Innovations in Incidence Geometry
Volume 11 (2010), Pages 35–67
ISSN 1781-6475

A new proof for the uniqueness of Lyons' simple group

Matthias Grüninger

Abstract

In this paper we will provide a new proof for the uniqueness of the Lyons group using the 5-local building-like geometry of this group discovered by Kantor.

`Keywords:` sporadic groups, diagram geometries, buildings

`MSC 2000:` 20D05,20D08,51E24

1 Introduction

In 1972, R. Lyons examined groups having the property that the centralizer of an involution is isomorphic to the 2-cover of the alternating group A_{11} [6, 7]. He could gather a lot of information about groups having this property, but the questions if such groups exist and if two of them are isomorphic remained open. In 1973 Charles Sims [8] provided a computer-aided proof showing that there exists exactly one such group. This group is usually called the *Lyons group* (short Ly) or the *Lyons–Sims group* (short LyS) and is one of the 26 sporadic groups. Sims' proof has the disadvantage that it involves computations in the symmetric group of 8835156 letters which can only be done by computer. Therefore it does not provide any insight in the Lyons group. The first computerfree proof was done by Aschbacher and Segev in 1992 (see [1]). In 1981, William Kantor constructed a geometry Δ of rank three with diagram having $\mathsf{PG}(2,5)$ and the Cayley hexagon of order 5 as non-trivial residues such that Ly is a flag-transitive automorphism group of Δ (see [5]). This geometry can be regarded as the natural geometry of the Lyons group since it is extraordinary beautiful and almost classical. For this reason, one wishes to prove the existence and uniqueness of the Lyons group by using this geometry. The greatest obstacle is that Δ is not simply connected (the universal cover $\tilde{\Delta}$ of Δ is an infinite building), hence there is no canonical way to prove existence and uniqueness of the group from this geometry. In this paper, we will show that Δ is determined by certain thin subgeometries which are covered by apartments in $\tilde{\Delta}$. We will conclude that Ly is the univeral completion of a certain amalgam of rank three and hence uniquely determined.

We will use the following notation:

- For a set X, let $\mathcal{P}(X)$ be the power set of X, $\mathcal{P}_n(X) := \{Y \subseteq X; |Y| = n\}$ and $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$.
- For any prime power q we denote the desarguesian projective plane of order q by $\mathsf{PG}(2,q)$ and the Cayley hexagon of order q by $\mathbb{H}(q)$.
- If A, B and G are groups, $G = A.B$ means that G possesses a normal subgroup isomorphic to A such that the factor group is isomorphic to B. We will write $G = A : B$ if we emphasize that it is a split extension and $G = A \cdot B$ for a nonsplit extension.
- A cyclic group of order n is denoted by Z_n or simply by n.
- If p is a prime, a special group of order p^{n+k} with center of order p^n is denoted by p^{n+k}.
- $G = p^{n_1+n_2+\ldots+n_k}.H$ means that there is an ascending series of normal subgroups of G of order $p^{n_1}, p^{n_1+n_2}, \ldots, p^{n_1+n_2+\ldots+n_k}$ such that the last factor group is isomorphic to H and all other factor groups are elementary abelian.

This work is mostly contained in the author's PhD thesis (see [3]).

2 Preliminaries

2.1 Coverings of simplicial complexes

A *simplicial complex* is a pair $\Delta = (V, S)$ such that V is a nonempty set with $V \cap \mathcal{P}(V) = \emptyset$ and S is a subset of $\mathcal{P}^*(V)$ such that $\mathcal{P}_1(V) \subseteq S$ holds and $\sigma \in S$ and $\emptyset \neq \tau \subseteq \sigma$ implies $\tau \in S$. The set V is called the set of *vertices* of Δ and an element of S is called a *simplex* of Δ. We will say that $x, y \in V$ are *adjacent* (short $x \sim y$) if $\{x, y\}$ is a simplex and $x \neq y$. If $\Delta = (V, S),\ \Delta' = (V', S')$ are complexes, then Δ' is called a *subcomplex* of Δ if $V' \subseteq V$ and $S' \subseteq S$ holds. For $\sigma \in S$, the subcomplex $(\sigma, \mathcal{P}^*(\sigma))$ is simply called σ.

For a simplicial complex $\Delta = (V, S)$ let $\mathrm{Cl}(\Delta)$ be the complex $(V, \mathrm{Cl}(S))$, where $\mathrm{Cl}(S) := \{\sigma \in \mathcal{P}^*(V); x \sim y$ for all $x, y \in \sigma\}$. We will call Δ *complete* if $\Delta = \mathrm{Cl}(\Delta)$.

If $\Delta = (V, S), \Delta' = (V', S')$ are complexes, then a map $\varphi\colon V \to V'$ is called a *morphism* if $\varphi(S) \subseteq S'$ holds. We will write $\varphi\colon \Delta \to \Delta'$ instead of $\varphi\colon V \to V'$. φ is called an isomorphism if φ is bijective and φ^{-1} is also a morphism. Let $\mathrm{Aut}\,\Delta$ be the group of all automorphisms of Δ.

If σ is in S, we define the subcomplexes $\Delta_\sigma = (V_\sigma, S_\sigma),\ \mathrm{st}(\sigma) = (V_\sigma \cup \sigma, S'_\sigma)$, where $V_\sigma := \{v \in V \setminus \sigma; \sigma \cup \{v\} \in S\}, S_\sigma = \{\tau \in S; \tau \cap \sigma = \emptyset, \sigma \cup \tau \in S\}$ and $S'_\sigma := \mathcal{P}^*(\sigma) \cup S_\sigma \cup \{\tau \cup \rho; \tau \in S_\sigma, \rho \in \mathcal{P}^*(\sigma)\}$. The subcomplex Δ_σ is called the *residue* of σ in Δ and $\mathrm{st}(\sigma)$ is called the *star* of σ in Δ.

If Δ' is a subcomplex of Δ and G a subgroup of $\mathrm{Aut}\,\Delta$, let $G_{(\Delta')}$ be the subgroup of all elements in $G_{\Delta'}$ acting trivially on Δ'. The factor group $G_{\Delta'}/G_{(\Delta')}$ is denoted by $G^{\Delta'}$. For a simplex σ, we simply write $G_{(\sigma)}$ instead of $G_{(\Delta_\sigma)}$ and G^σ instead of G^{Δ_σ}.

A path of lenght n in Δ is a sequence $\delta = (v_0, v_1, \ldots, v_n)$ with $\{v_i, v_{i+1}\} \in S$. We define $l(\delta) = n, o(\delta) = v_0,\ \mathrm{end}(\delta) = v_n$ and $\delta^{-1} = (v_n, v_{n-1}, \ldots, v_0)$. If $\gamma = (v_0, \ldots, v_n), \delta = (w_0, \ldots, w_m)$ are in $P(\Delta)$ with $v_n = \mathrm{end}(\gamma) = o(\delta) = w_0$, then set $\gamma\delta := (v_0, \ldots, v_n = w_0, w_1, \ldots, w_m)$. The set of all paths in Δ is denoted by $P(\Delta)$, the set of all paths with origin v_0 by $P(\Delta)(v_0, *)$, the set of all paths with end v_n by $P(\Delta)(*, v_n)$, and $P(\Delta)(v_0, v_n)$ is $P(\Delta)(v_0, *) \cap P(\Delta)(*, v_n)$. We say that Δ is connected if $P(\Delta)(v_0, v_1) \neq \emptyset$ for all $v_0, v_1 \in V$. The maximal connected subcomplexes of Δ are called the *components* of Δ.

Two paths γ, δ are said to be *elementary homotopic* if there are a simplex σ and paths $\gamma_1, \gamma_2, \gamma'_2$ and γ_3 with $\gamma_2, \gamma'_2 \in P(\sigma)$ and $\gamma = \gamma_1\gamma_2\gamma_3,\ \delta = \gamma_1\gamma'_2\gamma_3$. Two paths γ and δ are called homotopic if there is a sequence $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_k = \delta$ such that γ_i and γ_{i+1} are elementary-homotopic for all i. Homotopy is an equivalence relation; we denote by $[\gamma]$ the equivalence class of γ. It is straightforward that homotopic paths have the same origin and the same end.

For a vertex $v_0 \in V$ we define the *fundamental group of Δ relative to v_0* in the following way: Set $\pi_1(\Delta, v_0) := \{[\gamma]; \gamma \in P(\Delta)(v_0, v_0)\}$ and $[\gamma] \cdot [\delta] := [\gamma\delta]$. for $\gamma, \delta \in P(\Delta)(v_0, v_0)$. It is easily seen that this multiplication defines a group structure on $\pi_1(\Delta, v_0)$ with $[(v_0)]$ as neutral element and $[\gamma]^{-1} = [\gamma^{-1}]$. Furthermore, if $v_1 \in V$ and $\delta \in P(v_0, v_1)$, then $[\gamma] \mapsto [\delta^{-1}\gamma\delta]$ is an isomorphism between $\pi_1(\Delta, v_0)$ and $\pi_1(\Delta, v_1)$. If Δ is connected, then the isomorphism type of the fundamental group does not depend on the choice of the vertex v_0. In this case we set $\pi_1(\Delta)$ as a group isomorphic to $\pi_1(\Delta_1, v_0)$ for any $v_0 \in V$.

A complex Δ is said to be *simply-connected* if it is connected and $\pi_1(\Delta) = 1$.

A map $\varphi\colon \tilde{\Delta} \to \Delta$ between simplicial complexes $\tilde{\Delta}$ and Δ is called a *covering* if φ is surjective and induces an isomorphism from $\mathrm{st}(\tilde{\sigma})$ to $\mathrm{st}(\varphi(\tilde{\sigma}))$ for all simplices $\tilde{\sigma}$ of $\tilde{\Delta}$.

For a covering $\varphi\colon \tilde{\Delta} \to \Delta$, a path $\gamma = (v_0, \ldots, v_n) \in P(\Delta)$ and a vertex $\tilde{v}_0 \in \varphi^{-1}(v_0)$ there exists exactly one path $\tilde{\gamma} \in P(\tilde{\Delta})(\tilde{v}_0, *)$ with $\varphi(\tilde{\gamma}) = \gamma$. We say that $\tilde{\gamma}$ is a lift of γ.

A map $g \in \operatorname{Aut}\tilde{\Delta}$ is called a *deck transformation of* φ if $\varphi \circ g = \varphi$. The deck transformations of φ form a subgroup of $\operatorname{Aut}\tilde{\Delta}$ denoted by $\operatorname{Aut}\tilde{\Delta}_\varphi$. The normalizer of $\operatorname{Aut}\tilde{\Delta}_\varphi$ in $\operatorname{Aut}\tilde{\Delta}$ is denoted by $\operatorname{Aut}(\tilde{\Delta}, \varphi)$. If $\tilde{\Delta}$ is connected, then it is easily seen that $\operatorname{Aut}\tilde{\Delta}_\varphi$ operates freely on $\varphi^{-1}(v)$ for all $v \in V$. We call the covering φ *normal* if this action is transitive for all $v \in V$.

If Δ is connected, then a covering $\varphi\colon \tilde{\Delta} \to \Delta$ is called *universal* if $\tilde{\Delta}$ is connected and if the following property holds: if $\psi\colon \hat{\Delta} \to \Delta$ is another covering, then there exists a covering $\phi\colon \tilde{\Delta} \to \hat{\Delta}$ with $\varphi = \psi \circ \phi$.

The proof of the following theorem is standard.

Theorem 2.1. *If Δ is a connected complex, then there exists up to isomorphism exactly one universal covering $\varphi\colon \tilde{\Delta} \to \Delta$. This covering is normal, $\pi_1(\Delta) \cong \operatorname{Aut}\tilde{\Delta}_\varphi$ and $\operatorname{Aut}\Delta \cong \operatorname{Aut}(\tilde{\Delta}, \varphi)/\operatorname{Aut}\tilde{\Delta}_\varphi$. Furthermore, $\tilde{\Delta}$ is simply-connected.*

Definition 2.2. Let $\Delta = (V, S)$ be a simplicial complex. Set

$$E_o(\Delta) := \big\{(x, y) \in V^2; \{x, y\} \in S\big\}$$

and let G be a group. A 1-cocycle from Δ to G is a map $\mu\colon E_o(\Delta) \to G$ such that $\mu(x, y)\mu(y, z) = \mu(x, z)$ for all $x, y, z \in V$ with $\{x, y, z\} \in S$. The set of all 1-cocycles from Δ to G is denoted by $Z^1(\Delta, G)$.

If Δ is a simplicial complex and $\mu\colon E_o \to G$ is a 1-cocycle, we can define a new complex $(\Delta \times G)_\mu$ by taking $\Delta \times G$ as vertex set and all nonempty sets of the form $\{(x_1, g), (x_2, \mu(x_2, x_1)g), \ldots, (x_n, \mu(x_n, x_1)g)\}$ with $g \in G$ and $\{x_1, \ldots, x_n\} \in S$ as simplices. (Note that this is independent of the choice of x_1.)

The proof of the following theorem is very easy.

Theorem 2.3. *Let $\varphi\colon \tilde{\Delta} \to \Delta$ be a normal covering and $G := \operatorname{Aut}\tilde{\Delta}_\varphi$. For each $x \in V$ choose an element $\tilde{x} \in \varphi^{-1}(x)$. Then, for $\{x, y\} \in S$ there exists a unique element $\mu(y, x) \in G$ such that $\tilde{x}$ and $\tilde{y}^{\mu(y,x)}$ are adjacent. The mapping $\mu\colon E_o(\Delta) \to G\colon (x, y) \mapsto \mu(x, y)$ is a 1-cocycle and $(x, g) \mapsto \tilde{x}^g$ defines an isomorphism between $(\Delta \times G)_\mu$ and $\tilde{\Delta}$. If $\tilde{\Delta}$ is connected, then φ is an isomorphism iff $\mu(x, y) = 1$ for all pairs of adjacent vertices x and y.*

A *geometry* of rank n is a pair (Δ, τ) where $\Delta = (V, S)$ is a connected, complete simplicial complex and $\tau\colon V \to \{1, \ldots, n\}$ is a map such that for every

$\sigma \in S$ with $\dim \sigma < n-2$ the residue Δ_σ is connected and the following property holds: $\tau|\sigma$ is bijective or there exist at least two different simplices σ_1, σ_2 such that $\sigma \subseteq \sigma_1 \cap \sigma_2$ and $\tau|\sigma_i$ is bijective for $i = 1, 2$.

In a geometry, simplices are called *flags*, and two adjacent but different vertices are called *incident*. We set $\mathrm{type}(\sigma) := \tau(\sigma), \mathrm{cotype}(\sigma) := \{1, \ldots, n\} \setminus \mathrm{type}(\sigma), \mathrm{rank}(\sigma) := |\,\mathrm{type}(\sigma)|$ and $\mathrm{corank}(\sigma) := n - \mathrm{rank}(\sigma)$. If $\mathrm{corank}(\sigma) \geq 2$, then the residue Δ_σ is a geometry of rank equal to $\mathrm{corank}(\sigma)$.

2.2 Groups of Type Ly

Definition 2.4. A finite group G is called a group of type Ly if there is an involution $t \in G$ not contained in $Z^*(G)$ such that $C_G(t)$ is isomorphic to $2 \cdot \mathsf{A}_{11}$ (the double cover of the alternating group on eleven letters).

We present here some facts about groups of type Ly and the Chevalley group $\mathsf{G}_2(5)$.

Theorem 2.5. *Let G be a group of type Ly. Then the following statements hold:*

(a) *G is simple (see [6, 2.1(e)]).*

(b) *The order of G is $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$ ([6, 3.2]).*

(c) *There is a unique conjugacy class of involutions and elements of order 4 in G, respectively. If t is an element of order 4 in G, then the image of t in $\mathsf{A}_{11} \cong C_G(t^2)/\langle t^2 \rangle$ is a double transposition ([6, 2.1]).*

(d) *There are exactly two conjugacy classes of elements of order 3 in G, which are called $3A$ and $3B$. If t is an involution in G and x is an element of order 3 in $C_G(t)$, then x is in $3A$ if and only if x corresponds to a 3-cycle in $\mathsf{A}_{11} \cong C_G(t)/\langle t \rangle$. In this case the normalizer of $\langle x \rangle$ in G is isomorphic to $3 \cdot \mathrm{Aut}\,\mathsf{McL}$. If x is in $3B$, then the normalizer of $\langle x \rangle$ is a group of type $3^6 : (2 \cdot \mathsf{A}_5.2)$. ([6, 2.2–2.6]).*

(e) *There are two conjugacy classes of elements of order 5, called $5A$ and $5B$. If t is an involution and x is an element of order 5 in $C_G(t)$, then x is in $5A$ if and only if the image of x in A_{11} is a 5-cycle. The normalizer of an element in $5A$ is a group of type $5^{1+4} : (4 \cdot \mathsf{S}_6)$, while the normalizer of an element in $5B$ is of type $(5 \times 5^3) : \mathsf{S}_3$ (cf. [6, 2.9–2.16] and [7]).*

(f) *The group G has exactly 53 conjugacy classes. The character table of G is uniquely determined (see [6, Table II, pp. 557–559] with conjugacy class 25 instead of 5_3, or [2, p. 175]).*

(g) *There is up to conjugation a unique subgroup H of G with $H \cong \mathsf{G}_2(5)$. The group G has rank 5 on the set of cosets of H. The non-trivial two-point* *stabilizers are isomorphic to* $5^{1+4} : (4 \cdot \mathsf{S}_4)$, $\mathsf{PSU}(3,3)$, $2 \cdot (\mathsf{A}_5 \times \mathsf{A}_4).2$ *and* $3 : \mathsf{PGL}(2,7)$*. If* χ *is the permutation character belonging to* H/G*, then* $\chi = 1a + 45694a + 1534500a + 3029266a + 4226695a$ *(notation as in [2]). See [6, 5.4–5.7].*

(h) *If* $H \leq G$ *is isomorphic to* $\mathsf{G}_2(5)$ *and* $t \in H$ *is an involution, an element in* $3A$ *or an element in* $3B$*, then* $t^H = t^G \cap H$*.*

(i) *If* t *is an involution in* H*, then* $C_H(t)$ *is a group of type* $2 \cdot (\mathsf{A}_5 \times \mathsf{A}_5).2$*. The proof of Lemma 5.3 in [6] implies that* $C_H(t)/\langle t \rangle$*, regarded as subgroup of* $\mathsf{A}_{11} \cong C_G(t)/\langle t \rangle$*, has two orbits on the set* $\{1, \ldots, 11\}$*. These orbits have size* 5 *and* 6*. It acts as* S_5 *on the orbit of size* 5 *and as* $\mathsf{PGL}(2,5)$ *on the orbit of size* 6*.*

(j) *If* t *is an involution in* H*, then* t *fixes exactly* 42 *points and* 42 *lines in* $\mathbb{H}(5)$ *(This can be seen by regarding the permutation characters of* $\mathsf{G}_2(5)$ *on the set of points resp. lines in* $\mathbb{H}(5)$*. We can label these points resp. lines by* p_i, p_{ij} *resp.* l_i, l_{ij} *with* $1 \leq i, j \leq 6$ *such that* p_i *is incident to* l_{ij}*,* l_i *is incident to* p_{ij} *and* p_{ij} *is incident to* l_{ji}*. The group* $C_H(t)$ *acts as* $\mathsf{PGL}(2,5)$ *on both sets* $\{l_i; 1 \leq i \leq 6\}$ *and* $\{p_i; 1 \leq i \leq 6\}$*. The kernels of these two actions are isomorphic to* $\mathsf{SL}(2,5)$ *and intersect in* $\langle t \rangle$*.*

(k) *If* $t \in H$ *is an element in* $3A$*, then* $N_H(\langle t \rangle) \cong 3 \cdot \mathsf{U}_3(5).2$ *and* t *fixes exactly* 126 *lines and no points in* $\mathbb{H}(5)$*. The lines fixed by* t *form a spread in* $\mathbb{H}(5)$*.*

(l) *If* $t \in H$ *is an element in* $3B$*, then* t *fixes no lines and exactly* 6 *points in* $\mathbb{H}(5)$*.*

(m) *Up to conjugacy, there is a unique subgroup* $T \leq H$ *of type* 4×4*. If* A *is the set of all elements in* $\mathbb{H}(5)$ *fixed by* T*, then* A *is an apartment in* $\mathbb{H}(5)$*.*

2.3 Amalgams

An amalgam of rank n of groups consists of a collection of groups $(G_J)_{\emptyset \neq J \subseteq \{1,\ldots n\}}$ and homomorphisms $(\varphi_{J,K} \colon G_K \to G_J)_{\emptyset \neq J \subset K \subseteq \{1,\ldots n\}}$ such that $\varphi_{J,K} \circ \varphi_{K,L} = \varphi_{J,L}$ for $J \subset K \subset L$ holds. In our case, G_K will always be a subgroup of G_J for $J \subseteq K$, and $\varphi_{J,K}$ will always be the inclusion. For $i_1, \ldots, i_m \in \{1, \ldots, n\}$, we will write $G_{i_1 i_2 \ldots i_m}$ instead of $G_{\{i_1, i_2, \ldots, i_m\}}$.

A *completion* of an amalgam $\mathcal{A} = ((G_J)_J, (\varphi_{J,K})_{J \subseteq K})$ is a group G and a collection of maps $\psi_J \colon G_J \to G$ such that $\psi_J \circ \varphi_{J,K} = \psi_K$ for all $J \subset K$. A completion $(G, (\psi_J)_J)$ is called *faithful* if all maps ψ_J are injective.

A completion $(G, (\psi_J)_J)$ of an amalgam $\mathcal{A} = (G_J)_J$ is called *universal* if the following condition holds: For any other completion $(H, (\pi_J)_J)$ of $\mathcal{A}$ there exists a homomorphism $\Phi \colon G \to H$ with $\Phi \circ \psi_J = \pi_J$ for all J. Universal completions always exist and are unique up to isomorphism (but they are not always faithful). See for instance [10, 1.1].

If $\mathcal{A} = ((G_J)_J, (\varphi_{J,K})_{J\subset K})$ and $\overline{\mathcal{A}} = ((\overline{G}_J)_J, (\rho_{J,K})_{J\subset K})$ are amalgams of groups, then they are called *isomorphic* if there exist isomorphisms $\phi_J : G_J \to \overline{G}_J$ such that $\phi_J \circ \varphi_{J,K} = \rho_{J,K} \circ \phi_K$ for all $J \subseteq K$. One easily sees that isomorphic amalgams have isomorphic universal completions.

3 The 5-local geometry of the Lyons group

3.1 Construction

We briefly describe the construction of the 5-local geometry of a group of type Ly, for more details see [5] or [3]. Let G be a group of type Ly and H be a subgroup of G isomorphic to $\mathsf{G}_2(5)$. We set $\mathfrak{P} := H/G$; this will be the set of points in our geometry. Then G has rank 5 on $\mathfrak{P}$ with non-trivial double point stabilizers of type $5^{1+4} : 4 \cdot \mathsf{S}_4,\ \mathsf{PSU}(3,3),\ 2\cdot(\mathsf{A}_4 \times \mathsf{A}_5).2$ and $3 : \mathsf{PGL}(2,7)$. For a point x, we denote the corresponding orbits $\Gamma(x),\ \Gamma_2(x),\ \Gamma_3(x)$ and $\Gamma_4(x)$, respectively (or just $\Gamma_2,\ \Gamma_3$ etc.). The graph having vertex set $\mathfrak{P}$ with x, y adjacent iff $y \in \Gamma(x)$ will also be denoted by Γ.

Let y be in $\Gamma(x)$ and $R := O_5(G_{xy}) \cong 5^{1+4}$. If l is the set of fixed points of R in $\mathfrak{P}$, then $|l| = 6$, $N_G(R) = G_l$ is a group of type $5^{1+4} : 4.\mathsf{S}_6$ and $G^l = \mathsf{S}_6$. We set $\mathfrak{L} := \{l^g; g \in G\}$. This will be the set of lines in our geometry. Let l be a line and x a point in l. Then there exists a unique line L in the Cayley hexagon $\mathbb{H}(5)$ with $G_{l,x} = G_{L,x}$. If P is a point in $\mathbb{H}(5)$ incident to L and E is the subgroup of $G_{x,L,P}$ fixing all points in $\mathbb{H}(5)$ collinear to P, then E is elementary abelian of order 5^3. Let π be set of fixed points of E in $\mathfrak{P}$ and $\mathfrak{F} := \{\pi^g; g \in G\}$. The members of $\mathfrak{F}$ will be the planes in our geometry.

Now let Δ be the geometry with $\mathfrak{P} \mathbin{\dot\cup} \mathfrak{L} \mathbin{\dot\cup} \mathfrak{F}$ as vertes set, symmetrized inclusion as incidence relation and the natural type function. Then we have:

Theorem 3.1. (a) *If x is a point, then $G_x \cong \mathsf{G}_2(5)$ and Δ_x is isomorphic to $\mathbb{H}(5)$. The planes incident to x correspond to the points in $\mathbb{H}(5)$ and the lines incident to x correspond to the lines in $\mathbb{H}(5)$.*

(b) *If l is a line, then Δ_l is a generalized digon, $G_l = 5^{1+4} : (4\cdot\mathsf{S}_6)$ and $G^l = \mathsf{S}_6$. The actions on the sets of points resp. planes in Δ_l are not isomorphic.*

(c) *If π is a plane, then Δ_π is a projective plane of order 5. We have $G_\pi \cong 5^3 \cdot \mathsf{SL}(3,5)$ and $G^\pi \cong \mathsf{SL}(3,5)$.*

(d) *If σ is a maximal simplex in Δ, then G_σ is a group of type* $5^{1+4+1} : (4 \times 4)$. *The group G acts transitively on the set of maximal simplices of Δ.*

The geometry Δ is a geometry with affine diagram $\widetilde{G}_2 =$. If $\varphi\colon \tilde{\Delta} \to \Delta$ is the universal covering of Δ, then $\tilde{\Delta}$ is a building having the same diagram as Δ by a theorem of J. Tits (see [9, Theorem 1]). Since apartments in $\tilde{\Delta}$ are infinite, $\tilde{\Delta}$ is infinite. Thus $\pi_1(\Delta)$ is infinite and Δ and $\tilde{\Delta}$ are not isomorphic.

Clearly, G is a subgroup of the automorphism group of Δ. By Theorem 2.1, there exists a subgroup $\tilde{G}$ of $\mathrm{Aut}(\tilde{\Delta}, \varphi)$ containing $\Pi := \mathrm{Aut}\,\tilde{\Delta}_\varphi$ such that $\tilde{G}/\Pi \cong G$. Using the following theorem, we can deduce that G and $\tilde{G}$ are the full automorphism groups of Δ and $\tilde{\Delta}$ respectively.

Theorem 3.2. *Let Ω be a geometry having diagram with non-trivial residues isomorphic to* $\mathsf{PG}(2,5)$ *and* $\mathbb{H}(5)$ *such that for every point p the planes of Ω_p correspond to the points in $\mathbb{H}(5)$. Suppose X is a subgroup of* $\mathrm{Aut}\,\Omega$ *acting transitively on the set of maximal simplices such that* $X^p \cong \mathsf{G}_2(5)$ *for a point p,* $X^l \cong \mathsf{S}_6$ *for a line l and* $X^E = \mathsf{SL}(3,5)$ *for a plane E. Then* $\mathrm{Aut}\,\Omega = X$ *and* $X_{(p)} = 1$ *for every point p.*

Proof. Set $A := \mathrm{Aut}\,\Omega$ and let p be a point in Ω. Then, by Frattini, we have $A = XA_p$. Furthermore, since X_p is the full automorphism group of Ω_p, we get $A_p = X_p A_{(p)}$, hence $A = XA_{(p)}$. Set $U := A_{(p)}$. We will show $U = 1$.

If l is a line in Ω_p, then $U \le A_l$, hence $A_l = UX_l$. If K is the kernel of the action of A_l on the set of planes in Ω_l, then $U \le K$ and $X_{(l)}U = K \trianglelefteq A_l$. Hence K fixes p; since A_l is transitive on the set of points in Ω_l, we get $K = A_{(l)}$. So U fixes every point collinear to p.

If E is a plane in Ω_p, then U fixes every point in E, so we get $U \le A_{(E)}$. If l is a line in E, then every point in Ω_l is fixed by U, so again we get $U \le A_{(l)}$.

Now let l be a line in Ω_p and $p \ne q$ a point on l. We have proved that U fixes every element in Ω_q having distance at most 3 to l in $\Omega_q \cong \mathbb{H}(5)$. If $\alpha = (E_1, l_1, E_2, l, E_3, l_3, E_4)$ is an ordered root in Ω_q, then the image of U in A^q is contained in the root subgroup U_α. Hence we get $|U : U \cap A_{(q)}| \le 5$ and $U \cap A_{(q)} = U \cap A_{(E_1)}$, since U_α is sharply transitive on the set of lines in Ω_{q,E_1} different from l_1.

Suppose $U \cap A_{(q)} \ne U$. Then we have $|U : U \cap A_{(q)}| = 5$. If x is in $U \setminus A_{(q)}$, then the image of x in A^{E_1} is an elation with axis l_1. If r is the center of x, then r is incident to l_1. So every element in U fixes every line in E_1 incident to the point r. But $X^{E_2}_{E_2,p,l_1} \cong \mathsf{GL}(2,5)$ acts transitively on the point set of Ω_{l_1} and on the set of planes in Ω_{l_1} different from E_2. Hence, $X_{E_1,E_2,l_1,p}$ is still transitive on the set of points incident to l_1. Since U is normalized by this group, the elements of U fix every line in E_1, a contradiction.

We have proved $A_{(p)} \leq A_{(q)}$, and by symmetry equality holds. Therefore, by induction, $A_{(p)} = A_{(q)}$ for every point q. Hence $A_{(p)}$ must be trivial. □

Corollary 3.3. $G = \mathrm{Aut}\,\Delta$ *and* $\tilde{G} = \mathrm{Aut}\,\tilde{\Delta}$.

The crucial point in the proof of the theorem is that an element in $\mathrm{Aut}\,\Omega$ which leaves invariant all points incident to a line automatically fixes all planes incident to this line and vice versa. This is a very unusual situation. For example, if Ω is a projective space of dimension 3 over a field $\mathbb{K}$ and L is a line in Ω, then the stabilizer of L in $\mathsf{PGL}(4, \mathbb{K})$ acts as a group of type $\mathsf{PGL}(2, \mathbb{K}) \times \mathsf{PGL}(2, \mathbb{K})$ on the generalized digon Ω_L, and one factor of this group fixes all points incident to L and the other factor fixes all planes incident to L.

The automorphism group of the building $\tilde{\Delta}$ is relatively small; it does act transitively on the set of maximal flags, but the stabilizer of a maximal flag is finite. Therefore, the automorphism group of $\tilde{\Delta}$ does not possess a BN-pair and $\tilde{\Delta}$ is not a classical building.

3.2 Apartments in Δ

If G is a group of type Ly, then there is exactly one conjugacy class of subgroups isomorphic to $\mathsf{Z}_4 \times \mathsf{Z}_4$. If T is such a group, let A be the set of fixed elements of T in Δ. Then A is a thin subgeometry of Δ containing exactly 12 points, 24 planes and 36 lines. The points of A can be labeled by the set $\{1, 2, 3, 4\} \times \{a, b, c\}$, two points are collinear if and only if both coordinates are different. Now lines and planes can be identified as sets of two respectively three collinear points (see Figure 1; this description of A can be found on [5, p. 246]).

We call every G-conjugate of A an *apartment* of Δ. This is justified since it is easily seen that every connected component $\tilde{A}$ of $\varphi^{-1}(A)$ is an apartment in $\tilde{\Delta}$.

If N is the normalizer of T in G, then $N/T \cong \mathsf{S}_4 \times \mathsf{S}_3$ acts transitively on the set of maximal flags of A. This action can be described by the natural action of $\mathsf{S}_4 \times \mathsf{S}_3$ on the set $\{1, 2, 3, 4\} \times \{a, b, c\}$.

Let $\mathfrak{A}$ be set of all apartments of Δ. For any subset X of Δ let $\mathfrak{A}_X$ be the set of all apartments containing X.

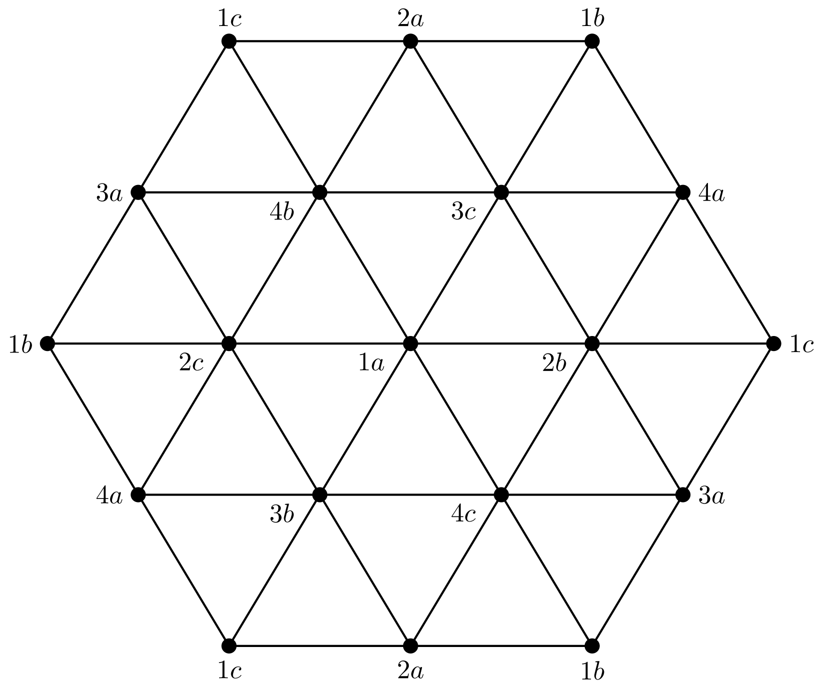

Figure 1: The thin subgeometry A

4 Closed paths of small length in Δ

4.1 The diameter of Δ

Lemma 4.1. *There are three orbits of triples (x, y, z) such that y is collinear to x and z but x, y are not contained in a common plane.*

(I) *In this case $d(yx, yz) = 4$ and $G_{x,y,z} = 5 : (4 \times 4)$.*

(IIa) *In this case $d(yx, yz) = 6$ and $G_{x,y,z} = 4 \cdot \mathsf{S}_4$.*

(IIb) *In this case $d(yx, yz) = 6$ and $G_{x,y,z} = 3 : 8$.*

Here, d refers to the distance function of Δ_y (and not the one of Δ).

Proof. G_y acts transitively on both $\{(l, m); l$ and m lines in $\Delta_y,\ d(l, m) = 4\}$ and $\{(l, m); l$ and m lines in $\Delta_y,\ d(l, m) = 6\}$. The stabilizer of a pair of lines is in the first case a group of type $5^3 : (4 \times 4)$, in the second case a group isomorphic to $\mathsf{GL}(2, 5)$. In the first case the stabilizer of a pair (l, m) acts transitively on the set of pairs (x, z), where x is a point on l and z a point on m and both are different from y. Hence we get our first orbit.

Now if l and m are lines in Δ_y having maximal distance, then it is easily seen that $G_{y,l,m}$ has two orbits on the set $\{(x,z); x$ point on l, z point on m, $x \neq y \neq z\}$ and that the stabilizers are isomorphic to $4 \cdot \mathsf{S}_4$ and $3 : 8$, respectively. Hence the claim follows. □

Lemma 4.2. *If* (x,y,z) *is a path of type* (I)*, then* x *and* z *are in relation* Γ_3.

Proof. Since the order of $G_{x,y,z}$ is divisible by 5, either $z \in \Gamma(x)$ or $z \in \Gamma_3(x)$ holds. Now let T be a subgroup of type $\mathsf{Z}_4 \times \mathsf{Z}_4$ in G_{xz} and $A := \mathrm{Fix}_\Delta(T)$. Then x and z cannot be collinear since in this case the line xz would also be in A, which can easily be recognized as impossible. □

Lemma 4.3. *Let* z *be in* $\Gamma_3(x)$.

(a) *There are exactly six planes* $\pi_1, \ldots, \pi_6$ *in* Δ_x *and six planes* $\pi'_1, \ldots, \pi'_6$ *in* Δ_z *such that* π_i *and* π'_i *are incident to a common line* l_i.

(b) *The planes* $\pi_1, \ldots, \pi_6$ *and* $\pi'_1, \ldots, \pi'_6$ *have pairwise maximal distance in* Δ_x *and* Δ_z*, respectively.*

(c) *Every point in* $\Gamma(x) \cap \Gamma(z)$ *is incident to one of the lines* $l_1, \ldots, l_6$.

Proof. (a) Let (x,y,z) be a path of type (I) and let (yx, π, l, π', yz) be the unique shortest path in Δ_y between yx and yz. Now $G_{x,y,z} \leq G_{x,z,\pi} \leq G_{x,z}$, the first group is isomorphic to $5 : (4 \times 4)$, the second to $5 : (4 \cdot \mathsf{S}_4)$ and the third to $2 \cdot (\mathsf{A}_5 \times \mathsf{A}_4).2$. Since G is transitive on the set of paths of type (I), the claim follows.

(b) $G_{x,z}$ acts as $\mathsf{PGL}(2,5)$ and hence 2-transitively on each set $\{\pi_1, \ldots, \pi_6\}$ and $\{\pi'_1, \ldots, \pi'_6\}$. If i and j are different, then the order of G_{π_i,π_j} is divisible by 3. Hence these two planes must have maximal distance.

(c) Suppose there is a point $y \in \Gamma(x) \cap \Gamma(z)$ such that (x,y,z) is a path of type (IIa) or (IIb). Let t be the central involution in G_{xz} and s the central involution in $G_{x,y,z}$. If $s \neq t$, then s corresponds in $C_G(t)/\langle t \rangle \cong \mathsf{A}_{11}$ to a product of four disjoint transpositions (see Theorem 2.5(c)). In $G_{x,y,z}$, there is an element of the conjugacy class $3A$ (i.e. an element corresponding to 3-cycle in $\mathsf{A}_{11} \cong C_G(t)/\langle t \rangle$) centralizing s. Now $C_{G_x}(t)$ fixes a partition of type $(5,6)$ and acts as S_5 on the set of 5 letters and as $\mathsf{PGL}(2,5)$ on the set of 6 letters (see Theorem 2.5(i)). Therefore the 3 fixed letters of s must be in the orbit of size 5. But this implies that s cannot be a square in $G_{x,y,z} \leq C_{G_x}(t)$, a contradiction.

Now if $s = t$, then by Theorem 2.5(j), there is a plane π_i having distance at most 3 to xy. Clearly, $d(\pi_i, xy) = 1$ is a contradiction, and if $d(xy, \pi_i) = 3$, then there is a point w in l_i (and hence collinear to z) such that (w, y, z) is a path of type (I), also a contradiction. □

Lemma 4.4. *If x, y and z are three pairwise collinear points, then there is a plane π incident to all three of them.*

Proof. Suppose not. If $d(yx, yz) = 4$, then (x, y, z) would be a path of type (I) and hence z would be in $\Gamma_3(x)$. So we must have $d(xy, xz) = d(yx, yz) = 6$ in Δ_x and Δ_y, respectively. Let $(xy, \pi_1, l_1, \pi_2, l_2, \pi_3, xz)$ be a path in Δ_x connecting xy and xz. Then there is a path $(xz, \pi_3, l_3, \pi_4, l_4, \pi_5, yz)$ connecting xz and yz in Δ_z. There is a point w lying on both l_2 and l_3. We now have that $y \in \Gamma_3(w)$ and that π_2 and π_4 are two of the six planes in Δ_w containing a line whose points are all collinear to y. But π_2 and π_4 have distance 4, a contradiction. □

Let $\Lambda := \mathrm{Cl}(\Gamma)$ be the complex whose vertices are the points of Δ and whose simplices are the sets of pairwise collinear points. Then Λ contains all information about Δ since the planes of Δ can be identified as the maximal simplices in Λ and the lines in Δ correspond to the 5-dimensional simplices contained in exactly six maximal simplices. Now we see that every covering of Δ corresponds to a covering of Λ and vice versa and that $\pi_1(\Delta) \cong \pi_1(\Lambda)$ holds.

Lemma 4.5. *If (x, y, z) is a path of type* (IIa)*, then $z \in \Gamma_2(x)$.*

Proof. We know that z must be in $\Gamma_2(x)$ or in $\Gamma_4(x)$, and since 32 is a divisor of $|G_{x,y,z}|$, the latter possibility can be excluded. □

Lemma 4.6. *If (x, y, z) is a path of type* (IIb)*, then $z \in \Gamma_4(x)$.*

Proof. Again, $z \in \Gamma_2(x) \cup \Gamma_4(x)$ must hold. Suppose that $z \in \Gamma_2(x)$. Let g be an element of order 4 in $H := G_{x,y,z} \cong 3 : 8$. Then there are exactly two lines l_1 and l_2 in Δ_x which are fixed pointwise by g. With Theorem 2.5(j), it can be easily seen that these two lines are the only lines in Δ_x fixed by H. Hence y must be incident to one of these lines, say to l_1. Set $J := N_{G_{x,z}}(\langle g \rangle)$. Since g commutes with an element of order 3 in $G_{x,y,z} \leq G_{x,z} \cong \mathsf{PSU}(3,3)$, we have $H \leq J \cong 4 \cdot \mathsf{S}_4$ (see [2]). So there is an element $a \in J \setminus H$ which fixes l_1. The point $y^a \neq y$ is incident to l_1 and belongs to $\Gamma(x) \cap \Gamma(z)$. But now y, y^a and z are pairwise collinear and therefore must be incident to a common plane π. Since x lies on $yy^a = l_1$, x and z are collinear, a contradiction. Thus $z \in \Gamma_4(x)$. □

Since G acts transitively on the sets of paths of type (I), (IIa) and (IIb), we have shown:

Theorem 4.7. *For all points x and y there is point collinear to both of them. In particular, the diameter of Γ is 2. If $d(x,y) = 2$, then G_{xy} acts transitively on $\Gamma(x) \cap \Gamma(y)$.*

Lemma 4.8. *Let x be a point and $y \in \Gamma_2(x)$. Let Ψ_{xy} be the geometry of rank 2 having $\Gamma(x) \cap \Gamma(y)$ as set of points and $\mathfrak{A}_{xy}$ as set of lines and canonical incidence relation. Then Ψ_{xy} is G_{xy}-isomophic to the Cayley hexagon $\mathbb{H}(2)$.*

Proof. Let z be a point $\Gamma(x) \cap \Gamma(y)$ and A an apartment containing x, y and z. Then $G_{x,y} \cong \mathsf{PSU}(3,3) \cong G_2(2)'$, $G_{x,y,A} = (4 \times 4).\mathsf{S}_3$, $G_{x,y,z} = 4 \cdot \mathsf{S}_4$ and $G_{x,y,z,A} = (4 \times 4).2$. Since G_{xy} acts transitively on the set of maximal flags in Ψ_{xy}, the claim follows. □

Lemma 4.9. *Let x and y be two points in relation Γ_4, $H := G_{xy} \cong 3 : \mathsf{PGL}(2,7)$ and g be an element of order 3 in $O_3(H)$. Then g is a $3A$-element and $O_3(H) = \langle g \rangle$ is the kernel of the action of $G_{x,y}$ on $\Gamma(x) \cap \Gamma(y)$. If z and w are two different points in $\Gamma(x) \cap \Gamma(y)$, then xw and xz have maximal distance in Δ_x.*

Proof. Let S be a Sylow 7-subgroup of H. Then $N_H(S)$ contains a 3-sylow subgroup P of H. By [6, Proposition 2.8] all $3A$-Elements in P are contained in $C_H(S) \cap P = O_3(H)$. For $z \in \Gamma(x) \cap \Gamma(y)$, H_z fixes a line in Δ_x, hence a 3-element in H_z must be a $3A$-element, and therefore $O_3(H) \leq H_z$. Thus the first claim follows.

Now if w and z are distinct points in $\Gamma(x) \cap \Gamma(y)$, then w and z can neither be collinear nor in relation Γ_3 (since 3 divides $|G_{x,y,z,w}|$), hence $d(xw, xz) = 6$ must hold. □

Lemma 4.10. *Let x be a point and $y \in \Gamma_3(x)$.*

(a) *If z is in $\Gamma(x) \cap \Gamma(y)$, then $\Gamma(x) \cap \Gamma(y)$ contains five points from each $\Gamma(z)$ and $\Gamma_3(z)$ and 25 points from $\Gamma_2(z)$.*

(b) *Define a graph structure on $\Gamma(x) \cap \Gamma(y)$ such that z and w are adjacent if and only if x, y, z and w are contained in a common apartment or w and z are collinear. Then this graph is connected.*

Proof. Let $\pi_1, \ldots, \pi_6$ be the six planes in Δ_x as in Lemma 4.3.

(a) Suppose z is contained in π_1. If w is another point in $\Gamma(x) \cap \Gamma(y)$, then z and w are collinear exactly if w is also in π_1. Hence we have 5 points in $\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$.

In each plane $\pi_2, \ldots, \pi_6$ there exists exactly one point in $\Gamma(x) \cap \Gamma(y)$ which is in relation Γ_3 to z, so we have $|\Gamma(x) \cap \Gamma(y) \cap \Gamma_3(z)| = 5$. The other points must be in relation Γ_2 to z by Lemma 4.9.

(b) For all $i \neq j$ there is an apartment containing x, y, π_i and π_j. Hence the claim follows. □

Lemma 4.11. *Let y be in $\Gamma_4(x)$.*

(a) *If z is in $\Gamma(x) \cap \Gamma(y)$, then there is a point $w \in \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)$.*

(b) *The graph having $\Gamma(x) \cap \Gamma(y)$ as vertex set with z and w adjacent if and only if they are in relation Γ_2 is connected.*

Proof. (a) Let $H := G_{x,y}$ and g be an element of order 3 in $O_3(H)$. Then g is a $3A$-element by Lemma 4.9, hence g fixes exactly 750 points in $\Gamma_4(x)$ since $N_{G_x}(\langle g\rangle) \cong 3 \cdot \mathsf{U}_3(5).2$. Let t be an involution in H and $l_1, \dots l_6$ the 6 lines in Δ_x fixed by t pointwise. By Theorem 2.5(j), we have $C_{G_x}(t)/\langle t\rangle \cong (\mathsf{A}_5 \times \mathsf{PGL}(2,5)).2$, and the points on the l_i different from x can be labeled by the set $\{1, \dots, 5\} \times \{1, \dots, 6\}$ where (i, j) is incident to l_j such that the action of $C_{G_x}(t)/\langle t\rangle$ on the set of these points corresponds to the natural action of this group on the set $\{1, \dots, 5\} \times \{1, \dots, 6\}$.

In $C_{G_x}(t)/\langle t\rangle$, g corresponds to an element $(s, 1)$ with s a 3-cycle. We can assume $s = (123)$. Then g fixes all of the lines $l_1, \dots, l_6$ and every point with first coordinate 4 or 5. For $1 \leq i \leq 6$ set $y_i := (4, i)$. Now g fixes 126 lines in Δ_{y_i}, l_i and 125 others. There are exactly 3 fixed points of g on each of these 125 lines. These points are y_i, the unique point in $\Gamma_2(x)$ on this line and one other point in $\Gamma_4(x)$. So we get up to $6 \cdot 125 = 750$ fixed points of g in $\Gamma_4(x)$.

Suppose that these points are all different. Then these points are all the 750 fixed points of g in $\Gamma_4(x)$. We can assume that y is collinear to y_1. But if h is a $3B$-element in $C_{G_{x,y}}(t)$, then h fixes none of the lines $l_1, \dots, l_6$, and since h centralizes g, y_1^h must be a point with first coordinate 4. Hence y^h cannot be y, a contradiction.

We conclude that there must be a point $z \in \Gamma_4(x)$ and $1 \leq i < j \leq 6$ such that z is collinear to y_i and y_j. Now $C_{G_x}(t) \cap G_{y_i,y_j} = 2 \cdot (\mathsf{A}_4 \times 2).2$. Since 32 is a divisor of $|G_{y_i,y_j}|$, the points y_i and y_j are in relation Γ_2. Since G_x is transitive on $\Gamma_4(x)$, the claim follows.

(b) Now by Lemma 4.8, if $w, z \in \Gamma(x) \cap \Gamma(y)$ with $w \in \Gamma_2(z)$, then $H_z = 3:8$ and $H_{z,w} = 3$, hence z has exactly 8 neighbours in $\Gamma(x) \cap \Gamma(y)$. If $H_z < J < H$, then $|J : H_z|$ must be 2. Since the connected component of z in $\Gamma(x) \cap \Gamma(y)$ is a block of size at least 9, we see that $\Gamma(x) \cap \Gamma(y)$ must be connected. □

4.2 The action of a point stabilizer on the sets of planes and lines in Δ

Lemma 4.12. *Let $J_1, J_2 \leq \mathsf{SL}(3,5)$ be isomorphic to S_4 and A_4, respectively. Then the following statements hold.*

(a) *Both J_1 and J_2 have exactly 5 orbits on the point set of $\mathsf{PG}(2,5)$:*

Orbit	$\mathfrak{O}$	Ex_1	Ex_2	Inn_1	Inn_2
Number of points	6	3	12	4	6
Stabilizer of a representative in J_1	Z_4	D_4	Z_2	S_3	Z_2^2
Stabilizer of a representative in J_2	Z_2	V_4	1	Z_3	Z_2

Table 1: Action of J_1 and J_2 on the points of $\mathsf{PG}(2,5)$

(b) *Both J_1 and J_2 have exactly 5 orbits on the set of lines of $\mathsf{PG}(2,5)$:*

Orbit	$\mathfrak{T}$	Sec_1	Sec_2	Pas_1	Pas_2
Number of lines	6	3	12	4	6
Stabilizer of a representative in J_1	Z_4	D_4	Z_2	S_3	Z_2^2
Stabilizer of a representative in J_2	Z_2	V_4	1	Z_3	Z_2

Table 2: Action of J_1 and J_2 on the lines of $\mathsf{PG}(2,5)$

(c) *Table 3 shows to how many points/lines on each orbit a line/point is incident:*

	$\mathfrak{T}$	Sec_1	Sec_2	Pas_1	Pas_2
$\mathfrak{O}$	$1/1$	$1/2$	$4/2$	$0/0$	$0/0$
Ex_1	$2/1$	$2/2$	$0/0$	$0/0$	$2/1$
Ex_2	$2/4$	$0/0$	$2/2$	$1/3$	$1/2$
Inn_1	$0/0$	$0/0$	$3/1$	$0/0$	$3/2$
Inn_2	$0/0$	$1/2$	$2/1$	$2/3$	$1/1$

Table 3: Incidence between points and lines in each orbit

For example, the entry in the first column of the second row means that a point of Ex_1 is incident to exactly two lines in $\mathfrak{T}$ and that a line in $\mathfrak{T}$ contains exactly one point in Ex_1.

Proof. In $\mathsf{SL}(3,5)$, there is only one conjugacy class of subgroups isomorphic to $\mathsf{Z}_2 \times \mathsf{Z}_2$. The centralizer of such a group is a group isomorphic to $\mathsf{Z}_4 \times \mathsf{Z}_4$ and the normalizer a split extension of the centralizer by a group isomorphic to S_3. Every complement of the centralizer equals the normalizer of a Sylow 3-subgroup in the normalizer, hence there is a unique conjugacy class of complements. It follows that there is only one conjugacy class of subgroups isomorphic to S_4 and A_4, respectively. So we can assume that J_1 is the subgroup of monomial matrices having 1 and -1 as entries and that $J_2 = J_1'$. Now all the claims can be easily verified. □

For the rest of this section, let x be a point and $H := G_x$. We will use the abbrevations $\Gamma, \Gamma_2, \Gamma_3$ and Γ_4 to represent $\Gamma(x), \Gamma_2(x), \Gamma_3(x)$ and $\Gamma_4(x)$ respectively.

Theorem 4.13. *The group H has exactly seven orbits on $\mathfrak{F}$. The stabilizer of a representative of each orbit is listed in Table 4:*

Orbit	Stabilizer
$\mathfrak{F}_1(x)$	$5^{3+2} : \mathsf{GL}(2,5)$
$\mathfrak{F}_2(x)$	$5^{1+2} : (4 \cdot \mathsf{S}_4)$
$\mathfrak{F}_3(x)$	$5 : (4 \times 4)$
$\mathfrak{F}_4(x)$	$31 : 3$
$\mathfrak{F}_5(x)$	S_4
$\mathfrak{F}_6(x)$	S_4
$\mathfrak{F}_7(x)$	A_4

Table 4: Stabilizers of a representative of each orbit of H on $\mathfrak{F}$

Instead of $\mathfrak{F}_i(x)$ we will simply write $\mathfrak{F}_i$.

Proof. The planes in $\mathfrak{F}_1$ are the planes incident to x, those in $\mathfrak{F}_2$ contain exactly one line whose points are all collinear to x and those in $\mathfrak{F}_3$ contain exactly one point collinear to x. Now 31 is a divisor of $|G|, |H|$ and $|\mathsf{SL}_3(5)|$. A Sylow 31-subgroup has index 3 in its normalizer in $\mathsf{SL}_3(5)$ and index 6 in its normalizer in G and H. Looking at the permutation character of G on the set of points in Δ, one sees that an element of order 31 fixes exactly one point. We conclude that if $g \in G$ has order 31, then there are exactly two $\langle g \rangle$-invariant planes in Δ which are conjugate under $N_G(\langle g \rangle)$. Therefore, we have exactly one orbit of planes whose stabilizer is a group of type $31 : 3$. We call this orbit $\mathfrak{F}_4$.

If $g \in G$ is of order 5 or 6 and π is a plane fixed by g, then there is a fixed point of g in π. By counting the number of fixed points of such an element, we see that if π is a plane outside $\bigcup_{i=1}^{4} \mathfrak{F}_i$, then $G_{x,\pi}$ contains no element of order 5 or 6. Hence $G_{x,\pi}$ is a $\{2,3\}$-group. Moreover, $G_{x,\pi}$ contains no group of type 4×4, because in the other case x and π would be contained in a common apartment. But all planes in a common apartment with x are in $\mathfrak{F}_1(x) \cup \mathfrak{F}_2(x) \cup \mathfrak{F}_3(x)$ (this can be seen in Figure 1 on page 44). Therefore $|G_{x,\pi}| \in \{6, 8, 12, 16, 24\}$. If $G_{x,\pi}$ has order 12 or 24, then $G_{x,\pi}$ is isomorphic to S_4 resp. A_4 because these are the only groups of order 12 resp. 24 containing no element of order 6. There are exactly $2^5 \cdot 3^2 \cdot 5^6 \cdot 7 \cdot 31$ planes in $\mathfrak{F} \setminus \bigcup_{i=1}^{4} \mathfrak{F}_i$. So we see that there are five possibilities:

(a) There is only one other orbit with stabilizer of order 6.

(b) There are two other orbits with stabilizers both of order 12, hence isomorphic to A_4.

(c) There are two other orbits with stabilizers of order 24 and 8.

(d) There are three other orbits with stabilizers of order 16, 16 and 24.

(e) There are three other orbits with stabilizers isomorphic to S_4, S_4 and A_4.

Counting the number of fixed points of an element in the conjugacy class $3B$, we see that the cases (c) and (d) cannot hold.

Now let y be a point in $\Gamma_4(x)$ and $z \in \Gamma(x) \cap \Gamma(y)$. Then there is another point $w \in \Gamma(x) \cap \Gamma(y)$ such that $3 : 8 \cong G_{x,y,z} = G_{x,y,z,w}$. Let π be a plane in Δ_y having distance 3 to both yz and yw. Then $|G_{x,y,z,w,\pi}| = 4$. Since there is an element in $G_{x,y}$ which interchanges w and z, we conclude that 8 is a divisor of $|G_{x,y,\pi}|$. Since y is in $\Gamma_4(x)$, π cannot be in $\bigcup_{i=1}^{4} \mathfrak{F}_i(x)$. Hence cases (a) and (b) can be excluded and (e) must hold. □

We will later see that we can distinguish the orbits $\mathfrak{F}_5$ and $\mathfrak{F}_6$ by the convention that a plane in $\mathfrak{F}_5$ contains Γ_2-points whereas a plane in $\mathfrak{F}_6$ does not. Lemma 4.12 and Theorem 4.13 imply that there are 16 G_x-orbits of pairs (y, π) where π is a plane in $\bigcup_{i=4}^{7} \mathfrak{F}_i$ and y is incident to π.

Lemma 4.14. *Let π be a plane in $\mathfrak{F}_3$. Then π contains exactly one point in $\Gamma(x)$ and five points in both $\Gamma_2(x)$ and $\Gamma_3(x)$. There is a line l^* in π which contains all points in $\Gamma_2(x)$. All planes incident to l^* are in $\mathfrak{F}_3(x)$. Moreover, G_{x,l^*} is isomorphic to $\mathsf{GL}(2,5)$ and acts transitively on the set of these planes.*

Proof. Let y be the unique point in π collinear to x and $l = \mathrm{proj}_{yx} \pi$ (in Δ_y). Then all points on l different from y are in Γ_3. The points on π outside l are in

$\Gamma_2 \cup \Gamma_4$, and there is exactly one point from Γ_2 on each line incident to $\{y, \pi\}$ different from l. Therefore the first claim is proved.

Set $\pi' := \mathrm{proj}_{xy}\, l$. Since $G^{l}_{l,\pi,\pi',y} = \mathsf{Z}_4$, there is a unique point z on l different from y such that $G_{l,\pi,\pi',y} = G_{l,\pi,\pi',y,z}$. Of course z is in $\Gamma_3(x)$. Let $y_1, \ldots, y_5$ be the five Γ_2-points in π and set $l_{ij} = y_i y_j$ for $i \neq j$. Now, $G_{x,\pi}$ acts 2-transitively on the set $\{y_1, \ldots, y_5\}$, hence this group acts transitively on the set of the l_{ij}. Every line l_{ij} intersects l in a point different from y since lines incident to y carry at most one point in $\Gamma_2(x)$. Suppose there is a pair (i, j) such that $l_{ij} \cap l$ is not z. Because $G_{x,\pi}$ acts transitively on the points on l different from y and z, 4 divides $|\{l_{ij}; i \neq j\}|$. But $G_{x,\pi}$ acts 2-transitively on the points y_i, and therefore every line l_{ij} contains either two, three or five points in Γ_2. Thus there is no possibility that the number of lines l_{ij} is divisible by 4, a contradiction. Hence l_{ij} and l intersect in z for all pairs (i, j), and by the 2-transitivity of $G_{x,\pi}$ on the set $\{y_1, \ldots, y_5\}$ we see that $l^* := \{z\} \cup \{y_1, \ldots, y_5\}$ is a line.

We see $G_{x,l^*,\pi} = G_{x,\pi} = 5 : (4 \times 4)$. If π' is another plane incident to l^*, then either 5 divides $|G_{x,l^*,\pi'}|$ or $G_{x,l^*,\pi'}$ contains an abelian group of type $(4, 4)$. By Theorem 4.13 every plane incident to l^* must be in $\mathfrak{F}_3$, and if $g \in G_x$ such that l^* is in π^g, then g must be in G_{x,l^*}. Hence $|G_{x,l^*}| = 480$ and $|G^{l^*}_{x,l^*}| = 120$, therefore G_{x,l^*} must be isomorphic to $\mathsf{GL}(2, 5)$. □

Lemma 4.15. *Let π be a plane in $\mathfrak{F}_4$ and l a line incident to π. Then there is a plane π' incident to l which is not in $\mathfrak{F}_4$.*

Proof. Let y be a point in π. Then there is a line m in Δ_y containing a point collinear to x. Now π cannot be incident to m because all points in π are G_x-conjugate.

There is a path of minimal length between m and π in Δ_y, and in this path there must be at least one line which is incident to a plane in $\mathfrak{F}_4$ and to a plane outside $\mathfrak{F}_4$. Thus the claim follows. □

Lemma 4.16. *For $y \in \Gamma_2(x)$ there are exactly three H_y-orbits of planes incident to y. Planes in the first orbit are in $\mathfrak{F}_3$, planes in second orbit are in $\mathfrak{F}_5$ and planes in the last orbit are in $\mathfrak{F}_7$. In the second case we have $y \in \mathfrak{O}(\pi)$. In the third case, $y \in \mathrm{Inn}_1(\pi)$ holds.*

Proof. Let χ be the permutation character of G_y on the set of planes incident to y. Then χ is known by the character table of $\mathsf{G}_2(5)$ since this set corresponds to the point set of $\mathbb{H}(5)$. Up to conjugacy there is a unique subgroup in $\mathsf{G}_2(5)$ isomorphic to $\mathsf{PSU}(3, 3)$. For a plane π in Δ_y we compute $|(G_y)_\pi \backslash G_y / H_y| = \langle \chi, 1_{H_y}^{G_y} \rangle = \langle \chi_{H_y}, 1_{H_y} \rangle = 3$. Therefore we have proved the first part of the claim.

There is exactly one orbit of planes whose elements are in $\mathfrak{F}_3$. This orbit contains exactly 378 planes. By Lemma 4.12 the other orbits have size $3024, 2016, 1512, 1008$ or 756. However, 2016 and 1512 is the only possibility to choose two of these numbers such that their sum is $3906 - 378 = 3528$. Thus there is one orbit such that $H_{y,\pi} = \mathsf{Z}_3$ for all planes π in this orbit. Hence elements in this orbit are either contained in $\mathfrak{F}_4$ or $\mathfrak{F}_7$. Lemma 4.15 implies that there cannot be a line whose points are all in Γ_2. We conclude that the Inn_1-points in a $\mathfrak{F}_7$-plane must be points in Γ_2.

Now let z be in $\Gamma(x) \cap \Gamma(y)$. Lemma 4.8 shows together with Lemma 4.10 that there are exactly 30 points $w \in \Gamma(x) \cap \Gamma(y)$ such that $d(yz, yw) = 4$ in Δ_y holds. If π is a plane incident to yz, l the unique line in π containing all Γ_2-points in π, then $G_{l,x} \cong \mathsf{GL}(2,5)$ operates transitively on the set of planes incident to l (Lemma 4.14). On each plane apart from π there is exactly one point in $\Gamma(x)$ different from z. These points are all different, hence we get all the $6 \cdot 5 = 30$ points w in $\Gamma(x) \cap \Gamma(y)$ for which $d(yw, yz) = 4$ in Δ_y holds.

Now let m be a line which is different from l and yz and incident to π and y. Set $J := G_{x,y,m,\pi}$. Then J is cyclic of order 4, and J fixes exactly one other plane π' incident to m. We see that π' cannot contain a point w in $\Gamma(x)$ because in this case we would have $d(yz, yw) = 4$ in Δ_y, which would imply that the intersection of π and π' is either y or l.

So π' cannot be a plane in $\mathfrak{F}_3$, hence $G_{x,y,\pi'} = \mathsf{Z}_4$. Therefore π' is a plane in $\mathfrak{F}_5$ (by convention) and y is in $\mathfrak{O}(\pi')$. □

Lemma 4.17. *If y is a point in $\Gamma_4(x)$, then there are exactly ten H_y-orbits on the sets of lines and planes incident to y, respectively.*

Proof. Since $\mathsf{U}_3(5)$ contains a unique conjugacy class of subgroups isomorphic to $\mathsf{PGL}(2,7)$ (see [2]), there is up to conjugacy exactly one subgroup of type $3 : \mathsf{PGL}(2,7)$ in $\mathsf{G}_2(5)$, hence the claim can be verified using the character table of this group. □

Theorem 4.18. (a) *If π is a plane in $\mathfrak{F}_4$, then all points in π are Γ_4-points.*

(b) *If $\pi \in \mathfrak{F}_5$, then the points in $\mathfrak{O}$ are in Γ_2, the points in $\mathrm{Ex}_1 \cup \mathrm{Inn}_2$ are in Γ_3 and the points in $\mathrm{Ex}_2 \cup \mathrm{Inn}_1$ are in Γ_4.*

(c) *If $\pi \in \mathfrak{F}_6$, then the points in $\mathrm{Ex}_1 \cup \mathrm{Ex}_2 \cup \mathrm{Inn}_2$ are in Γ_4 and the points in $\mathfrak{O} \cup \mathrm{Inn}_1$ are in Γ_3.*

(d) *If π is in $\mathfrak{F}_7$, then the points in Inn_1 are in Γ_2, the points in Inn_2 are in Γ_3 and the points in $\mathrm{Ex}_1 \cup \mathrm{Ex}_2 \cup \mathfrak{O}$ are in Γ_4.*

Proof. Let $y \in \Gamma_4(x)$ and $z \in \Gamma(x) \cap \Gamma(y)$. Since $G_{x,y,z}$ has index 2 in its stabilizer in G_{xy}, there exists another point $w \in \Gamma(x) \cap \Gamma(z)$ such that $H_{y,z} = H_{y,w} \cong 3:8$. If π is a plane in Δ_y with $d(\pi, yz) = d(\pi, yw) = 3$, then we have $|H_{y,\pi}| = 8$. Set $l := \mathrm{proj}_\pi\, yz$ and $\pi' := \mathrm{proj}_l\, yz$ (here, proj means the projection in Δ_y). Similarly, set $m := \mathrm{proj}_{\pi'}\, xz$ (here, proj means the projection in Δ_z). For $a := l \cap m$ in π', we have that a is a Γ_3-point in π and that $H_{a,\pi}$ contains a cyclic group of order 4. Hence π must be in $\mathfrak{F}_6$, a must be a point in $\mathfrak{O}$ and y must be a Ex_1-point.

Let now be y in Γ_3. We have found an orbit of $\mathfrak{F}_6$-planes in Δ_y having length $2880/4 = 720$. Furthermore, in $\mathfrak{F}_2 \cup \mathfrak{F}_3$, there are exactly 186 planes incident to y. By Lemma 4.16 and Lemma 4.17 we know that there must be exactly four other orbits in $\bigcup_{i=4}^{7} \mathfrak{F}_i$. The possible sizes of these orbits are $360, 480, 720, 960, 1440$ and 2880. Since $3000 \equiv 8 \mod 16$, one orbit must have size 360. The only possibility to choose three of these numbers such that their sum adds up to 2640 without using 960 twice is $2640 = 1440 + 720 + 480$. We conclude that the Ex_1-points in a $\mathfrak{F}_5$-plane are in Γ_3 and that all points in $\mathfrak{F}_4$-plane and the Ex_2-points in $\mathfrak{F}_7$-plane are in Γ_4.

Now let π be in $\mathfrak{F}_3$ and l a line in π containing exactly one Γ_2-point and one Γ_3-point. Then $K := H_{\pi,l}$ is cyclic of order 4. Hence there is exactly one other plane π' fixed by K which is incident to l. Now π' must belong to $\mathfrak{F}_5$. Hence the Ex_2-points in a $\mathfrak{F}_5$-plane are points in Γ_4. The other four planes incident to l must be $\mathfrak{F}_7$-planes. Let π'' be such a plane. Since l contains exactly one point in $\mathrm{Inn}_1(\pi'') \subset \Gamma_2(x)$, l must be a line in $\mathrm{Sec}_2(\pi'')$. Therefore the points in $\mathfrak{O}(\pi'')$ must be points in Γ_4 and the points in $\mathrm{Inn}_2(\pi'')$ must be Γ_3-points. So the Ex_2-points in a $\mathfrak{F}_6$-plane must be Γ_4-points.

Now let π be a $\mathfrak{F}_4$-plane. Then all points in π are Γ_4-points and $H_\pi = 31:3$ operates transitively on both lines and points of π. If l is a line in π and $K := H_l$, then K_π is cyclic of order 3. Now Lemma 4.15 implies that $|K| < 18$. Every involution in G_l centralizing an element of order 3 in G_l is cointained in $G_{(l)}$, hence H_l cannot contain an element of order 6. There is no element of order 5 in K, since in the other case there would be a point y in l such that 5 is a divisor of K_y, a contradiction to $y \in \Gamma_4$. Now suppose $|K| = 3$. Then l would be a Sec_2-line or a Pas_1-line in a $\mathfrak{F}_7$-plane, a contradiction since in this case l would be incident to a point in Γ_2 or Γ_3.

Suppose $|K| = 9$. Then l is a Pas_1-line in $\mathfrak{F}_7$-plane and we get the same contradiction. Suppose $K \cong \mathsf{A}_4$. In this case there would be two planes incident to l fixed by K, surely a contradiction. Hence K must be isomorphic to S_3. So there is one plane π in Δ_l with $K_\pi = \mathsf{S}_3$, two planes with $K_\pi = \mathsf{Z}_3$ and three planes with $K_\pi = \mathsf{Z}_2$. Hence l is incident to one plane in $\mathfrak{F}_5 \cup \mathfrak{F}_6$, two planes in $\mathfrak{F}_4$ and three planes in $\mathfrak{F}_7$. In the last case l is a $\mathfrak{T}$-line. Therefore the Ex_1-points in a $\mathfrak{F}_7$-plane are Γ_4-points.

Let π be a $\mathfrak{F}_7$-plane and l a Pas_1-line in π. Then l is incident to three points in each Γ_3 and Γ_4. $H_{\pi,l}$ acts transitively on both sets of points. Again, set $K := H_l$. Then $3 \leq |K| \leq 18$. Just like before on easily sees that there is no element of order 5 or 6 in K and K is not isomorphic to A_4. If π is a $\mathfrak{F}_7$-plane, then the lines in Pas_1 are the only lines in π incident to exactly 3 points in both Γ_3 and Γ_4. Furthermore, the group H operates transitively on the set $\{(\pi, m); \pi \in \mathfrak{F}_7, m \in \mathrm{Pas}_1(\pi)\}$. Therefore all $\mathfrak{F}_7$-planes incident to l are H_l-conjugate. We conclude that the order of K can neither be 3 nor 9. $|K| = 18$ is not possible either since in this case an involution in K would not fix any plane in Δ_l. Therefore $K \cong \mathsf{S}_3$ must hold. If π is a plane in Δ_l with $K_\pi = \mathsf{Z}_2$, then π must be in $\mathfrak{F}_6$ and l is a Sec_2-line in π. Therefore either the Inn_1-points or the Inn_2-points in π are in Γ_3. Hence there are two possibilities left:

(I) If π is a $\mathfrak{F}_5$-plane, then the Inn_1-points are in Γ_3 and the Inn_2-points are in Γ_4. If π is a $\mathfrak{F}_6$-plane, then the Inn_1-points are in Γ_4 and the Inn_2-points are in Γ_3.

(II) If π is a $\mathfrak{F}_5$-plane, then the Inn_1-points are in Γ_4 and the Inn_2-points are in Γ_3. If π is a $\mathfrak{F}_6$-plane, then the Inn_1-points are in Γ_3 and the Inn_2-points are in Γ_4.

Suppose (I) holds. Let l be a Sec_1-line in a plane from $\mathfrak{F}_5$ and set $K := H_l$. Then l is incident to two points from each Γ_2, Γ_3 and Γ_4. Hence every plane π incident to l must be a $\mathfrak{F}_5$-plane and l must be a Sec_1-line in π. Therefore $|K| = 48$ since H is transitive on the set $\{(\pi, m); m \in \mathfrak{F}_5, \pi \in \mathrm{Sec}_1(\pi)\}$. But K has three orbits of size two on the set of points in Δ_l, and so an element of order 3 in K must be in $G_{(l)}$, surely a contradiction. Hence case (II) must hold and we are done. □

Theorem 4.19. (a) *H has exactly thirteen orbits on the set of lines in Δ_l as listed in Table 5.*

(b) *For $l \in \mathfrak{L}_8$, H_l has two orbits of size 3 on the point set of Δ_l, and for $l \in \mathfrak{L}_{13}$, H_l fixes one Γ_4-point in Δ_l and acts transitively on the others. In all other cases the points and planes in the same H-orbit are H_l-conjugate.*

Proof. From Theorem 4.18 and its proof we know that there is in each case just one orbit of lines with point distribution as follows: $1\,\Gamma_2, 1\,\Gamma_3, 4\,\Gamma_4$, $3\,\Gamma_3, 3\,\Gamma_4$ and $6\,\Gamma_4$. The type of H_l is clear in all these cases. Furthermore there is in each case just one orbit of lines with the following point distribution: $1\,\Gamma_3$, $5\,\Gamma_4$ and $2\,\Gamma_2, 4\,\Gamma_3$. In the second case we have $|H_l| = 6 \cdot |H_{l,\pi}|$ for all $\pi \in \mathfrak{F}(l)$. With this information we can determine the structure of the stabilizers.

Orbit	Characterization	Points incident to a representative	Planes incident to a representative	Stabilizer in H of a representative
$\mathfrak{L}_1$		$x, 5\,\Gamma$	$6\,\mathfrak{F}_1$	$5^{1+4} : \mathsf{GL}(2,5)$
$\mathfrak{L}_2$		$6\,\Gamma$	$1\,\mathfrak{F}_1, 5\,\mathfrak{F}_2$	$5^4 : (4 \times 4)$
$\mathfrak{L}_3$		$1\,\Gamma, 5\,\Gamma_3$	$1\,\mathfrak{F}_2, 5\,\mathfrak{F}_3$	$5^2 : (4 \times 4)$
$\mathfrak{L}_4$		$5\,\Gamma_2, 1\,\Gamma_3$	$6\,\mathfrak{F}_3$	$\mathsf{GL}(2,5)$
$\mathfrak{L}_5$		$1\,\Gamma, 1\,\Gamma_2, 4\,\Gamma_4$	$6\,\mathfrak{F}_3$	$4 \cdot \mathsf{S}_4$
$\mathfrak{L}_6$	$\mathfrak{T} \in \mathfrak{F}_6$	$1\,\Gamma_3, 5\,\Gamma_4$	$1\,\mathfrak{F}_3, 5\,\mathfrak{F}_6$	$5 : 4$
$\mathfrak{L}_7$	$\mathfrak{T}$ in $\mathfrak{F}_5$, Sec_2 in $\mathfrak{F}_7$	$1\,\Gamma_2, 1\,\Gamma_3, 4\,\Gamma_4$	$1\,\mathfrak{F}_3, 1\,\mathfrak{F}_5, 4\,\mathfrak{F}_7$	4
$\mathfrak{L}_8$	Pas_1 in $\mathfrak{F}_6$, $\mathfrak{T}$ in $\mathfrak{F}_7$	$6\,\Gamma_4$	$2\,\mathfrak{F}_4, 1\,\mathfrak{F}_6, 3\,\mathfrak{F}_7$	S_3
$\mathfrak{L}_9$	Sec_1 in $\mathfrak{F}_5$	$2\,\Gamma_2, 4\,\Gamma_3$	$6\,\mathfrak{F}_5$	$\mathsf{GL}(2,3)$
$\mathfrak{L}_{10}$	Sec_2 in $\mathfrak{F}_5$, Pas_2 in $\mathfrak{F}_7$	$2\,\Gamma_2, 1\,\Gamma_3, 3\,\Gamma_4$	$3\,\mathfrak{F}_5, 3\,\mathfrak{F}_7$	S_3
$\mathfrak{L}_{11}$	Pas_1 in $\mathfrak{F}_5$ and $\mathfrak{F}_7$, Sec_2 in $\mathfrak{F}_6$	$3\,\Gamma_3, 3\,\Gamma_4$	$1\,\mathfrak{F}_5, 3\,\mathfrak{F}_6, 2\,\mathfrak{F}_7$	S_3
$\mathfrak{L}_{12}$	Pas_2 in $\mathfrak{F}_5$, Sec_1 in $\mathfrak{F}_6$	$2\,\Gamma_3, 4\,\Gamma_4$	$4\,\mathfrak{F}_5, 2\,\mathfrak{F}_6$	$\mathsf{Z}_8 : \mathsf{Z}_2$
$\mathfrak{L}_{13}$	Pas_2 in $\mathfrak{F}_6$, Sec_1 in $\mathfrak{F}_7$	$2\,\Gamma_3, 4\,\Gamma_4$	$3\,\mathfrak{F}_6, 3\,\mathfrak{F}_7$	D_6

Table 5: Orbits of H on the set of lines in Δ_l

Choose $l \in \mathrm{Sec}_2(\mathfrak{F}_5)$. Then l is incident to two points in Γ_2, one point in Γ_3 and three points in Γ_4. Suppose all planes in Δ_l are $\mathfrak{F}_5$-planes. Then $|H_l| = 12$. Since $H_{(l)}$ is trivial, H_l cannot contain an element of order 6, hence we have $H_l \cong \mathsf{A}_4$. But H_l operates transitively on the set of the two Γ_2-points in Δ_l, a contradiction. Hence there exists a plane $\pi \in \mathfrak{F}_7$ such that $l \in \mathrm{Pas}_2(\pi)$. We conclude $H_l \cong \mathsf{S}_3$.

The following lines contain exactly two points in Γ_3 and four points in Γ_4: $\mathrm{Pas}_2(\mathfrak{F}_5), \mathrm{Sec}_1(\mathfrak{F}_6), \mathrm{Pas}_2(\mathfrak{F}_6)$ and $\mathrm{Sec}_1(\mathfrak{F}_7)$. For $\pi \in \mathfrak{F}_6$ and $l \in \mathrm{Sec}_1(\pi)$ we have $H_{\pi,l} \cong \mathsf{D}_4$ and $H^l_{\pi,l} \cong \mathsf{V}_4$. In the other cases one has $H_{\pi,l} \cong \mathsf{V}_4$ and $H^l_{\pi,l} \cong \mathsf{Z}_2$. Take $l \in \mathrm{Pas}_2(\mathfrak{F}_6) \cup \mathrm{Sec}_1(\mathfrak{F}_7)$. Suppose H_l is transitive on the set of planes incident to l. Then we get $|H_l| = 24$ and $|H^l| = 12$. Since H^l acts transitively on the set of the two Γ_3-points incident to l, there is a normal subgoup of index 2

and hence a normal subgroup of order 3 in H_l. We conclude $H_l \cong \mathsf{D}_6$. But in this case H_l must contain an element inducing a transposition on the point set of l. The square of such an element is an element in $G_{(l)}$ having order divisible by 4. This is a contradiction since we have $H_{(l)} = 2$.

Suppose $\mathrm{Sec}_1(\mathfrak{F}_7)$ and $\mathrm{Pas}_2(\mathfrak{F}_6)$ are contained in different H-orbits. Then there is an orbit consisting of $\mathrm{Pas}_2(\mathfrak{F}_5)$ and exactly one of these sets. For $l \in \mathrm{Pas}_2(\mathfrak{F}_5)$ we get $|H_l| = 4 \cdot 3 = 12$, hence H^l acts as S_3 on the four Γ_4-points of l. But this is a contradiction because if π' is a $\mathfrak{F}_5$-plane incident to l, then $H_{l,\pi'}$ fixes no Γ_4-point in l.

We see that there are exactly three different possiblities left for the H-orbits on lines with four Γ_4- and two Γ_3-points:

(i) $\mathrm{Sec}_1(\mathfrak{F}_7) \cup \mathrm{Pas}_2(\mathfrak{F}_6),\ \mathrm{Sec}_1(\mathfrak{F}_6) \cup \mathrm{Pas}_2(\mathfrak{F}_5)$.

(ii) $\mathrm{Sec}_1(\mathfrak{F}_7) \cup \mathrm{Pas}_2(\mathfrak{F}_5) \cup \mathrm{Pas}_2(\mathfrak{F}_6),\ \mathrm{Sec}_1(\mathfrak{F}_6)$.

(iii) $\mathrm{Sec}_1(\mathfrak{F}_7) \cup \mathrm{Pas}_2(\mathfrak{F}_6),\ \mathrm{Sec}_1(\mathfrak{F}_6), \mathrm{Pas}_2(\mathfrak{F}_5)$.

Let y be in $\Gamma_4(x)$. Suppose case (iii) holds. For $l \in \mathrm{Sec}_2(\mathfrak{F}_7) \cup \mathrm{Pas}_2(\mathfrak{F}_6)$ one sees that $H_l^l = \mathsf{S}_3$ must hold, hence H_l has two orbits on the set of Γ_4-points in l. Therefore H_y has at least eleven orbits on the set of lines incident to l, a contradiction to Lemma 4.17. Suppose case (ii) holds. Choose $l \in \mathrm{Sec}_1(\mathfrak{F}_7) \cup \mathrm{Pas}_2(\mathfrak{F}_6) \cup \mathrm{Pas}_2(\mathfrak{F}_5)$. Then $|H_l| = 8$ and $|H^l| = 4$. Suppose that there is an element $a \in H_l$ with $o(a) = 4$. Because H_l-orbits on the set of planes in Δ_l have size two, $a^2 \in H_{(l)}$ must hold. Furthermore a does not fix any plane incident to l, hence a corresponds to a product of three disjoint transpositions. This is a contradiction since if we set $Z := Z(O_5(G_l))$, then a must induce an automorphism of order 4 of Z, therefore a^2 cannot be an involution in $G_{(l)}$. We conclude that H_l is elementary abelian of order 8. This is also a contradiction, since G_l does not contain an elementary abelian subgroup of order 8. So case (i) must hold. We see that there are exactly thirteen orbits. We are left to determine the stabilizer of a line in $\mathfrak{L}_{12}$ and in $\mathfrak{L}_{13}$.

For $l \in \mathfrak{L}_{12}$ we have $|H_l| = 16$ and $|H_{(l)}| = 2$. Since $H_{(l)} = \mathsf{Z}_2$ holds, every element of H_l induces an even permutation on the sets of points and planes in Δ_l. We conclude that H_l is contained in a subgroup K of G_l with $K \cong 4 \cdot \mathsf{A}_6$ and H^l is isomorphic to a Sylow 2-subgroup of A_6. Hence H_l contains an element corresponding to a permutation of type $(2,4)$ in A_6. Such an element has order 8 in G_l. Because H_l contains an abelian group of type $(2,2)$, this extension must split. (It is not so easy to determine the exact type of H_l, but this is not important.)

If l is in $\mathfrak{L}_{13}$ then $|H_l| = 12$. H_l possesses a normal subgroup of order 2 and an elementary abelian Sylow 2-subgroup. If t is an involution in $H_l \setminus H_{(l)}$ and $s \in H_l$ an element of order 3, then s and t cannot commute. Therefore $H_l \cong \mathsf{D}_6$

must hold. $\square$

Table 6 gives information to how many points of each orbit a point is collinear. This information can already be found in [5], but without a proof.

	p	Γ	Γ_2	Γ_3	Γ_4
p		19530			
Γ	1	154	3125	3750	12500
Γ_2		63	2520	4599	12348
Γ_3		36	2190	4544	12720
Γ_4		42	1056	4452	12978

Table 6: Collinearity of points of each orbit

For example, a point in Γ_2 is collinear to 63 points from Γ, 2520 points from Γ_2, 4544 points from Γ_3 and 12398 points from Γ_4.

5 Coverings of Δ which induce automorphisms on apartments

Theorem 5.1. *Let $\theta\colon \Delta^* \to \Delta$ be a covering such that Δ^* is connected. Suppose further that θ induces an isomorphism from Σ^* to Σ for every apartment Σ of Δ and every connected component Σ^* of $\theta^{-1}(\Sigma)$. Then θ itself is an isomorphism.*

Proof. Let $\varphi\colon \tilde{\Delta} \to \Delta$ be the universal covering of Δ. Then there is a covering $\zeta\colon \tilde{\Delta} \to \Delta^*$ such that $\theta \circ \zeta = \varphi$. Set $\Pi := \operatorname{Aut} \tilde{\Delta}_\varphi \cong \pi_1(\Delta)$ and $\Pi_0 := \operatorname{Aut} \tilde{\Delta}_\zeta \cong \pi_1(\Delta^*)$. Then $\Pi_0 \subseteq \Pi$ and we have to show that equality holds. Let Σ be an apartment in Δ, $\tilde{\Sigma}$ an apartment in $\tilde{\Delta}$ with $\varphi(\tilde{\Sigma}) = \Sigma$ and $\Sigma^* := \zeta(\tilde{\Sigma})$. For $v \in \tilde{\Sigma}$ and $g \in \Pi_{\tilde{\Sigma}}$ we have $\zeta(v), \zeta(v^g) \in \Sigma^*$, hence $\theta \circ \zeta(v) = \varphi(v) = \varphi(v^g) = \theta \circ \zeta(v^g)$. Since $\theta|\Sigma^*$ is injective, we conclude $\zeta(v^g) = \zeta(v)$, therefore $g \in \Pi_0$. So we have $\Pi_{\tilde{\Sigma}} \leq \Pi_0$. The same holds for every $\tilde{G}$-conjugate of $\tilde{\Sigma}$. Hence we can prove the theorem by showing $\Pi = \langle \Pi_{\tilde{\Sigma}}^g ; g \in \tilde{G} \rangle$. We may assume $\Pi_0 = \langle \Pi_{\tilde{\Sigma}}^g ; g \in \tilde{G} \rangle$. Since Π_0 is normal in Π, θ is a normal covering with $B := \Pi/\Pi_0$ as group of deck transformations. Moreover, Π_0 is even normal in $\tilde{G}$, hence $G^* := \operatorname{Aut} \Delta^* = \tilde{G}/\Pi_0$ acts transitively on the set of maximal flags in Δ^*.

Let Γ^* be the graph having the point set of Δ^* as vertex set such that two collinear points are adjacent. Set Λ^* as the associated clique complex. Then θ induces a covering from Λ^* to Λ which we will also call θ. We will show that this map is an isomorphism. Choose a point $x \in \mathfrak{P}$ and a point x^* in the preimage of x. We set $E_o := \{(y,z) \in \mathfrak{P}^2; y$ and z are collinear$\}$. For every $y \in \mathfrak{P}$ we select a preimage y^* of y with $d(x,y) = d(x^*, y^*)$. Hence we get a $\mu \in Z^1(\Lambda, B)$ such that $(y^*)^{\mu(y,z)}$ and z^* are adjacent for all $(y,z) \in E_o$. If y is adjacent to x, then we get $\mu(x,y) = 1$. If $d(x,y) = 2$, then there exists a point z adjacent to both x and y with $\mu(x,y) = \mu(y,z) = 1$.

We need some lemmata to finish the proof.

Lemma 5.2. *If z is in $\Gamma_2(x) \cup \Gamma_3(x)$, then $\mu(y,z) = 1$ for all $y \in \Gamma(x) \cap \Gamma(z)$.*

Proof. There is a $y_0 \in \Gamma(x) \cap \Gamma(z)$ with $\mu(y_0, z) = 1$. Suppose that x, z, y and y_0 are contained in a common apartment A. In this case the closed path (x, y_0, z, y, x) can be lifted to a closed path having x^* as origin and end. Now it is easily seen that $\mu(y,z) = 1$ holds. If y_0 and y are collinear, then $\mu(y,z) = \mu(y,y_0)\mu(y_0,z) = \mu(y,x)\mu(x,y_0) = 1$. By Lemma 4.8 and Lemma 4.10, there is always a chain $y_0, y_1, \ldots, y_n = y$ in $\Gamma(x) \cap \Gamma(z)$ such that y_i, y_{i+1} are collinear or x, z, y_i and y_{i+1} are contained in a common apartment, so the statement can be proved by induction. □

Since G acts transitively on the vertex set of Γ and G^* acts transitively on the vertex set of Γ^*, we conclude:

Lemma 5.3. *If $y, z \in \mathfrak{P}$ such that z is in $\Gamma_2(y) \cup \Gamma_3(y)$, there exists a unique element $z^+ \in \theta^{-1}(z)$ such that $d(y^*, z^+) = 2$. It is $z^+ = (z^*)^{\mu(z,w)\mu(w,y)}$ for any $w \in \Gamma(y) \cap \Gamma(z)$.*

Lemma 5.4. *For all $z \in \Gamma_4(x)$ and $y \in \Gamma(x) \cap \Gamma(z)$ one has $\mu(y,z) = 1$.*

Proof. Again, there exists an element $y_0 \in \Gamma(x) \cap \Gamma(z)$ with $\mu(y,z) = 1$. Suppose y and y_0 are in relation Γ_2. Then $y^+ = (y^*)^{\mu(y,x)\mu(x,y_0)} = y^*$ and $y^+ = (y^*)^{\mu(y,z)\mu(z,y_0)} = (y^*)^{\mu(y,z)}$, hence $\mu(y,z) = 1$. By Lemma 4.11, for any y in $\Gamma(x) \cap \Gamma(z)$ there is a chain $y_0, y_1, \ldots, y_n = y$ such that y_i and y_{i+1} are in relation Γ_2, so again we are done by induction. □

The two groups G_x and $G^*_{x^*}$ are naturally isomorphic, hence we can identify these two groups and regard G_x as a subgroup of G^*.

Lemma 5.5. *For all $g \in G_x$ and all $(y,z) \in E_o$ we have $\mu(y^g, z^g) = \mu(x,y)^g$.*

Proof. We have shown that for all $y \in \mathfrak{P}$ the element y^* is the unique element in the preimage of y with $d(x,y) = d(x^*, y^*)$. For $g \in G_x$ we have $d(x^*, y^*) = d((x^*)^g, (y^*)^g) = d(x^*, (y^*)^g)$ and $d(x, y^g) = d(x^*, (y^g)^*)$. Since $\theta((y^*)^g) = \theta(y^*)^g = y^g$, we conclude $(y^*)^g = (y^g)^*$. If (y,z) is in E_o, then we get $(y^*)^{\mu(y,z)} \sim z^*$, hence $(y^*)^{\mu(y,z)g} \sim (z^*)^g$ and finally $((y^g)^*)^{\mu(y,z)^g} \sim (z^g)^*$. So the statement follows. □

We now continue the proof of Theorem 5.1. We define for every plane π in Δ an equivalence relation $\perp = \perp_\pi$ on the set of points in π such that $y \perp z$ holds if and only if $\mu(y,z) = 1$. Since μ is a 1-cocycle, this relation is really an equivalence relation. By Lemma 5.5 this relation is $G_{x,\pi}$-invariant. If π is a plane in $\bigcup_{i=1}^{3} \mathfrak{F}_i$, then there is a point y in π collinear to x, hence $\mu(y,z) = 1$ for all points $z \in \pi$. By Lemma 5.2 and Lemma 5.4 all points in π are in relation $\perp$ in this case.

Now suppose $\pi \in \mathfrak{F}_5 \cup \mathfrak{F}_6,\ y \in \mathrm{Ex}_2(\pi)$ and let t_1, t_2 be the two lines in $\mathfrak{T}(\pi)$ incident to y. If y_i is a point incident to t_i, then y and y_i are contained in a common plane in $\mathfrak{F}_3$ (see Theorem 4.19), hence we get $y \perp y_i$. Hence the equivalence class of y contains two points in $\mathrm{Ex}_1(\pi)$. But $G_{x,\pi}$ acts primitively on the set of these points, hence the equivalence class of y contains all points in $\mathrm{Ex}_1(\pi)$. So this class is invariant under $G_{x,\pi}$ and contains all points in $\mathfrak{O}(\pi), \mathrm{Ex}_1(\pi)$ and $\mathrm{Ex}_2(\pi)$.

Now let π be a plane in $\mathfrak{F}_7$, y a point in $\mathfrak{O}(\pi)$ and l_1, l_2 two different lines in $\mathrm{Sec}_2(\pi)$ incident to y. Then again by Theorem 4.19 we have that for all points y_i incident to l_i the two points y and y_i are contained in a common plane in $\mathfrak{F}_3$, hence $y \perp y_i$. Therefore the equivalence class of y contains two points in $\mathrm{Inn}_1(\pi)$. But again, $G_{x,\pi}$ acts primitively on the set of these points. We conclude that the equivalence class of y contains all points in $\mathfrak{O}(\pi), \mathrm{Inn}_1(\pi), \mathrm{Inn}_2(\pi)$ and $\mathrm{Ex}_2(\pi)$.

Suppose now that π is a plane in $\mathfrak{F}_4$ and l a line in π. Then there is plane $\pi' \in \mathfrak{F}_7$ such that l is a $\mathfrak{T}$-line in π'. Hence there are two points y, z in l with $y \perp z$. Since $G_{x,\pi}$ is primitive on the set of points in π, we conclude $y \perp z$ for all points y and z in π.

Is π again a plane in $\mathfrak{F}_7$, t a $\mathfrak{T}$-line in π and y, z two points on l with $y \in \mathfrak{O}(\pi)$ and $z \in \mathrm{Ex}_1(\pi)$, then $y \perp z$. We conclude $y \perp z$ for all points y, z in π.

Let π be again a plane in $\mathfrak{F}_5$ and l a line in $\mathrm{Sec}_2(\pi)$. Then there is a plane π' in $\mathfrak{F}_7$ such that l is in $\mathrm{Pas}_2(\pi')$. Therefore we get $y \perp z$ for all y, z incident to l. Now l contains points in $\mathfrak{O}(\pi), \mathrm{Ex}_2(\pi), \mathrm{Inn}_1(\pi)$ and $\mathrm{Inn}_2(\pi)$, hence $y \perp z$ for all points y and z in π.

Finally let π be again a plane in $\mathfrak{F}_6$ and l a line in $\mathrm{Sec}_2(\pi)$. Then $l \in \mathrm{Pas}_1(\pi')$ for some plane π' in $\mathfrak{F}_7$. We conclude $y \perp z$ for all points y, z on l. Since l contains points in $\mathfrak{O}(\pi), \mathrm{Ex}_2(\pi), \mathrm{Inn}_1(\pi)$ and $\mathrm{Inn}_2(\pi)$, we finally get $y \perp z$ for all points y, z on π.

We have shown that $\mu(y,z) = 1$ holds for all pairs of adjacent points y and z. Since Δ^* is connected, θ must be an isomorphism by Theorem 2.3. $\square$

6 Amalgams of type Ly and the uniqueness of the Lyons group

We will now apply our result to prove the uniqueness of the Lyons group.

Definition 6.1. Let G be a group of type Ly, Δ its 5-local geometry, x, y two distinct collinear points in Δ and Σ an apartment containing both points. Set $G_1 = G_x,\ G_2 = G_{\{x,y\}},\ G_3 = G_\Sigma,\ G_{ij} = G_i \cap G_j$ for $1 \le i < j \le 3$ and $G_{123} = G_1 \cap G_2 \cap G_3$. For $\emptyset \neq J \subset K \subseteq \{1, 2, 3\}$, let $\phi_{J,K}$ be the inclusion map of G_K in G_J. Let $\mathcal{A}$ be the amalgam consisting of these groups and homomorphisms. Then $\mathcal{A}$ is called an amalgam of type Ly.

If $\mathcal{A}$ is such an amalgam, then $G_1 \cong \mathsf{G}_2(5), G_2 \cong 5^{1+4} : (4 \cdot \mathsf{S}_4.2),\ G_3 = (4 \times 4).(\mathsf{S}_4 \times \mathsf{S}_3),\ G_{12} = 5^{1+4} : (4 \cdot \mathsf{S}_4),\ G_{13} = (4 \times 4) : \mathsf{D}_6, G_{23} = (4 \times 4).\mathsf{V}_4$ and $G_{123} = (4 \times 4).2$.

A priori, it is not clear if there is only one amalgam of type Ly (up to isomorphism). To prove the uniqueness of the Lyons group, we first have to show that two amalgams of type Ly are isomorphic.

Lemma 6.2. *If $\mathcal{A} = (G_J)_J$ and $\overline{\mathcal{A}} = (\overline{G}_J)_J$ are amalgams of type Ly, then there are isomorphisms $\phi\colon G_i \to \overline{G}_i$ such that $\phi_i(G_{ij}) = \overline{G}_{ij}$ and $\phi_i(G_{123}) = \overline{G}_{123}$ for all $1 \le i, j \le 3, i \neq j$.*

Proof. This is clear for G_1. Proposition 6.8 will show that $G_2 \cong \operatorname{Aut} G_{12} \cong \operatorname{Aut} \overline{G}_{12} = \overline{G}_2$. Since G_{23} is a Sylow 2-subgroup of G_2 and G_{12} is normal in G_2, one can choose an isomorphism between G_2 and $\overline{G}_2$ such that G_{12} is mapped onto $\overline{G}_{12}$, G_{23} is mapped onto $\overline{G}_{23}$ and $G_{123} = G_{12} \cap G_{23}$ is mapped onto $\overline{G}_{12} \cap \overline{G}_{23} = \overline{G}_{123}$.

Both G_3 and $\overline{G}_3$ are extensions of an abelian group T resp. $\overline{T}$ of type $(4, 4)$ by a group isomorphic to $\mathsf{S}_4 \times \mathsf{S}_3$. Since $G_{13} \cong \overline{G}_{13}, G_{23} \cong \overline{G}_{23}$, $G_3 = \langle G_{13}, G_{23} \rangle$ and $\overline{G}_3 = \langle \overline{G}_{13}, \overline{G}_{23} \rangle$, the action of G_3 on T is isomorphic to the action $\overline{G}_3$ on $\overline{T}$. Moreover, their Sylow 2-subgroups are isomorphic to a Sylow 2-subgroup of $2 \cdot \mathsf{A}_{11}$. Now one can easily deduce from Gaschütz' Theorem (see [4, I.17.4]) that G_3 and $\overline{G}_3$ must be isomorphic. Both G_3 and $\overline{G}_3$ act transitively on the chambers of a geometry described in section 3.2. Thus there is an isomorphism $\phi_3\colon G_3 \to \overline{G}_3$ having the desired properties. □

From now on let $\mathcal{A} = (G_J)_J$ be a fixed amalgam of type Ly. We will need some facts about the automorphism groups of the groups G_J involved in $\mathcal{A}$. We first treat the groups contained in G_3. Recall that G_3 is an extension of an abelian group T of type $(4, 4)$ with a group $W \cong \mathsf{S}_4 \times \operatorname{Sym}\{a, b, c\}$. By regarding a Sylow 2-subgroup of G_3 which is isomorphic to a Sylow 2-subgroup of $2 \cdot \mathsf{A}_{11}$, one sees that this extension is non-split.

Let $\mathcal{W}$ be the automorphism group of T, $M = \Phi(T)$ the Frattini subgroup of T and $\mathcal{W}_0 := C_{\mathcal{W}}(M)$. Then $\mathcal{W}_0 = C_{\mathcal{W}}(T/M), \mathcal{W}/\mathcal{W}_0 \cong \mathsf{S}_3$ and $\mathcal{W}_0 \cong \mathrm{Hom}(T/M, M)$ is elementary abelian of order 2^4. The center of $\mathcal{W}$ is contained in $\mathcal{W}_0$ and generated by the map $\rho\colon T \to T\colon t \mapsto t^{-1}$. One sees easily that the $\mathcal{W}$-module $\mathcal{W}_0$ is the direct sum of two submodules $\mathcal{W}_1$ and $\mathcal{W}_2$ such that $\mathcal{W}_1$ and M are isomorphic as $\mathcal{W}$-modules, $\mathcal{W}_2$ contains $Z(\mathcal{W})$ and $\mathcal{W}$ acts trivially on $\mathcal{W}_2/Z(\mathcal{W})$.

Set $W_1 := O_2(W) \cong \mathsf{V}_4$ and let W_2 be the unique normal subgroup of W isomorphic to S_3. Then $W_2' = C_W(T)$ and W_1W_2 is the preimage of $\mathcal{W}_0$ in W. The image of W_1 in $\mathcal{W}$ is $\mathcal{W}_1$ and W_2 is mapped onto $Z(\mathcal{W})$. Regarding the description of apartments of Δ (see section 3.2) we can assume that the image of G_{123} in W is $\langle((34), 1)\rangle$, that the image of G_{13} is

$$W_3 := \{(g, h) \in \mathsf{S}_4 \times \mathrm{Sym}\{a, b, c\}; 1^g = 1, a^h = a\}$$

and that the image of G_{23} is $\langle((34), 1), ((12), (ab)\rangle$. We see that W_2W_3 is complement of W_1 in W.

Lemma 6.3. *The center of G_{123} is cyclic of order 4 and equals the center of G_{23}.*

Proof. The group G_{123} is contained in a line stabilizer in $\mathsf{G}_2(5)$ which is isomorphic to $5^{1+4} : \mathsf{GL}(2, 5)$. Hence G_{123} is isomorphic to the group of all monomial matrices in $\mathsf{GL}(2, 5)$ and therefore $Z(G_{123}) \cong \mathsf{Z}_4$. Now G_{23} is a Sylow 2-subgroup in the normalizer of a group generated by a $5A$-element, and so one sees $Z(G_{123}) = Z(G_{23})$. □

Lemma 6.4. *There is exactly one non-trivial automorphism $\beta \in \mathrm{Aut}(G_{23})$ which centralizes G_{123}. This automorphism is induced by an automorphism α of G_3 which centralizes G_{13}.*

Proof. By [4, I.17.1], the group of all automorphism of G_{23} which centralize G_{123} (and hence G_{23}/G_{123} because this group has order 2) is isomorphic to $Z^1(G_{23}/G_{123}, Z(G_{123}))$. Since $\mathsf{Z}_4 \cong Z(G_{123}) = Z(G_{23})$, this group has order 2. Thus the first claim follows.

Since W_1 and M are isomorphic W-modules, there exists a W-isomorphism $\varphi\colon W_1 \to M$. Set $f\colon W \to T\colon f(xy) = \varphi(y)$ for $x \in W_2W_3$ and $y \in W_1$. This map f is well defined because W_2W_3 is a complement of W_1 in W. Since φ is W-homomorphism, we have $f \in Z^1(W, T)$. So f defines an automorphism α of G_3 by $x^\alpha = xf(Tx)$ (see [4, I.17.1]). Now α centralizes G_{13} since f vanishes on $G_{13}/T \le W_2W_3$, but it does not centralize G_{23} since $f((12)(34)(ab)) =$

$\varphi((12)(34)) \neq 1$. Hence the restriction of α on G_{23} is the unique non-trivial automorphism of G_{23} which centralizes G_{123}.. □

Lemma 6.5. *There exists an automorphism ϵ of G_3 which centralizes G_3/T (and hence normalizes every subgroup of G_3 containing T) and inverts every element of T.*

Proof. Set $C := C_G(T)$ and $X := G_3/C$. Then C is the direct product of T and a cyclic group K of order 3 (the A_3 part in the decomposition $G_3/T \cong \mathsf{S}_4 \times \mathsf{S}_3$). Since K has a complement in G_3 (just take the preimage of $\mathsf{S}_4 \times \langle(12)\rangle$ in G_3), there is a $f \in Z^2(X, C)$ taking values in T which determines the extension of C by X.

Let σ be the automorphism of C with $t^\sigma = t$ for all $t \in T$ and $s^\sigma = s^{-1}$ for all $s \in K$. We identify X with its image in $\operatorname{Aut} C$ and set $Y := \langle X, \sigma \rangle$. Then $Y = X \times \langle \sigma \rangle$ since σ is contained in $Z(\operatorname{Aut} C)$ but not in X. Since σ acts trivially on T and f takes only values in T, the map $\tilde{f} \colon Y \times Y \to C$ with $\tilde{f}((x_1, s_1), (x_2, s_2)) = f(x_1, x_2)$ for $x_1, x_2 \in X, s_1, s_2 \in \langle\sigma\rangle$ is a 2-cocycle. Let H be the extension of C by Y with $\tilde{f}$. Then G_3 can be identified with the preimage of X in H. There is an element $s \in X$ such that s inverts every element in C. Therefore, $\tau := s\sigma \in Y$ centralizes K and inverts every element in T. The preimage of $\langle\tau\rangle$ in H/T is a cyclic normal subgroup of H/T having order 6. Hence there is a preimage τ' of τ in the center of H/T. If ϵ is a preimage of τ' in H, then ϵ induces an autormorphism on G_3 with the desired properties. □

Proposition 6.6. *Every automorphism of G_{13} which normalizes G_{123} is induced by an automorphism of G_3 which normalizes G_{23}.*

Proof. The group G_{13} is the normalizer of a maximal torus in $\mathsf{G}_2(5)$, hence G_{13} is a semidirect product of T by a group U isomorphic to D_6 which acts faithfully on T. We can therefore regard U as a subgroup of $\mathcal{W}$. Now U is the normalizer of a Sylow 3-subgroup of G_{13}, which implies that all complements of T in G_{13} are conjugate. We see that an automorphism of G_{13} which centralizes T and G_{13}/T must be an inner automorphism of G_{13} induced by an element in T. Hence $\operatorname{Aut} G_{13}$ is the semidirect product of T by $N_{\mathcal{W}}(U)$.

One easily sees that $N_{\mathcal{W}}(U) = U\mathcal{W}_2$ and $U \cap \mathcal{W}_0 = U \cap \mathcal{W}_2 = Z(\mathcal{W})$ holds. If U_0 is the image of G_{123} in $\mathcal{W}$, then the normalizer of U_0 in $\mathcal{W}$ is $U_0 Z(\mathcal{W}) C_{U_0}(\mathcal{W}_1)$. It follows $N_{\mathcal{W}}(U) \cap N_{\mathcal{W}}(U_0) = U_0 Z(\mathcal{W})$. This is just the image of $N_{G_{13}}(G_{123})$ in $\mathcal{W}$.

Let $g \in G_{13}$ be an element whose image in $\mathcal{W}$ generates $Z(\mathcal{W})$. Then every element of T is inverted by g. By Lemma 6.5, there is an automorphism ϵ of G_3 which normalizes G_{13} and G_{23} and inverts every element of T. We conclude that there is a $t \in T$ with $h^g = h^{t\epsilon}$ for all $h \in G_{13}$. Now the claim follows. □

If H is a line stabilizer in $\mathsf{G}_2(5)$ and $V = O_5(H)/Z(O_5(H))$, then the action of $H'/O_5(H) \cong \mathsf{SL}(2,5)$ on V is irreducible. Therefore, this action is isomorphic to the action of $\mathsf{SL}(2,5)$ on the space of all homogeneous polynomials of degree 3 in two indeterminates over $\mathbb{F}_5$ since this up to isomorphism the unique 4-dimensional $\mathsf{SL}(2,5)$-module over $\mathbb{F}_5$. Moreover, one can easily see that all irreducible representations of $\mathsf{GL}(2,5)$ over $\mathbb{F}_5$ which extend this representation are conjugate by $\operatorname{Aut}\mathsf{GL}(2,5)$. Therefore the following lemma will be useful.

Lemma 6.7. *Let H be a subgroup of index 5 in $\mathsf{GL}(2,5)$ and let P be the space of all homogenous polynomials of degree 3 in $\mathbb{F}_5[X,Y]$. Then the action of H on P is absolutely irreducible.*

Proof. Since there is only one conjugacy class of subgroups of index 5 in $\mathsf{GL}(2,5)$, we can assume that H is the normalizer of the quaternion group generated by $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 4 \\ 1 & 0 \end{smallmatrix}\right)$. Then H is generated by all diagonal matrices and the matrix $\alpha = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right)$.

Suppose the action of H on P is not irreducible. Since H acts semi-simple on P, there is an H-invariant decompositon $P = P_1 \oplus P_2$ of P with P_1, P_2 proper subspaces of P. Now set $\beta := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Then the image of β in $\mathsf{GL}(P)$ has four distinct eigenvalues. All eigenvectors of β are monomials. Since $\beta|P_1$ and $\beta|P_2$ are diagonalizable, these two spaces must be spanned by monomials. Hence we can assume $X^3 \in P_1$. But this would mean $(X+Y)^3 = (X^3)^\alpha \in P_1$, hence $(X+Y)^3$ is the sum of at most 3 monomials, a contradiction.

By Schur's Lemma $\mathbb{E} := C_{\mathrm{End}(P)}(H)$ is a field of order 5^d, where d is either $1, 2$ or 4. If β is as above, then $C_P(\beta)$ is a $\mathbb{E}$-subspace of P, hence $\dim_{\mathbb{F}} C_P(\beta)$ is divisible by d. We conclude that d must be 1. Thus we have $C_{\mathsf{GL}(P)}(H) = Z(\mathsf{GL}(P))$ as desired. □

Proposition 6.8. *Every automorphism of G_{12} is induced by an element of of G_2.*

Proof. Set $R := O_5(G_{12})$, $V := R/Z(R)$, let D be a complement of R in G_{12} and set $S := N_{G_2}(D)$. Then S is a complement of R in G_2. The central factor group V of R is a vector space over $\mathbb{F}_5$, and the commutator map defines a nondegenerate symplectic form f of V which is preserved projectively by S. The whole outer automorphism group of R is $\mathsf{GSp}(V,f)$ (the group of all automorphisms of V preserving f projectively), and the outer automorphism group of G_{12} is given by $N_{GSp(V,f)}(D)/D$. By Lemma 6.7 we have $N_{\mathsf{GL}(V)}(D)/Z \leq \operatorname{Aut} D$ for $Z = Z(D) = Z(\mathsf{GL}(V))$.

Let $A := C_{\operatorname{Aut} D}(Z)$ and $A_0 := C_A(D/Z)$. Since $D/Z \cong \mathsf{S}_4 \cong \operatorname{Aut}\mathsf{S}_4$, we conclude $A = \operatorname{Inn}(D)A_0$. A simple commutator argument shows that $A_0 \cong \operatorname{Hom}(D/Z, Z) \cong \operatorname{Hom}(\mathsf{S}_4, \mathsf{Z}_4)$ is cyclic of order 2. Of course, $N_{\mathsf{GL}(V)}(D)$ centralizes Z, and therefore $N_{\mathsf{GL}(V)}(D)/Z$ is a subgroup of A and $N_{\mathsf{GL}(V)}(D)/D$

is isomorphic to a subgroup of A_0. Hence we conclude $S = N_{\mathsf{GL}(V)}(D)$ and $\operatorname{Aut} G_{12} \cong G_2$. □

Theorem 6.9. *Up to isomorphism, there is only one amalgam of type* Ly.

Proof. Suppose $\mathcal{A} = (G_J)_J$ and $\overline{\mathcal{A}} = (\overline{G})_J$ are two amalgams of type Ly. Then by Lemma 6.2 there are isomorphism $\phi_i \colon G_i \to \overline{G}_i$ for $i = 1, 2, 3$ such that $\phi_i(G_J) = \overline{G}_J$ for J containing i.

The map $\phi_3^{-1} \circ \phi_1 \colon G_{13} \to G_{13}$ is an automorphism of G_{13} normalizing G_{123}. By Proposition 6.6, there is an automorphism γ of G_3 which induces $\phi_3^{-1} \circ \phi_1$ on G_{13} and normalizes G_{23}. We replace ϕ_3 by $\phi_3^* := \phi_3 \circ \gamma$. Then ϕ_3^* equals ϕ_1 on G_{13} and stills map G_{23} to $\overline{G}_{23}$.

By Lemma 6.8, there is an element $a \in G_2$ such that $\phi_2^{-1}(\phi_1(x)) = x^a$ for all $x \in G_{12}$. Again we replace ϕ_2 by ϕ_2^* with $\phi_2^*(x) = \phi_2(x^a)$ for $x \in G_2$. Then $\phi_1(x) = \phi_2^*(x)$ for all $x \in G_{12}$. Note that ϕ_2^* maps G_{23} to $\overline{G}_{23}$ since a normalizes G_{123} and G_{23} is the normalizer of G_{123} in G_2.

Now ϕ_3^* and ϕ_2^* define an automorphism β of G_{23}. Since $\phi_1(x) = \phi_2^*(x) = \phi_3^*(x)$ for $x \in G_{123}$, we see that β centralizes G_{123}. By Lemma 6.4, there is an automorphism α of G_3 extending β which centralizes G_{13}. We replace ϕ_3^* by $\phi_3^{**} = \phi_3^* \circ \alpha$. Now the three automorphisms ϕ_1, ϕ_2^* and ϕ_3^{**} take the same values on intersections and thus define an isomorphism between $\mathcal{A}$ and $\overline{\mathcal{A}}$. □

From now on let $\hat{G}$ be the universal completion of $\mathcal{A}$; since G is a faithful completion of $\mathcal{A}$, $\hat{G}$ is also faithful. Hence G_1, G_2 and G_3 can be regarded as subgroups of $\hat{G}$.

Since $\hat{G}$ is the universal completion of $\mathcal{A}$, there is an epimorphism $\xi \colon \hat{G} \to G$ such that $\xi|G_i$ is an isomorphism for $i = 1, 2, 3$. Set $B := \ker \xi$. Let t be an involution in G_{23} such that $G_2 = G_{12}\langle t \rangle$.

Let $\hat{\Gamma}$ be the graph having vertex set $\hat{\mathfrak{P}} := G_1/\hat{G}$ such that $G_1 g$ and $G_1 thg$ for $g \in \hat{G}$ and $h \in G_1$ are joined by an edge. Furthermore, let $\hat{\Sigma}$ be the graph with $\{G_1 g; g \in G_3\}$ as vertex set and all pairs $\{G_1 g, G_1 thg\}$ with $g \in G_3$ and $h \in G_{23}$ as edges. Then $\hat{\Sigma}$ is a full subgraph of $\hat{\Gamma}$. Define $\zeta \colon \hat{\Gamma} \to \Gamma$ by $\zeta(G_1 g) := x^{\xi(g)}$. It is $\hat{G} \leq \operatorname{Aut} \hat{\Gamma}$ and $\zeta(v^g) = \zeta(v)^{\xi(g)}$ for all $g \in \hat{G}$ and $v \in \hat{\mathfrak{P}}$.

Lemma 6.10. (a) *For all $g \in \hat{G}$, ζ induces an isomorphism from $\hat{\Sigma}^g$ to $\Sigma^{\xi(g)}$.*

(b) *The map ζ induces a covering from $\operatorname{Cl}(\hat{\Gamma})$ to $\operatorname{Cl}(\Gamma)$.*

Proof. (a) Σ and $\hat{\Sigma}$ are both isomorphic to the graph having G_{13}/G_3 as vertex set and $\{\{G_{13}g, G_{13}thg\}; g \in G_3, h \in G_{13}\}$ as set of edges. Therefore the claim follows.

(b) Clearly, ζ is a surjective morphism from $\hat{\Gamma}$ to Γ. For $g \in G, \hat{g} \in \xi^{-1}(g)$ and $h \in G_1$ the vertex $G_1th\hat{g}$ is the unique element in the preimage of G_1th which is adjacent to $G_1\hat{g}$. Thus ζ is a covering.

Let $\hat{x}$ be in $\zeta^{-1}(x) \cap \hat{\Sigma}$ and let $\hat{y}, \hat{z}$ be adjacent in $\hat{\Gamma}_{\hat{x}}$ such that $y := \zeta(\hat{y})$ and $z := \zeta(\hat{z})$ are adjacent. Suppose first that x, y, z are not incident to a common line in Δ. Then there is a $g \in G$ such that x, y, z are in Σ^g. Because all apartments containing x are G_1-conjugate, we can assume that g is in G_1. Since ζ is a covering from $\hat{\Gamma}$ to Γ, there is for both y and z exactly one preimage $\hat{y} \in \zeta^{-1}(y)$ and $\hat{z} \in \zeta^{-1}(z)$ which are adjacent to $\hat{x}$. Now ζ induces an isomorphism from $\hat{\Sigma}^g$ to Σ^g, hence $\hat{y}$ and $\hat{z}$ are in $\hat{\Sigma}^g$. Because $y \sim z$ holds in Σ^g, we can conclude that $\hat{y}$ and $\hat{z}$ are adjacent in $\hat{\Sigma}^g$ and therefore in $\hat{\Gamma}$.

Since $\hat{G}$ acts transitively on $\hat{\mathfrak{P}}$ and since $\zeta(v^g) = \zeta(v)^{\xi(g)}$ for all $v \in \hat{\mathfrak{P}}$ and all $g \in \hat{G}$ holds, we have just shown: If y, z and $w \in \mathfrak{P}$ are pairwise collinear, but not collinear in Δ, and if $\hat{y} \in \zeta^{-1}(y),\ \hat{z} \in \zeta^{-1}(z) \cap \hat{\Gamma}_{\hat{y}}$ and $\hat{w} \in \zeta^{-1}(w) \cap \hat{\Gamma}_{\hat{y}}$, then $\hat{z} \sim \hat{w}$.

Now we suppose, x, y and z are incident to a common line in Δ. Let π be a plane incident to the line xy and choose a point w which is incident to π but not to xy. Then there is a uniquely determined point $\hat{w} \in \zeta^{-1}(w) \cap \hat{\Gamma}_{\hat{x}}$. Neither x, y and w nor x, z and w are incident to a common line, hence $\hat{w} \sim \hat{y}$ and $\hat{w} \sim \hat{z}$. Now y, z and w are not collinear, therefore $\hat{y} \sim \hat{z}$. Hence, ζ induces a covering from $\mathrm{Cl}(\hat{\Gamma})$ to $\mathrm{Cl}(\Gamma)$. □

Theorem 6.11. *The map ζ is an isomorphism.*

Proof. Let $\hat{\Delta}$ be the geometry whose points are the elements of $\hat{\mathfrak{P}}$, whose planes are the maximal cliques in $\hat{\Gamma}$ and whose lines are the cliques of size six in $\hat{\Gamma}$ which are contained in exactly six maximal cliques. Then ζ induces a covering from $\hat{\Delta}$ to Δ which maps apartments in $\hat{\Delta}$ isomorphically on apartments in Δ. Therefore this map is an isomorphism itself by Theorem 5.1. □

Corollary 6.12. $G \cong \hat{G}$.

Proof. The claim follows since $B = \ker \xi$ acts regularly on each preimage under ξ. □

Theorem 6.9 and Corollary 6.12 now imply:

Theorem 6.13. *Up to isomorphism, there is at most one group of type* Ly.

References

[1] **M. Aschbacher** and **Y. Segev**, The uniqueness of groups of Lyons type, *J. Amer. Math. Soc.* **5** (1992), 75–98.

[2] **J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker** and **R. A. Wilson**, *Atlas of Finite Simple groups. Maximal subgroups and ordinary characters for simple groups*, Oxford Univ. Press, Eynsham, 1985.

[3] **M. Grüninger**, *Die Geometrie der Lyonsgruppe*, Dissertation, Universität Tübingen, 2007.

[4] **B. Huppert**, *Endliche Gruppen I*, Grundlehren Math. Wiss. **134**, Springer Verlag, Berlin, Heidelberg, New York, 1967.

[5] **W. M. Kantor**, Some geometries that are almost buildings, *European J. Combin.* **2** (1981), 239–247.

[6] **R. Lyons**, Evidence for a new finite simple group, *J. Algebra* **20** (1972), 540–569.

[7] ______, Errata: Evidence for a new finite simple group, *J. Algebra* **34** (1975), 188–189.

[8] **C. Sims**, The existence and uniqeness of the Lyons' group, **in** *Finite Groups '72*, Proc. Gainesville Conf. 1972 (1973), pp. 138–141.

[9] **J. Tits**, A local approach to Buildings, **in** *The Geometric Vein*, The Coxeter Festschrift (1981), pp. 519–547.

[10] ______, Buildings and group amalgamations, **in** *Proc. Int. Conf.*, St. Andrews/Scotland 1985, London Math. Soc. Lectures Note Ser. **121** (1986), pp. 110–127.

Matthias Grüninger

Fakultät für Mathematik, Universität Bielefeld, Universitätsstrasse 25, 33501 Bielefeld, Germany

e-mail: matthias.grueninger@web.de