Innovations in Incidence Geometry
Volume 11 (2010), Pages 69–97
ISSN 1781-6475

Disc structure of certain chamber graphs

P. J. Rowley

Abstract

The discs of chamber graphs for group geometries, including certain minimal parabolic geometries, maximal p-local geometries, Petersen geometries, GABs and Buekenhout geometries, are investigated.

Keywords: chamber graphs, group geometries, minimal parabolic geometries, maximal p-local geometries, Buekenhout geometries
MSC 2000: 05E20

1 Introduction

A chamber system over the set I consists of a set $\mathcal{C}$ and a system $(\mathcal{P}_i)_{i\in I}$ of partitions of $\mathcal{C}$ indexed by I. The elements of $\mathcal{C}$ are called chambers and, for brevity, the system $(\mathcal{C}, (\mathcal{P}_i)_{i\in I})$ will often be referred to as the chamber system $\mathcal{C}$. Two chambers which are both in the same member of $\mathcal{P}_j$ for some $j \in I$ are said to be adjacent chambers. The chamber graph of the chamber system $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and two (distinct) chambers are adjacent (in the chamber graph) if they are adjacent chambers in $\mathcal{C}$. (Note that a chamber is not adjacent to itself in the chamber graph.) If two chambers c, c' are both in the same member of $\mathcal{P}_i$ $(i \in I)$, then we say they are i-adjacent, and denote this by $c \sim_i c'$ (or $c' \sim_i c$). The rank of the chamber system is the cardinality of I. An automorphism of $\mathcal{C}$ is a permutation σ of $\mathcal{C}$ which preserves each of the partitions $\mathcal{P}_i$, that is, whenever $c \sim_i c'$ $(c, c' \in \mathcal{C}, i \in I)$ then $c\sigma \sim_i c'\sigma$.

We shall now concentrate on the following situation.

Hypothesis 1.1. *$\mathcal{C}$ is a chamber system over I and $G \leq \operatorname{Aut}\mathcal{C}$ is such that*

(i) *G is transitive on $\mathcal{C}$; and*

(ii) *for each $i \in I$, G is transitive on the members of the partition $\mathcal{P}_i$.*

Suppose Hypothesis 1.1 holds, and let c_0 be a fixed chamber of $\mathcal{C}$. Let B denote the stabilizer in G of c_0, and for each $i \in I$ let P_i be the stabilizer in G of the member of $\mathcal{P}_i$ to which c_0 belongs. Observe that $B \leq \bigcap_{i\in I} P_i$. We may now identify $\mathcal{C}$ with the set of (right) cosets of B in G with, for each $i \in I$, the members of $\mathcal{P}_i$ being the sets of cosets of B which are contained in a coset of P_i. In other words, for chambers Bg and Bh, Bg and Bh are i-adjacent whenever $gh^{-1} \in P_i$. Such a chamber system will be denoted by $\mathcal{C}(G; B, (P_i))$. Further, we note that the valency of the chamber graph of $\mathcal{C}$ is one less than the number of cosets of B in $\bigcup_{i\in I} P_i$ (counting multiplicities if we have $P_i = P_j$, for $i \neq j$).

Conversely, if we start with a group G, a subgroup B of G and a collection of subgroups P_i of G $(i \in I)$ each containing B we may define a chamber system $\mathcal{C}$ by taking the (right) cosets of B as chambers and the partition $\mathcal{P}_i$ to be given by taking right cosets of B contained in a right coset of P_i. Now letting G act by right multiplication on the chambers of $\mathcal{C}$, it is easily checked that Hypothesis 1.1 holds for $\mathcal{C}$ with $G/\operatorname{core}_G B$ playing the role of G.

A rich source of chamber systems is provided by geometries. We recall that a geometry (over the set I) is a triple $(\Gamma, \tau, *)$ where Γ is a set, τ is an onto map from Γ to I and $*$ is a symmetric relation on Γ with the property that for $x, y \in \Gamma$ $x * y$ implies $\tau(x) \neq \tau(y)$. The relation $*$ is called the incidence relation and $x \in \Gamma$ is said to have type i if $\tau(x) = i$. As is customary we shall just say Γ is a geometry. A flag F of Γ is a set of pairwise incident elements of Γ — the type of F, denoted $\tau(F)$, is the set $\{\tau(x) \mid x \in F\}$. The rank of Γ is $|I|$ and the rank of a flag F is $|\tau(F)|(= |F|)$. Now let $\mathcal{F}$ denote the set of maximal flags of Γ — a flag F is maximal if its rank is $|I|$. For $i \in I$ and F, $F' \in \mathcal{F}$ we define F and F' to be i-adjacent if either $F = F'$ or the rank of the flag $F \cap F'$ is $|I| - 1$ and $i \notin \tau(F \cap F')$. This yields a partition $\mathcal{P}_i$ of $\mathcal{F}$; note that a member of $\mathcal{P}_i$ consists of all maximal flags containing some fixed flag of type $I \setminus \{i\}$. So $\mathcal{F}$ is a chamber system — we shall call this the chamber system of Γ. ($\mathcal{F}$ is sometimes referred to as the flag complex of Γ.) An automorphism of the geometry Γ is a permutation σ of Γ for which $x * y$ implies $x\sigma * y\sigma$ and $\tau(x) = i$ implies $\tau(x\sigma) = i$ (where $x, y \in \Gamma$, $i \in I$). Now further suppose that G is a subgroup of $\operatorname{Aut}\Gamma$ with G acting flag transitively on Γ (that is, if F and F' are flags of Γ with $\tau(F) = \tau(F')$, then there exists $g \in G$ such that $Fg = F'$). Then we see that $G \leq \operatorname{Aut}\mathcal{F}$ and that Hypothesis 1.1 holds for G and $\mathcal{F}$. Thus, as discussed earlier, we may study the chamber system $\mathcal{F}$ within G.

Buildings afford an extensive supply of geometries and hence of chamber systems. In fact the theory of buildings may be developed in the language of chamber systems (see [9] and [18] for more on this). In this approach the chamber graph underpins (pun intended) much of the conceptual framework (for example, galleries, connectedness and thin subgeometries). An outgrowth of Tits's pioneering work on buildings was the study of more general geometries — usually ones associated with sporadic simple groups but also those arising from "small" Lie type groups of mixed characteristic. We will, from now on, rather loosely, refer to this mixed bag of geometries as the "sporadic group geometries". This programme was initiated by Buekenhout [2, 3] in the late seventies. Since then sporadic group geometries have received considerable attention — some in the form of characterization theorems, some more concerned with delving into geometric properties of particular geometries. However, compared to the chamber graph of a building, there has been very little work on the chamber graphs of the chamber systems associated with the sporadic group geometries.

In this paper we gather, numerical data concerning chamber graphs for a variety of sporadic group geometries, including minimal parabolic geometries [16], maximal p-local geometries [15], Petersen geometries [8, 9], GABs [10] and various Buekenhout geometries [4]. All the geometries we consider will come equipped with a flag transitive automorphism group G and we will usually study the chamber graph via $\mathcal{C}(G; B, (P_i))$. Moreover we will only be studying connected chamber graphs (this is equivalent to the condition $G = \langle P_i \mid i \in I\rangle$). We will mostly examine rank 3 and 4 geometries, though we also include one or two "notorious" rank 2 systems.

Before proceeding further, we need some notation. For c_0 a fixed chamber of a chamber system $\mathcal{C}$, $D_i(c_0)$ $(i \in \mathbb{N})$ is the set of chambers at distance i from c_0 in the chamber graph of $\mathcal{C}$. We shall call $D_i(c_0)$ the ith disc (of c_0).

Many ideas and results concerning geometries have taken buildings as their inspiration. So let us pause for a moment and consider the chamber graph of $\mathcal{C}$ where $\mathcal{C}$ is the chamber system associated with the building which arises from a finite group G of Lie type over $GF(q)$. Let c_0 be a fixed chamber of $\mathcal{C}$. Now $c \in D_i(c_0)$ if and only if $\delta(c_0, c) = w$ where w is an element of Weyl group W of G and the length of w in W is i. (δ is the W-distance function — see [14, Chapter 3] for further details of this approach to buildings.) For $w \in W$, U_w acts simply transitively on the set of chambers such that $\delta(c_0, c) = w$. (U_w is a certain subgroup of $B = \mathrm{Stab}_G\, c_0$ and $|U_w| = q^{\ell(w)}$ — again see [14, pp. 75,76]). Since

$$D_i(c_0) = \bigcup_{\substack{w \in W\\ l(w)=i}} \{c \mid \delta(c_0, c) = w\}\,,$$

the number of chambers in $D_i(c_0)$ is

$$q^{\ell(w)} \times \text{ (size of the } i\text{th disc in the chamber system for } W).$$

The diameter d of $\mathcal{C}$ is the Coxeter number of W and $|D_d(c_0)| = |U|$ where U is the unipotent radical of B. (This is because there is a unique $w_0 \in W$ with

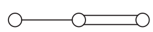
$\ell(w_0) = d$ and $U_{w_0} = U$.) So in particular, we have that B acts transitively on $D_d(c_0)$. In the chamber systems analyzed in this paper, this property is rarely observed. However, there are some interesting instances when this property does occur — for example in the M_{24} maximal 2-local geometry [17].

So, looking at the building case, we see that the sizes of discs, particularly the last disc and the diameter of the chamber graph of a sporadic group geometry are potentially interesting pieces of information relating to the group and the geometry. It is these features of the chamber graph that we focus upon here. Much of the data has been obtained using MAGMA [5] and extends to chamber systems with up to about 400,000 chambers.

The aim of this exercise in data collection is to highlight those geometries deserving of further detailed study. Indeed, in [17], combinatorial descriptions of the discs for the M_{24} maximal 2-local geometry are obtained by hand — the sizes of the discs agree with those given here in section 2.22 (Geometry 1)!

Section 2 tabulates the disc sizes of various geometries together with some additional observations. The geometries we study are described either in terms of some combinatorial structure or by means of an appropriate diagram [3, 4]. For a rank n geometry we shall take $I = \{0, 1, \ldots, n-1\}$. We use $G_{i,\ldots,i_r}$, where $\{i_1, \ldots, i_r\} \subseteq I$, to denote the stabilizer in G of a flag of type $\{i_1, \ldots, i_r\}$, and put $B = G_{0\cdots n-1} = G_0 \cap G_1 \cap \ldots \cap G_{n-1}$. Since we utilize the group in our calculations we give $G_{i_1\cdots i_r}$ for all subsets $\{i_1, \ldots, i_r\}$ of I.

Throughout we use the Atlas [6] conventions and terminology when describing groups except that we use $\mathsf{Dih}(n)$, $\mathsf{Sym}(n)$ and $\mathsf{Alt}(n)$ to denote, respectively, the dihedral group of order n, the symmetric group and alternating group of degree n. Thus we shall (usually) only describe the groups $G_{i,\ldots,i_r}$ up to "shape".

In section 3 we give some hand calculations for the  $\mathsf{Alt}(7)$-geometry. This geometry, over the years has attracted a good deal of attention [12, 13, 14]. These calculations uncover the structure of the last disc — the chamber graph has diameter 5 and $D_5(c_0)$ consists of 104 chambers.

2 Disc structures

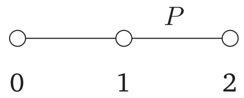
2.1 Group $G = \mathsf{L}_2(11)$

$|G| = 2^2 \cdot 3 \cdot 5 \cdot 11 = 660$

GEOMETRY: Petersen Geometry

NUMBER OF CHAMBERS: 330

DIAMETER: 9



$G_0 \cong \mathsf{Alt}(5)$, $G_1 \cong \mathsf{Dih}(12) \cong G_2$,
$G_{01} \cong \mathsf{Sym}(3)$, $G_{02} \cong 2^2 \cong G_{12}$,
$B \cong 2$ (see [8, p. 944]).

DISC	1	2	3	4	5	6	7	8	9
SIZE	4	8	15	26	42	58	76	68	32

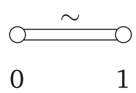
2.2 Group $G = \hat{S}_6 (\cong 3{\cdot}\mathsf{Sym}(6))$

$|G| = 2^4 \cdot 3^3 \cdot 5 = 2{,}160$

GEOMETRY: 3-fold cover of the $\mathsf{Sp}_4(2)$-quadrangle

NUMBER OF CHAMBERS: 135

DIAMETER: 8



$G_0 \cong 2^3\mathsf{Sym}(3) \cong G_1$,
$B \cong \mathsf{Dih}(8) \times 2$.
This geometry appears as a residue in the minimal parabolic geometries for M_{24}, $\cdot 1$, M, He and Fi_{24}' — see [16].

DISC	1	2	3	4	5	6	7	8
SIZE	4	8	16	32	48	16	8	2

2.3 Group $G = \mathsf{Alt}(7)$

$|G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2{,}520$

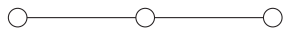
1. GEOMETRY: NUMBER OF CHAMBERS: 315
DIAMETER: 8

$G_0 \cong \mathsf{Sym}(4) \cong G_1$,
$B \cong \mathsf{Dih}(8)$.

Biduads $(ab)(cd)$ and triduads $(ab)(cd)(ef)$ are unordered pairs and triples of disjoint duads of a 7-element set. This rank 2 geometry also appears as a residue in a number of sporadic geometries; see [16].

DISC	1	2	3	4	5	6	7	8
SIZE	4	8	16	32	56	72	98	28

2. GEOMETRY: NUMBER OF CHAMBERS: 315
C_3-Geometry for $\mathsf{Alt}(7)$ DIAMETER: 5



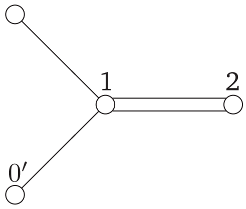
$G_0 \cong \mathsf{Alt}(6)$, $G_1 \cong (3 \times \mathsf{Alt}(4))2$, $G_2 \cong \mathsf{L}_3(2)$,
$G_{01} \cong G_{02} \cong G_{12} \cong \mathsf{Sym}(4)$,
$B \cong \mathsf{Dih}(8)$.

The points are the points of a 7-element set Ω, the lines are all 3-element subsets of Ω and the planes are one $\mathsf{Alt}(7)$-orbit of $PG(2,2)$ on Ω. See [16] and [12]. This geometry receives further attention in section 3.

DISC	1	2	3	4	5
SIZE	6	20	56	128	104

3. GEOMETRY: NUMBER OF CHAMBERS: 315
DIAMETER: 5

This chamber system is something of a hybrid of the chamber system in Example 2. Starting with the chamber system $\mathcal{C}$ of the $\mathsf{Alt}(7)$ C_3-geometry, we choose a fixed $\mu \in \mathsf{Sym}(7) \setminus \mathsf{Alt}(7)$ and then define two chambers c, d of $\mathcal{C}$ to be $0'$- adjacent if $c\mu$ and $d\mu$ are 0-adjacent. Together with the other 0-, 1- ,2-adjacencies of $\mathcal{C}$, this delivers a rank 4 chamber system with diagram

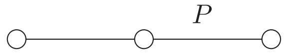


(See [14, p. 54].)

$B \cong \mathsf{Dih}(8)$,
$P_i \cong \mathsf{Sym}(4)$ for $i \in \{0, 0', 1, 2\}$.

DISC	1	2	3	4	5
SIZE	8	26	88	120	72

4. GEOMETRY: Petersen Geometry
NUMBER OF CHAMBERS: 630
DIAMETER: 11



$G_0 \cong \mathsf{Sym}(5)$, $G_1 \cong (3 \times 2^2)2$, $G_2 \cong \mathsf{Sym}(4)$,
$G_{01} \cong \mathsf{Dih}(12)$, $G_{02} \cong \mathsf{Dih}(8) \cong G_{12}$,
$B \cong 2^2$ (see [8, p. 945]).

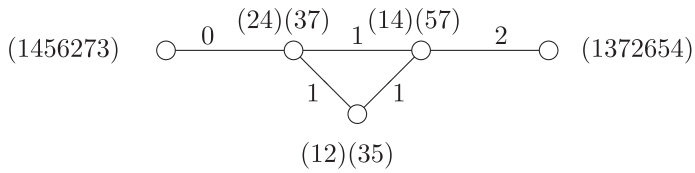
DISC	1	2	3	4	5	6	7	8	9	10	11
SIZE	4	8	15	26	42	58	76	104	136	144	16

5. GEOMETRY: Chamber system of type $\tilde{A}_2$
NUMBER OF CHAMBERS: $2{,}520$
DIAMETER: 9

$B = 1$,
$P_i \cong 3$, $i \in \{0, 1, 2\}$. (See [14, p. 53].)

DISC	1	2	3	4	5	6	7	8	9
SIZE	6	24	72	192	468	851	737	164	5

Remark 2.1. Taking $P_0 = \langle(123)(456)\rangle$, $P_1 = \langle(124)(375)\rangle$ and $P_2 = \langle(153)(276)\rangle$ (and noting that the chambers are just the elements of $\mathsf{Alt}(7)$ and $c_0 = 1$), $D_9(c_0)$ looks as follows:

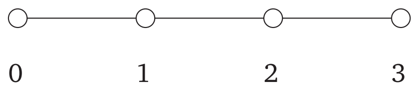


The labels on the edges indicate the i-adjacency.

2.4 Group $G = \mathsf{Sym}(7)$

GEOMETRY: Number 17 of [4]

$|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5{,}040$
NUMBER OF CHAMBERS: 840
DIAMETER: 10



The type 0 objects are the elements of a 7-element set Ω and the objects of type 1, 2, 3 are, respectively, all the 2-, 3- and 4-element subsets of Ω.

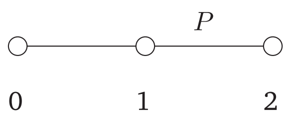
$G_0 \cong \mathsf{Sym}(6)$, $G_1 \cong 2 \times \mathsf{Sym}(5)$, $G_2 \cong \mathsf{Sym}(3) \times \mathsf{Sym}(4) \cong G_3$,
$G_{01} \cong \mathsf{Sym}(5)$, $G_{02} \cong 2 \times \mathsf{Sym}(4) \cong G_{12}$,
$G_{03} \cong \mathsf{Sym}(3) \times \mathsf{Sym}(3) \cong G_{23}$, $G_{13} \cong 2^2 \times \mathsf{Sym}(3)$,
$G_{012} \cong \mathsf{Sym}(4)$, $G_{013} \cong G_{023} \cong G_{123} \cong 2 \times \mathsf{Sym}(3)$,
$B \cong \mathsf{Sym}(3)$.

DISC	1	2	3	4	5	6	7	8	9	10
SIZE	6	17	39	68	102	136	147	135	108	81

2.5 Group $G = \mathsf{L}_2(25)$

GEOMETRY: Petersen Geometry

$|G| = 2^3 \cdot 3 \cdot 5^2 \cdot 13 = 7{,}800$
NUMBER OF CHAMBERS: $1{,}950$
DIAMETER: 18



$G_0 \cong \mathsf{Sym}(5)$, $G_1 \cong \mathsf{Dih}(24)$, $G_2 \cong \mathsf{Sym}(4)$,
$G_{01} \cong \mathsf{Dih}(12)$, $G_{01} \cong G_{12} \cong \mathsf{Dih}(8)$,
$B \cong 2^2$. (See [8, p. 944].)

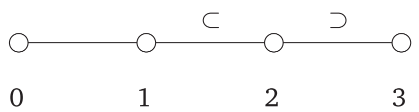
DISC	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
SIZE	4	8	15	26	42	58	76	104	136	176	192	216	256	256	232	100	44	8

2.6 Group $G = \mathsf{M}_{11}$

$|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11 = 7{,}920$

1. GEOMETRY: Number 27 of [4].

NUMBER OF CHAMBERS: $2{,}640$
DIAMETER: 11



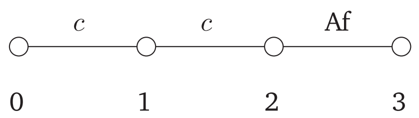
Considering M_{11} acting 3-transitively on the 12-element set Ω, we may describe the geometry thus. The objects of type 0 and 1 are, respectively all 1- and 2- subsets of Ω; those of type 2 are 3-element subsets of Ω of the form $\mathrm{Fix}_{\Omega}(g)$ where g is an element of order 3 in M_{11} and those of type 3 are one "half" of a total (that is, a 6-element subset of the $6 \mid 6$ partition). Incidence is symmetized containment.

$G_0 \cong \mathsf{L}_2(11)$, $G_1 \cong \mathsf{Sym}(5)$, $G_2 \cong 3(\mathsf{Sym}(3) \times 2)$, $G_3 \cong \mathsf{Alt}(6)$,
$G_{01} \cong \mathsf{Alt}(5) \cong G_{03}$, $G_{02} \cong 2 \times \mathsf{Sym}(3) \cong G_{12}$, $G_{13} \cong \mathsf{Sym}(4)$,
$G_{23} \cong 3^2 : 2$,
$G_{012} \cong G_{023} \cong G_{123} \cong \mathsf{Sym}(3)$, $G_{013} \cong \mathsf{Alt}(4)$,
$B \cong 3$.

DISC	1	2	3	4	5	6	7	8	9	10	11
SIZE	6	19	51	106	204	327	426	534	549	393	24

2. GEOMETRY:

NUMBER OF CHAMBERS: $3{,}960$
DIAMETER: 10

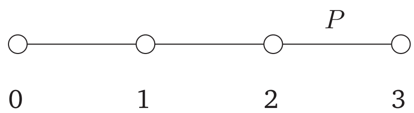


Regarding M_{11} as the stabilizer of the $S(12, 6, 5)$ Steiner system on a 12-element set Ω and an element α of Ω, the geometry may be described as follows. The objects of type 0,1,2 are, respectively, all 1-, 2- and 3-element subsets of $\Omega \setminus \{\alpha\}$ and those of type 3 all the hexads of $S(12,6,5)$ containing α. See [13, pp. 72 and 94].

$G_0 \cong \mathsf{M}_{10}$, $G_1 \cong \mathsf{M}_9 : 2$, $G_2 \cong 2^{\cdot}\mathsf{Sym}(4)$, $G_3 \cong \mathsf{Sym}(5)$,
$G_{01} \cong 3^2 : Q_8$, $G_{02} \cong \mathsf{SDih}(16) \cong G_{12}$, $G_{03} \cong \mathsf{Sym}(4)$,
$G_{13} \cong 2 \times \mathsf{Sym}(3) \cong G_{23}$, $G_{012} \cong Q_8$, $G_{013} \cong \mathsf{Sym}(3)$, $G_{023} \cong 2^2 \cong G_{123}$,
$B \cong 2$.
($\mathsf{SDih}(n)$ denotes the semidihedral group of order n.)

DISC	1	2	3	4	5	6	7	8	9	10
SIZE	7	26	73	155	300	494	636	756	864	648

3. GEOMETRY: Petersen Geometry   NUMBER OF CHAMBERS: 3,960
DIAMETER: 15



Again starting with M_{11} acting 3-transitively on a 12-element set Ω, we take as our objects of type 0, 1, 2 to be, respectively, all 1-, 2- and 3- subset of Ω and objects of type 3 to be 4 subsets of the form $\mathrm{Fix}_{\Omega}(g)$ where g is an involution of M_{11}. Incidence being symmeterized inclusion.

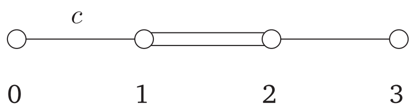
$G_0 \cong \mathsf{L}_2(11)$, $G_1 \cong \mathsf{Sym}(5)$, $G_2 \cong 3(\mathsf{Sym}(3) \times 2)$,
$G_3 \cong 2^{\cdot}\mathsf{Sym}(4)$, $G_{01} \cong \mathsf{Alt}(5)$, $G_{02} \cong G_{03} \cong G_{12} \cong G_{23} \cong 2 \times \mathsf{Sym}(3)$,
$G_{13} \cong \mathsf{Dih}(8)$, $G_{012} \cong \mathsf{Sym}(3)$, $G_{013} \cong G_{023} \cong G_{123} \cong 2^2$, $B \cong 2$.
(See [8, p. 954].)

DISC	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
SIZE	5	13	28	55	101	171	278	406	516	578	612	590	446	144	16

2.7 Group $G = A_8$

GEOMETRY:

$|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 = 20{,}160$
NUMBER OF CHAMBERS: 2,520
DIAMETER: 8



Let Ω be an 8-element set. Then the objects of type 0,1,2 are, respectively the elements, duads and 4^2 partitions of Ω. The 35 4^2 partitions of Ω may be identified with the lines of projective 3-space (see [13, Proposition 1]). Objects of type 3 are the points of the projective 3-space which may be identified with a set of seven 2^4 partitions of Ω (there are $105 = 7 \times 15$ 2^4 partitions of Ω). These seven 2^4 partitions may also be viewed as the non-identity elements of $O_2(G_3)$. For the definition of incidence and more details see [12, Section 3].

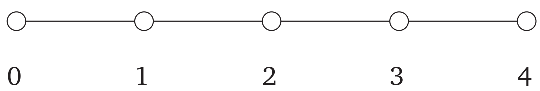
$G_0 \cong \mathsf{Alt}(7)$, $G_1 \cong \mathsf{Sym}(6)$, $G_2 \cong (\mathsf{Alt}(4) \times \mathsf{Alt}(4)) : 2^2$,
$G_3 \cong 2^3 : \mathsf{L}_3(2)$,
$G_{01} \cong \mathsf{Alt}(6)$, $G_{02} \cong (3 \times \mathsf{Alt}(4)):2$, $G_{03} \cong \mathsf{L}_3(2)$,
$G_{12} \cong 2^3:\mathsf{Sym}(3)$, $G_{23} \cong 2^3:\mathsf{Sym}(4)$, $G_{13} \cong \mathsf{Sym}(4) \times 2$,
$G_{012} \cong G_{013} \cong G_{023} \cong \mathsf{Sym}(4)$, $G_{123} \cong \mathsf{Dih}(8) \times 2$,
$B \cong \mathsf{Dih}(8)$.

DISC	1	2	3	4	5	6	7	8
SIZE	7	28	92	256	488	720	744	184

2.8 Group $G = \mathsf{U}_4(2)$

GEOMETRY: Example 6 in [11, Table 4]

$|G| = 2^6 \cdot 3^4 \cdot 5 = 25{,}920$
NUMBER OF CHAMBERS: 25,920
DIAMETER: 12



For each $i \in I$, $G_J \cong 3$ for $J = I \setminus \{i\}$ and $B = 1$. (See [11] for the other stabilizers.)

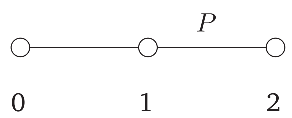
Disc	1	2	3	4	5	6	7	8	9	10	11	12
Size	10	60	260	855	2190	4510	6930	6542	3325	1150	85	2

2.9 Group $G = \mathsf{M}_{12}$

$|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11 = 95{,}040$

1. Geometry: Number 5 of [4]

Number of Chambers: $1{,}320$
Diameter: 6



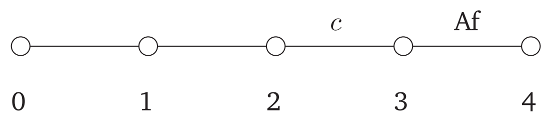
With G acting (5-transitively) on the 12-element set Ω, we take all 1-, 2- and 3-sets of Ω to be the objects of type 0,1,2 of the geometry respectively.

$G_0 \cong \mathsf{M}_{11}$, $G_1 \cong \mathsf{M}_{10} : 2$, $G_2 \cong 3^2 : 2\mathsf{Sym}(4)$,
$G_{01} \cong \mathsf{M}_{10}$, $G_{02} \cong 3^2 : \mathsf{SDih}(16) \cong G_{12}$,
$B \cong 3^2 : Q_8$.

Disc	1	2	3	4	5	6
Size	11	29	118	189	243	729

2. Geometry:

Number of Chambers: $47{,}520$
Diameter: 15



With Ω a 12-element set, the objects of type 0, 1, 2, 3 and 4 are, respectively 1-, 2-, 3-, 4-subsets of Ω and the hexads of the Steiner system $S(12, 6, 5)$. See [13, pp. 72 and 94].

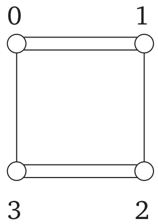
$G_0 \cong \mathsf{M}_{11}$, $G_1 \cong \mathsf{Alt}(6)^{\cdot}2^2 \cong G_4$, $G_2 \cong 3^2 : 2\mathsf{Sym}(4)$, $G_3 \cong 2_+^{1+4} : \mathsf{Sym}(3)$, $G_{0123} \cong Q_8$, $G_{0124} \cong \mathsf{Sym}(3)$, $G_{0134} \cong G_{0234} \cong G_{1234} \cong 2^2$, $B \cong 2$.

Disc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Size	8	34	107	263	574	1116	1887	2934	4280	5692	6504	6840	6912	6480	3888

2.10 Group $G = \mathbf{U}_3(5)$

$|G| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7 = 126{,}000$

1. Geometry:



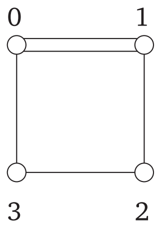
Number of Chambers: 15,750
Diameter: 10

See [7] and [12].

$G_0 \cong G_1 \cong G_2 \cong G_3 \cong \mathsf{Alt}(7)$, $G_{01} \cong G_{23} \cong \mathsf{Alt}(6)$,
$G_{02} \cong G_{13} \cong (3 \times \mathsf{Alt}(4)) : 2$, $G_{03} \cong G_{12} \cong \mathsf{L}_3(2)$,
$G_{012} \cong G_{013} \cong G_{023} \cong G_{123} \cong \mathsf{Sym}(4)$, $B \cong \mathsf{Dih}(8)$.

Disc	1	2	3	4	5	6	7	8	9	10
Size	8	40	176	704	2080	4748	5680	2060	252	1

2. Geometry:



Number of Chambers: 15,750
Diameter: 10

See [7] and [19].

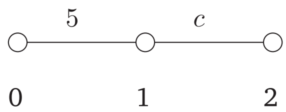
$G_0 \cong G_1 \cong G_2 \cong G_3 \cong \mathsf{Alt}(7)$, $G_{01} \cong G_{03} \cong \mathsf{L}_3(2)$,
$G_{02} \cong G_{13} \cong (3 \times \mathsf{Alt}(4)) : 2$, $G_{12} \cong \mathsf{L}_3(2)$, $G_{23} \cong \mathsf{Alt}(6)$,
$G_{012} \cong G_{013} \cong G_{023} \cong G_{123} \cong \mathsf{Sym}(4)$, $B \cong \mathsf{Dih}(8)$.

DISC	1	2	3	4	5	6	7	8	9	10
SIZE	8	40	168	624	1840	4628	6776	1620	44	1

2.11 Group $G = \mathsf{J}_1$

$|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = 175{,}500$
NUMBER OF CHAMBERS: 29,260
DIAMETER: 12

GEOMETRY:
Number 28 of [4]



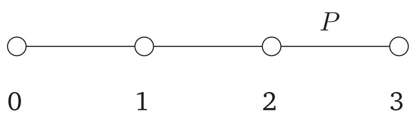
$G_0 \cong \mathsf{L}_2(11)$, $G_1 \cong 2 \times \mathsf{Alt}(5)$, $G_2 \cong \mathsf{Sym}(3) \times \mathsf{Dih}(10)$,
$G_{01} \cong \mathsf{Alt}(5)$, $G_{02} \cong G_{12} \cong 2 \times \mathsf{Sym}(3)$,
$B \cong \mathsf{Sym}(3)$.

DISC	1	2	3	4	5	6	7	8	9	10	11	12
SIZE	11	29	119	209	379	1260	2124	3960	9402	8196	3102	468

2.12 Group $G = \mathsf{Alt}(9)$

$|G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7 = 1{,}811{,}440$
NUMBER OF CHAMBERS: 22,680
DIAMETER: 18

GEOMETRY:
Petersen Geometry



See [8].

$G_0 \cong \mathsf{Sym}(7)$, $G_1 \cong 2.\mathsf{Sym}(5)$, $G_2 \cong \mathsf{Sym}(4) \times \mathsf{Sym}(3)$, $G_3 \cong 2^3\mathsf{Sym}(4)$,
$|G_{012}| = 2^3 3$, $|G_{013}| = |G_{023}| = |G_{123}| = 2^4$,
$|B| = 2^3$.

DISC	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
SIZE	5	13	28	55	101	171	278	442	692	1038	1372	1724	2160	2760	3408	4032	3344	1056

2.13 Group $G = \mathsf{M}_{22}$

$|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 443{,}520$

1. GEOMETRY:

NUMBER OF CHAMBERS: $3{,}465$
DIAMETER: 5

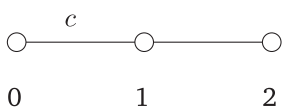
If Ω is a 22-element set upon which G acts transitively, then the objects of type 0 and 1 are, respectively, the two element subsets (duads) of Ω and the triduads of Ω (that is, 2^3 partition of hexads of the Steiner system $S(22, 3, 6)$ on Ω). This is the minimal parabolic geometry for M_{22} (see [16]).

$G_0 \cong 2^4 : \mathsf{Sym}(5)$, $G_1 \cong 2^6\mathsf{Sym}(3)$,
$G_{01} \cong 2^4 : \mathsf{Dih}(8)$.

DISC	1	2	3	4	5
SIZE	16	56	432	1040	1920

2. GEOMETRY:
Number 43 of [4]

NUMBER OF CHAMBERS: $2{,}310$
DIAMETER: 6



Assuming Ω is as in Geometry 1 above, the objects of type 0, 1, 2 are respectively the elements, duads and hexads of Ω.

$G_0 \cong \mathsf{L}_3(4)$, $G_1 \cong 2^4 : \mathsf{Sym}(5)$, $G_2 \cong 2^4 : \mathsf{Alt}(6)$,
$G_{01} \cong G_{02} \cong 2^4 : \mathsf{Alt}(5)$, $G_{12} \cong 2^4 : \mathsf{Sym}(4)$,
$B \cong 2^4 : \mathsf{Alt}(4)$.

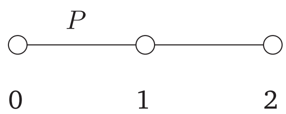
DISC	1	2	3	4	5	6
SIZE	9	44	144	320	768	1024

2.14 Group $G = \mathrm{Aut}\,\mathsf{M}_{22}$

$|G| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 887{,}040$

GEOMETRY:
Petersen Geometry

NUMBER OF CHAMBERS: 6,930
DIAMETER: 13



It is convenient to describe this geometry by beginning with a 24-element set Ω, assumed to be equipped with the Steiner system $S(24, 8, 5)$. Now fix a duad D (2-element subset) of Ω. Then $\mathrm{Stab}_{\mathsf{M}_{24}} D \cong \mathrm{Aut}\,\mathsf{M}_{22}$ and objects of type 0,1,2 of the geometry are, respectively, all octads in $\Omega \setminus D$, all trios which have D contained in one of its octads and all sextets which have D contained in one of its tetrads. Incidence being given by compatibility of partitions (see [9]).

$G_0 \cong (2^3 : \mathsf{L}_3(2)) \times 2$, $G_1 \cong 2^{1+4}(2^2 \times \mathsf{Sym}(3)$,
$G_2 \cong 2^5 : \mathsf{Sym}(5)$, $G_{01} \cong G_{02} \cong (2^3 : \mathsf{Sym}(4)) \times 2$,
$G_{12} \cong 2^5 : \mathsf{Dih}(8)$, $B \cong (2^3 : \mathsf{Dih}(8)) \times 2$.

DISC	1	2	3	4	5	6	7	8	9	10	11	12	13
SIZE	5	14	30	56	112	200	320	512	800	1248	1808	1696	128

Remark 2.2. $D_{13}(c_0)$ is a B-orbit.

2.15 Group $G = 3\mathsf{M}_{22}$

$|G| = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 1{,}330{,}560$

GEOMETRY:

NUMBER OF CHAMBERS: 10,395
DIAMETER: 8

This is a 3-fold cover of the minimal parabolic geometry for M_{22} (see section 2.13).

$G_0 \cong 2^4 : \mathsf{Sym}(5)$, $G_1 \cong 2^6\mathsf{Sym}(3)$, $B \cong 2^4 : \mathsf{Dih}(8)$.

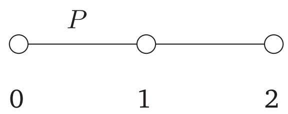
Disc	1	2	3	4	5	6	7	8
Size	16	56	432	1056	3632	4304	872	26

2.16 Group $G = 3\mathsf{M}_{22}2$

$|G| = 2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 2{,}661{,}120$

Geometry:

Number of Chambers: 20,790
Diameter: 24



A 3-fold cover of the Petersen geometry given in section 2.14 (see also [9]).
$G_0 \cong (2^3 : \mathsf{L}_3(2)) \times 2$, $G_1 \cong 2^{1+4}(2^2 \times \mathsf{Sym}(3))$,
$G_2 \cong 2^5 : \mathsf{Sym}(5)$, $G_{01} \cong G_{02} \cong (2^3 : \mathsf{Sym}(4)) \times 2$,
$G_{12} \cong 2^5 : \mathsf{Dih}(8)$, $B \cong (2^3 : \mathsf{Dih}(8)) \times 2$.

Disc	1	2	3	4	5	6	7	8	9	10
Size	5	14	30	56	112	200	320	512	800	1248

Disc	11	12	13	14	15	16	17	18	19	20
Size	1808	2368	3008	3968	3456	1216	736	464	248	120

Disc	21	22	23	24
Size	60	28	10	2

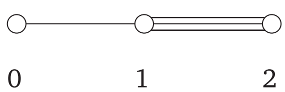
Remark 2.3. Note that the sizes here of $D_i(c_0)$ for $1 \leq i \leq 11$ coincide with those in section 2.14.

2.17 Group $G = G_2(3)$

$|G| = 2^6 \cdot 3^6 \cdot 7 \cdot 13 = 4{,}245{,}696$

1. Geometry:

Number of Chambers: 66,339
Diameter: 13



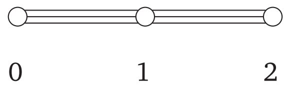
See [1].

$G_0 \cong 2^3 : \mathsf{L}_3(2)$, $G_1 \cong 2_+^{1+4} : 3^2.2$,
$G_3 \cong G_2(2)$, $G_{01} \cong G_{02} \cong G_{12} \cong 2^5.\mathsf{Sym}(3)$,
$B \cong 2^{3 \cdot}\mathsf{Dih}(8)$.

DISC	1	2	3	4	5	6	7	8	9	10	11	12	13
SIZE	6	20	56	144	384	960	2176	4864	10368	10972	21248	6976	64

2. GEOMETRY:

NUMBER OF CHAMBERS: 66,339
DIAMETER: 12



See [1].

$G_0 \cong G_2(2) \cong G_2$, $G_1 \cong 2_+^{1+4} : 3^2.2$,
$G_{01} \cong G_{02} \cong G_{12} \cong 2^5.\mathsf{Sym}(3)$,
$B \cong 2^{3 \cdot}\mathsf{Dih}(8)$.

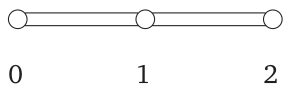
DISC	1	2	3	4	5	6	7	8	9	10	11	12
SIZE	6	20	64	208	600	1728	4640	10368	17920	20416	9472	896

2.18 Group $G = \mathbf{U}_4(3)2$

$|G| = 2^8 \cdot 3^6 \cdot 5 \cdot 7 = 6{,}531{,}840$

GEOMETRY:

NUMBER OF CHAMBERS: 25,515
DIAMETER: 10



This geometry is an example of a GAB — see [10, Section 3] for details.

$G_0 \cong 2^4.\mathsf{Sym}(6)$, $G_1 \cong [2^6.].(\mathsf{Sym}(3) \times \mathsf{Sym}(3))$, $G_2 \cong 2^5.\mathsf{Alt}(6)$,
$G_{01} \cong G_{02} \cong G_{12} \cong [2^6].\mathsf{Sym}(3)$,
$B \cong 2^4.(\mathsf{Dih}(8) \times 2)$.

Disc	1	2	3	4	5	6	7	8	9	10
Size	6	20	64	176	416	1024	2432	5120	9088	7168

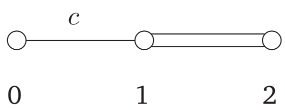
2.19 Group $G = \mathsf{U}_5(2)$

$|G| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11 = 13{,}685{,}760$

Geometry:

Number of Chambers: 28,160

Diameter: 11



Number 20 of [4] (with $n = 5$).

$G_0 \cong 3 \times \mathsf{U}_4(2)$, $G_1 \cong \mathsf{Sym}(3) \times (3^{1+2} : \mathsf{SL}_2(3))$, $G_2 \cong 3^4.\mathsf{Sym}(5)$,
$G_{01} \cong 3^4.\mathsf{SL}_2(3)$, $G_{02} \cong 3^4.\mathsf{Sym}(4)$, $G_{12} \cong 3^5 : 2^2$,
$B = 3^5 : 2$.

Disc	1	2	3	4	5	6	7	8	9	10	11
Size	7	27	99	270	594	1431	3051	5427	8019	8262	972

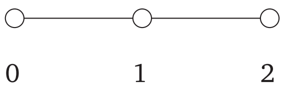
2.20 Group $G = \mathsf{M}_{23}$

$|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 10{,}200{,}960$

1. Geometry:

Number of Chambers: 79,695

Diameter: 7



Minimal parabolic "1-geometry" for M_{23} (see [16]).

$G_0 \cong \mathsf{M}_{22}$, $G_1 \cong 2^4(3 \times \mathsf{Alt}(5))2$, $G_2 \cong 2^4\mathsf{L}_3(2)$,
$G_{01} \cong 2^4\mathsf{Sym}(5)$, $G_{02} \cong 2^4\mathsf{Sym}(4) \cong G_{12}$,
$B \cong 2^4\mathsf{Dih}(8)$.

Disc	1	2	3	4	5	6	7
Size	18	92	664	3104	10728	36032	29056

Remark 2.4. M_{23} has two (non-isomorphic) minimal parabolic geometries which are locally isomorphic (meaning all their residues are isomorphic). Globally they differ with the choice of an $\mathsf{L}_3(2)$-conjugacy class

within $\mathsf{Alt}(7)$ — so producing two possible choices for G_2 contained in $H = 2^4\mathsf{Alt}(7)$ (the stabilizer in M_{23} of a heptad). In one case the $\mathsf{L}_3(2)$ leaves a 1-space (of $O_2(H)$) invariant and in the other a 3-space (of $O_2(H)$); the former we refer to as the "1-geometry" and the latter, dealt with next, the "3-geometry".

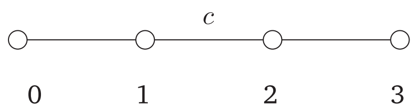
2. GEOMETRY: NUMBER OF CHAMBERS: 79,695
DIAMETER: 7

Minimal parabolic "3-geometry" for M_{23} (see [16]) — object stabilizers as for the "1- geometry".

DISC	1	2	3	4	5	6	7
SIZE	18	92	664	3104	10728	36544	28544

Remark 2.5. The disc sizes of the 1-geometry and the 3-geometry differ only in discs 6 and 7 — the 1-geometry has 512 fewer chambers in the sixth disc and 512 more in the seventh disc (than the 3-geometry).

3. GEOMETRY: NUMBER OF CHAMBERS: 53,130
DIAMETER: 10

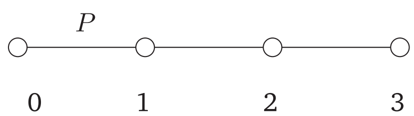


Number 44 of [4].

$G_0 \cong \mathsf{M}_{22}$, $G_1 \cong: \mathsf{L}_3(4)2$, $G_2 \cong 2^4 : (3 \times \mathsf{Alt}(5)) : 2$,
$G_3 \cong 2^4 : \mathsf{Alt}(7)$, $G_{012} \cong G_{013} \cong 2^4\mathsf{Alt}(5)$, $G_{023} \cong G_{123} \cong 2^4\mathsf{Sym}(4)$,
$B \cong 2^4\mathsf{Alt}(4)$.

DISC	1	2	3	4	5	6	7	8	9	10
SIZE	10	54	201	560	1552	3392	5376	9216	16384	16384

4. GEOMETRY: NUMBER OF CHAMBERS: 159,390
Petersen Geometry DIAMETER: 14



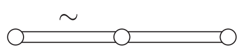
$G_0 \cong \mathsf{Alt}(8)$, $G_1 \cong 2^3(\mathsf{L}_3(2) \times 2)$, $G_2 \cong 2^4(3 \times \mathsf{Alt}(5))2$,
$G_3 \cong \mathsf{M}_{22}$,
$G_{012} \cong G_{013} \cong G_{023} \cong 2^3 : \mathsf{Sym}(4)$, $G_{123} \cong 2^4 : \mathsf{Dih}(8)$,
$B \cong 2^3 : \mathsf{Dih}(8)$.

DISC	1	2	3	4	5	6	7	8	9	10	11	12	13	14
SIZE	7	28	86	220	512	1128	2432	5152	10528	21024	38528	51840	26304	1600

2.21 Group $G = 3\mathsf{U}_4(3)2$

GEOMETRY:

$|G| = 2^8 \cdot 3^7 \cdot 5 \cdot 7 = 19{,}595{,}520$
NUMBER OF CHAMBERS: 76,545
DIAMETER: 11



0 1 2

This geometry is a triple cover of the geometry in section 2.18.
$G_0 \cong 2^4 : \mathsf{Sym}(6)$, $G_1 \cong [2^6](\mathsf{Sym}(3) \times \mathsf{Sym}(3))$, $G_2 \cong 2^5.3\mathsf{Alt}(6)$,
$G_{01} \cong G_{02} \cong G_{12} \cong [2^7]\mathsf{Sym}(3)$, $B \cong 2^4(\mathsf{Dih}(8) \times 2)$.

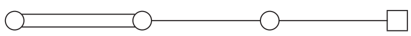
DISC	1	2	3	4	5	6	7	8	9	10	11
SIZE	6	20	64	192	528	1424	3848	9658	19812	27680	13312

2.22 Group $G = \mathsf{M}_{24}$

$|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244{,}823{,}040$

1. GEOMETRY:
   Maximal 2-local geometry (see [15])

NUMBER OF CHAMBERS: 79,695
DIAMETER: 10



0 1 2

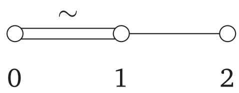
$G_0 \cong 2^4 : \mathsf{Alt}(8)$, $G_1 \cong 2^6 : (\mathsf{L}_3(2) \times \mathsf{Sym}(3))$, $G_2 \cong 2^6 : (3^{\cdot}\mathsf{Sym}(6))$,
$G_{01} \cong 2^6 : (\mathsf{L}_3(2) \times 2)$, $G_{02} \cong 2^6 : (3^{\cdot}(\mathsf{Sym}(4) \times 2)) \cong G_{12}$,
$B \cong [2^9] : \mathsf{Sym}(3)$.

DISC	1	2	3	4	5	6	7	8	9	10
SIZE	10	44	184	544	1536	4800	10368	22272	38400	1536

Remark 2.6. B is transitive on $D_{10}(c_0)$ (see [17]).

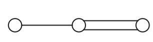
2. GEOMETRY: Minimal parabolic geometry (see [16])

NUMBER OF CHAMBERS: 239,085
DIAMETER: 17

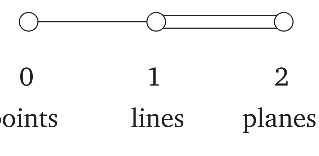


$G_0 \cong 2^4 : 2^3 : \mathsf{L}_3(2)$, $G_1 \cong 2^{6+2}(\mathsf{Sym}(3) \times \mathsf{Sym}(3))$,
$G_2 \cong 2^6 : (3^{\cdot}\mathsf{Sym}(6))$,
$G_{01} \cong G_{02} \cong G_{23} \cong [2^9] : \mathsf{Sym}(3)$,
$B \cong 2^6 : (\mathsf{Dih}(8) \times 2)$.

DISC	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
SIZE	6	20	56	144	368	848	1800	3810	8040	16920	32832	55200	62336	47616	6656	2048	384

3 The last disc of the  Alt(7)-geometry.

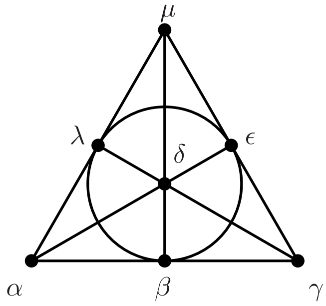
Let $G = \mathsf{Alt}(7)$ act upon the set $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$ and let Γ be the $\mathsf{Alt}(7)$-geometry with diagram



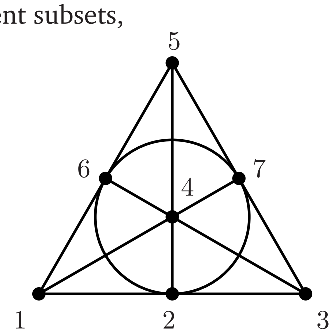
whose description we now give. The points, lines and planes of Γ (that is, objects of type 1, 2 and 3) are, respectively, the elements of Ω, all the 3-element subsets of Ω and one A_7-orbit of projective planes defined on Ω. (So there are

7 points, 35 lines and 15 planes.) For a point p, a line l and a plane P, $p * l$ whenever $p \in l$, $p * P$ and whenever $p \in P$ and $l * P$ whenever l is a line in the plane P

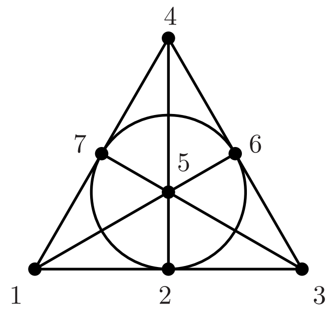
We use the following labelled projective plane P (where $\Omega = \{\alpha, \beta, \gamma, \delta, \epsilon, \lambda, \mu\}$)



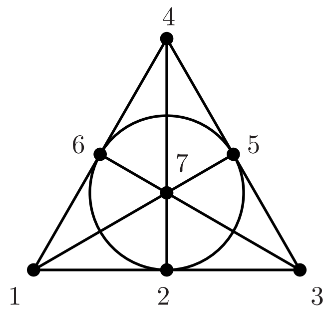
to also stand for the chamber $\{\{\alpha\}, \{\alpha, \beta, \gamma\}, P\}$. So the left-most label of the bottom line and the bottom line give the incident point and line of the maximal flag. Note that, as a projective plane is defined by its seven lines, that is 3-element subsets,



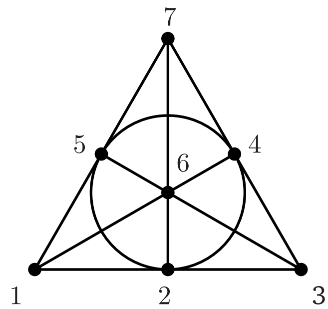
and



for example, denote the same chamber. Equally



and



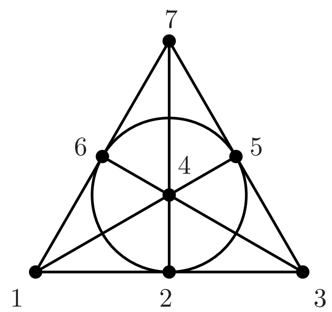
are two chambers which have the same point and line — they are 3-adjacent.

In the chamber system $\mathcal{C}$ obtained from the flag complex of Γ, two different chambers are 1-adjacent if they have the same line and plane, are 2-adjacent if they have the same point and plane and 3-adjacent if they have the same point and line.

So as to view $\mathcal{C}$ from a group theoretic perspective, we set

$$H_1 = \mathrm{Stab}_G\{1\}, \qquad H_2 = \mathrm{Stab}_G\{1,2,3\},$$
$$B = \langle (23)(67), (23)(4657)\rangle \quad \text{and} \quad H_3 = \langle B, (625)(143)\rangle.$$

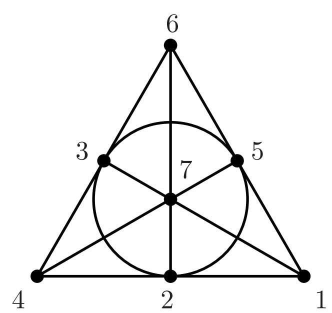
It is straightforward to check that H_3 is the stabilizer in G of the projective plane P,



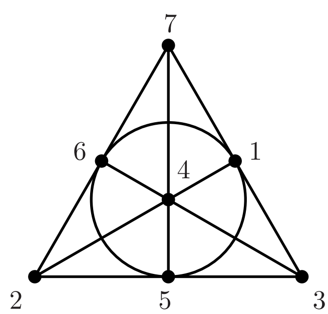
and that B is the stabilizer in G of the chamber $c_0 = \{\{1\}, \{1,2,3\}, P\}$. If we set $P_1 = H_2 \cap H_3$, $P_2 = H_1 \cap H_3$ and $P_3 = H_1 \cap H_2$, then we obtain the chamber system $(G; B, (P_i))$ over $I = \{1,2,3\}$ isomorphic to $\mathcal{C}$. We now investigate $D_5(c_0)$, the last disc of $\mathcal{C}$; there are 104 chambers in $D_5(c_0)$ (see section 2.3, Geometry 2).

The group B has 13 orbits on $D_5(c_0)$ and acts simply transitively on $D_5(c_0)$; orbit representatives $c_1, ..., c_{13}$ are given below.

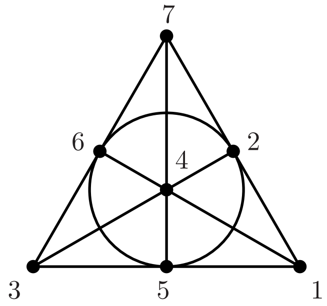
c_1



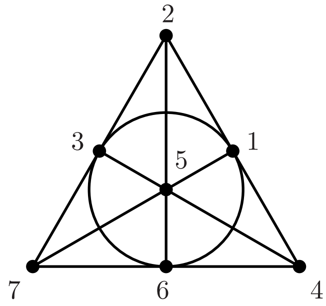
c_2



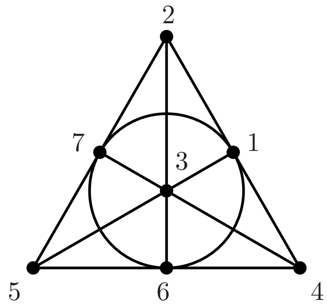
c_3



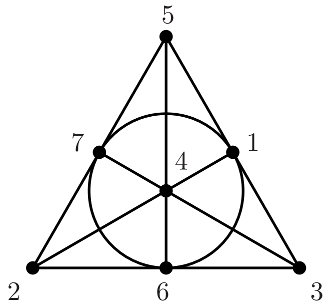
c_4



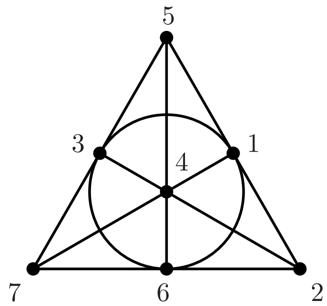
c_5



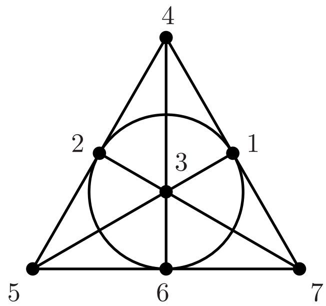
c_6



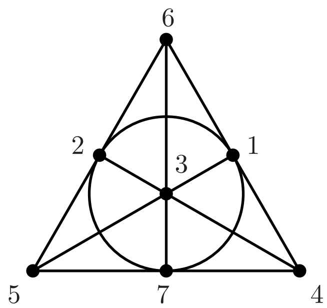
c_7



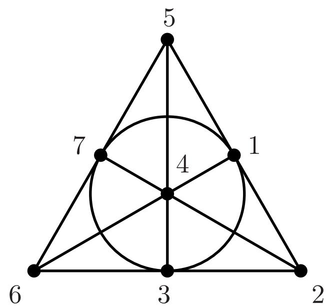
c_8



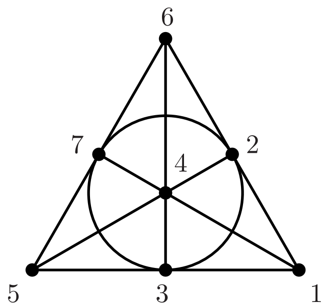
c_9



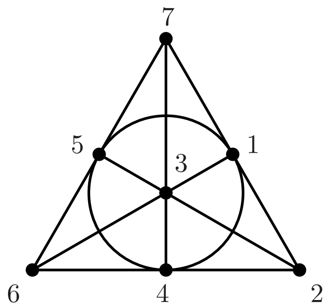
c_{10}



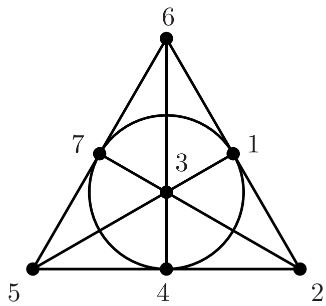
c_{11}



c_{12}



c_{13}



We next examine the induced subgraph (from the chamber graph) on $D_5(c_0)$. We name the elements of B as follows:

$$x_1 = (1), \quad x_2 = (23)(67), \quad x_3 = (47)(65), \quad x_4 = (23)(4657),$$
$$x_5 = (46)(57), \quad x_6 = (23)(4756), \quad x_7 = (45)(67), \quad x_8 = (23)(45).$$

In Table 1, the number j in brackets after each chamber indicates that the chamber is j-adjacent to c_i.

Remark 3.1. Using the action of elements of B, from Table 1 we may obtain the edge set for the induced graph on $D_5(c_0)$.

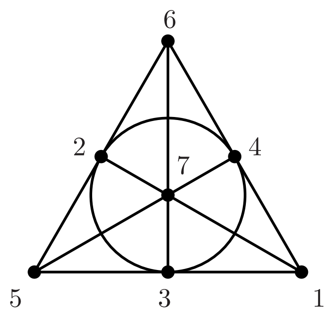
Theorem 3.2. *The induced subgraph on $D_5(c_0)$ is a connected graph.*

Proof. Let $\mathcal{E}$ denote the connected component of c_8 (see p. 93) in the subgraph $D_5(c_0)$. Also put $E = \mathrm{Stab}_B(\mathcal{E})$. From 1, c_8 and $c_8(23)(67)$ are 3-adjacent and so $c_8(23)(67) \in \mathcal{E}$. Hence $(23)(67) \in E$. Also, by Table 1, c_{11} is 2-adjacent to c_8 and c_{11} is 2-adjacent to

c_i	Chambers in $D_5(c_0)$ adjacent to c_i
c_1	$c_3x_s(1)$; $c_9x_8(2)$; $c_{11}x_8(3)$.
c_2	$c_3x_8(3)$; $c_6x_3(3)$; $c_{10}x_4(2)$.
c_3	$c_1x_8(1)$; $c_2x_8(2)$.
c_4	$c_5x_5(3)$; $c_8x_8(1)$; $c_9x_3(1)$; $c_9x_4(3)$; $c_{10}x_7(2)$.
c_5	$c_4x_5(3)$; $c_5x_8(1)$; $c_9x_2(3)$; $c_{12}x_3(2)$.
c_6	$c_2x_3(2)$; $c_6x_8(1)$.
c_7	$c_{13}x_3(1)$; $c_{13}x_5(2)$.
c_8	$c_4x_8(1)$; $c_8x_2(2)$; $c_9x_6(1)$; $c_{11}x_1 = c_{11}(2)$; $c_{13}x_1 = c_{13}(2)$.
c_9	$c_1x_8(2)$; $c_4x_3(1)$; $c_4x_6(3)$; $c_5x_2(3)$; $c_8x_4(1)$.
c_{10}	$c_2x_6(1)$; $c_4x_7(2)$; $c_{10}x_8(3)$.
c_{11}	$c_1x_8(3)$; $c_8x_1 = c_8(2)$; $c_{13}x_1 = c_{13}(2)$.
c_{12}	$c_5x_3(2)$.
c_{13}	$c_7x_3(1)$; $c_7x_5(3)$; $c_8x_1(2)$; $c_{11}x_1 = c_{11}(2)$.

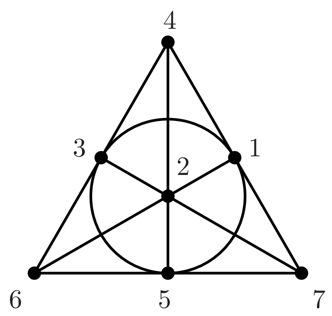
Table 1: Adjacency of chambers in $D_5(c_0)$

$c_1x_8 =$



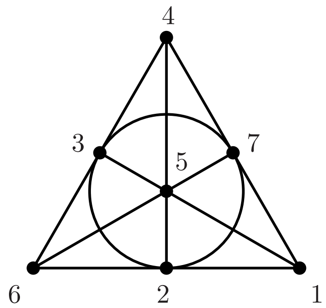
Therefore $c_1x_8 \in \mathcal{E}$. Again from Table 1, c_8 is 1-adjacent to

$c_9x_6 =$



Now c_9x_6 is 2-adjacent to

$c_1x_5 =$



So $c_1x_5 \in \mathcal{E}$. Since $(23)(4657)$ sends c_1x_5 to c_1x_8, $(23)(4657) \in E$. Thus, as $B = \langle(23)(67),\ (23)(4657)\rangle, E = B$. Inspecting Table 1 we see that there is a path in $D_5(c_0)$ from c_8 to a chamber in each $c_i^B(i \in \{1, ..., 13\})$. Therefore $\mathcal{E} = D_5(c_0)$, and this completes the proof. □

References

[1] **M. Aschbacher and S. D. Smith**, Tits geometries over $GF(2)$ defined by groups over $GF(3)$, *Comm. Algebra* **11** (1983), 1675–1684.

[2] **F. Buekenhout**, The geometry of diagrams, *Geom. Dedicata* **8** (1979), 253–257.

[3] ______, Diagrams for geometries and groups, *J. Combin. Theory Ser. A*, **27** (1979), 121–151.

[4] ______, Diagrams for sporadic groups, **in** *Finite Groups — Coming of Age*, J. McKay, ed., Contemp. Math. **45** (1985), pp. 1–32.

[5] **J. J. Cannon and C. Playoust**, *An Introduction to Algebraic Programming with MAGMA*, Draft, 1997.

[6] **J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker** and **R. A. Wilson**, *Atlas of Finite Groups. Maximal subgroups and ordinary characters for simple groups*, Oxford Univ. Press, Eynsham, 1985.

[7] **S. Heiss**, Two sporadic geometries related to the Hoffman-Singleton graph, *J. Combin. Theory Ser. A* **53** (1990), 68–80.

[8] **A. A. Ivanov and S. V. Shpectorov**, Geometries for sporadic groups related to the Petersen graph I, *Comm. Algebra* **16** (1988), 925–953.

[9] ______, Geometries for sporadic groups related to the Petersen graph II, *European J. Combin.* **10** (1989), 347–362.

[10] **W. M. Kantor**, Some Geometries that are Almost Buildings, *European J. Combin.* **2** (1981), 239–247.

[11] **E. A. Komissartschik and S. V. Tsaranov**, Construction of finite groups amalgams and geometries. Geometries of the group $\mathsf{U}_4(2)$, *Comm. Algebra* **18** (1990), 1071-1117.

[12] **A. Neumaier**, Some sporadic geometries related to $PG(3,2)$, *Arch. Math.*, **42** (1984), 89-96.

[13] **A. Pasini**, *Diagram Geometries*, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994.

[14] **M. A. Ronan**, *Lectures on Buildings*, Persp. Math., Academic Press, Boston, 1989.

[15] **M. A. Ronan** and **S. D. Smith**, 2-local geometries for some sporadic groups, **in** *The Santa Cruz Conference on Finite Groups* (Univ. California, Santa Cruz, 1979), *Proc. Symp. Pure Math.* **37** (1980), pp. 283–289.

[16] **M. A. Ronan and G. Stroth**, Minimal parabolic geometries for the sporadic groups, *European J. Combin.* **5** (1984), 59–91.

[17] **P. Rowley**, The Chamber graph of the M_{24}-maximal 2-local geometry, *J. Comput. Math.* **12** (2009), 120–143.

[18] **J. Tits**, A local approach to buildings. **in** *The Geometric Vein*, The Coxeter Festschrift (1981), 519–547.

[19] **M. Wester**, Endliche fahnentransitive Tits-Geometrien and ihre universellen Überlagerungen, *Mitt. Math. Sem. Giessen* **170** (1985), 143 pp.

P. J. Rowley

School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, United Kingdom

e-mail: Peter.J.Rowley@manchester.ac.uk