi, yw) = 3 \), then we have \( |H_{y, \pi}| = 8 \).

Set \( l := \text{proj}_x yz \) and \( \pi' := \text{proj}_x yz \) (here, \( \text{proj} \) means the projection in \( \Delta_y \)). Similarly, set \( m := \text{proj}_x xz \) (here, \( \text{proj} \) means the projection in \( \Delta_z \)).

For \( a := l \cap m \) in \( \pi' \), we have that \( a \) is a \( \Gamma_3 \)-point in \( \pi \) and that \( H_{a, \pi} \) contains a cyclic group of order 4. Hence \( \pi \) must be in \( \mathfrak{H}_6 \), and \( a \) must be a point in \( \mathcal{D} \) and \( y \) must be a \( \text{Ex}_1 \)-point.

Let now be \( y \in \Gamma_3 \). We have found an orbit of \( \mathfrak{H}_6 \)-planes in \( \Delta_y \) having length \( 2880/4 = 720 \). Furthermore, in \( \mathfrak{H}_2 \cup \mathfrak{H}_3 \), there are exactly 186 planes incident to \( y \).

By Lemma 4.16 and Lemma 4.17 we know that there must be exactly four other orbits in \( \bigcup_{i=4}^9 \mathfrak{H}_i \). The possible sizes of these orbits are 360, 480, 720, 960, 1440, and 2880. Since 3000 \( \equiv 8 \) mod 16, one orbit must have size 360. The only possibility to choose three of these numbers such that their sum adds up to 2640 without using 960 twice is 2640 = 1440 + 720 + 480. We conclude that the \( \text{Ex}_1 \)-points in a \( \mathfrak{H}_5 \)-plane are in \( \Gamma_3 \) and that all points in \( \mathfrak{H}_1 \)-plane and the \( \text{Ex}_2 \)-points in a \( \mathfrak{H}_7 \)-plane are in \( \Gamma_4 \).

Now let \( \pi \) be in \( \mathfrak{H}_3 \) and \( l \) a line in \( \pi \) containing exactly one \( \Gamma_2 \)-point and one \( \Gamma_3 \)-point. Then \( K := H_{\pi, l} \) is cyclic of order 4. Hence there is exactly one other plane \( \pi' \) fixed by \( K \) which is incident to \( l \). Now \( \pi' \) must belong to \( \mathfrak{H}_5 \). Hence the \( \text{Ex}_2 \)-points in a \( \mathfrak{H}_5 \)-plane are points in \( \Gamma_4 \). The other four planes incident to \( l \) must be \( \mathfrak{H}_7 \)-planes. Let \( \pi'' \) be such a plane. Since \( l \) contains exactly one point in \( \text{Inn}_1(\pi'') \subset \Gamma_2(x) \), \( l \) must be a line in \( \text{Sec}_2(\pi'') \). Therefore the points in \( \mathcal{D}(\pi'') \) must be points in \( \Gamma_4 \) and the points in \( \text{Inn}_2(\pi'') \) must be \( \Gamma_3 \)-points. So the \( \text{Ex}_2 \)-points in a \( \mathfrak{H}_6 \)-plane must be \( \Gamma_4 \)-points.

Now let \( \pi \) be a \( \mathfrak{H}_4 \)-plane. Then all points in \( \pi \) are \( \Gamma_4 \)-points and \( H_\pi = 31 : 3 \) operates transitively on both lines and points of \( \pi \). If \( l \) is a line in \( \pi \) and \( K := H_l \), then \( K_\pi \) is cyclic of order 3. Now Lemma 4.15 implies that \( |K| < 18 \). Every involution in \( G_l \) centralizing an element of order 3 in \( G_l \) is coïntained in \( G(l) \), hence \( H_l \) cannot contain an element of order 6. There is no element of order 5 in \( K_\pi \), since in the other case there would be a point \( y \) in \( l \) such that 5 is a divisor of \( K_\pi \), a contradiction to \( y \in \Gamma_4 \).

Now suppose \( |K| = 3 \). Then \( l \) would be a \( \text{Sec}_2 \)-line or a \( \text{Pas}_1 \)-line in a \( \mathfrak{H}_7 \)-plane, a contradiction since in this case \( l \) would be incident to a point in \( \Gamma_2 \) or \( \Gamma_3 \).

Suppose \( |K| = 9 \). Then \( l \) is a \( \text{Pas}_1 \)-line in a \( \mathfrak{H}_7 \)-plane and we get the same contradiction. Suppose \( K \cong A_4 \). In this case there would be two planes incident to \( l \) fixed by \( K \), surely a contradiction. Hence \( K \) must be isomorphic to \( S_3 \).

So there is one plane \( \pi \) in \( \Delta \) with \( K_\pi = S_3 \), two planes with \( K_\pi = Z_3 \) and three planes with \( K_\pi = Z_2 \). Hence \( l \) is incident to one plane in \( \mathfrak{H}_4 \cup \mathfrak{H}_6 \), two planes in \( \mathfrak{H}_4 \) and three planes in \( \mathfrak{H}_7 \). In the last case \( l \) is a \( \mathfrak{T}_1 \)-line. Therefore the \( \text{Ex}_1 \)-points
Let \( \pi \) be a \( 3_7 \)-plane and \( l \) a \( \text{Pas}_1 \)-line in \( \pi \). Then \( l \) is incident to three points in each \( \Gamma_3 \) and \( \Gamma_4 \). \( H_{\pi,l} \) acts transitively on both sets of points. Again, set \( K := H_l \). Then \( 3 \leq |K| \leq 18 \). Just like before on easily sees that there is no element of order 5 or 6 in \( K \) and \( K \) is not isomorphic to \( A_4 \). If \( \pi \) is a \( 3_7 \)-plane, then the lines in \( \text{Pas}_1 \) are the only lines in \( \pi \) incident to exactly 3 points in both \( \Gamma_3 \) and \( \Gamma_4 \). Furthermore, the group \( H \) operates transitively on the set \( \{ (\pi, m); \pi \in 3_7, m \in \text{Pas}_1(\pi) \} \). Therefore all \( 3_7 \)-planes incident to \( l \) are \( H_l \)-conjugate. We conclude that the order of \( K \) can neither be 3 nor 9, \( |K| = 18 \) is not possible either since in this case an involution in \( K \) would not fix any plane in \( \Delta_l \). Therefore \( K \cong S_3 \) must hold. If \( \pi \) is a plane in \( \Delta_l \) with \( K_{\pi} = Z_2 \), then \( \pi \) must be in \( 3_6 \) and \( l \) is a \( \text{Sec}_2 \)-line in \( \pi \). Therefore either the \( \text{Inn}_1 \)-points or the \( \text{Inn}_2 \)-points in \( \pi \) are in \( \Gamma_3 \). Hence there are two possibilities left:

(I) If \( \pi \) is a \( 3_5 \)-plane, then the \( \text{Inn}_1 \)-points are in \( \Gamma_3 \) and the \( \text{Inn}_2 \)-points are in \( \Gamma_4 \). If \( \pi \) is a \( 3_6 \)-plane, then the \( \text{Inn}_1 \)-points are in \( \Gamma_4 \) and the \( \text{Inn}_2 \)-points are in \( \Gamma_3 \).

(II) If \( \pi \) is a \( 3_5 \)-plane, then the \( \text{Inn}_1 \)-points are in \( \Gamma_4 \) and the \( \text{Inn}_2 \)-points are in \( \Gamma_3 \). If \( \pi \) is a \( 3_6 \)-plane, then the \( \text{Inn}_1 \)-points are in \( \Gamma_3 \) and the \( \text{Inn}_2 \)-points are in \( \Gamma_4 \).

Suppose (I) holds. Let \( l \) be a \( \text{Sec}_1 \)-line in a plane from \( 3_5 \) and set \( K := H_l \). Then \( l \) is incident to two points from each \( \Gamma_2, \Gamma_3 \) and \( \Gamma_4 \). Hence every plane \( \pi \) incident to \( l \) must be a \( 3_5 \)-plane and \( l \) must be a \( \text{Sec}_1 \)-line in \( \pi \). Therefore \( |K| = 48 \) since \( H \) is transitive on the set \( \{ (\pi, m); m \in 3_5, \pi \in \text{Sec}_1(\pi) \} \). But \( K \) has three orbits of size two on the set of points in \( \Delta_l \), and so an element of order 3 in \( K \) must be in \( G_{\pi} \), surely a contradiction. Hence case (II) must hold and we are done.

**Theorem 4.19.**

(a) \( H \) has exactly thirteen orbits on the set of lines in \( \Delta_l \) as listed in Table 5.

(b) For \( l \in \mathcal{L}_8, H_l \) has two orbits of size 3 on the point set of \( \Delta_l \), and for \( l \in \mathcal{L}_{13}, H_l \) fixes one \( \Gamma_4 \)-point in \( \Delta_l \) and acts transitively on the others. In all other cases the points and planes in the same \( H \)-orbit are \( H_l \)-conjugate.

**Proof.** From Theorem 4.18 and its proof we know that there is in each case just one orbit of lines with point distribution as follows: 1 \( \Gamma_2 \), 1 \( \Gamma_3 \), 4 \( \Gamma_4 \), 3 \( \Gamma_3 \), 3 \( \Gamma_4 \) and 6 \( \Gamma_4 \). The type of \( H_l \) is clear in all these cases. Furthermore there is in each case just one orbit of lines with the following point distribution: 1 \( \Gamma_3 \), 5 \( \Gamma_4 \) and 2 \( \Gamma_2, 4 \Gamma_3 \). In the second case we have \( |H_l| = 6 \cdot |H_{l,\pi}| \) for all \( \pi \in 3(l) \). With this information we can determine the structure of the stabilizers.
<table>
<thead>
<tr>
<th>Orbit</th>
<th>Characterization</th>
<th>Points incident to a representative</th>
<th>Planes incident to a representative</th>
<th>Stabilizer in $H$ of a representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}_1$</td>
<td>$\times, 5 \Gamma$</td>
<td>$6 \mathcal{F}_1$</td>
<td>$5^{1+4} : \text{GL}(2,5)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}_2$</td>
<td>$\Gamma$</td>
<td>$1 \mathcal{F}_1, 5 \mathcal{F}_2$</td>
<td>$5^{4} : (4 \times 4)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}_3$</td>
<td>$1 \Gamma, 5 \Gamma_3$</td>
<td>$1 \mathcal{F}_2, 5 \mathcal{F}_3$</td>
<td>$5^{2} : (4 \times 4)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}_4$</td>
<td>$5 \Gamma_2, 1 \Gamma_3$</td>
<td>$6 \mathcal{F}_3$</td>
<td>$\text{GL}(2,5)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}_5$</td>
<td>$1 \Gamma, 1 \Gamma_2, 4 \Gamma_4$</td>
<td>$6 \mathcal{F}_3$</td>
<td>$4 \cdot S_4$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{L}_6$</td>
<td>$\mathcal{T} \in \mathcal{F}_6$</td>
<td>$1 \Gamma_3, 5 \Gamma_4$</td>
<td>$1 \mathcal{F}_3, 5 \mathcal{F}_6$</td>
<td>$5 : 4$</td>
</tr>
<tr>
<td>$\mathcal{L}_7$</td>
<td>$\mathcal{T}$ in $\mathcal{F}_5$, Sec in $\mathcal{F}_7$</td>
<td>$1 \Gamma_2, 1 \Gamma_3, 4 \Gamma_4$</td>
<td>$1 \mathcal{F}_3, 1 \mathcal{F}_5, 4 \mathcal{F}_7$</td>
<td>$4$</td>
</tr>
<tr>
<td>$\mathcal{L}_8$</td>
<td>Sec in $\mathcal{F}_5$, Pas in $\mathcal{F}_7$</td>
<td>$6 \Gamma_4$</td>
<td>$2 \mathcal{F}_4, 1 \mathcal{F}_6, 3 \mathcal{F}_7$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>$\mathcal{L}_9$</td>
<td>Sec in $\mathcal{F}_5$, Pas in $\mathcal{F}_7$</td>
<td>$2 \Gamma_2, 4 \Gamma_3$</td>
<td>$6 \mathcal{F}_5$</td>
<td>$\text{GL}(2,3)$</td>
</tr>
<tr>
<td>$\mathcal{L}_{10}$</td>
<td>Sec in $\mathcal{F}_5$, Pas in $\mathcal{F}_7$</td>
<td>$2 \Gamma_2, 1 \Gamma_3, 3 \Gamma_4$</td>
<td>$3 \mathcal{F}_5, 3 \mathcal{F}_7$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>$\mathcal{L}_{11}$</td>
<td>Pas in $\mathcal{F}_5$ and Sec in $\mathcal{F}_6$, Sec in $\mathcal{F}_7$</td>
<td>$3 \Gamma_3, 3 \Gamma_4$</td>
<td>$1 \mathcal{F}_5, 3 \mathcal{F}_6, 2 \mathcal{F}_7$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>$\mathcal{L}_{12}$</td>
<td>Pas in $\mathcal{F}_5$, Sec in $\mathcal{F}_6$</td>
<td>$2 \Gamma_2, 4 \Gamma_4$</td>
<td>$4 \mathcal{F}_5, 2 \mathcal{F}_6$</td>
<td>$Z_8 : Z_2$</td>
</tr>
<tr>
<td>$\mathcal{L}_{13}$</td>
<td>Pas in $\mathcal{F}_6$, Sec in $\mathcal{F}_7$</td>
<td>$2 \Gamma_3, 4 \Gamma_4$</td>
<td>$3 \mathcal{F}_6, 3 \mathcal{F}_7$</td>
<td>$D_6$</td>
</tr>
</tbody>
</table>

Table 5: Orbits of $H$ on the set of lines in $\Delta_1$

Choose $l \in \text{Sec}_2(\mathcal{F}_5)$. Then $l$ is incident to two points in $\Gamma_2$, one point in $\Gamma_3$ and three points in $\Gamma_4$. Suppose all planes in $\Delta_1$ are $\mathcal{F}_5$-planes. Then $|H_l| = 12$. Since $H_l$ is trivial, $H_l$ cannot contain an element of order 6, hence we have $H_l \cong A_4$. But $H_l$ operates transitively on the set of the two $\Gamma_2$-points in $\Delta_1$, a contradiction. Hence there exists a plane $\pi \in \mathcal{F}_7$ such that $l \in \text{Pas}_2(\pi)$. We conclude $H_l \cong S_3$.

The following lines contain exactly two points in $\Gamma_3$ and four points in $\Gamma_4$: $\text{Pas}_2(\mathcal{F}_5), \text{Sec}_1(\mathcal{F}_6), \text{Pas}_2(\mathcal{F}_6)$ and $\text{Sec}_1(\mathcal{F}_7)$. For $\pi \in \mathcal{F}_6$ and $l \in \text{Sec}_1(\pi)$ we have $H_{\pi,l} \cong D_4$ and $H_{\pi,l}^2 \cong V_4$. In the other cases one has $H_{\pi,l} \cong V_4$ and $H_{\pi,l}^2 \cong Z_2$. Take $l \in \text{Pas}_2(\mathcal{F}_6) \cup \text{Sec}_1(\mathcal{F}_7)$. Suppose $H_l$ is transitive on the set of planes incident to $l$. Then we get $|H_l| = 24$ and $|H_l^2| = 12$. Since $H_l^2$ acts transitively on the set of the two $\Gamma_3$-points incident to $l$, there is a normal subgroup of index 2
and hence a normal subgroup of order 3 in $H_l$. We conclude $H_l \cong D_6$. But in this
case $H_l$ must contain an element inducing a transposition on the point set
of $l$. The square of such an element is an element in $G_{(i)}$ having order divisible
by 4. This is a contradiction since we have $H_{(i)} = 2$.

Suppose $Sec_1(\mathfrak{F}_7)$ and $Pas_2(\mathfrak{F}_6)$ are contained in different $H$-orbits. Then
there is an orbit consisting of $Pas_2(\mathfrak{F}_5)$ and exactly one of these sets. For $l \in
Pas_2(\mathfrak{F}_5)$ we get $|H_l| = 4 \cdot 3 = 12$, hence $H^l$ acts as $S_3$ on the four $\Gamma_4$-points of $l$.
But this is a contradiction because if $\pi'$ is a $\mathfrak{F}_5$-plane incident to $l$, then $H_{l, \pi'}$ fixes no $\Gamma_4$-point in $l$.

We see that there are exactly three different possibilities left for the $H$-orbits
on lines with four $\Gamma_4$- and two $\Gamma_3$-points:

(i) $Sec_1(\mathfrak{F}_7) \cup Pas_2(\mathfrak{F}_6)$, $Sec_1(\mathfrak{F}_6) \cup Pas_2(\mathfrak{F}_5)$.

(ii) $Sec_1(\mathfrak{F}_7) \cup Pas_2(\mathfrak{F}_5) \cup Pas_2(\mathfrak{F}_6)$, $Sec_1(\mathfrak{F}_6)$.

(iii) $Sec_1(\mathfrak{F}_7) \cup Pas_2(\mathfrak{F}_6)$, $Sec_1(\mathfrak{F}_6), Pas_2(\mathfrak{F}_5)$.

Let $y$ be in $\Gamma_4(x)$. Suppose case (iii) holds. For $l \in Sec_2(\mathfrak{F}_7) \cup Pas_2(\mathfrak{F}_6)$ one
sees that $H^l_1 = S_3$ must hold, hence $H_l$ has two orbits on the set of $\Gamma_4$-points
in $l$. Therefore $H_y$ has at least eleven orbits on the set of lines incident to $l$, a
contradiction to Lemma 4.17. Suppose case (ii) holds. Choose $l \in Sec_1(\mathfrak{F}_7) \cup
Pas_2(\mathfrak{F}_6) \cup Pas_2(\mathfrak{F}_5)$. Then $|H_l| = 8$ and $|H^l| = 4$. Suppose that there is an
element $a \in H_l$ with $o(a) = 4$. Because $H_l$-orbits on the set of lines in $\Delta_i$
have size two, $a^2 \in H_{(i)}$ must hold. Furthermore $a$ does not fix any plane
incident to $l$, hence $a$ corresponds to a product of three disjoint transpositions.
This is a contradiction since if we set $Z := Z(Out(G_{(i)}))$, then $a$ must induce an
automorphism of order 4 of $Z$, therefore $a^2$ cannot be an involution in $G_{(i)}$.
We conclude that $H_l$ is elementary abelian of order 8. This is also a contradiction,
since $G_l$ does not contain an elementary abelian subgroup of order 8. So case
(i) must hold. We see that there are exactly thirteen orbits. We are left to
determine the stabilizer of a line in $\Sigma_{12}$ and in $\Sigma_{13}$.

For $l \in \Sigma_{12}$ we have $|H_l| = 16$ and $|H_{(i)}| = 2$. Since $H_{(l)} = Z_2$ holds,
every element of $H_l$ induces an even permutation on the sets of points and
planes in $\Delta_i$. We conclude that $H_l$ is contained in a subgroup $K$ of $G_l$ with
$K \cong 4 \cdot A_6$ and $H^l$ is isomorphic to a Sylow 2-subgroup of $A_6$. Hence $H_l$
contains an element corresponding to a permutation of type (2, 4) in $A_6$. Such
an element has order 8 in $G_l$. Because $H_l$ contains an abelian group of type
$(2, 2)$, this extension must split. (It is not so easy to determine the exact type
of $H_l$, but this is not important.)

If $l$ is in $\Sigma_{13}$ then $|H_l| = 12$. $H_l$ possesses a normal subgroup of order 2 and
an elementary abelian Sylow 2-subgroup. If $t$ is an involution in $H_l \setminus H_{(i)}$ and
$s \in H_l$ an element of order 3, then $s$ and $t$ cannot commute. Therefore $H_l \cong D_6$.
Table 6 gives information to how many points of each orbit a point is collinear. This information can already be found in [5], but without a proof.

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Table 6: Collinearity of points of each orbit

For example, a point in \( \Gamma_2 \) is collinear to 63 points from \( \Gamma_2 \), 2520 points from \( \Gamma_2 \), 4544 points from \( \Gamma_3 \) and 12348 points from \( \Gamma_4 \).

5 Coverings of \( \Delta \) which induce automorphisms on apartments

Theorem 5.1. Let \( \theta : \Delta^* \to \Delta \) be a covering such that \( \Delta^* \) is connected. Suppose further that \( \theta \) induces an isomorphism from \( \Sigma^* \) to \( \Sigma \) for every apartment \( \Sigma \) of \( \Delta \) and every connected component \( \Sigma^* \) of \( \theta^{-1}(\Sigma) \). Then \( \theta \) itself is an isomorphism.

Proof. Let \( \varphi : \hat{\Delta} \to \Delta \) be the universal covering of \( \Delta \). Then there is a covering \( \zeta : \hat{\Delta} \to \Delta^* \) such that \( \varphi(\hat{\Sigma}) = \Sigma \) and \( \Sigma^* := \zeta(\hat{\Sigma}) \). For \( v \in \Sigma \) and \( g \in \Pi_{\hat{\Sigma}} \) we have \( \zeta(v), \zeta(v^g) \in \Sigma^* \), hence \( \theta(\zeta(v)) = \varphi(v) = \theta(\zeta(v^g)) \). Since \( \theta|\Sigma^* \) is injective, we conclude \( \zeta(v^g) = \zeta(v) \), therefore \( g \in \Pi_0 \). So we have \( \Pi_{\hat{\Sigma}} \leq \Pi_0 \). The same holds for every \( \hat{G} \)-conjugate of \( \hat{\Sigma} \). Hence we can prove the theorem by showing \( \Pi = \langle \Pi_{\hat{\Sigma}}^G; g \in \hat{G} \rangle \). We may assume \( \Pi_0 = \langle \Pi_{\hat{\Sigma}}^G; g \in \hat{G} \rangle \). Since \( \Pi_0 \) is normal in \( \Pi \), \( \theta \) is a normal covering with \( B := \Pi/\Pi_0 \) as group of deck transformations. Moreover, \( \Pi_0 \) is even normal in \( \hat{G} \), hence \( G^* := \text{Aut } \Delta^* = \hat{G}/\Pi_0 \) acts transitively on the set of maximal flags in \( \Delta^* \).

Let \( \Gamma^* \) be the graph having the point set of \( \Delta^* \) as vertex set such that two collinear points are adjacent. Set \( \Lambda^* \) as the associated clique complex. Then \( \theta \) induces a covering from \( \Lambda^* \) to \( \Lambda \) which we will also call \( \theta \). We will show that this map is an isomorphism. Choose a point \( x \in \Psi \) and a point \( x^* \) in the preimage
of $x$. We set $E_{o} := \{(y, z) \in \mathcal{P}^{2}; y$ and $z$ are collinear$\}$. For every $y \in \mathcal{P}$ we select a preimage $y^{*}$ of $y$ with $d(x, y) = d(x^{*}, y^{*})$. Hence we get a $\mu \in Z^{1}(\Lambda, B)$ such that $(y^{*})^{\mu(y,z)}$ and $z^{*}$ are adjacent for all $(y, z) \in E_{o}$. If $y$ is adjacent to $x$, then we get $\mu(x, y) = 1$. If $d(x, y) = 2$, then there exists a point $z$ adjacent to both $x$ and $y$ with $\mu(x, y) = \mu(y, z) = 1$.

We need some lemmata to finish the proof.

Lemma 5.2. If $z$ is in $\Gamma_{2}(x) \cup \Gamma_{3}(x)$, then $\mu(y, z) = 1$ for all $y \in \Gamma(x) \cap \Gamma(z)$.

Proof. There is a $y_{0} \in \Gamma(x) \cap \Gamma(z)$ with $\mu(y_{0}, z) = 1$. Suppose that $x, z, y$ and $y_{0}$ are contained in a common apartment $A$. In this case the closed path $(x, y_{0}, z, y, x)$ can be lifted to a closed path having $x^{*}$ as origin and end. Now it is easily seen that $\mu(y, z) = 1$ holds. If $y_{0}$ and $y$ are collinear, then $\mu(y, z) = \mu(y, y_{0})\mu(y_{0}, z) = \mu(y, x)\mu(x, y_{0}) = 1$. By Lemma 4.8 and Lemma 4.10, there is always a chain $y_{0}, y_{1}, \ldots, y_{n} = y$ in $\Gamma(x) \cap \Gamma(z)$ such that $y_{i}, y_{i+1}$ are collinear or $x, z, y$, and $y_{i+1}$ are contained in a common apartment, so the statement can be proved by induction.

Since $G$ acts transitively on the vertex set of $\Gamma$ and $G^{*}$ acts transitively on the vertex set of $\Gamma^{*}$, we conclude:

Lemma 5.3. If $y, z \in \mathcal{P}$ such that $z$ is in $\Gamma_{2}(y) \cup \Gamma_{3}(y)$, there exists a unique element $z^{+} \in \theta^{-1}(z)$ such that $d(y^{*}, z^{+}) = 2$. It is $z^{+} = (z^{*})^{\mu(z, w)}\mu(w, y)$ for any $w \in \Gamma(y) \cap \Gamma(z)$.

Lemma 5.4. For all $z \in \Gamma_{4}(x)$ and $y \in \Gamma(x) \cap \Gamma(z)$ one has $\mu(y, z) = 1$.

Proof. Again, there exists an element $y_{0} \in \Gamma(x) \cap \Gamma(z)$ with $\mu(y, z) = 1$. Suppose $y$ and $y_{0}$ are in relation $\Gamma_{2}$. Then $y^{*} = (y^{*})^{\mu(x, y)}\mu(x, y_{0}) = y^{*}$ and $y^{+} = (y^{*})^{\mu(y_{0}, z)}\mu(z, y_{0}) = (y^{*})^{\mu(y, z)}$, hence $\mu(y, z) = 1$. By Lemma 4.11, for any $y$ in $\Gamma(x) \cap \Gamma(z)$ there is a chain $y_{0}, y_{1}, \ldots, y_{n} = y$ such that $y_{i}$ and $y_{i+1}$ are in relation $\Gamma_{2}$, so again we are done by induction.

The two groups $G_{x}$ and $G_{x}^{*}$ are naturally isomorphic, hence we can identify these two groups and regard $G_{x}$ as a subgroup of $G^{*}$.

Lemma 5.5. For all $g \in G_{x}$ and all $(y, z) \in E_{o}$ we have $\mu(y^{g}, z^{g}) = \mu(x, y)^{g}$.

Proof. We have shown that for all $y \in \mathcal{P}$ the element $y^{*}$ is the unique element in the preimage of $y$ with $d(x, y) = d(x^{*}, y^{*})$. For $g \in G_{x}$ we have $d(x^{g}, y^{g}) = d((x^{*})^{g}, (y^{*})^{g}) = d(x^{*}, (y^{g})^{*})$ and $d(x, y^{g}) = d(x^{g}, (y^{g})^{*})$. Since $\theta((y^{g})^{*}) = \theta(y^{g})^{*} = y^{g}$, we conclude $(y^{*})^{g} = (y^{g})^{*}$. If $(y, z)$ is in $E_{o}$, then we get $(y^{*})^{\mu(y,z)} \sim z^{*}$, hence $(y^{g})^{\mu(y,z)} \sim (z^{*})^{g}$ and finally $((y^{g})^{*})^{\mu(y,z)} \sim (z^{g})^{*}$. So the statement follows.
We now continue the proof of Theorem 5.1. We define for every plane \( \pi \) in \( \Delta \) an equivalence relation \( \perp = \perp_\pi \) on the set of points in \( \pi \) such that \( y \perp z \) holds if and only if \( \mu(y, z) = 1 \). Since \( \mu \) is a 1-cocycle, this relation is really an equivalence relation. By Lemma 5.5 this relation is \( G_{x, \pi} \)-invariant. If \( \pi \) is a plane in \( \bigcup_{i=1}^3 \mathfrak{F}_i \), then there is a point \( y \) in \( \pi \) collinear to \( x_i \), hence \( \mu(y, z) = 1 \) for all points \( z \in \pi \). By Lemma 5.2 and Lemma 5.4 all points in \( \pi \) are in relation \( \perp \) in this case.

Now suppose \( \pi \in \mathfrak{F}_5 \cup \mathfrak{F}_6 \), \( y \in \text{Ex}_2(\pi) \) and let \( t_1, t_2 \) be the two lines in \( \mathfrak{F}(\pi) \) incident to \( y \). If \( y_i \) is a point incident to \( t_i \), then \( y \) and \( y_i \) are contained in a common plane in \( \mathfrak{F}_3 \) (see Theorem 4.19), hence we get \( y \perp y_i \). Hence the equivalence class of \( y \) contains two points in \( \text{Ex}_1(\pi) \). But \( G_{x, \pi} \) acts primitively on the set of these points, hence the equivalence class of \( y \) contains all points in \( \text{Ex}_1(\pi) \). So this class is invariant under \( G_{x, \pi} \) and contains all points in \( \mathfrak{D}(\pi), \text{Ex}_1(\pi) \) and \( \text{Ex}_2(\pi) \).

Now let \( \pi \) be a plane in \( \mathfrak{F}_7 \), \( y \) a point in \( \mathfrak{D}(\pi) \) and \( l_1, l_2 \) two different lines in \( \text{Sec}_2(\pi) \) incident to \( y \). Then again by Theorem 4.19 we have that for all points \( y_i \) incident to \( l_i \) the two points \( y \) and \( y_i \) are contained in a common plane in \( \mathfrak{F}_3 \), hence \( y \perp y_i \). Therefore the equivalence class of \( y \) contains two points in \( \text{Inn}_1(\pi) \). But again, \( G_{x, \pi} \) acts primitively on the set of these points. We conclude that the equivalence class of \( y \) contains all points in \( \mathfrak{D}(\pi), \text{Inn}_1(\pi), \text{Inn}_2(\pi) \) and \( \text{Ex}_2(\pi) \).

Suppose now that \( \pi \) is a plane in \( \mathfrak{F}_4 \) and \( l \) a line in \( \pi \). Then there is a plane \( \pi' \in \mathfrak{F}_7 \) such that \( l \) is a \( \mathfrak{S} \)-line in \( \pi' \). Hence there are two points \( y, z \) in \( \pi \) with \( y \perp z \). Since \( G_{x, \pi} \) is primitive on the set of points in \( \pi \), we conclude \( y \perp z \) for all points \( y \) and \( z \) in \( \pi \).

Is \( \pi \) again a plane in \( \mathfrak{F}_7 \), \( l \) a \( \mathfrak{S} \)-line in \( \pi \) and \( y, z \) two points on \( l \) with \( y \in \mathfrak{D}(\pi) \) and \( z \in \text{Ex}_1(\pi) \), then \( y \perp z \). We conclude \( y \perp z \) for all points \( y, z \) in \( \pi \).

Let \( \pi \) be again a plane in \( \mathfrak{F}_7 \) and \( l \) a line in \( \text{Sec}_2(\pi) \). Then there is a plane \( \pi' \) in \( \mathfrak{F}_7 \) such that \( l \) is in \( \text{Pas}_2(\pi') \). Therefore we get \( y \perp z \) for all \( y, z \) incident to \( l \).

Now \( l \) contains points in \( \mathfrak{D}(\pi), \text{Ex}_2(\pi), \text{Inn}_1(\pi) \) and \( \text{Inn}_2(\pi) \), hence \( y \perp z \) for all points \( y \) and \( z \) in \( \pi \).

Finally let \( \pi \) be again a plane in \( \mathfrak{F}_6 \) and \( l \) a line in \( \text{Sec}_2(\pi) \). Then \( l \in \text{Pas}_3(\pi') \) for some plane \( \pi' \) in \( \mathfrak{F}_7 \). We conclude \( y \perp z \) for all points \( y, z \) on \( l \). Since \( l \) contains points in \( \mathfrak{D}(\pi), \text{Ex}_2(\pi), \text{Inn}_1(\pi) \) and \( \text{Inn}_2(\pi) \), we finally get \( y \perp z \) for all points \( y, z \) on \( \pi \).

We have shown that \( \mu(y, z) = 1 \) holds for all pairs of adjacent points \( y \) and \( z \). Since \( \Delta^* \) is connected, \( \theta \) must be an isomorphism by Theorem 2.3. \( \square \)
6 Amalgams of type Ly and the uniqueness of the Lyons group

We will now apply our result to prove the uniqueness of the Lyons group.

**Definition 6.1.** Let \( G \) be a group of type \( \text{Ly} \), \( \Delta \) its 5-local geometry, and \( \Sigma \) an apartment containing both points. Set \( G_1 = G_x, G_2 = G_{(x,y)}, G_3 = G_y, G_{ij} = G_i \cap G_j \) for \( 1 \leq i < j \leq 3 \) and \( G_{123} = G_1 \cap G_2 \cap G_3 \). For \( \emptyset \neq J \subset K \subseteq \{1,2,3\} \), let \( \phi_{J,K} \) be the inclusion map of \( G_K \) in \( G_J \). Let \( \mathcal{A} \) be the amalgam consisting of these groups and homomorphisms. Then \( \mathcal{A} \) is called an amalgam of type \( \text{Ly} \).

If \( \mathcal{A} \) is such an amalgam, then \( G_1 \cong G_2(5), G_2 \cong 5^{1+4} : (4 \cdot S_4), G_3 = (4 \times 4), (S_3 \times S_3), G_{12} = 5^{1+4} : (4 \cdot S_4), G_{13} = (4 \times 4) : S_6, G_{23} = (4 \times 4) : V_4 \) and \( G_{123} = (4 \times 4) : 2 \).

A priori, it is not clear if there is only one amalgam of type \( \text{Ly} \) (up to isomorphism). To prove the uniqueness of the Lyons group, we first have to show that two amalgams of type \( \text{Ly} \) are isomorphic.

**Lemma 6.2.** If \( \mathcal{A} = (G_j)_J \) and \( \mathcal{\overline{A}} = (\overline{G}_j)_J \) are amalgams of type \( \text{Ly} \), then there are isomorphisms \( \phi : G_i \to \overline{G}_i \) such that \( \phi(G_{ij}) = \overline{G}_{ij} \) and \( \phi(G_{123}) = \overline{G}_{123} \) for all \( 1 \leq i, j \leq 3, i \neq j \).

**Proof.** This is clear for \( G_1 \). Proposition 6.8 will show that \( G_2 \cong \text{Aut} G_{12} \cong \text{Aut} \overline{G}_{12} = \overline{G}_2 \). Since \( G_{23} \) is a Sylow 2-subgroup of \( G_2 \) and \( G_{12} \) is normal in \( G_2 \), one can choose an isomorphism between \( G_2 \) and \( \overline{G}_2 \) such that \( G_{12} \) is mapped onto \( \overline{G}_{12} \), \( G_{23} \) is mapped onto \( \overline{G}_{23} \) and \( G_{123} = G_{12} \cap G_{23} \) is mapped onto \( \overline{G}_{12} \cap \overline{G}_{23} = \overline{G}_{123} \).

Both \( G_3 \) and \( \overline{G}_3 \) are extensions of an abelian group \( T \) resp. \( T \) of type \( (4,4) \) by a group isomorphic to \( S_3 \times S_3 \). Since \( G_{13} \cong \overline{G}_{13}, G_{23} \cong \overline{G}_{23}, G_3 = (G_{13}, G_{23}) \) and \( \overline{G}_3 = (\overline{G}_{13}, \overline{G}_{23}) \), the action of \( G_3 \) on \( T \) is isomorphic to the action \( \overline{G}_3 \) on \( T \). Moreover, their Sylow 2-subgroups are isomorphic to a Sylow 2-subgroup of \( 2 \cdot A_4 \). Now one can easily deduce from Gaschütz' Theorem (see [4, I.17.4]) that \( G_3 \) and \( \overline{G}_3 \) must be isomorphic. Both \( G_3 \) and \( \overline{G}_3 \) act transitively on the chambers of a geometry described in section 3.2. Thus there is an isomorphism \( \phi_3 : G_3 \to \overline{G}_3 \) having the desired properties. \( \square \)

From now on let \( \mathcal{A} = (G_j)_J \) be a fixed amalgam of type \( \text{Ly} \). We will need some facts about the automorphism groups of the groups \( G_j \) involved in \( \mathcal{A} \). We first treat the groups contained in \( G_3 \). Recall that \( G_3 \) is an extension of an abelian group \( T \) of type \( (4,4) \) with a group \( W \cong S_4 \times \text{Sym}(a,b,c) \). By regarding
a Sylow 2-subgroup of $G_3$ which is isomorphic to a Sylow 2-subgroup of $2 \cdot A_{11}$, one sees that this extension is non-split.

Let $W$ be the automorphism group of $T$, $M = \Phi(T)$ the Frattini subgroup of $T$ and $W_0 := C_W(M)$. Then $W_0 = C_W(T/M), W/W_0 \cong S_3$ and $W_0 \cong \text{Hom}(T/M, M)$ is elementary abelian of order $2^4$. The center of $W$ is contained in $W_0$ and generated by the map $\rho: T \to T$: $t \mapsto t^{-1}$. One sees easily that the $W$-module $W_0$ is the direct sum of two submodules $W_1$ and $W_2$ such that $W_1$ and $M$ are isomorphic as $W$-modules, $W_2$ contains $Z(W)$ and $W$ acts trivially on $W_2/Z(W)$.

Set $W_1 := O_2(W) \cong V_4$ and let $W_2$ be the unique normal subgroup of $W$ isomorphic to $S_3$. Then $W_2 = C_W(T)$ and $W_1 W_2$ is the preimage of $W_0$ in $W$. The image of $W_1$ in $W$ is $W_1$ and $W_2$ is mapped onto $Z(W)$. Regarding the description of apartments of $\Delta$ (see section 3.2) we can assume that the image of $G_{123}$ in $W$ is $\langle (34), 1 \rangle$, that the image of $G_{13}$ is

$$W_3 := \{(g, h) \in S_4 \times \text{Sym}\{a, b, c\}; 1^g = 1, a^h = a\}$$

and that the image of $G_{23}$ is $\langle (34), 1 \rangle, (\langle 12 \rangle, (ab))$. We see that $W_2 W_3$ is complement of $W_1$ in $W$.

**Lemma 6.3.** The center of $G_{123}$ is cyclic of order 4 and equals the center of $G_{23}$.

**Proof.** The group $G_{123}$ is contained in a line stabilizer in $G_2(5)$ which is isomorphic to $5^{1+4} : \text{GL}(2, 5)$. Hence $G_{123}$ is isomorphic to the group of all monomial matrices in $\text{GL}(2, 5)$ and therefore $Z(G_{123}) \cong Z_4$. Now $G_{23}$ is a Sylow 2-subgroup in the normalizer of a group generated by a $5A$-element, and so one sees $Z(G_{123}) = Z(G_{23})$. \hfill $\square$

**Lemma 6.4.** There is exactly one non-trivial automorphism $\beta \in \text{Aut}(G_{23})$ which centralizes $G_{123}$. This automorphism is induced by an automorphism $\alpha$ of $G_3$ which centralizes $G_{13}$.

**Proof.** By [4, I.17.1], the group of all automorphism of $G_{23}$ which centralize $G_{123}$ (and hence $G_{23}/G_{13}$ because this group has order 2) is isomorphic to $Z^1(G_{23})G_{123}/Z(G_{123})$. Since $Z_4 \cong Z(G_{123}) = Z(G_{23})$, this group has order 2. Thus the first claim follows.

Since $W_1$ and $M$ are isomorphic $W$-modules, there exists a $W$-isomorphism $\varphi: W_1 \to M$. Set $f: W \to T$: $f(xy) = \varphi(y)$ for $x \in W_2 W_3$ and $y \in W_1$. This map $f$ is well defined because $W_2 W_3$ is a complement of $W_1$ in $W$. Since $\varphi$ is $W$-homomorphism, we have $f \in Z^1(W, T)$. So $f$ defines an automorphism $\alpha$ of $G_3$ by $x^\alpha = x f(Tx)$ (see [4, I.17.1]). Now $\alpha$ centralizes $G_{13}$ since $f$ vanishes on $G_{13}/T \leq W_2 W_3$, but it does not centralize $G_{23}$ since $f(\langle 12 \rangle(34)(ab)) = \ldots$
Lemma 6.5. There exists an automorphism $\epsilon$ of $G_3$ which centralizes $G_3/T$ (and hence normalizes every subgroup of $G_3$ containing $T$) and inverts every element of $T$.

**Proof.** Set $C := C_G(T)$ and $X := G_3/C$. Then $C$ is the direct product of $T$ and a cyclic group $K$ of order 3 (the $A_3$ part in the decomposition $G_3/T \cong S_4 \times S_3$). Since $K$ has a complement in $G_3$ (just take the preimage of $S_4 \times \langle (12) \rangle$ in $G_3$), there is a $f \in Z^2(X, C)$ taking values in $T$ which determines the extension of $C$ by $X$.

Let $\sigma$ be the automorphism of $C$ with $t^\sigma = t$ for all $t \in T$ and $s^\sigma = s^{-1}$ for all $s \in K$. We identify $X$ with its image in $\text{Aut} C$ and set $Y := \langle X, \sigma \rangle$. Then $Y = X \times \langle \sigma \rangle$ since $\sigma$ is contained in $Z(\text{Aut} C)$ but not in $X$. Since $\sigma$ acts trivially on $T$ and $f$ takes only values in $T$, the map $\tilde{f} : Y \times Y \to C$ with $\tilde{f}((x_1, s_1), (x_2, s_2)) = f(x_1, x_2)$ for $x_1, x_2 \in X, s_1, s_2 \in \langle \sigma \rangle$ is a 2-cocycle. Let $H$ be the extension of $C$ by $Y$ with $\tilde{f}$. Then $G_3$ can be identified with the preimage of $X$ in $H$. There is an element $s \in X$ such that $s$ inverts every element in $C$. Therefore, $\tau := s\sigma \in Y$ centralizes $K$ and inverts every element in $T$. The preimage of $\langle \tau \rangle$ in $H/T$ is a cyclic normal subgroup of $H/T$ having order 6. Hence there is a preimage $\tau'$ of $\tau$ in the center of $H/T$. If $\epsilon$ is a preimage of $\tau'$ in $H$, then $\epsilon$ induces an automorphism on $G_3$ with the desired properties. □

**Proposition 6.6.** Every automorphism of $G_{13}$ which normalizes $G_{123}$ is induced by an automorphism of $G_3$ which normalizes $G_{23}$.

**Proof.** The group $G_{13}$ is the normalizer of a maximal torus in $G_3(5)$, hence $G_{13}$ is a semidirect product of $T$ by a group $U$ isomorphic to $D_6$ which acts faithfully on $T$. We can therefore regard $U$ as a subgroup of $W$. Now $U$ is the normalizer of a Sylow 3-subgroup of $G_{13}$, which implies that all complements of $T$ in $G_{13}$ are conjugate. We see that an automorphism of $G_{13}$ which centralizes $T$ and $G_{13}/T$ must be an inner automorphism of $G_{13}$ induced by an element in $T$. Hence $\text{Aut} G_{13}$ is the semidirect product of $T$ by $N_W(U)$.

One easily sees that $N_W(U) = UW_2$ and $U \cap W_0 = U \cap W_2 = Z(W)$ holds. If $U_0$ is the image of $G_{123}$ in $W$, then the normalizer of $U_0$ in $W$ is $U_0Z(W)C_{U_0}(W_1)$. It follows $N_W(U) \cap N_W(U_0) = U_0Z(W)$. This is just the image of $N_G_{13}(G_{123})$ in $W$.

Let $g \in G_{13}$ be an element whose image in $W$ generates $Z(W)$. Then every element of $T$ is inverted by $g$. By Lemma 6.5, there is an automorphism $\epsilon$ of $G_3$ which normalizes $G_{13}$ and $G_{23}$ and inverts every element of $T$. We conclude that there is a $t \in T$ with $h^\sigma = h^{t\epsilon}$ for all $h \in G_{13}$. Now the claim follows. □

\(\varphi((12)(34)) \neq 1\). Hence the restriction of $\alpha$ on $G_{23}$ is the unique non-trivial automorphism of $G_{23}$ which centralizes $G_{123}$. □
If $H$ is a line stabilizer in $G_2(5)$ and $V = O_5(H)/Z(O_5(H))$, then the action of $H'/O_5(H) \cong \text{SL}(2, 5)$ on $V$ is irreducible. Therefore, this action is isomorphic to the action of $\text{SL}(2, 5)$ on the space of all homogeneous polynomials of degree 3 in two indeterminates over $\mathbb{F}_5$ since this up to isomorphism the unique 4-dimensional $\text{SL}(2, 5)$-module over $\mathbb{F}_5$. Moreover, one can easily see that all irreducible representations of $\text{GL}(2, 5)$ over $\mathbb{F}_5$ which extend this representation are conjugate by $\text{Aut}(\text{GL}(2, 5))$. Therefore the following lemma will be useful.

**Lemma 6.7.** Let $H$ be a subgroup of index 5 in $\text{GL}(2, 5)$ and let $P$ be the space of all homogeneous polynomials of degree 3 in $\mathbb{F}_5[X, Y]$. Then the action of $H$ on $P$ is absolutely irreducible.

**Proof.** Since there is only one conjugacy class of subgroups of index 5 in $\text{GL}(2, 5)$, we can assume that $H$ is the normalizer of the quaternion group generated by $(\frac{2}{0} \frac{0}{2})$ and $(\frac{0}{1} \frac{1}{0})$. Then $H$ is generated by all diagonal matrices and the matrix $\alpha = (\frac{1}{0} \frac{0}{1})$.

Suppose the action of $H$ on $P$ is not irreducible. Since $H$ acts semi-simple on $P$, there is an $H$-invariant decompositon $P = P_1 \oplus P_2$ of $P$ with $P_1, P_2$ proper subspaces of $P$. Now set $\beta := (\frac{1}{0} \frac{0}{1})$. Then the image of $\beta$ in $\text{GL}(P)$ has four distinct eigenvalues. All eigenvectors of $\beta$ are monomials. Since $\beta|P_1$ and $\beta|P_2$ are diagonalizable, these two spaces must be spanned by monomials. Hence we can assume $X^3 \in P_1$. But this would mean $(X + Y)^3 = (X^3)^\alpha \in P_1$, hence $(X + Y)^3$ is the sum of at most 3 monomials, a contradiction.

By Schur’s Lemma $\mathbb{E} := C_{\text{End}(P)}(H)$ is a field of order $5^d$, where $d$ is either 1, 2 or 4. If $\beta$ is as above, then $C_P(\beta)$ is a $\mathbb{E}$-subspace of $P$, hence $\dim_{\mathbb{E}} C_P(\beta)$ is divisible by $d$. We conclude that $d$ must be 1. Thus we have $C_{\text{GL}(P)}(H) = Z(\text{GL}(P))$ as desired. \qed

**Proposition 6.8.** Every automorphism of $G_{12}$ is induced by an element of $G_{12}$.

**Proof.** Set $R := O_5(G_{12}), V := R/Z(R)$, let $D$ be a complement of $R$ in $G_{12}$ and set $S := N_{G_2}(D)$. Then $S$ is a complement of $R$ in $G_2$. The central factor group $V$ of $R$ is a vector space over $\mathbb{F}_5$, and the commutator map defines a nondegenerate symplectic form $f$ of $V$ which is preserved projectively by $S$. The whole outer automorphism group of $R$ is $\text{GSp}(V, f)$ (the group of all automorphisms of $V$ preserving $f$ projectively), and the outer automorphism group of $G_{12}$ is given by $N_{\text{GSp}(V, f)}(D)/D$. By Lemma 6.7 we have $N_{\text{GL}(V)}(D)/Z \leq \text{Aut} D$ for $Z = Z(D) = Z(\text{GL}(V))$.

Let $A := C_{\text{Aut} D}(Z)$ and $A_0 := C_A(D/Z)$. Since $D/Z \cong S_4 \cong \text{Aut} S_4$, we conclude $A = \text{Im}(D)A_0$. A simple commutator argument shows that $A_0 \cong \text{Hom}(D/Z, Z) \cong \text{Hom}(S_4, Z_4)$ is cyclic of order 2. Of course, $N_{\text{GL}(V)}(D)$ centralizes $Z$, and therefore $N_{\text{GL}(V)}(D)/Z$ is a subgroup of $A$ and $N_{\text{GL}(V)}(D)/D$
is isomorphic to a subgroup of $A_6$. Hence we conclude $S = N_{\text{GL}(V)}(D)$ and $\text{Aut } G_{12} \cong G_2$. \qedhere

**Theorem 6.9.** Up to isomorphism, there is only one amalgam of type $\text{Ly}$. 

**Proof.** Suppose $\mathcal{A} = (G_i)_J$ and $\mathfrak{A} = (\overline{G}_i)_J$ are two amalgams of type $\text{Ly}$. Then by Lemma 6.2 there are isomorphism $\phi_i : G_i \to \overline{G}_i$ for $i = 1, 2, 3$ such that $\phi_i(G_J) = \overline{G}_J$ for $J$ containing $i$.

The map $\phi_3^{-1} \circ \phi_1 : G_{13} \to G_{13}$ is an automorphism of $G_{13}$ normalizing $G_{123}$. By Proposition 6.6, there is an automorphism $\gamma$ of $G_3$ which induces $\phi_3^{-1} \circ \phi_1$ on $G_{13}$ and normalizes $G_{23}$. We replace $\phi_1$ by $\phi_3^* := \phi_3 \circ \gamma$. Then $\phi_3^*$ equals $\phi_1$ on $G_{13}$ and stills map $G_{23}$ to $G_{23}$.

By Lemma 6.8, there is an element $a \in G_2$ such that $\phi_3^{-1}(\phi_1(x)) = x^a$ for all $x \in G_{12}$. Again we replace $\phi_2$ by $\phi_3^*$ with $\phi_3^*(x) = \phi_2(x^a)$ for $x \in G_2$. Then $\phi_1(x) = \phi_3^*(x)$ for all $x \in G_{12}$. Note that $\phi_3^*$ maps $G_{23}$ to $G_{23}$ since $a$ normalizes $G_{123}$ and $G_{23}$ is the normalizer of $G_{123}$ in $G_2$. 

Now $\phi_3^*$ and $\phi_2^*$ define an automorphism $\beta$ of $G_{23}$. Since $\phi_1(x) = \phi_3^*(x) = \phi_2^*(x)$ for $x \in G_{12}$, we see that $\beta$ centralizes $G_{123}$. By Lemma 6.4, there is an automorphism $\alpha$ of $G_3$ extending $\beta$ which centralizes $G_{13}$. We replace $\phi_3^*$ by $\phi_3^{**} = \phi_3^* \circ \alpha$. Now the three automorphisms $\phi_1, \phi_2^*$ and $\phi_3^{**}$ take the same values on intersections and thus define an isomorphism between $\mathcal{A}$ and $\mathfrak{A}$. \qedhere

From now on let $\hat{G}$ be the universal completion of $\mathcal{A}$; since $G$ is a faithful completion of $\mathcal{A}$, $\hat{G}$ is also faithful. Hence $G_1, G_2$ and $G_3$ can be regarded as subgroups of $\hat{G}$.

Since $\hat{G}$ is the universal completion of $\mathcal{A}$, there is an epimorphism $\xi : \hat{G} \to G$ such that $\xi|G_i$ is an isomorphism for $i = 1, 2, 3$. Set $B := \ker \xi$. Let $t$ be an involution in $G_{23}$ such that $G_2 = G_{123}(t)$.

Let $\hat{\Gamma}$ be the graph having vertex set $\hat{\Psi} := G_1/\hat{G}$ such that $G_1g$ and $G_1thg$ for $g \in \hat{G}$ and $h \in G_1$ are joined by an edge. Furthermore, let $\Sigma$ be the graph with \{ $G_1g; g \in G_3$\} as vertex set and all pairs \{ $G_1g, G_1thg$\} with $g \in G_3$ and $h \in G_{23}$ as edges. Then $\Sigma$ is a full subgraph of $\hat{\Gamma}$. Define $\zeta : \hat{\Gamma} \to \Gamma$ by $\zeta(G_1g) := x^\xi(g)$. It is $\hat{G} \leq \text{Aut } \hat{\Gamma}$ and $\zeta(v^g) = \zeta(v)^\xi(g)$ for all $g \in \hat{G}$ and $v \in \hat{\Psi}$.

**Lemma 6.10.** (a) For all $g \in \hat{G}$, $\zeta$ induces an isomorphism from $\Sigma^g$ to $\Sigma^{\zeta(g)}$.

(b) The map $\zeta$ induces a covering from $\text{Cl}(\hat{\Gamma})$ to $\text{Cl}(\Gamma)$.

**Proof.** (a) $\Sigma$ and $\hat{\Sigma}$ are both isomorphic to the graph having $G_{13}/G_3$ as vertex set and \{ $G_{13}g, G_{13}thg$; $g \in G_3, h \in G_{13}$\} as set of edges. Therefore the claim follows.
(b) Clearly, $\zeta$ is a surjective morphism from $\hat{\Gamma}$ to $\Gamma$. For $g \in G$, $\hat{g} \in \xi^{-1}(g)$ and $h \in G_1$ the vertex $G_1 \hat{g} h$ is the unique element in the preimage of $G_1 \hat{g} h$ which is adjacent to $G_1 \hat{g}$. Thus $\zeta$ is a covering.

Let $\hat{x}$ be in $\zeta^{-1}(x) \cap \Sigma$ and let $\hat{y}, \hat{z}$ be adjacent in $\hat{\Gamma}$ such that $y := \zeta(\hat{y})$ and $z := \zeta(\hat{z})$ are adjacent. Suppose first that $x, y, z$ are not incident to a common line in $\Delta$. Then there is a $g \in G$ such that $x, y, z$ are in $\Sigma^g$. Because all apartments containing $x$ are $G_1$-conjugate, we can assume that $g$ is in $G_1$. Since $\zeta$ is a covering from $\hat{\Gamma}$ to $\Gamma$, there is for both $y$ and $z$ exactly one preimage $\hat{y} \in \zeta^{-1}(y)$ and $\hat{z} \in \zeta^{-1}(z)$ which are adjacent to $\hat{x}$. Now $\zeta$ induces an isomorphism from $\hat{\Sigma}^g$ to $\Sigma^g$, hence $\hat{y}$ and $\hat{z}$ are in $\hat{\Sigma}^g$.

Because $y \sim z$ holds in $\Sigma^g$, we can conclude that $\hat{y}$ and $\hat{z}$ are adjacent in $\hat{\Sigma}^g$ and therefore in $\hat{\Gamma}$.

Now we suppose, $x, y$ and $z$ are incident to a common line in $\Delta$. Let $\pi$ be a plane incident to the line $xy$ and choose a point $w$ which is incident to $\pi$ but not to $xy$. Then there is a uniquely determined point $\hat{w} \in \zeta^{-1}(w) \cap \hat{\Gamma}$.

Neither $x, y$ and $w$ nor $x, z$ and $w$ are incident to a common line, hence $\hat{w} \sim \hat{y}$ and $\hat{w} \sim \hat{z}$. Now $y, z$ and $w$ are not collinear, therefore $\hat{y} \sim \hat{z}$. Hence, $\zeta$ induces a covering from $\text{Cl}(\hat{\Gamma})$ to $\text{Cl}(\Gamma)$.

**Theorem 6.11.** The map $\zeta$ is an isomorphism.

**Proof.** Let $\hat{\Delta}$ be the geometry whose points are the elements of $\hat{\mathcal{P}}$, whose planes are the maximal cliques in $\hat{\Gamma}$ and whose lines are the cliques of size six in $\hat{\Gamma}$ which are contained in exactly six maximal cliques. Then $\zeta$ induces a covering from $\hat{\Delta}$ to $\Delta$ which maps apartments in $\hat{\Delta}$ isomorphically on apartments in $\Delta$. Therefore this map is an isomorphism itself by Theorem 5.1.

**Corollary 6.12.** $G \cong \hat{G}$.

**Proof.** The claim follows since $B = \ker \xi$ acts regularly on each preimage under $\xi$.

Theorem 6.9 and Corollary 6.12 now imply:

**Theorem 6.13.** Up to isomorphism, there is at most one group of type $\text{Ly}$. 

References


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