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Two projectively generated subsets of the Hermitian surface

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Abstract

Using a variation of Seydewitz's method of projective generation of quadrics we define two algebraic surfaces of $\mathsf{PG}(3,q^2)$, called elliptic $\mathcal{Q}_F$-sets and semi-hyperbolic $\mathcal{Q}_F$-sets, and we show that these surfaces are contained in the Hermitian surface of $\mathsf{PG}(3,q^2)$. Also, we characterize a semi-hyperbolic $\mathcal{Q}_F$-set as the intersection of two Hermitian surfaces. Finally we describe all possible configurations of the absolute set of an α-correlation in $\mathsf{PG}(2,q^2)$, where α is the involutory automorphism of $\mathsf{GF}(q^2)$.

Keywords: Hermitian surfaces, collineations, correlations.

MSC 2000: 51E20, 05B25.

1 Introduction

Let A and B be two distinct points of a three-dimensional projective space. Let $\mathcal{S}_A$ be the star of lines through A, let $\mathcal{S}_B^*$ be the star of planes through B and let Φ be a projectivity between $\mathcal{S}_A$ and $\mathcal{S}_B^*$. In 1847 Franz Seydewitz proved that quadrics may be generated as the set of points of intersection of corresponding elements under Φ (see e.g. [14]). In this paper we define two algebraic surfaces of $\mathsf{PG}(3,q^2)$ by using a variation of Seydewitz's projective generation of quadrics by means of a suitable collineation instead of a projectivity.

A *Hermitian variety* of a Desarguesian projective space $\mathsf{PG}(n,q^2)$ of order q^2, q any prime power, is the set of absolute points of a unitary polarity. A Hermitian variety of a projective plane $\mathsf{PG}(2,q^2)$ is called a *Hermitian curve* and a Hermitian variety of a projective space $\mathsf{PG}(3,q^2)$ is called a *Hermitian surface*. A point P on a Hermitian variety $\mathcal{H}$ is *singular* if any line through P either intersects $\mathcal{H}$

just in P or it is contained in $\mathcal{H}$. A Hermitian variety of $\mathsf{PG}(n,q^2)$ is called *degenerate* if it contains at least a singular point, otherwise it is called *non-degenerate*. The *vertex* of $\mathcal{H}$ is the set of all singular points of $\mathcal{H}$ and it is denoted by $V(\mathcal{H})$. It easily follows that $V(\mathcal{H})$ is a projective subspace of $\mathsf{PG}(n,q^2)$. The *rank* $r(\mathcal{H})$ of $\mathcal{H}$ is defined as $r(\mathcal{H}) = n - \dim(V(\mathcal{H}))$. Notice that a Hermitian curve of rank 2 is a Baer subpencil of lines of $\mathsf{PG}(2,q^2)$ and that a Hermitian curve of rank 1 is a line counted $q+1$ times. Moreover a Hermitian surface of $\mathsf{PG}(3,q^2)$ of rank 3 is a cone with vertex a point projecting a non-degenerate Hermitian curve, a Hermitian surface of rank 2 is a Baer subpencil of planes, a Hermitian surface of rank 1 is a plane repeated $q+1$ times. A non-degenerate Hermitian curve $\mathcal{H}$ of $\mathsf{PG}(2,q^2)$ has q^3+1 points; every line meets $\mathcal{H}$ in a Baer subline or in exactly one point. Through each point of $\mathcal{H}$ there is exactly one tangent line and through each point not on $\mathcal{H}$ there are exactly $q+1$ tangent lines, that form a Baer subpencil of lines. A non-degenerate Hermitian surface $\mathcal{H}$ of $\mathsf{PG}(3,q^2)$ has $(q^2+1)(q^3+1)$ points, every line intersects $\mathcal{H}$ in 1, $q+1$ or q^2+1 points. Every $(q+1)$-secant line intersects $\mathcal{H}$ in a Baer subline. Every plane intersects $\mathcal{H}$ either in a non-degenerate Hermitian curve or in a Baer subpencil of lines. More details about Hermitian varieties can be found in [8].

Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the pencils of lines with vertices two distinct points A and B in $\mathsf{PG}(2,q^2)$. Let α_F be the involutory automorphism of $\mathsf{GF}(q^2)$ and let Φ be an α_F-collineation between $\mathcal{P}_A$ and $\mathcal{P}_B$. If Φ does not map the line AB onto the line BA, then the set of points of intersections of corresponding lines under Φ is called a $\mathcal{C}_F$-set (see [4]). If Φ maps the line AB onto the line BA, then the set of points of intersections of corresponding lines under Φ is called a degenerate $\mathcal{C}_F$-set (see [5]).

Every $\mathcal{C}_F$-set has q^2+1 points, it is of type $(0,1,2,q+1)$ with respect to lines of $\mathsf{PG}(2,q^2)$ and every $(q+1)$-secant intersects such a set in a Baer subline. The $(q+1)$-secant lines number $q-1$ and all contain a common point C not on the $\mathcal{C}_F$-set. Those lines, together with the lines CA and CB, form a Baer subpencil. Every $\mathcal{C}_F$-set is projectively equivalent to the set of $\mathsf{GF}(q^2)$-rational points of algebraic curve with equation $x_1x_2^q - x_3^{q+1} = 0$. Under the André–Bruck–Bose representation a $\mathcal{C}_F$-set corresponds with an elliptic quadric contained in a suitable hyperplane of $\mathsf{PG}(4,q)$.

Every degenerate $\mathcal{C}_F$-set has $2q^2+1$ points, it is of type $(1,2,q+1,q^2+1)$ with respect to lines of $\mathsf{PG}(2,q^2)$ and every $(q+1)$-secant intersects such a set in a Baer subline. Moreover every degenerate $\mathcal{C}_F$-set is the union of the line AB and a Baer subplane meeting the line AB in a Baer subline. The points A and B are called the *vertices* of a $\mathcal{C}_F$-set (degenerate or not).

Let $\mathcal{P}$ and $\mathcal{P}'$ be two Baer subpencils of lines of $\mathsf{PG}(2,q^2)$ with vertices V and V' respectively and let $\mathcal{C}$ be the set of points of intersection between the lines of

$\mathcal{P}$ and the lines of $\mathcal{P}'$. If the line VV' belongs to $\mathcal{P}$ and not to $\mathcal{P}'$, then $\mathcal{C}$ is called a *K-set*. Under the André–Bruck–Bose representation to a K-set corresponds a quadratic cone contained in a suitable hyperplane of $\mathsf{PG}(4, q)$.

If the line VV' belongs neither to $\mathcal{P}$ nor to $\mathcal{P}'$, then $\mathcal{C}$ is called an H-set. Under the André–Bruck–Bose representation to an H-set corresponds an hyperbolic quadric contained in a suitable hyperplane of $\mathsf{PG}(4, q)$.

Every K-set $\mathcal{C}$ has $q^2 + q + 1$ points, it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\mathsf{PG}(2, q^2)$ and every $(q + 1)$-secant intersects $\mathcal{C}$ in a Baer subline. Every H-set $\mathcal{C}$ has $(q + 1)^2$ points, it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\mathsf{PG}(2, q^2)$ and every $(q + 1)$-secant intersects $\mathcal{C}$ in a Baer subline.

Finally let $\mathcal{H}$ be a non-degenerate Hermitian curve and let $\mathcal{P}$ be a Baer subpencil with vertex V on $\mathcal{H}$ containing the tangent line to $\mathcal{H}$ at V. A *Γ-set* $\mathcal{C}$ is the set of points of intersection between $\mathcal{H}$ and the lines of $\mathcal{P}$. It has $q^2 + 1$ points, it is of type $(0, 1, 2, q + 1)$ with respect to lines of $\mathsf{PG}(2, q^2)$ and every $(q + 1)$-secant intersects $\mathcal{C}$ in a Baer subline. In the following we will make use of the Barlotti–Cofman representation of $\mathsf{PG}(3, q^2)$ in $\mathsf{PG}(6, q)$.

A *spread* of $\mathsf{PG}(3, q)$ is a partition of $\mathsf{PG}(3, q)$ into lines. A spread $\mathcal{S}$ of $\mathsf{PG}(3, q)$ is a *regular* spread if the regulus containing any three distinct lines of $\mathcal{S}$ is contained in $\mathcal{S}$. A 1-*spread* of $\mathsf{PG}(5, q)$ is a partition of $\mathsf{PG}(5, q)$ into lines. A 1-spread $\mathcal{S}$ of $\mathsf{PG}(5, q)$ is a *normal* spread if $\mathcal{S}$ induces a spread in any subspace generated by two distinct lines of $\mathcal{S}$ (see [13, 10]). Let H_∞ be a hyperplane of $\mathsf{PG}(3, q^2)$, that we consider as the hyperplane at infinity. Let Σ_∞ be a hyperplane of $\mathsf{PG}(6, q)$ and let $\mathcal{S}$ be a normal spread of Σ_∞. The points of the affine space $\mathsf{PG}(3, q^2) \setminus H_\infty$ are represented by the points of $\mathsf{PG}(6, q) \setminus \Sigma_\infty$. The points of H_∞ are represented by the elements of the spread $\mathcal{S}$. The lines of $\mathsf{PG}(3, q^2) \setminus H_\infty$ are represented by planes of $\mathsf{PG}(6, q)$ intersecting Σ_∞ exactly in an element of $\mathcal{S}$. The lines of H_∞ are represented by the 3-dimensional subspaces containing two elements of $\mathcal{S}$. The incidence relation of $\mathsf{PG}(3, q^2)$ is represented by set theoretic inclusion in $\mathsf{PG}(6, q)$.

From now on if P is a point of $\mathsf{PG}(3, q^2)$, the corresponding point or line of $\mathsf{PG}(6, q)$ will be denoted by P^*. The same notation will be used for subsets of $\mathsf{PG}(3, q^2)$. For more details about the Barlotti–Cofman representation see [1].

A parabolic quadric $Q(4, q)$ of a four-dimensional subspace contained in Σ_∞, is called an *R-quadric* if $Q(4, q)$ contains a regulus contained in $\mathcal{S}$.

A *correlation* v of a projective plane π is a one-to-one mapping of its points onto its lines and its lines onto its points, such that $P \in \ell \iff v(\ell) \in v(P)$ for every flag (P, ℓ). A correlation v of $\mathsf{PG}(2, q^2)$ maps a point $P = \langle (y_1, y_2, y_3) \rangle$ onto the following line:

$$v(P) = \{\langle (x_1, x_2, x_3) \rangle : (x_1, x_2, x_3) A (\alpha(y_1), \alpha(y_2), \alpha(y_3))^T = 0\},$$

where A is a non-singular 3×3 matrix over $\mathsf{GF}(q^2)$ and α is an automorphism of $\mathsf{GF}(q^2)$ called the *companion automorphism* of v. The map v will be called an α-correlation. An *absolute point* of v is a point P such that $P \in v(P)$. The *absolute set* of v is the set of its absolute points.

2 Definitions

Let A and B be two distinct points of a three-dimensional projective space $\mathsf{PG}(3, q^2)$ over the Galois field $\mathsf{GF}(q^2)$, q any prime power. Let α_F be the involutory automorphism of $\mathsf{GF}(q^2)$, given by $\alpha_F : x \in \mathsf{GF}(q^2) \mapsto x^q \in \mathsf{GF}(q^2)$, and let Φ be an α_F-collineation between the star $\mathcal{S}_A$ of lines through A and the star $\mathcal{S}_B^*$ of planes through B, mapping the line AB onto a plane not containing the line AB. Without loss of generality we may assume that $A = \langle(0,0,0,1)\rangle$, $B = \langle(0,0,1,1)\rangle$ and that Φ maps the line through A and $\langle(y_1, y_1, y_3, 0)\rangle$ onto the plane through B with equation $b_1x_1 + b_2x_2 + b_3x_3 - b_3x_4 = 0$, where

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} {y_1}^q \\ {y_2}^q \\ {y_3}^q \end{pmatrix},$$

and (a_{ij}) is a non-singular matrix over $\mathsf{GF}(q^2)$.

Put $U_1 = \langle(0,0,0,1)\rangle$, $U_2 = \langle(0,1,0,0)\rangle$, $U_3 = \langle(0,0,1,0)\rangle$. We may assume that the line AU_1 is mapped onto the plane $x_1 = 0$, that the line AU_2 is mapped onto the plane $x_2 = 0$ and that the line AU_3 is mapped onto the plane $x_3 - x_4 = 0$. Under these assumptions it follows readily that $a_{21} = a_{31} = a_{12} = a_{32} = a_{13} = a_{23} = 0$. Hence $a_{11}a_{22}a_{33} \neq 0$. Let ℓ be a line through A and let $\langle(y_1, y_2, y_3, 0)\rangle)$ be a point of $\ell \setminus \{A\}$. If $y_3 = 0$ and $a_{11}y_1^{q+1} + a_{22}y_2^{q+1} = 0$, then the line ℓ is contained in the plane $\Phi(\ell)$. If either $y_3 \neq 0$ or $a_{11}y_1^{q+1} + a_{22}y_2^{q+1} \neq 0$, then $\ell \cap \Phi(\ell)$ is a point with homogeneous coordinates

$$(a_{33}y_3^q y_1,\ a_{33}y_3^q y_2,\ a_{33}y_3^{q+1},\ a_{11}y_1^{q+1} + a_{22}y_2^{q+1} + a_{33}y_3^{q+1}),$$

and this gives a parametric representation of the set $\mathcal{Q}$ of points of intersection of corresponding elements under Φ. Therefore the locus $\mathcal{Q}$ has an equation of the form

$$ax_1^{q+1} + bx_2^{q+1} + x_3^{q+1} - x_4x_3^q = 0,$$

where $a = \frac{a_{11}}{a_{33}} \neq 0$ and $b = \frac{a_{22}}{a_{33}} \neq 0$.

The line AB is mapped under Φ onto a plane π_B not through A. Moreover, the lines through A, mapped under Φ onto the planes of the pencil with axis AB, are the lines of a pencil $\mathcal{P}_A$ contained in a plane π_A not through B. So the α_F-collineation $\Phi_{\pi_A} : \ell \in \mathcal{P}_A \mapsto \Phi(\ell) \cap \pi_A \in \mathcal{P}_A$ can be defined. The lines through A contained in $\mathcal{Q}$ are exactly the lines fixed by Φ_{π_A}, so they number $0, 1, 2$ or $q+1$ and form in the last case a Baer subpencil of $\mathcal{P}_A$ (see [3]). Observe that the line $r = AU_1$ is not fixed under Φ_{π_A} and that a line t of $\mathcal{P}_A \setminus \{r\}$ through A and a point $\langle (y_1, y_2, 0, 0) \rangle$, is mapped onto the plane $ay_1^q x_1 + by_2^q x_2 = 0$. So t is fixed by Φ_{π_A} if and only if $(y_1 {y_2}^{-1})^{q+1} = -ba^{-1}$. This equation, in the unknown $y_1 {y_2}^{-1}$, has $q+1$ distinct roots over $\mathsf{GF}(q^2)$ if and only if $ba^{-1} \in \mathsf{GF}(q)$, otherwise the equation has no root over $\mathsf{GF}(q^2)$ (see e.g. [12, p. 102]). It follows that the set $\mathcal{Q}$ intersects the plane π_A either in a Baer subpencil of lines with vertex A or exactly in A, hence π_A is the tangent plane to $\mathcal{Q}$ at A. Similarly, the set $\mathcal{Q}$ intersects π_B either in a Baer subpencil of lines with vertex B or exactly in B, so π_B is the tangent plane to $\mathcal{Q}$ at B.

If $\mathcal{Q}$ intersects the tangent plane π_A (respectively the tangent plane π_B) in a Baer subpencil of lines with vertex A (respectively B), then $\mathcal{Q}$ is called a *semi-hyperbolic $\mathcal{Q}_F$-set*. If $\mathcal{Q}$ intersects the tangent plane π_A (respectively the tangent plane π_B) just in A (respectively just in B), then $\mathcal{Q}$ is called an *elliptic $\mathcal{Q}_F$-set*. The points A and B are called the *vertices* of $\mathcal{Q}$ and the line $\pi_A \cap \pi_B$ is called the *axis* of $\mathcal{Q}$. The set $\mathcal{Q}$ is then a semi-hyperbolic $\mathcal{Q}_F$-set if and only if $ba^{-1} \in \mathsf{GF}(q)$. The set $\mathcal{Q}$ is an elliptic $\mathcal{Q}_F$-set if and only if $ba^{-1} \notin \mathsf{GF}(q)$.

3 Semi-hyperbolic $\mathcal{Q}_F$-sets

Let $\mathcal{Q}$ be a semi-hyperbolic $\mathcal{Q}_F$-set, then $ba^{-1} \in \mathsf{GF}(q)$. It follows that there exists an element λ in $\mathsf{GF}(q)$ such that $b = a\lambda$, and then the equation of $\mathcal{Q}$ has the form $ax_1^{q+1} + a\lambda x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0$. Since λ belongs to $\mathsf{GF}(q)$, it follows that there exists an element $\rho \in \mathsf{GF}(q^2)$ such that $\rho^{q+1} = \lambda$. Hence, via the projectivity ${x_1}' = x_1, {x_2}' = \rho x_2, {x_3}' = x_3, {x_4}' = x_4$, the equation of the locus $\mathcal{Q}$ becomes $ax_1^{q+1} + ax_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0$. Assuming that the point $\langle (1, 0, 1, 0) \rangle$ belongs to $\mathcal{Q}$, the equation of this set has the form $-x_1^{q+1} - x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0$. Let δ be an element of $\mathsf{GF}(q^2)$ such that $\delta^{q+1} = -1$. Hence, via the projectivity ${x_1}' = \delta x_1, {x_2}' = \delta x_2, {x_3}' = x_3, {x_4}' = x_4$, the equation of a semi-hyperbolic $\mathcal{Q}_F$-set has the canonical form

$$x_1^{q+1} + x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0.$$

Proposition 3.1. *Let $\mathcal{Q}$ be a semi-hyperbolic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$ with vertices A and B. The set $\mathcal{Q}$ has $q^4 + q^3 + q^2 + 1$ points. Every line of $\mathsf{PG}(3, q^2)$ intersects $\mathcal{Q}$ in $0, 1, 2, q+1$ or $q^2 + 1$ points and every $(q+1)$-secant meets $\mathcal{Q}$ in a Baer subline. Moreover every line contained in $\mathcal{Q}$ contains either A or B.*

Proof. Every line of the pencil $\mathcal{P}_A$ either intersects $\mathcal{Q}$ exactly in A or it is one of the $q+1$ lines through A contained in $\mathcal{Q}$. Every line ℓ through A, not in π_A, intersects $\mathcal{Q}$ in two points, namely A and $\ell \cap \Phi(\ell)$. Similarly, every line through B either is contained in $\mathcal{Q}$ or intersects $\mathcal{Q}$ exactly in B or intersects $\mathcal{Q}$ in two distinct points. It follows that $\mathcal{Q}$ has $q^4+q^3+q^2+1$ points. Let ℓ be a line of $\mathsf{PG}(3,q^2)$ neither through A nor through B. If there exists a point R on ℓ such that $\ell \subseteq \Phi(AR)$, then the axis of the pencil of planes $\{\Phi(AP) : P \in \ell\}$ intersects ℓ in a point R' possibly coincident to R. It follows that for every point $P \in \ell$, distinct from R and distinct from R', the plane $\Phi(AP)$ cannot contain P, so $\mathcal{Q} \cap \ell = \{R, R'\}$. If the line ℓ is not contained in any plane of the pencil $\{\Phi(AP) : P \in \ell\}$, then Φ induces an α_F-collineation of the line ℓ into itself defined by

$$\phi_\ell \colon P \in \ell \mapsto \Phi(AP) \cap \ell \in \ell.$$

The points of the line ℓ which belong to $\mathcal{Q}$ are exactly all the fixed points of ϕ_ℓ. The system of fixed points of ϕ_ℓ is one of the following (see [3]): the empty set, a single point, a pair of distinct points or a subline formed by all the points of ℓ coordinatized over the subfield $\mathrm{Fix}(\alpha_F) = \{x \in \mathsf{GF}(q^2) : x^q = x\} = \mathsf{GF}(q)$, with respect to a suitable basis of ℓ. In the last case this set is a Baer subline of the line ℓ. From these arguments it follows that every line of $\mathsf{PG}(3,q^2)$, neither through A nor through B, intersects $\mathcal{Q}$ in $0, 1, 2$ or $q+1$ points. $\square$

Proposition 3.2. *Every semi-hyperbolic $\mathcal{Q}_F$-set is the union of $q-1$ non-degenerate Hermitian curves with two Baer subpencils of lines with vertices A and B, all with a common Baer subline, such that the planes containing the $q-1$ non-degenerate Hermitian curves, together with π_A and π_B, form a Baer subpencil of planes with axis the line $\pi_A \cap \pi_B$.*

Proof. Let $\mathcal{Q}$ be a semi-hyperbolic $\mathcal{Q}_F$-set. W.l.o.g. we may assume that $\mathcal{Q}$ has canonical equation $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0$. Let $\mathcal{F}$ be the set of all the planes of the pencil with axis $\pi_A \cap \pi_B$, different from π_A and from π_B. A plane of $\mathcal{F}$ has equation $x_4 = kx_3$, $k \in \mathsf{GF}(q^2) \setminus \{1\}$. The intersection of this plane with $\mathcal{Q}$ is a set $\mathcal{C}_k$ with equations $x_4 = kx_3$ and $x_1^{q+1} + x_2^{q+1} + (1-k)x_3^{q+1} = 0$. The set $\mathcal{C}_k$ is a non-degenerate Hermitian curve of the plane $x_4 = kx_3$ if and only if $1-k \in \mathsf{GF}(q)$ and so if and only if $k \in \mathsf{GF}(q)$. Therefore $\mathcal{Q}$ intersects $q-1$ planes of $\mathcal{F}$ in a non-degenerate Hermitian curve and intersects the planes π_A and π_B in two Baer subpencils of lines with vertices A and B, respectively. Observe that the set of the $q-1$ planes with equations $x_4 = kx_3$, $k \in \mathsf{GF}(q) \setminus \{1\}$, together with π_A and π_B, form a Baer subpencil of planes with axis the line $\pi_A \cap \pi_B$. Furthermore, it is clear that $\mathcal{Q} \cap \pi_A \cap \pi_B$ is a Baer subline contained in every non-degenerate Hermitian curve $\mathcal{C}_k$ and contained in each one of the two Baer subpencils of lines $\mathcal{Q} \cap \pi_A$ and $\mathcal{Q} \cap \pi_B$. Finally, the set $\mathcal{Q}$ contains the set

$$\mathcal{Q}' = \bigcup_{k \in \mathsf{GF}(q) \setminus \{1\}} \mathcal{C}_k \cup (\mathcal{Q} \cap \pi_A) \cup (\mathcal{Q} \cap \pi_B)$$

and since $|\mathcal{Q}'| = q^4 + q^3 + q^2 + 1 = |\mathcal{Q}|$, it follows that $\mathcal{Q} = \mathcal{Q}'$ as requested. □

We will need the following well known result on polynomials over $\mathsf{GF}(q^2)$.

Lemma 3.3. *The polynomial $x^q + x + 1$ has q roots over $\mathsf{GF}(q^2)$.*

Proof. Consider the map $f : x \in \mathsf{GF}(q^2) \mapsto x^q + x \in \mathsf{GF}(q)$. For any $y \in \mathsf{GF}(q)$ there exist at most q elements x of $\mathsf{GF}(q^2)$ such that $x^q + x = y$. From the cardinalities of $\mathsf{GF}(q^2)$ and $\mathsf{GF}(q)$, it follows that for any $y \in \mathsf{GF}(q)$ there exist exactly q elements of $\mathsf{GF}(q^2)$ which are mapped onto y under f. So the equation $x^q + x = -1$ has q roots over $\mathsf{GF}(q^2)$. □

Proposition 3.4. *Every semi-hyperbolic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$ is contained in a non-degenerate Hermitian surface.*

Proof. Let $\mathcal{Q}$ be a semi-hyperbolic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$. Without loss of generality we may assume that $\mathcal{Q}$ has canonical equation $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0$. By Lemma 3.3 there exists an element σ of $\mathsf{GF}(q^2)$ satisfying the condition $\sigma^q + \sigma + 1 = 0$. It follows that $\mathcal{Q}$ is contained in the non-degenerate Hermitian surface with equation $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + \sigma x_3 x_4^q + \sigma^q x_4 x_3^q = 0$. □

Proposition 3.5. *Every semi-hyperbolic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$ is the intersection of a non-degenerate Hermitian surface with a Baer subpencil of planes.*

Proof. Let $\mathcal{Q}$ be a semi-hyperbolic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$. By Propositions 3.2 and 3.4, $\mathcal{Q}$ is a set of $q^4 + q^3 + q^2 + 1$ points contained in the intersection of a non-degenerate Hermitian surface $\mathcal{H}$ with a Baer subpencil of planes $\mathcal{P}$. From [6] it is known that there exists only one configuration for the intersection of $\mathcal{H}$ and $\mathcal{P}$ with more than $q^4 + q^3 + q^2 + 1$ points. Such a configuration is the union of $q + 1$ Baer subpencils of lines with a common Baer subline such that the vertices of these pencils form a Baer subline. Since this configuration cannot contain a semi-hyperbolic $\mathcal{Q}_F$-set, it follows that $\mathcal{H} \cap \mathcal{P} = \mathcal{Q}$. □

Proposition 3.6. *Every plane of $\mathsf{PG}(3, q^2)$ intersects a semi-hyperbolic $\mathcal{Q}_F$-set in one of the following: a non-degenerate Hermitian curve, a Baer subpencil of lines, a pair of distinct lines, a $\mathcal{C}_F$-set, a degenerate $\mathcal{C}_F$-set, a K-set, an H-set, a Γ-set, a Baer subline.*

Proof. Let π be a plane of $\mathsf{PG}(3, q^2)$ and let $\mathcal{Q}$ be a semi-hyperbolic $\mathcal{Q}_F$-set with vertices A and B. By Proposition 3.5 $\mathcal{Q}$ is the intersection of a non-degenerate Hermitian surface $\mathcal{H}$ with a Baer subpencil of planes $\mathcal{P}$ with axis the line $\pi_A \cap \pi_B$. If π is a plane of $\mathcal{P}$, then π intersects $\mathcal{Q}$ either in a non-degenerate Hermitian curve or in a Baer subpencil of lines with vertex A or B. If π belongs to the pencil of planes with axis $\pi_A \cap \pi_B$ and does not belong to $\mathcal{P}$, then $\pi \cap \mathcal{Q}$ is a Baer subline of the line $\pi_A \cap \pi_B$. If π is a plane not containing the line $\pi_A \cap \pi_B$ then $\pi \cap \mathcal{Q} = \mathcal{P} \cap \pi \cap \mathcal{H}$ and so $\pi \cap \mathcal{Q}$ is the intersection between the Hermitian curve $\pi \cap \mathcal{H}$ (possibly degenerate of rank 2) with the Baer subpencil of lines $\pi \cap \mathcal{P}$ (degenerate Hermitian curve of rank 2). In [6] the intersection between two distinct, possibly degenerate, Hermitian curves has been studied and it has been proved that, if $\pi \cap \mathcal{H} \neq \pi \cap \mathcal{P}$, then $\pi \cap \mathcal{Q}$ is one of the following: a $\mathcal{C}_F$-set, a degenerate $\mathcal{C}_F$-set, an H-set, a K-set, a Γ-set, a pair of distinct lines. Finally, if $\pi \cap \mathcal{H} = \pi \cap \mathcal{P}$ then $\pi \cap \mathcal{Q}$ is a Baer subpencil of lines, hence either $\pi = \pi_A$ or $\pi = \pi_B$, which is not possible in this case. □

Proposition 3.7. *Let $\mathcal{H}$ be a non-degenerate Hermitian curve of a plane π of $\mathsf{PG}(3, q^2)$ and let A, B be two points not on π such that the point $AB \cap \pi$ is not on $\mathcal{H}$. Then there exists a unique semi-hyperbolic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$, with vertices A and B, that meets π in the Hermitian curve $\mathcal{H}$.*

Proof. Let u be the polarity associated with $\mathcal{H}$. The α_F-collineation

$$\Phi\colon \ell \in \mathcal{S}_A \mapsto \langle u(\ell \cap \pi), B\rangle \in \mathcal{S}_B^*,$$

maps the line AB onto a plane not through AB. Let $P = \pi \cap AB$, every line joining A with a point of $u(P) \cap \mathcal{H}$ is contained in the corresponding plane under Φ. It follows that Φ generates a semi-hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ of $\mathsf{PG}(3, q^2)$, with vertices A and B and axis $u(AB \cap \pi)$, that meets π in $\mathcal{H}$. In order to prove the uniqueness observe that there is a bijection Ψ between the set of the α_F-correlations of the plane π and the set of the α_F-collineations between $\mathcal{S}_A$ and $\mathcal{S}_B^*$. Indeed, Ψ maps the α_F-correlation v of π onto the α_F-collineation Φ_v defined by $\Phi_v\colon \ell \in \mathcal{S}_A \mapsto \langle v(\ell \cap \pi), B\rangle \in \mathcal{S}_B^*$. Hence Φ_u is the unique α_F-collineation between $\mathcal{S}_A$ and $\mathcal{S}_B^*$ such that every point of $\mathcal{H}$ is a point of intersection of corresponding elements. Therefore the semi-hyperbolic $\mathcal{Q}_F$-set with vertices A and B defined by Φ_u is the unique semi-hyperbolic $\mathcal{Q}_F$-set that intersects π exactly in $\mathcal{H}$. □

3.1 Representation of semi-hyperbolic $\mathcal{Q}_F$-sets in $\mathsf{PG}(6, q)$

We start with the Barlotti–Cofman representation of Baer subpencils of planes of $\mathsf{PG}(3, q^2)$.

Lemma 3.8. *Every Baer subpencil of planes of* $\mathsf{PG}(3,q^2)$ *containing* π_∞ *is represented by a hyperplane* H *of* $\mathsf{PG}(6,q)$ *different from* Σ_∞*. Conversely, every hyperplane of* $\mathsf{PG}(6,q)$*, different from* Σ_∞*, represents a Baer subpencil of planes containing* π_∞*.*

Proof. Let $\mathcal{P}$ be a Baer subpencil of planes of $\mathsf{PG}(3,q^2)$ containing π_∞ with axis ℓ and let m be a line skew to ℓ. Then $\mathcal{P}\cap m$ is a Baer subline m_0 which corresponds in $\mathsf{PG}(6,q)$ to a line m_0^* not contained in Σ_∞. The axis ℓ corresponds to a spread ℓ^* induced by $\mathcal{S}$ on a three-dimensional subspace S_ℓ contained in Σ_∞. Therefore $\mathcal{P}$ corresponds to the hyperplane (different from Σ_∞) spanned by m_0^* and S_ℓ. Conversely let H be a hyperplane of $\mathsf{PG}(6,q)$ different from Σ_∞. There exists a unique three-dimensional subspace S_ℓ of $H\cap\Sigma_\infty$ such that $\mathcal{S}$ induces a spread ℓ^* on S_ℓ. A line contained in H, skew to S_ℓ, represents a Baer subline m_0 contained in a line skew to ℓ. Hence H represents the Baer subpencil of planes $\{\langle \ell, P\rangle : P\in m_0\}$ with axis ℓ. □

Now we are able to prove that semi-hyperbolic $\mathcal{Q}_F$-sets of $\mathsf{PG}(3,q^2)$ correspond, under the Barlotti–Cofman representation, to hyperbolic quadrics contained in hyperplanes of $\mathsf{PG}(6,q)$, and viceversa.

Proposition 3.9. *Let* $\mathcal{Q}$ *be a semi-hyperbolic* $\mathcal{Q}_F$*-set of* $\mathsf{PG}(3,q^2)$*, with axis* ℓ *and let* π_∞ *be a plane such that* $\mathcal{Q}\cap\pi_\infty$ *is a non-degenerate Hermitian curve. Then in the Barlotti–Cofman representation with* π_∞ *as plane at infinity,* $\mathcal{Q}^*$ *is a hyperbolic quadric contained in a hyperplane of* $\mathsf{PG}(6,q)$*, meeting* Σ_∞ *in an* R*-quadric.*

Proof. From Proposition 3.5, $\mathcal{Q}$ is the intersection of a Baer subpencil of planes $\mathcal{P}$ with a non-degenerate Hermitian surface $\mathcal{H}$. Hence in $\mathsf{PG}(6,q)$ we have $\mathcal{Q}^* = \mathcal{P}^*\cap\mathcal{H}^*$. Since $\mathcal{H}\cap\pi_\infty = \mathcal{Q}\cap\pi_\infty$ is a non-degenerate Hermitian curve, it follows that $\mathcal{H}^*$ is a non-degenerate quadric $Q(6,q)$ of $\mathsf{PG}(6,q)$ (see e.g. [11]). From Lemma 3.8, $\mathcal{P}^*$ is a hyperplane of $\mathsf{PG}(6,q)$ different from Σ_∞. Hence $\mathcal{Q}^*$ is a quadric of $\mathcal{P}^*$ and since $|\mathcal{Q}^*| = (q+1)^2+(q^3-q)+(q^4+q^2) = q^4+q^3+2q^2+q+1$ it follows that $\mathcal{Q}^*$ is a hyperbolic quadric of $\mathcal{P}^*$. Finally, since $\mathcal{Q}\cap\pi_\infty$ is a Hermitian curve of π_∞ intersecting ℓ in a Baer subline ℓ_0, it follows that $\mathcal{Q}^*$ meets Σ_∞ in an R-quadric containing the regulus ℓ_0^* contained in $\mathcal{S}$. □

Proposition 3.10. *Every hyperbolic quadric contained in a hyperplane of* $\mathsf{PG}(6,q)$*, meeting* Σ_∞ *in an* R*-quadric, represents a semi-hyperbolic* $\mathcal{Q}_F$*-set.*

Proof. Let $Q^+(5,q)$ be a hyperbolic quadric contained in a hyperplane H of $\mathsf{PG}(6,q)$ meeting Σ_∞ in an R-quadric $Q(4,q)$. The hyperplane H represents a Baer subpencil of planes $\mathcal{P}$ with axis a line ℓ and $Q(4,q)$ represents a Hermitian curve $\mathcal{H}$ of the plane π_∞ (see [2]). Let $\mathcal{R}$ be the regulus contained in $Q(4,q)$ and contained in $\mathcal{S}$. The lines of $\mathcal{S}$ contained in the three-dimensional subspace S_ℓ

spanned by the lines of $\mathcal{R}$, represent all the points of ℓ. There exist exactly two four-dimensional subspaces of H containing S_ℓ which are tangent hyperplanes to $Q^+(5,q)$ at points say A^* and B^*. The line A^*B^* intersects Σ_∞ in a point not on $Q(4,q)$, so the line AB intersects π_∞ in a point not on $\mathcal{H}$. From Proposition 3.7 there exists a unique semi-hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ of $\mathsf{PG}(3,q^2)$, with vertices A and B, that meets π_∞ in the Hermitian curve $\mathcal{H}$. The set $\mathcal{Q}$ is represented by a hyperbolic quadric $\mathcal{Q}^*$ contained in the hyperplane $\langle Q(4,q), A^*, B^*\rangle = H$, meeting Σ_∞ in the R-quadric $Q(4,q)$ (see Proposition 3.9). Let m and m' be two lines of $\mathcal{R}$. The quadrics $Q^+(5,q)$ and $\mathcal{Q}^*$ both contain the quadric $Q(4,q)$ and the planes $\langle A^*, m\rangle$ and $\langle B^*, m'\rangle$. This gives 20 independent linear conditions satisfied by the equations of both quadrics. It follows that $Q^+(5,q) = \mathcal{Q}^*$. $\square$

3.2 Semi-hyperbolic $\mathcal{Q}_F$-sets and Hermitian surfaces

In Proposition 3.4 we proved that every semi-hyperbolic $\mathcal{Q}_F$-set is contained in a Hermitian surface. In this section we characterize a semi-hyperbolic $\mathcal{Q}_F$-set as the intersection of two Hermitian surfaces.

Let $\mathcal{H}_3$ be a non-degenerate Hermitian surface of $\mathsf{PG}(3,q^2)$, with associated polarity u_3 and let ℓ be a $(q+1)$-secant line to $\mathcal{H}_3$ meeting $\mathcal{H}_3$ in a Baer subline ℓ_0. Let π_∞ be a plane through ℓ, meeting $\mathcal{H}_3$ in a non-degenerate Hermitian curve $\mathcal{H}_2$ with associated polarity u_2, that we consider as the plane at infinity in the Barlotti–Cofman representation. Moreover let π_A and π_B be two tangent planes to $\mathcal{H}_3$ at points A and B (respectively) containing ℓ.

Proposition 3.11. *The intersection of the Hermitian surface $\mathcal{H}_3$ with the Baer subpencil of planes containing π_∞, π_A and π_B is the unique semi-hyperbolic $\mathcal{Q}_F$-set with vertices A and B containing the non degenerate Hermitian curve $\mathcal{H}_2$.*

Proof. Let $\mathcal{P}$ be the Baer subpencil of planes with axis ℓ containing π_∞, π_A and π_B. Observe that if $\mathcal{P}$ would contain three tangent planes to $\mathcal{H}_3$, then it would be coincident with the Baer subpencil of planes formed by the tangent planes to $\mathcal{H}_3$ through ℓ. It follows that every plane of $\mathcal{P}$ different from π_A and π_B intersects $\mathcal{H}_3$ in a non-degenerate Hermitian curve. Hence $|\mathcal{H}_3 \cap \mathcal{P}| = q^4+q^3+q^2+1$ and $\mathcal{H}_3^* \cap \mathcal{P}^*$ is a hyperbolic quadric of the hyperplane $\mathcal{P}^*$ meeting Σ_∞ in the R-quadric $\mathcal{H}_2^*$ (see Proof of Proposition 3.9). By Proposition 3.10 $\mathcal{H}_3^* \cap \mathcal{P}^*$ represents a semi-hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ of $\mathsf{PG}(3,q^2)$. A^* and B^* are the points of intersection of $\mathcal{H}_3^* \cap \mathcal{P}^*$ with the two tangent hyperplanes to $\mathcal{H}_3^* \cap \mathcal{P}^*$ through the subspace S_ℓ containing the lines of ℓ^*. Such points are the Barlotti–Cofman representation of the vertices of $\mathcal{Q}$ (see Proof of Proposition 3.10). Therefore $\mathcal{Q}$ is a semi-hyperbolic $\mathcal{Q}_F$-set with vertices A and B containing the non-degenerate Hermitian curve $\mathcal{H}_2$. From Proposition 3.7 the uniqueness of $\mathcal{Q}$ follows. $\square$

Proposition 3.12. *Let $\mathcal{H}_3$ and $\mathcal{H}'_3$ be two distinct non-degenerate Hermitian surfaces of $\mathsf{PG}(3, q^2)$, with associated polarities u_3 and u'_3, respectively. Let A, B be two distinct points of $\mathcal{H}_3 \cap \mathcal{H}'_3$ such that $B \notin u_3(A)$ and $B \notin u'_3(A)$. Then $\mathcal{H}_3 \cap \mathcal{H}'_3$ is a semi-hyperbolic $\mathcal{Q}_F$-set with vertices A and B if and only if the following conditions hold:*

(1) *u_3 and u'_3 agree on the points A and B.*

(2) *u_3 and u'_3 induce the same unitary polarity on a plane π_∞ containing the line $\ell = u_3(A) \cap u_3(B)$.*

Proof. Suppose that $\mathcal{H}_3 \cap \mathcal{H}'_3$ is a semi-hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ with vertices A and B. Since the two tangent planes π_A and π_B to $\mathcal{Q}$ at A and B (respectively) intersect $\mathcal{Q}$ in two Baer subpencils of planes with vertices A and B (respectively), it follows that $u_3(A) = u'_3(A) = \pi_A$ and $u_3(B) = u'_3(B) = \pi_B$. The axis of $\mathcal{Q}$ is the line $\ell = \pi_A \cap \pi_B$, hence $\mathcal{Q}$ contains $q-1$ non-degenerate Hermitian curves contained in $q-1$ planes through ℓ. Let $\mathcal{H}_2$ be one of such Hermitian curves, with associated polarity u_2, contained in a plane π_∞. Since $\mathcal{H}_2 \subseteq \mathcal{Q} = \mathcal{H}_3 \cap \mathcal{H}'_3$, it follows that u_3 and u'_3 induce on the plane π_∞ the polarity u_2. Conversely, from conditions (1) and (2) it follows that there exists a non-degenerate Hermitian curve $\mathcal{H}_2$ of a plane π_∞, with associated polarity u_2, contained in $\mathcal{H}_3 \cap \mathcal{H}'_3$ such that the line $\ell = u_3(A) \cap u_3(B)$ is contained in π_∞. Since $B \notin u_3(A)$ and $B \notin u'_3(A)$, the line AB is a $(q+1)$-secant to both $\mathcal{H}_3$ and $\mathcal{H}'_3$, hence $\ell = u_3(AB)$ is also a $(q+1)$-secant line to both $\mathcal{H}_3$ and $\mathcal{H}'_3$. Let $\mathcal{P}$ be the Baer subpencil of planes with axis ℓ containing the planes π_∞, $\pi_A = u_3(A)$ and $\pi_B = u_3(B)$. From Proposition 3.11 it follows that $\mathcal{H}_3 \cap \mathcal{P}$ is the unique semi-hyperbolic $\mathcal{Q}_F$-set $\mathcal{Q}$ with vertices A and B containing $\mathcal{H}_2$. In a similar way it can be shown that $\mathcal{H}'_3 \cap \mathcal{P} = \mathcal{Q}$, and so $\mathcal{Q} \subseteq \mathcal{H}_3 \cap \mathcal{H}'_3$. From [6] or [7] it is known that there exists only one configuration of the intersection of $\mathcal{H}_3$ and $\mathcal{H}'_3$ with more than $q^4 + q^3 + q^2 + 1$ points. This configuration has exactly $q^4 + 2q^3 + 1$ points and it is formed by the union of $q+1$ Baer subpencils of lines with a common Baer subline such that the vertices of these pencils form a Baer subline. Since such a configuration cannot contain a semi-hyperbolic $\mathcal{Q}_F$-set, it follows that $\mathcal{H}_3 \cap \mathcal{H}'_3 = \mathcal{Q}$, as requested. □

4 Elliptic $\mathcal{Q}_F$-sets

Let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set. Assume that the point $\langle(0,1,1,0)\rangle$ belongs to $\mathcal{Q}$, it follows that $b = -1$ hence $\mathcal{Q}$ has equation $ax_1^{q+1} - x_2^{q+1} + x_3^{q+1} - x_4x_3^q = 0$ and assuming that the point $\langle(1,1,1,\xi)\rangle$, ξ a primitive element of $\mathsf{GF}(q^2)$, belongs to $\mathcal{Q}$, it follows that $\mathcal{Q}$ has equation $\xi x_1^{q+1} - x_2^{q+1} + x_3^{q+1} - x_4x_3^q = 0$. Let δ be an element of $\mathsf{GF}(q^2)$ such that $\delta^{q+1} = -1$. Hence, via the projectivity

$x_1{}' = \delta x_1, x_2{}' = \delta x_2, x_3{}' = x_3, x_4{}' = x_4$, we may assume that the canonical form of the equation of an elliptic $\mathcal{Q}_F$-set is as follows:

$$\xi x_1^{q+1} + x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0.$$

Proposition 4.1. *Let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$ with vertices A and B. The set $\mathcal{Q}$ has $q^4 + 1$ points. Every line of $\mathsf{PG}(3, q^2)$ intersects $\mathcal{Q}$ in $0, 1, 2$ or $q + 1$ points and every $(q + 1)$-secant meets $\mathcal{Q}$ in a Baer subline.*

Proof. Every line of the pencil $\mathcal{P}_A$ intersects $\mathcal{Q}$ exactly in A. Every line ℓ through A, not in π_A, intersects $\mathcal{Q}$ in two points, namely A and $\ell \cap \Phi(\ell)$. Similarly, every line through B either intersects $\mathcal{Q}$ exactly in B or intersects $\mathcal{Q}$ in two distinct points. It follows that $\mathcal{Q}$ has $q^4 + 1$ points. Using the same proof as in Proposition 3.1 it follows that every line of $\mathsf{PG}(3, q^2)$, neither through A nor through B, intersects $\mathcal{Q}$ in $0, 1, 2$ or $q + 1$ points. □

Proposition 4.2. *Every elliptic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$ is contained in a non-degenerate Hermitian surface.*

Proof. Let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$. Without loss of generality we may assume that $\mathcal{Q}$ has canonical equation

$$\xi x_1^{q+1} + x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0.$$

By Lemma 3.3 there exists an element σ of $\mathsf{GF}(q^2)$ satisfying the condition $\sigma^q + \sigma + 1 = 0$. It follows that $\mathcal{Q}$ is contained in the Hermitian surface with equation

$$-(\xi\sigma + \xi^q\sigma^q)x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + \sigma x_3^q x_4 + \sigma^q x_3 x_4^q = 0.$$

Observe that for $\sigma \neq \frac{\xi^{q-1}}{1-\xi^{q-1}}$, the previous Hermitian surface is non-degenerate. □

Proposition 4.3. *Every elliptic $\mathcal{Q}_F$-set is the union of $q^3 - q^2$ Baer sublines contained in $q^3 - q^2$ lines through a common point V on no Baer subline with $q^2 + 1$ points forming a $\mathcal{C}_F$-set on a plane not containing V.*

Proof. Let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$. From the proof of previous proposition we have that $\mathcal{Q}$ is contained in the Hermitian cone Γ with equation

$$x_2^{q+1} + x_3^{q+1} + \frac{\xi^{q-1}}{1 - \xi^{q-1}} x_3^q x_4 + \frac{1}{\xi^{q-1} - 1} x_3 x_4^q = 0$$

with vertex the point $V = \langle (1, 0, 0, 0) \rangle$ and in a non-degenerate Hermitian surface $\mathcal{H}$ not containing V. It follows that $\mathcal{Q}$ is contained in the base $\mathcal{B}$ of the

Hermitian pencil $\mathcal{F}$ generated by Γ and $\mathcal{H}$. We will prove that $\mathcal{Q} = \mathcal{B}$. Let u be the polarity associated with $\mathcal{H}$. The plane $\pi = u(V)$ has equation $x_1 = 0$ and $\mathcal{B} \cap \pi$ contains the set $\mathcal{C}$ with equations $x_2^{q+1} + x_3^{q+1} - x_4 x_3^q = 0, x_1 = 0$. Via the projectivity $x_1' = x_1, x_2' = x_2, x_3' = x_3, x_4' = x_3 - x_4$ the equations of the set $\mathcal{C}$ become $x_2^{q+1} - x_4 x_3^q = 0, x_1 = 0$ and hence $\mathcal{C}$ is a $\mathcal{C}_F$-set. From [6, Theorem 2.2] we have that $\mathcal{B} \cap \pi$ is either a $\mathcal{C}_F$-set or a K-set or an H-set. Since H-sets and K-sets cannot contain a $\mathcal{C}_F$-set it follows that $\mathcal{B} \cap \pi$ is a $\mathcal{C}_F$-set. From [6, Theorem 3.1] we have that $|\mathcal{B}| = |\mathcal{Q}| = q^4 + 1$, hence $\mathcal{B} = \mathcal{Q}$ and $\mathcal{Q}$ is the union of $q^3 - q^2$ Baer sublines contained in $q^3 - q^2$ lines through a common point V on no Baer subline with $q^2 + 1$ points forming a $\mathcal{C}_F$-set on a plane not containing V. □

Proposition 4.4. *Every elliptic $\mathcal{Q}_F$-set of* $\mathsf{PG}(3, q^2)$ *is the intersection of two non-degenerate Hermitian surfaces.*

Proof. Let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set of $\mathsf{PG}(3, q^2)$. From the proof of the previous proposition $\mathcal{Q}$ is the base of an Hermitian pencil $\mathcal{F}$ generated by a Hermitian cone Γ and a non-degenerate Hermitian surface $\mathcal{H}$. Let $\mathcal{H}'$ be a non-degenerate Hermitian surface, different from $\mathcal{H}$. It follows that $\mathcal{Q} = \mathcal{H} \cap \mathcal{H}'$. □

Proposition 4.5. *Every plane through A (resp. B), different from π_A (resp. π_B) intersects an elliptic $\mathcal{Q}_F$-set in a $\mathcal{C}_F$-set.*

Proof. Let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set with vertices A and B generated by an α_F-collineation Φ between $\mathcal{S}_A$ and $\mathcal{S}_B^*$. Let π be a plane on A, different from π_A. The lines on A contained in π correspond, under Φ, to planes of a pencil with axis a line t, different from the line AB. Let $B' = \pi \cap t$; the collineation Φ induces an α_F-collineation ϕ_π mapping any line ℓ on A contained in π onto the line $\pi \cap \Phi(\ell)$ on B'. The points of $\pi \cap \mathcal{Q}$ are given by the points of intersection of corresponding lines under ϕ_π. Since the line AB' is not mapped onto itself it follows that $\pi \cap \mathcal{Q}$ is a $\mathcal{C}_F$-set with vertices A and B'. □

Proposition 4.6. *Every plane of* $\mathsf{PG}(3, q^2)$ *intersects an elliptic $\mathcal{Q}_F$-set in one of the following: a point, a Baer subline, a $\mathcal{C}_F$-set, an H-set, a K-set, a Γ-set, a complete $(q^2 - q + 1)$-arc.*

Proof. Let π be a plane of $\mathsf{PG}(3, q^2)$ and let $\mathcal{Q}$ be an elliptic $\mathcal{Q}_F$-set with vertices A and B, generated by an α_F-collineation $\Phi\colon \mathcal{S}_A \to \mathcal{S}_B^*$. By Proposition 4.4, the set $\mathcal{Q}$ is the intersection of two distinct non-degenerate Hermitian surfaces $\mathcal{H}$ and $\mathcal{H}'$. Let $\mathcal{C} = \mathcal{H} \cap \pi$ and let $\mathcal{C}' = \mathcal{H}' \cap \pi$. If $\mathcal{C}$ and $\mathcal{C}'$ are degenerate Hermitian curves of rank 2, then $\mathcal{C} \neq \mathcal{C}'$ since $\mathcal{Q}$ does not contain lines. If $\mathcal{C}$ and $\mathcal{C}'$ are non-degenerate Hermitian curves, then also $\mathcal{C} \neq \mathcal{C}'$. Indeed, if $\mathcal{C} = \mathcal{C}'$ then π contains neither A nor B (see Proposition 4.5). Let u be the polarity associated to $\mathcal{C}$, from the proof of Proposition 3.7 it follows that Φ is uniquely determined by u. If $\pi \cap AB$ is a point not on $\mathcal{C}$, then Φ defines a semi-hyperbolic $\mathcal{Q}_F$-set, a contradiction. If $\pi \cap AB$ is a point on $\mathcal{C}$, then Φ defines a set containing the line AB, a contradiction. Hence $\mathcal{C} \neq \mathcal{C}'$ and from [6], where the intersection between two distinct, possibly degenerate, Hermitian curves has been studied, it follows that $\pi \cap \mathcal{Q}$ is one of the following: a point, a Baer subline, a $\mathcal{C}_F$-set, an H-set, a K-set, a Γ-set, a complete $(q^2 - q + 1)$-arc. □

5 Absolute points of an α_F-correlation of $\mathsf{PG}(2, q^2)$

B. C. Kestenband proved in [9] that the correlations of $\mathsf{PG}(2, q^{2n})$ defined by diagonal matrices with companion automorphism $\alpha \colon x \in \mathsf{GF}(q^{2n}) \mapsto x^{q^m} \in \mathsf{GF}(q^{2n})$, where $(m, 2n) = 1$, have the following numbers of absolute points:

$$q^{2n} + q^{n+2} - q^{n+1} + 1 \text{ or } q^{2n} - q^{n+1} + q^n + 1 \text{ or } (q^n + 1)^2 \text{ for } n \text{ odd;}$$
$$q^{2n} - q^{n+2} + q^{n+1} + 1 \text{ or } q^{2n} + q^{n+1} - q^n + 1 \text{ or } (q^n - 1)^2 \text{ for } n \text{ even.}$$

Moreover some properties regarding the configurations of the absolute sets of these correlations are given.

In this section we will determine, independently from Kestenband's results, all the possible configurations for the absolute set of an α_F-correlation of the plane $\mathsf{PG}(2, q^2)$, where α_F is the involutory automorphism of $\mathsf{GF}(q^2)$.

Proposition 5.1. *Let v be an α_F-correlation of $\mathsf{PG}(2, q^2)$. The set of absolute points of v is one of the following: a point, a Baer subline, a complete $(q^2 - q + 1)$-arc, a $\mathcal{C}_F$-set, a Γ-set, a K-set, an H-set, a non-degenerate Hermitian curve.*

Proof. We may assume that the correlation v is defined on a plane π embedded in a projective space $\mathsf{PG}(3, q^2)$. Let $\mathcal{A}$ be the set of absolute points of v in π. It is well known that null systems exist only in odd dimensional projective spaces, hence $\mathcal{A} \neq \pi$. Let P be a point of $\pi \setminus \mathcal{A}$ and let A and B be two distinct points of $\mathsf{PG}(3, q^2) \setminus \pi$ such that $AB \cap \pi = P$. The correlation v induces a unique α_F-collineation Φ between $\mathcal{S}_A$ and $\mathcal{S}_B^*$ (see the proof of Proposition 3.7). Let $\mathcal{Q}$ be the set of points of intersections of corresponding elements under Φ. Then $\mathcal{A} = \mathcal{Q} \cap \pi$. The set $\mathcal{Q}$ is either a semi-hyperbolic $\mathcal{Q}_F$-set or an elliptic $\mathcal{Q}_F$-set and since A and B do not belong to π, the set $\pi \cap \mathcal{Q}$ does not contain lines. From Proposition 3.6 and Proposition 4.6 the assertion follows. □

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