Disc structure of certain chamber graphs

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Abstract

The discs of chamber graphs for group geometries, including certain minimal parabolic geometries, maximal $p$-local geometries, Petersen geometries, GABs and Buekenhout geometries, are investigated.

Keywords: chamber graphs, group geometries, minimal parabolic geometries, maximal $p$-local geometries, Buekenhout geometries

MSC 2000: 05E20

1 Introduction

A chamber system over the set $I$ consists of a set $C$ and a system $(\mathcal{P}_i)_{i \in I}$ of partitions of $C$ indexed by $I$. The elements of $C$ are called chambers and, for brevity, the system $(C, (\mathcal{P}_i)_{i \in I})$ will often be referred to as the chamber system $C$. Two chambers which are both in the same member of $\mathcal{P}_j$ for some $j \in I$ are said to be adjacent chambers. The chamber graph of the chamber system $C$ is the graph with vertex set $C$ and two (distinct) chambers are adjacent (in the chamber graph) if they are adjacent chambers in $C$. (Note that a chamber is not adjacent to itself in the chamber graph.) If two chambers $c, c'$ are both in the same member of $\mathcal{P}_i$ ($i \in I$), then we say they are $i$-adjacent, and denote this by $c \sim_i c'$ (or $c' \sim_i c$). The rank of the chamber system is the cardinality of $I$. An automorphism of $C$ is a permutation $\sigma$ of $C$ which preserves each of the partitions $\mathcal{P}_i$, that is, whenever $c \sim_i c'$ ($c, c' \in C, i \in I$) then $c\sigma \sim_i c'\sigma$.

We shall now concentrate on the following situation.

Hypothesis 1.1. $C$ is a chamber system over $I$ and $G \leq \text{Aut} C$ is such that

(i) $G$ is transitive on $C$; and

(ii) for each $i \in I$, $G$ is transitive on the members of the partition $\mathcal{P}_i$. 
Suppose Hypothesis 1.1 holds, and let \( c_0 \) be a fixed chamber of \( C \). Let \( B \) denote the stabilizer in \( G \) of \( c_0 \), and for each \( i \in I \) let \( P_i \) be the stabilizer in \( G \) of the member of \( P_i \) to which \( c_0 \) belongs. Observe that \( B \leq \bigcap_{i \in I} P_i \). We may now identify \( C \) with the set of (right) cosets of \( B \) in \( G \) with, for each \( i \in I \), the members of \( P_i \) being the sets of cosets of \( B \) which are contained in a coset of \( P_i \). In other words, for chambers \( Bg \) and \( Bh, Bg \) and \( Bh \) are \( i \)-adjacent whenever \( gh^{-1} \in P_i \). Such a chamber system will be denoted by \( C(G; B, \{ P_i \}) \). Further, we note that the valency of the chamber graph of \( C \) is one less than the number of cosets of \( B \) in \( \bigcup_{i \in I} P_i \) (counting multiplicities if we have \( P_i = P_j \), for \( i \neq j \)).

Conversely, if we start with a group \( G \), a subgroup \( B \) of \( G \) and a collection of subgroups \( P_i \) of \( G \) \((i \in I) \) each containing \( B \) we may define a chamber system \( C \) by taking the (right) cosets of \( B \) as chambers and the partition \( P_i \) to be given by taking right cosets of \( B \) contained in a right coset of \( P_i \). Now letting \( G \) act by right multiplication on the chambers of \( C \), it is easily checked that Hypothesis 1.1 holds for \( C \) with \( G/\text{core}_G B \) playing the role of \( G \).

A rich source of chamber systems is provided by geometries. We recall that a geometry (over the set \( I \)) is a triple \((\Gamma, \tau, \ast)\) where \( \Gamma \) is a set, \( \tau \) is an onto map from \( \Gamma \) to \( I \) and \( \ast \) is a symmetric relation on \( \Gamma \) with the property that for \( x, y \in \Gamma \) \( x \ast y \) implies \( \tau(x) \neq \tau(y) \). The relation \( \ast \) is called the incidence relation and \( x \in \Gamma \) is said to have type \( i \) if \( \tau(x) = i \). As is customary we shall just say \( \Gamma \) is a geometry. A flag \( F \) of \( \Gamma \) is a set of pairwise incident elements of \( \Gamma \) — the type of \( F \), denoted \( \tau(F) \), is the set \( \{ \tau(x) \mid x \in F \} \). The rank of \( \Gamma \) is \( |I| \) and the rank of a flag \( F \) is \( |\tau(F)| (= |F|) \). Now let \( \mathcal{F} \) denote the set of maximal flags of \( \Gamma \) — a flag \( F \) is maximal if its rank is \( |I| \). For \( i \in I \) and \( F, F' \in \mathcal{F} \) we define \( F \) and \( F' \) to be \( i \)-adjacent if either \( F = F' \) or the rank of the flag \( F \cap F' \) is \( |I| - 1 \) and \( i \notin \tau(F \cap F') \). This yields a partition \( \mathcal{P}_i \) of \( \mathcal{F} \); note that a member of \( \mathcal{P}_i \) consists of all maximal flags containing some fixed flag of type \( I \setminus \{ i \} \). So \( \mathcal{F} \) is a chamber system — we shall call this the chamber system of \( \Gamma \). (\( \mathcal{F} \) is sometimes referred to as the flag complex of \( \Gamma \).) An automorphism of the geometry \( \Gamma \) is a permutation \( \sigma \) of \( \Gamma \) for which \( x \ast y \) implies \( x\sigma \ast y\sigma \) and \( \tau(x) = i \) implies \( \tau(x\sigma) = i \) (where \( x, y \in \Gamma, i \in I \)). Now further suppose that \( G \) is a subgroup of \( \text{Aut} \Gamma \) with \( G \) acting flag transitively on \( \Gamma \) (that is, if \( F \) and \( F' \) are flags of \( \Gamma \) with \( \tau(F) = \tau(F') \), then there exists \( g \in G \) such that \( Fg = F' \)). Then we see that \( G \leq \text{Aut} \mathcal{F} \) and that Hypothesis 1.1 holds for \( G \) and \( \mathcal{F} \). Thus, as discussed earlier, we may study the chamber system \( \mathcal{F} \) within \( G \).

Buildings afford an extensive supply of geometries and hence of chamber systems. In fact the theory of buildings may be developed in the language of chamber systems (see [9] and [18] for more on this). In this approach the chamber graph underpins (pun intended) much of the conceptual framework (for example, galleries, connectedness and thin subgeometries). An outgrowth
of Tits’s pioneering work on buildings was the study of more general geometries — usually ones associated with sporadic simple groups but also those arising from “small” Lie type groups of mixed characteristic. We will, from now on, rather loosely, refer to this mixed bag of geometries as the “sporadic group geometries”. This programme was initiated by Buekenhout [2, 3] in the late seventies. Since then sporadic group geometries have received considerable attention — some in the form of characterization theorems, some more concerned with delving into geometric properties of particular geometries. However, compared to the chamber graph of a building, there has been very little work on the chamber graphs of the chamber systems associated with the sporadic group geometries.

In this paper we gather, numerical data concerning chamber graphs for a variety of sporadic group geometries, including minimal parabolic geometries [16], maximal \( p \)-local geometries [15], Petersen geometries [8, 9], GABs [10] and various Buekenhout geometries [4]. All the geometries we consider will come equipped with a flag transitive automorphism group \( G \) and we will usually study the chamber graph via \( C(G; B, (P_i)) \). Moreover we will only be studying connected chamber graphs (this is equivalent to the condition \( G = \langle P_i \mid i \in I \rangle \)). We will mostly examine rank 3 and 4 geometries, though we also include one or two “notorious” rank 2 systems.

Before proceeding further, we need some notation. For \( c_0 \) a fixed chamber of a chamber system \( C \), \( D_i(c_0) \) (\( i \in \mathbb{N} \)) is the set of chambers at distance \( i \) from \( c_0 \) in the chamber graph of \( C \). We shall call \( D_i(c_0) \) the \( i \)th disc (of \( c_0 \)).

Many ideas and results concerning geometries have taken buildings as their inspiration. So let us pause for a moment and consider the chamber graph of \( C \) where \( C \) is the chamber system associated with the building which arises from a finite group \( G \) of Lie type over \( GF(q) \). Let \( c_0 \) be a fixed chamber of \( C \). Now \( c \in D_i(c_0) \) if and only if \( \delta(c_0, c) = w \) where \( w \) is an element of Weyl group \( W \) of \( G \) and the length of \( w \) in \( W \) is \( i \). (\( \delta \) is the \( W \)-distance function — see [14, Chapter 3] for further details of this approach to buildings.) For \( w \in W \), \( U_w \) acts simply transitively on the set of chambers such that \( \delta(c_0, c) = w \). (\( U_w \) is a certain subgroup of \( B = \text{Stab}_G c_0 \) and \( |U_w| = q^{\ell(w)} \) — again see [14, pp. 75,76]). Since

\[
D_i(c_0) = \bigcup_{w \in W, \ell(w) = i} \{ c \mid \delta(c_0, c) = w \},
\]

the number of chambers in \( D_i(c_0) \) is

\[
q^{\ell(w)} \times \text{(size of the } i \text{th disc in the chamber system for } W \text{).}
\]

The diameter \( d \) of \( C \) is the Coxeter number of \( W \) and \(|D_d(c_0)| = |U|\) where \( U \) is the unipotent radical of \( B \). (This is because there is a unique \( w_0 \in W \) with
\( \ell (v_0) = d \) and \( U_{v_0} = U \). So in particular, we have that \( B \) acts transitively on \( D_2(c_0) \). In the chamber systems analyzed in this paper, this property is rarely observed. However, there are some interesting instances when this property does occur — for example in the \( M_{24} \) maximal 2-local geometry [17].

So, looking at the building case, we see that the sizes of discs, particularly the last disc and the diameter of the chamber graph of a sporadic group geometry are potentially interesting pieces of information relating to the group and the geometry. It is these features of the chamber graph that we focus upon here. Much of the data has been obtained using MAGMA [5] and extends to chamber systems with up to about 400,000 chambers.

The aim of this exercise in data collection is to highlight those geometries deserving of further detailed study. Indeed, in [17], combinatorial descriptions of the discs for the \( M_{24} \) maximal 2-local geometry are obtained by hand — the sizes of the discs agree with those given here in section 2.22 (Geometry 1)!

Section 2 tabulates the disc sizes of various geometries together with some additional observations. The geometries we study are described either in terms of some combinatorial structure or by means of an appropriate diagram [3, 4]. For a rank \( n \) geometry we shall take \( I = \{0, 1, \ldots, n-1\} \). We use \( G_{i_1, \ldots, i_r} \), where \( \{i_1, \ldots, i_r\} \subseteq I \), to denote the stabilizer in \( G \) of a flag of type \( \{i_1, \ldots, i_r\} \), and put \( B = G_0 \cap G_1 \cap \ldots \cap G_{n-1} \). Since we utilize the group in our calculations we give \( G_{i_1, \ldots, i_r} \) for all subsets \( \{i_1, \ldots, i_r\} \) of \( I \).

Throughout we use the Atlas [6] conventions and terminology when describing groups except that we use \( \text{Dih}(n) \), \( \text{Sym}(n) \) and \( \text{Alt}(n) \) to denote, respectively, the dihedral group of order \( n \), the symmetric group and alternating group of degree \( n \). Thus we shall (usually) only describe the groups \( G_{i_1, \ldots, i_r} \) up to “shape”.

In section 3 we give some hand calculations for the \( \text{Alt}(7) \)-geometry. This geometry, over the years has attracted a good deal of attention [12, 13, 14]. These calculations uncover the structure of the last disc — the chamber graph has diameter 5 and \( D_5(c_0) \) consists of 104 chambers.
2 Disc structures

2.1 Group $G = L_2(11)$

| $G$ | $|G| = 2^2 \cdot 3 \cdot 5 \cdot 11 = 660$ |
|-----|-------------------------------------|
| GEOMETRY: Petersen Geometry | Number of Chambers: 330 |
| Diameter: 9 |

$G_0 \cong \text{Alt}(5), G_1 \cong \text{Dih}(12) \cong G_2,$
$G_{01} \cong \text{Sym}(3), G_{02} \cong 2^2 \cong G_{12},$
$B \cong 2$ (see [8, p. 944]).

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2.2 Group $G = \hat{S}_6(\cong 3\cdot \text{Sym}(6))$

| $G$ | $|G| = 2^4 \cdot 3^3 \cdot 5 = 2,160$ |
|-----|-------------------------------------|
| GEOMETRY: 3-fold cover of the | Number of Chambers: 135 |
| $\text{Sp}_4(2)$-quadrangle | Diameter: 8 |

$G_0 \cong 2^3\text{Sym}(3) \cong G_1,$
$B \cong \text{Dih}(8) \times 2.$

This geometry appears as a residue in the minimal parabolic geometries for $M_{24},$
$\cdot 1, M, \text{He}$ and $\text{Fi}'_{24}$ — see [16].

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2.3 Group $G = \text{Alt}(7)$

| $G$ | $|G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2,520$ |
|-----|-------------------------------------|

1. **Geometry:**

   **Number of Chambers:** 315  
   **Diameter:** 8

   ![Diagram](image)

   \[ G_0 \cong \text{Sym}(4) \cong G_1, \]
   \[ B \cong \text{Dih}(8). \]

   Biduads \((ab)(cd)\) and triduads \((ab)(cd)(ef)\) are unordered pairs and triples of disjoint duads of a 7-element set. This rank 2 geometry also appears as a residue in a number of sporadic geometries; see [16].

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<td>56</td>
<td>72</td>
<td>98</td>
<td>28</td>
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2. **Geometry:**

   **Number of Chambers:** 315  
   **Diameter:** 5

   \[ C_3\text{-Geometry for } \text{Alt}(7) \]

   ![Diagram](image)

   \[ G_0 \cong \text{Alt}(6), \ G_1 \cong (3 \times \text{Alt}(4))2, \ G_2 \cong \text{L}_3(2), \]
   \[ G_{01} \cong G_{02} \cong G_{12} \cong \text{Sym}(4), \]
   \[ B \cong \text{Dih}(8). \]

   The points are the points of a 7-element set \( \Omega \), the lines are all 3-element subsets of \( \Omega \) and the planes are one \( \text{Alt}(7) \)-orbit of \( \text{PG}(2, 2) \) on \( \Omega \). See [16] and [12]. This geometry receives further attention in section 3.

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<td>6</td>
<td>20</td>
<td>56</td>
<td>128</td>
<td>104</td>
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3. **Geometry:**

   **Number of Chambers:** 315  
   **Diameter:** 5

   This chamber system is something of a hybrid of the chamber system in Example 2. Starting with the chamber system \( \mathcal{C} \) of the \( \text{Alt}(7) \) \( C_3 \)-geometry, we choose a fixed \( \mu \in \text{Sym}(7) \setminus \text{Alt}(7) \) and then define two chambers \( c, d \) of \( \mathcal{C} \) to be \( 0' \)-adjacent if \( c\mu \) and \( d\mu \) are \( 0 \)-adjacent. Together with the other \( 0-, 1-, 2 \)-adjacencies of \( \mathcal{C} \), this delivers a rank 4 chamber system with diagram
Disc structure of certain chamber graphs

\[ B \cong \text{Dih}(8), \]
\[ P_i \cong \text{Sym}(4) \text{ for } i \in \{0, 0', 1, 2\}. \]

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<td>26</td>
<td>88</td>
<td>120</td>
<td>72</td>
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4. GEOMETRY: Petersen Geometry

\[ \text{Number of Chambers: } 630 \]
\[ \text{Diameter: } 11 \]

\[ G_0 \cong \text{Sym}(5), G_1 \cong (3 \times 2^2)2, G_2 \cong \text{Sym}(4), \]
\[ G_{01} \cong \text{Dih}(12), G_{02} \cong \text{Dih}(8) \cong G_{12}, \]
\[ B \cong 2^2 \text{ (see [8, p. 945]).} \]

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<td>144</td>
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5. GEOMETRY: Chamber system of type \( A_2 \)

\[ \text{Number of Chambers: } 2,520 \]
\[ \text{Diameter: } 9 \]

\[ B = 1, \]
\[ P_i \cong 3, i \in \{0, 1, 2\}. \text{ (See [14, p. 53].)} \]

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Remark 2.1. Taking \( P_0 = \langle (123)(456) \rangle, P_1 = \langle (124)(375) \rangle \) and \( P_2 = \langle (153)(276) \rangle \) (and noting that the chambers are just the elements of \( \text{Alt}(7) \) and \( c_0 = 1 \)), \( D_9(c_0) \) looks as follows:
The labels on the edges indicate the $i$-adjacency.

2.4 Group $G = \text{Sym}(7)$

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<thead>
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<th>$G$</th>
<th>$\text{Sym}(7)$</th>
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<tr>
<td>$</td>
<td>G</td>
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<td>GeometrY: Number 17 of [4]</td>
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<tr>
<td>Number of Chambers: 840</td>
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<td>Diameter: 10</td>
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The type 0 objects are the elements of a 7-element set $\Omega$ and the objects of type 1, 2, 3 are, respectively, all the 2-, 3- and 4-element subsets of $\Omega$.

$G_0 \cong \text{Sym}(6)$, $G_1 \cong 2 \times \text{Sym}(5)$, $G_2 \cong \text{Sym}(3) \times \text{Sym}(4) \cong G_3$,

$G_{01} \cong \text{Sym}(5)$, $G_{02} \cong 2 \times \text{Sym}(4) \cong G_{12}$,

$G_{03} \cong \text{Sym}(3) \times \text{Sym}(3) \cong G_{23}, G_{13} \cong 2^2 \times \text{Sym}(3)$,

$G_{012} \cong \text{Sym}(4)$, $G_{013} \cong G_{023} \cong G_{123} \cong 2 \times \text{Sym}(3)$,

$B \cong \text{Sym}(3)$.

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2.5 Group $G = L_2(25)$

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<th>$\text{L}_2(25)$</th>
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<td>$</td>
<td>G</td>
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<tr>
<td>GeometrY: Petersen Geometry</td>
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<td>Number of Chambers: 1,950</td>
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<td>Diameter: 18</td>
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$G_0 \cong \text{Sym}(5)$, $G_1 \cong \text{Dih}(24)$, $G_2 \cong \text{Sym}(4)$,

$G_{01} \cong \text{Dih}(12)$, $G_{01} \cong G_{12} \cong \text{Dih}(8)$,

$B \cong 2^2$. (See [8, p. 944].)
2.6 Group $G = M_{11}$

$|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11 = 7,920$

1. GEOMETRY: Number 27 of [4].

Number of Chambers: 2,640
Diameter: 11

Considering $M_{11}$ acting 3-transitively on the 12-element set $\Omega$, we may describe the geometry thus. The objects of type 0 and 1 are, respectively, all 1- and 2- subsets of $\Omega$; those of type 2 are 3-element subsets of $\Omega$ of the form $\text{Fix}_\Omega(g)$ where $g$ is an element of order 3 in $M_{11}$ and those of type 3 are one “half” of a total (that is, a 6-element subset of the 6 | 6 partition).

Incidence is symmetized containment.

$G_0 \cong L_2(11)$, $G_1 \cong \text{Sym}(5)$, $G_2 \cong 3(\text{Sym}(3) \times 2)$, $G_3 \cong \text{Alt}(6)$,
$G_{01} \cong \text{Alt}(5)$ $G_{03} \cong G_{02} \cong 2 \times \text{Sym}(3) \cong G_{12}$, $G_{13} \cong \text{Sym}(4)$,
$G_{23} \cong 3^2 : 2$,
$G_{012} \cong G_{023} \cong G_{123} \cong \text{Sym}(3)$, $G_{013} \cong \text{Alt}(4)$,
$B \cong 3$.

2. GEOMETRY: Number of Chambers: 3,960
Diameter: 10

Regarding $M_{11}$ as the stabilizer of the $S(12,6,5)$ Steiner system on a 12-element set $\Omega$ and an element $\alpha$ of $\Omega$, the geometry may be described
as follows. The objects of type 0,1,2 are, respectively, all 1-, 2- and 3-
element subsets of $\Omega \setminus \{\alpha\}$ and those of type 3 all the hexads of $S(12,6,5)$
containing $\alpha$. See [13, pp. 72 and 94].

\[ G_0 \cong M_{10}, \ G_1 \cong M_9 : 2, \ G_2 \cong 2 \cdot \text{Sym}(4), \ G_3 \cong \text{Sym}(5), \]
\[ G_{01} \cong 3^2 : Q_8, \ G_{02} \cong \text{SDih}(16) \cong G_{12}, \ G_{03} \cong \text{Sym}(4), \]
\[ G_{13} \cong 2 \cdot \text{Sym}(3) \cong G_{23}, \ G_{012} \cong Q_8, \ G_{013} \cong \text{Sym}(3), \ G_{023} \cong 2^2 \cong G_{123}, \]
\[ G_{012} \cong \text{Q}_8, \ G_{013} \cong \text{Sym}(3), \ G_{023} \cong 2 \cdot \text{Sym}(3), \]
\[ G_{023} \cong 2 \cdot \text{Sym}(3), \ G_{123} \cong 2 \cdot \text{Dih}(8), \]
\[ G_{123} \cong 2 \cdot \text{Sym}(8). \]

(See [8, p. 954].)

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3. **GEOMETRY: Petersen Geometry**

NUMBER OF CHAMBERS: 3,960

DIAMETER: 15

![Petersen Graph]

Again starting with $M_{11}$ acting 3-transitively on a 12-element set $\Omega$, we take as our objects of type 0, 1, 2 to be, respectively, all 1-, 2- and 3- subset of $\Omega$ and objects of type 3 to be 4 subsets of the form $\text{Fix}_3(g)$ where $g$ is an involution of $M_{11}$. Incidence being symmeterized inclusion.

\[ G_0 \cong L_2(11), \ G_1 \cong \text{Sym}(5), \ G_2 \cong 3(\text{Sym}(3) \times 2), \]
\[ G_3 \cong 2 \cdot \text{Sym}(4), \ G_{01} \cong \text{Alt}(5), \ G_{02} \cong G_{03} \cong G_{12} \cong G_{23} \cong 2 \times \text{Sym}(3), \]
\[ G_{13} \cong \text{Dih}(8), \ G_{012} \cong \text{Sym}(3), \ G_{013} \cong G_{023} \cong G_{123} \cong 2^2, B \cong 2. \]

(See [8, p. 954].)

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</table>
2.7 Group $G = A_8$

**Geometry:**

$|G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7 = 20,160$

Number of Chambers: 2,520

Diameter: 8

Let $\Omega$ be an 8-element set. Then the objects of type 0,1,2 are, respectively the elements, duads and $4^2$ partitions of $\Omega$. The 35 $4^2$ partitions of $\Omega$ may be identified with the lines of projective 3-space (see [13, Proposition 1]). Objects of type 3 are the points of the projective 3-space which may be identified with a set of seven $2^4$ partitions of $\Omega$ (there are $105 = 7 \times 15$ $2^4$ partitions of $\Omega$). These seven $2^4$ partitions may also be viewed as the non-identity elements of $O_2(G_3)$.

For the definition of incidence and more details see [12, Section 3].

$G_0 \cong \text{Alt}(7)$, $G_1 \cong \text{Sym}(6)$, $G_2 \cong (\text{Alt}(4) \times \text{Alt}(4)) : 2^2$,

$G_3 \cong 2^3 : \text{L}_3(2)$,

$G_{01} \cong \text{Alt}(6)$, $G_{02} \cong (3 \times \text{Alt}(4)) : 2$, $G_{03} \cong \text{L}_3(2)$,

$G_{12} \cong 2^4 : \text{Sym}(3)$, $G_{23} \cong 2^4 : \text{Sym}(4)$, $G_{13} \cong \text{Sym}(4) \times 2$,

$G_{012} \cong G_{013} \cong G_{023} \cong \text{Sym}(4)$, $G_{123} \cong \text{Dih}(8) \times 2$,

$B \cong \text{Dih}(8)$.

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2.8 Group $G = U_4(2)$

**Geometry:** Example 6 in [11, Table 4]

$|G| = 2^6 \cdot 3^4 \cdot 5 = 25,920$

Number of Chambers: 25,920

Diameter: 12

For each $i \in I$, $G_i \cong 3$ for $J = I \setminus \{i\}$ and $B = 1$. (See [11] for the other stabilizers.)
2.9 Group $G = \text{M}_{12}$

$|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11 = 95,040$

1. **Geometry:** Number 5 of [4]  

   **Number of Chambers:** 1,320  
   **Diameter:** 6

   ![Diagram](image)

   With $G$ acting (5-transitively) on the 12-element set $\Omega$, we take all 1-, 2- and 3-sets of $\Omega$ to be the objects of type $0, 1, 2$ of the geometry respectively.

   $G_0 \cong \text{M}_{11}$, $G_1 \cong \text{M}_{10} : 2$, $G_2 \cong 3^2 : 2\text{Sym}(4)$,  
   $G_{01} \cong \text{M}_{10}$, $G_{02} \cong 3^2 : \text{SDih}(16) \cong G_{12}$,  
   $B \cong 3^2 : Q_8$.

   ![Diagram](image)

2. **Geometry:**  

   **Number of Chambers:** 47,520  
   **Diameter:** 15

   ![Diagram](image)

   With $\Omega$ a 12-element set, the objects of type $0, 1, 2, 3$ and 4 are, respectively 1-, 2-, 3-, 4-subsets of $\Omega$ and the hexads of the Steiner system $S(12, 6, 5)$. See [13, pp. 72 and 94].

   $G_0 \cong \text{M}_{11}$, $G_1 \cong \text{Alt}(6) \cong 2^2 \cong G_4$, $G_2 \cong 3^2 : 2\text{Sym}(4)$, $G_3 \cong 2_4^{1+4} : \text{Sym}(3)$,  
   $G_{0123} \cong Q_8$, $G_{0124} \cong \text{Sym}(3)$, $G_{0134} \cong G_{0234} \cong G_{1234} \cong 2^2$, $B \cong 2$. 

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Disc structure of certain chamber graphs

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2.10 Group $G = U_3(5)$

$|G| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7 = 126,000$

1. Geometry:

Number of Chambers: 15,750
Diameter: 10

See [7] and [12].

$G_0 \cong G_1 \cong G_2 \cong G_3 \cong \text{Alt}(7), G_{01} \cong G_{23} \cong \text{Alt}(6),$  
$G_{02} \cong G_{13} \cong (3 \times \text{Alt}(4)) : 2, G_{03} \cong G_{12} \cong L_3(2),$  
$G_{012} \cong G_{013} \cong G_{023} \cong G_{123} \cong \text{Sym}(4), B \cong \text{Dih}(8).$

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2. Geometry:

Number of Chambers: 15,750
Diameter: 10

See [7] and [19].

$G_0 \cong G_1 \cong G_2 \cong G_3 \cong \text{Alt}(7), G_{01} \cong G_{03} \cong L_3(2),$  
$G_{02} \cong G_{13} \cong (3 \times \text{Alt}(4)) : 2, G_{12} \cong L_3(2), G_{23} \cong \text{Alt}(6),$  
$G_{012} \cong G_{013} \cong G_{023} \cong G_{123} \cong \text{Sym}(4), B \cong \text{Dih}(8).$
2.11 Group $G = J_1$

GEOMETRY: Number 28 of [4]  

Number of Chambers: 29,260  
Diameter: 12

\[ G_0 \cong L_2(11), \quad G_1 \cong 2 \times \text{Alt}(5), \quad G_2 \cong \text{Sym}(3) \times \text{Dih}(10), \]
\[ G_{01} \cong \text{Alt}(5), \quad G_{02} \cong G_{12} \cong 2 \times \text{Sym}(3), \]
\[ B \cong \text{Sym}(3). \]

\[
\begin{array}{ccccccccccc}
\text{DISC} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{SIZE} & 8 & 40 & 168 & 624 & 1840 & 4628 & 6776 & 1620 & 44 & 1 \\
\end{array}
\]

2.12 Group $G = \text{Alt}(9)$

GEOMETRY: Petersen Geometry  

Number of Chambers: 22,680  
Diameter: 18

\[ G_0 \cong \text{Sym}(7), \quad G_1 \cong 2 \times \text{Sym}(5), \quad G_2 \cong \text{Sym}(4) \times \text{Sym}(3), \quad G_3 \cong 2^3 \text{Sym}(4), \]
\[ |G_{012}| = 2^3, |G_{013}| = |G_{023}| = |G_{123}| = 2^4, \]
\[ |B| = 2^3. \]
2.13 Group $G = M_{22}$

$|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 443,520$

1. Geometry:

Number of Chambers: 3,465
Diameter: 5

If $\Omega$ is a 22-element set upon which $G$ acts transitively, then the objects of type 0 and 1 are, respectively, the two element subsets (duads) of $\Omega$ and the triduads of $\Omega$ (that is, $2^3$ partition of hexads of the Steiner system $S(22,3,6)$ on $\Omega$). This is the minimal parabolic geometry for $M_{22}$ (see [16]).

$G_0 \cong 2^4 : \text{Sym}(5)$, $G_1 \cong 2^6 \text{Sym}(3)$, $G_{01} \cong 2^4 : \text{Dih}(8)$.

2. Geometry:

Number 43 of [4]
Number of Chambers: 2,310
Diameter: 6

Assuming $\Omega$ is as in Geometry 1 above, the objects of type 0, 1, 2 are respectively the elements, duads and hexads of $\Omega$.

$G_0 \cong L_3(4)$, $G_1 \cong 2^4 : \text{Sym}(5)$, $G_2 \cong 2^4 : \text{Alt}(6)$, $G_{01} \cong G_{02} \cong 2^4 : \text{Alt}(5)$, $G_{12} \cong 2^4 : \text{Sym}(4)$, $B \cong 2^4 : \text{Alt}(4)$.
2.14 Group $G = \text{Aut} \ M_{22}$

| $|G|$ = $2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 887,040$ |

**Geometry:**

Petersen Geometry

- Number of Chambers: 6,930
- Diameter: 13

It is convenient to describe this geometry by beginning with a 24-element set $\Omega$, assumed to be equipped with the Steiner system $S(24, 8, 5)$. Now fix a duad $D$ (2-element subset) of $\Omega$. Then $\text{Stab}_{M_{22}} D \cong \text{Aut} \ M_{22}$ and objects of type 0, 1, 2 of the geometry are, respectively, all octads in $\Omega \setminus D$, all trios which have $D$ contained in one of its octads and all sextets which have $D$ contained in one of its tetrads. Incidence being given by compatibility of partitions (see [9]).

\[ G_0 \cong (2^3 : L_3(2)) \times 2, \quad G_1 \cong 2^{1+4}(2^2 \times \text{Sym}(3)), \]
\[ G_2 \cong 2^5 : \text{Sym}(5), \quad G_{01} \cong G_{02} \cong (2^3 : \text{Sym}(4)) \times 2, \]
\[ G_{12} \cong 2^5 : \text{Dih}(8), \quad B \cong (2^3 : \text{Dih}(8)) \times 2. \]

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<tbody>
<tr>
<td>1808</td>
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**Remark 2.2.** $D_{13}(e_0)$ is a $B$-orbit.

2.15 Group $G = 3M_{22}$

| $|G|$ = $2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 1,330,560$ |

**Geometry:**

- Number of Chambers: 10,395
- Diameter: 8

This is a 3-fold cover of the minimal parabolic geometry for $M_{22}$ (see section 2.13).

\[ G_0 \cong 2^1 : \text{Sym}(5), \quad G_1 \cong 2^6 \text{Sym}(3), \quad B \cong 2^4 : \text{Dih}(8). \]
Disc structure of certain chamber graphs

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2.16 Group $G = 3M_{22}2$

Geometry:

Number of Chambers: 20,790
Diameter: 24

A 3-fold cover of the Petersen geometry given in section 2.14 (see also [9]).

$G_0 \cong (2^3 : L_3(2)) \times 2$, $G_1 \cong 2^{1+4}(2^2 \times \text{Sym}(3))$,
$G_2 \cong 2^6 : \text{Sym}(5)$, $G_{01} \cong G_{02} \cong (2^3 : \text{Sym}(4)) \times 2$,
$G_{12} \cong 2^5 : \text{Dih}(8)$, $B \cong (2^3 : \text{Dih}(8)) \times 2$.

Remark 2.3. Note that the sizes here of $D_i(c_0)$ for $1 \leq i \leq 11$ coincide with those in section 2.14.

2.17 Group $G = G_2(3)$

$|G| = 2^6 \cdot 3^6 \cdot 7 \cdot 13 = 4,245,696$

1. Geometry:

Number of Chambers: 66,339
Diameter: 13

See [1].
\[ G_0 \cong 2^3 : L_3(2), \ G_1 \cong 2^{1+4} : 3^2.2, \]
\[ G_3 \cong G_2(2), \ G_{01} \cong G_{12} \cong 2^5.\text{Sym}(3), \]
\[ B \cong 2^3 \text{Dih}(8). \]

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\[ 11 \ 12 \ 13 \]
\[ 21248 \ 6976 \ 64 \]

2. GEOMETRY:

**NUMBER OF CHAMBERS:** 66,339

**DIAMETER:** 12

\[
\begin{align*}
0 & \quad 1 & \quad 2 \\
\end{align*}
\]

See [1].

\[ G_0 \cong G_2(2) \cong G_2, \ G_1 \cong 2^{1+4} : 3^2.2, \]
\[ G_{01} \cong G_{02} \cong G_{12} \cong 2^5.\text{Sym}(3), \]
\[ B \cong 2^3 \text{Dih}(8). \]

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\[ 11 \ 12 \]
\[ 9472 \ 896 \]

2.18 Group \( G = U_4(3)2 \)

\[ |G| = 2^5 \cdot 3^5 \cdot 5 \cdot 7 = 6,531,840 \]

**GEOMETRY:**

**NUMBER OF CHAMBERS:** 25,515

**DIAMETER:** 10

\[
\begin{align*}
0 & \quad 1 & \quad 2 \\
\end{align*}
\]

This geometry is an example of a GAB — see [10, Section 3] for details.

\[ G_0 \cong 2^4.\text{Sym}(6), \ G_1 \cong [2^6].(\text{Sym}(3) \times \text{Sym}(3)), \ G_2 \cong 2^5.\text{Alt}(6), \]
\[ G_{01} \cong G_{02} \cong G_{12} \cong [2^6].\text{Sym}(3), \]
\[ B \cong 2^4.(\text{Dih}(8) \times 2). \]
2.19 Group $G = U_5(2)$

**Disc Structure:**

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**Geometric Properties:**

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Number 20 of [4] (with $n = 5$).

$G_0 \cong 3 \times U_4(2), \ G_1 \cong \text{Sym}(3) \times (3^{1+2} : SL_2(3)), \ G_2 \cong 3^4 \cdot \text{Sym}(5),$

$G_{01} \cong 3^4 \cdot \text{SL}_2(3), \ G_{02} \cong 3^4 \cdot \text{Sym}(4), \ G_{12} \cong 3^5 : 2^4,$

$B \cong 3^5 : 2.$

2.20 Group $G = M_{23}$

**Disc Structure:**

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**Geometric Properties:**

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<td>36032</td>
<td>29056</td>
</tr>
</tbody>
</table>

**Remarks:**

- $G_0 \cong M_{22}, \ G_1 \cong 2^4(3 \times \text{Alt}(5))2, \ G_2 \cong 2^4 \text{L}_3(2),$

$G_{01} \cong 2^6 \text{Sym}(5), \ G_{02} \cong 2^6 \text{Sym}(4) \cong G_{12},$

$B \cong 2^4 \text{Dih}(8).$

**Remark 2.4.** $M_{23}$ has two (non-isomorphic) minimal parabolic geometries which are locally isomorphic (meaning all their residues are isomorphic). Globally they differ with the choice of an $\text{L}_3(2)$-conjugacy class.
within $\text{Alt}(7)$ — so producing two possible choices for $G_2$ contained in $H = 2^4\text{Alt}(7)$ (the stabilizer in $M_{23}$ of a heptad). In one case the $L_3(2)$ leaves a 1-space (of $O_2(H)$) invariant and in the other a 3-space (of $O_2(H)$); the former we refer to as the “1-geometry” and the latter, dealt with next, the “3-geometry”.

2. **Geometry:**

<table>
<thead>
<tr>
<th>Disc</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>18</td>
<td>92</td>
<td>664</td>
<td>3104</td>
<td>10728</td>
<td>36544</td>
<td>28544</td>
</tr>
</tbody>
</table>

**Remark 2.5.** The disc sizes of the 1-geometry and the 3-geometry differ only in discs 6 and 7 — the 1-geometry has 512 fewer chambers in the sixth disc and 512 more in the seventh disc (than the 3-geometry).

3. **Geometry:**

Number 44 of [4].

$G_0 \cong M_{22}$, $G_1 \cong L_3(4)2$, $G_2 \cong 2^4 : (3 \times \text{Alt}(5)) : 2$,

$G_3 \cong 2^4 \cdot \text{Alt}(7)$, $G_{012} \cong G_{013} \cong 2^4\text{Alt}(5)$, $G_{023} \cong G_{123} \cong 2^4\text{Sym}(4)$,

$B \cong 2^4\text{Alt}(4)$.

<table>
<thead>
<tr>
<th>Disc</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
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<th>10</th>
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</thead>
<tbody>
<tr>
<td>Size</td>
<td>10</td>
<td>54</td>
<td>201</td>
<td>560</td>
<td>1552</td>
<td>3392</td>
<td>5376</td>
<td>9216</td>
<td>16384</td>
<td>16384</td>
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</table>

4. **Geometry:**

**Petersen Geometry**

<table>
<thead>
<tr>
<th>Disc</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>10</td>
<td>54</td>
<td>201</td>
<td>560</td>
<td>1552</td>
<td>3392</td>
<td>5376</td>
</tr>
</tbody>
</table>

**Petersen Geometry**

Number of Chambers: 159,390

Diameter: 14
Disc structure of certain chamber graphs

\[ G_0 \cong \text{Alt}(8), \ G_1 \cong 2^4(L_3(2) \times 2), \ G_2 \cong 2^4(3 \times \text{Alt}(5))2, \]
\[ G_3 \cong \text{M}_{22}, \]
\[ G_{012} \cong G_{013} \cong G_{023} \cong 2^3: \text{Sym}(4), \ G_{123} \cong 2^4: \text{Dih}(8), \]
\[ B \cong 2^3: \text{Dih}(8). \]

<table>
<thead>
<tr>
<th>DISC</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIZE</td>
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<td>28</td>
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<td>220</td>
<td>512</td>
<td>1128</td>
<td>2432</td>
<td>5152</td>
<td>10528</td>
<td>21024</td>
</tr>
</tbody>
</table>

<table>
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<tr>
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<th>12</th>
<th>13</th>
<th>14</th>
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<tbody>
<tr>
<td></td>
<td>38528</td>
<td>51840</td>
<td>26304</td>
<td>1600</td>
</tr>
</tbody>
</table>

2.21 Group \( G = 3U_4(3)2 \)

\[ |G| = 2^8 \cdot 3^7 \cdot 5 \cdot 7 = 19,595,520 \]

**Geometry:**

Number of Chambers: 76,545

Diameter: 11

This geometry is a triple cover of the geometry in section 2.18.

\[ G_0 \cong 2^4: \text{Sym}(6), \ G_1 \cong [2^6](\text{Sym}(3) \times \text{Sym}(3)), \ G_2 \cong 2^5.3\text{Alt}(6), \]
\[ G_{01} \cong G_{02} \cong G_{12} \cong [2^7]\text{Sym}(3), \ B \cong 2^4(\text{Dih}(8) \times 2). \]

<table>
<thead>
<tr>
<th>DISC</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIZE</td>
<td>6</td>
<td>20</td>
<td>64</td>
<td>192</td>
<td>528</td>
<td>1424</td>
<td>3848</td>
<td>9658</td>
<td>19812</td>
<td>27680</td>
</tr>
</tbody>
</table>

| SIZE | 11 | 13312 |

2.22 Group \( G = \text{M}_{24} \)

\[ |G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244,823,040 \]

1. **Geometry:**

Number of Chambers: 79,695

Diameter: 10

Maximal 2-local geometry (see [15])
\[ G_0 \cong 2^4 : \text{Alt}(8), \quad G_1 \cong 2^6 : (L_3(2) \times \text{Sym}(3)), \quad G_2 \cong 2^6 : (3 \cdot \text{Sym}(6)), \]
\[ G_{01} \cong 2^6 : (L_3(2) \times 2), \quad G_{02} \cong 2^6 : (3 \cdot (\text{Sym}(4) \times 2)) \cong G_{12}, \]
\[ B \cong [2^9] : \text{Sym}(3). \]

\begin{verbatim}
<table>
<thead>
<tr>
<th>DISC</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>9</th>
<th>10</th>
</tr>
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<tbody>
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<td>544</td>
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<td>22272</td>
<td>38400</td>
<td>1536</td>
<td></td>
</tr>
</tbody>
</table>
\end{verbatim}

**Remark 2.6.** \(B\) is transitive on \(D_{10}(c_0)\) (see [17]).

2. **Geometry:**

- Minimal parabolic geometry (see [16])

\[ G_0 \cong 2^4 : 2^3 : L_3(2), \quad G_1 \cong 2^7 : (\text{Sym}(3) \times \text{Sym}(3)), \]
\[ G_2 \cong 2^6 : (3 \cdot \text{Sym}(6)), \]
\[ G_{01} \cong G_{02} \cong G_{23} \cong [2^9] : \text{Sym}(3), \]
\[ B \cong 2^6 : (\text{Dih}(8) \times 2). \]

\begin{verbatim}
<table>
<thead>
<tr>
<th>DISC</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
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<td>20</td>
<td>56</td>
<td>144</td>
<td>368</td>
<td>848</td>
<td>1800</td>
<td>3810</td>
<td>8040</td>
<td>16920</td>
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</table>
\end{verbatim}

\begin{verbatim}
<table>
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<tr>
<th></th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIZE</td>
<td>32832</td>
<td>55200</td>
<td>62336</td>
<td>47616</td>
<td>6656</td>
<td>2048</td>
<td>384</td>
</tr>
</tbody>
</table>
\end{verbatim}

3. **The last disc of the \textbf{-} Alt(7)-geometry.**

Let \(G = \text{Alt}(7)\) act upon the set \(\Omega = \{1, 2, 3, 4, 5, 6, 7\}\) and let \(\Gamma\) be the \text{Alt}(7)-geometry with diagram

\[ 0 \quad 1 \quad 2 \]

points lines planes

whose description we now give. The points, lines and planes of \(\Gamma\) (that is, objects of type 1, 2 and 3) are, respectively, the elements of \(\Omega\), all the 3-element subsets of \(\Omega\) and one \(A_7\)-orbit of projective planes defined on \(\Omega\). (So there are
7 points, 35 lines and 15 planes.) For a point \( p \), a line \( l \) and a plane \( P \), \( p \ast l \)
whenever \( p \in l \), \( p \ast P \) and whenever \( p \in P \) and \( l \ast P \) whenever \( l \) is a line in the
plane \( P \).

We use the following labelled projective plane \( P \) (where \( \Omega = \{\alpha, \beta, \gamma, \delta, \epsilon, \lambda, \mu\} \))

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\lambda & \delta & \epsilon \\
\mu & & \\
\end{array}
\]

to also stand for the chamber \( \{\alpha\}, \{\alpha, \beta, \gamma\}, P \). So the left-most label of the
bottom line and the bottom line give the incident point and line of the maximal flag. Note that, as a projective plane is defined by its seven lines, that is
3-element subsets,

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & & \\
\end{array}
\]

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & & \\
\end{array}
\]

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & & \\
\end{array}
\]

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & & \\
\end{array}
\]

for example, denote the same chamber. Equally

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & & \\
\end{array}
\]

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & & \\
\end{array}
\]

are two chambers which have the same point and line — they are 3-adjacent.
In the chamber system $C$ obtained from the flag complex of $\Gamma$, two different chambers are 1-adjacent if they have the same line and plane, are 2-adjacent if they have the same point and plane and 3-adjacent if they have the same point and line.

So as to view $C$ from a group theoretic perspective, we set

$$H_1 = \text{Stab}_G \{1\}, \quad H_2 = \text{Stab}_G \{1, 2, 3\},$$

$$B = \langle (23)(67), (23)(4657) \rangle \quad \text{and} \quad H_3 = \langle B, (625)(143) \rangle.$$ 

It is straightforward to check that $H_3$ is the stabilizer in $G$ of the projective plane $P$, and that $B$ is the stabilizer in $G$ of the chamber $c_0 = \{\{1\}, \{1, 2, 3\}, P\}$. If we set $P_1 = H_2 \cap H_3$, $P_2 = H_1 \cap H_3$ and $P_3 = H_1 \cap H_2$, then we obtain the chamber system $(G; B, (P_i))$ over $I = \{1, 2, 3\}$ isomorphic to $C$. We now investigate $D_5(c_0)$, the last disc of $C$; there are 104 chambers in $D_5(c_0)$ (see section 2.3, Geometry 2).

The group $B$ has 13 orbits on $D_5(c_0)$ and acts simply transitively on $D_5(c_0)$; orbit representatives $c_1, \ldots, c_{13}$ are given below.
Disc structure of certain chamber graphs

\[ c_3 \]

\[ c_4 \]

\[ c_5 \]

\[ c_6 \]

\[ c_7 \]

\[ c_8 \]

\[ c_9 \]

\[ c_{10} \]
We next examine the induced subgraph (from the chamber graph) on $D_5(c_0)$. We name the elements of $B$ as follows:

$x_1 = (1), \quad x_2 = (23)(67), \quad x_3 = (47)(65), \quad x_4 = (23)(4657), \quad x_5 = (46)(57), \quad x_6 = (23)(4756), \quad x_7 = (45)(67), \quad x_8 = (23)(45)$.

In Table 1, the number $j$ in brackets after each chamber indicates that the chamber is $j$-adjacent to $c_i$.

**Remark 3.1.** Using the action of elements of $B$, from Table 1 we may obtain the edge set for the induced graph on $D_5(c_0)$.

**Theorem 3.2.** The induced subgraph on $D_5(c_0)$ is a connected graph.

**Proof.** Let $E$ denote the connected component of $c_8$ (see p. 93) in the subgraph $D_5(c_0)$. Also put $E = \text{Stab}_B(E)$. From 1, $c_8$ and $c_8(23)(67)$ are 3-adjacent and so $c_8(23)(67) \in E$. Hence $(23)(67) \in E$. Also, by Table 1, $c_{11}$ is 2-adjacent to $c_8$ and $c_{11}$ is 2-adjacent to...
Disc structure of certain chamber graphs

Table 1: Adjacency of chambers in $D_5(c_0)$

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>Chambers in $D_5(c_0)$ adjacent to $c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$c_3x_8(1); c_9x_8(2); c_{11}x_8(3)$.</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$c_3x_8(3); c_6x_3(3); c_{10}x_4(2)$.</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$c_1x_8(1); c_2x_8(2)$.</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$c_5x_3(3); c_9x_3(1); c_9x_4(3); c_{10}x_7(2)$.</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$c_4x_3(3); c_5x_8(1); c_9x_2(3); c_{12}x_3(2)$.</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$c_2x_3(2); c_8x_8(1)$.</td>
</tr>
<tr>
<td>$c_7$</td>
<td>$c_{13}x_3(1); c_{13}x_5(2)$.</td>
</tr>
<tr>
<td>$c_8$</td>
<td>$c_4x_8(1); c_8x_2(2); c_9x_6(1); c_{11}x_1 = c_{11}(2); c_{13}x_1 = c_{13}(2)$.</td>
</tr>
<tr>
<td>$c_9$</td>
<td>$c_1x_8(2); c_4x_3(1); c_4x_6(3); c_5x_2(3); c_8x_4(1)$.</td>
</tr>
<tr>
<td>$c_{10}$</td>
<td>$c_2x_6(1); c_4x_7(2); c_{10}x_8(3)$.</td>
</tr>
<tr>
<td>$c_{11}$</td>
<td>$c_1x_8(3); c_8x_1 = c_8(2); c_{13}x_1 = c_{13}(2)$.</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>$c_5x_3(2)$.</td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>$c_7x_3(1); c_7x_5(3); c_8x_3(2); c_{11}x_1 = c_{11}(2)$.</td>
</tr>
</tbody>
</table>

Therefore $c_1x_8 \in \mathcal{E}$. Again from Table 1, $c_8$ is 1-adjacent to
Now $c_5x_6$ is 2-adjacent to

$$c_1x_5 = 3 \quad 5 \quad 7 \quad 4 \quad 1 \quad 2 \quad 6$$

So $c_1x_5 \in \mathcal{E}$. Since $(23)(4657)$ sends $c_1x_5$ to $c_1x_8$, $(23)(4657) \in E$. Thus, as $B = \langle (23)(67), (23)(4657) \rangle$, $E = B$. Inspecting Table 1 we see that there is a path in $D_5(c_0)$ from $c_8$ to a chamber in each $c_i^B (i \in \{1, \ldots, 13\})$. Therefore $\mathcal{E} = D_5(c_0)$, and this completes the proof. \hfill $\Box$

References


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